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CONSTRUCTIBLE CHARACTERS AND $b$-INVARIANT

by

CÉDRIC BONNAFÉ

Abstract. — If $W$ is a finite Coxeter group and $\varphi$ is a weight function, Lusztig has defined $\varphi$-constructible characters of $W$, as well as a partition of the set of irreducible characters of $W$ into Lusztig $\varphi$-families. We prove that every Lusztig $\varphi$-family contains a unique character with minimal $b$-invariant, and that every $\varphi$-constructible character has a unique irreducible constituent with minimal $b$-invariant. This generalizes Lusztig’s result about special characters to the case where $\varphi$ is not constant. This is compatible with some conjectures of Rouquier and the author about Calogero-Moser families and Calogero-Moser cellular characters.

Let $(W, S)$ be a finite Coxeter system and let $\varphi : S \to \mathbb{R}_{\geq 0}$ be a weight function that is, a map such that $\varphi(s) = \varphi(t)$ whenever $s$ and $t$ are conjugate in $W$. Associated with this datum, G. Lusztig has defined [Lu3, §22] a notion of constructible characters of $W$: it is conjectured that a character is constructible if and only if it is the character afforded by a Kazhdan-Lusztig left cell (defined using the weight function $\varphi$). These constructible characters depend heavily on $\varphi$ so we will call them the $\varphi$-constructible characters of $W$: the set of $\varphi$-constructible characters will be denoted by $\text{Cons}^\text{lus}_\varphi(W)$. We will also define a graph $\mathcal{G}_{\text{lus}, \varphi}$ as follows: the vertices of $\mathcal{G}_{\text{lus}, \varphi}$ are the irreducible characters and two irreducible characters $\chi$ and $\chi'$ are joined in this graph if there exists a $\varphi$-constructible character $\gamma$ of $W$ such that $\chi$ and $\chi'$ both occur as constituents of $\gamma$. The connected components of $\mathcal{G}_{\text{lus}, \varphi}$ (viewed as subsets of $\text{Irr}(W)$) will be called the Lusztig $\varphi$-families: the set of Lusztig $\varphi$-families will be denoted by $\text{Fam}^\text{lus}_\varphi(W)$. If $\mathcal{F} \in \text{Fam}^\text{lus}_\varphi(W)$, we denote by $\text{Cons}^\text{lus}_\varphi(\mathcal{F})$ the set of $\varphi$-constructible characters of $W$ all of whose irreducible components belong to $\mathcal{F}$.

On the other hand, using the theory of rational Cherednik algebras at $t = 0$ and the geometry of the Calogero-Moser space associated with $(W, \varphi)$, R. Rouquier and
the author (see [BoRo1] and [BoRo2]) have defined a notion of Calogero-Moser $\varphi$-cells of $W$, a notion of Calogero-Moser $\varphi$-cellular characters of $W$ (whose set is denoted by $\text{Cell}_{\varphi}^\text{CM}(W)$) and a notion of Calogero-Moser $\varphi$-families (whose set is denoted by $\text{Fam}_{\varphi}^\text{CM}(W)$).

**Conjecture (see [BoRo1], [BoRo2] and [GoMa]).** With the above notation,

$$\text{Cons}_{\varphi}^\text{Lus}(W) = \text{Cell}_{\varphi}^\text{CM}(W) \quad \text{and} \quad \text{Fam}_{\varphi}^\text{Lus}(W) = \text{Fam}_{\varphi}^\text{CM}(W)$$

for every weight function $\varphi : S \to \mathbb{R}_{>0}$.

The statement about families in this conjecture holds for classical Weyl groups thanks to a case-by-case analysis relying on [Lu3, §22] (for the computation of Lusztig $\varphi$-families), [GoMa] (for the computation of Calogero-Moser $\varphi$-families in type $A$ and $B$) and [Be2] (for the computation of the Calogero-Moser $\varphi$-families in type $D$). It also holds whenever $|S| = 2$ (see [Lu3, §17 and Lemma 22.2] and [Be1, §6.10]).

The statement about constructible characters is much more difficult to establish, as the computation of Calogero-Moser $\varphi$-cellular characters is at that time out of reach. It has been proved whenever the Calogero-Moser space associated with $(W, S, \varphi)$ is smooth [BoRo2, Theorem 14.4.1] (this includes the cases where $(W, S)$ is of type $A$, or of type $B$ for a large family of weight functions: in all these cases, the $\varphi$-constructible characters are the irreducible ones). It has also been checked by the author whenever $|S| = 2$ or $(W, S)$ is of type $B_3$ (unpublished).

Our aim in this paper is to show that this conjecture is compatible with properties of the $b$-invariant (as defined below). With each irreducible character $\chi$ of $W$ is associated its fake degree $f_\chi(t)$, using the invariant theory of $W$ (see for instance [BoRo2, Definition 1.5.7]). Let us denote by $b_\chi$ the valuation of $f_\chi(t)$: $b_\chi$ is called the $b$-invariant of $\chi$. Let $r_\chi$ denote the coefficient of $t^{b_\chi}$ in $f_\chi(t)$. In other words,

$$r_\chi \in \mathbb{N}^* \quad \text{and} \quad f_\chi(t) \equiv r_\chi t^{b_\chi} \mod t^{b_\chi+1}.$$

For instance, $b_1 = 0$ and $b_\varnothing$ is the number of reflections of $W$ (here, $\varnothing : W \to \{1, -1\}$ denotes the sign character). Also, $b_\chi = 1$ if and only if $\chi$ is an irreducible constituent of the canonical reflection representation of $W$. The following result is proved in [BoRo2, Theorems 9.6.1 and 12.3.14]:

**Theorem CM.** Let $\varphi : S \to \mathbb{R}_{>0}$ be a weight function. Then:

(a) If $\mathcal{F} \in \text{Fam}_{\varphi}^\text{CM}(W)$, then there exists a unique $\chi_{\mathcal{F}} \in \mathcal{F}$ with minimal $b$-invariant. Moreover, $r_{\chi_{\mathcal{F}}} = 1$.  

Constructible characters and $b$-invariants

(b) If $\gamma \in \text{Cell}_\varphi^{CM}(W)$, then there exists a unique irreducible constituent $\chi_\gamma$ of $\gamma$ with minimal $b$-invariant. Moreover, $r_{\chi_\gamma} = 1$.

The next theorem is proved in [Lu2, Theorem 5.25 and its proof] (see also [Lu1] for the first occurrence of the special representations):

**Theorem (Lusztig).** Assume that $\varphi$ is constant. Then:

(a) If $\mathcal{F} \in \text{Fam}_{\varphi}^{\text{Lus}}(W)$, then there exists a unique $\chi_{\mathcal{F}} \in \mathcal{F}$ with minimal $b$-invariant ($\chi_{\mathcal{F}}$ is called the special character of $\mathcal{F}$). Moreover, $r_{\chi_{\mathcal{F}}} = 1$.

(b) If $\gamma \in \text{Cons}_{\varphi}^{\text{Lus}}(W)$, then $\chi_{\mathcal{F}}$ is an irreducible constituent of $\gamma$ (and, of course, among the irreducible constituents of $\gamma$, $\chi_{\mathcal{F}}$ is the unique one with minimal $b$-invariant). Moreover, $\langle \gamma, \chi_{\mathcal{F}} \rangle = 1$.

It turns out that, for general $\varphi$, there might exist Lusztig $\varphi$-families $\mathcal{F}$ such that no element of $\mathcal{F}$ occurs as an irreducible constituent of all the $\varphi$-constructible characters in $\text{Cons}_{\varphi}^{\text{Lus}}(W)$ (this already occurs in type $B_n$, and the reader can also check this fact in type $F_4$, using the tables given by Geck [Ge, Table 2]). Nevertheless, we will prove in this paper the following result, which is compatible with the above conjecture and the above theorems:

**Theorem L.** Let $\varphi : S \to \mathbb{R}_{\geq 0}$ be a weight function. Then:

(a) If $\mathcal{F} \in \text{Fam}_{\varphi}^{\text{Lus}}(W)$, then there exists a unique $\chi_{\mathcal{F}} \in \mathcal{F}$ with minimal $b$-invariant. Moreover, $r_{\chi_{\mathcal{F}}} = 1$.

(b) If $\gamma \in \text{Cons}_{\varphi}^{\text{Lus}}(W)$, then there exists a unique irreducible constituent $\chi_\gamma$ of $\gamma$ with minimal $b$-invariant. Moreover, $r_{\chi_\gamma} = 1$ and $\langle \gamma, \chi_\gamma \rangle = 1$.

The proof of Theorem CM is general and conceptual, while our proof of Theorem L goes through a case-by-case analysis, based on Lusztig’s description of $\varphi$-constructible characters and Lusztig $\varphi$-families [Lu3, §22].

**Remark 0.** As the only irreducible Coxeter systems affording possibly unequal parameters are of type $I_2(2m)$, $F_4$ or $B_n$, and as $r_\chi = 1$ for any character $\chi$ in these groups, the statement “$r_\chi = 1$” in Theorem L(a) and (b) follows immediately from Lusztig’s Theorem. Therefore, we will prove only the statements about the minimality of the $b$-invariant and the scalar product. ■

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1. Proof of Theorem L

1.A. Reduction. — It is easily seen that the proof of Theorem L may be reduced to the case where \((W, S)\) is irreducible. If \(W\) is of type \(A_n, D_n, E_6, E_7, E_8, H_3, \) or \(H_4\), then \(\varphi\) is necessarily constant and Theorem L follows immediately from Lusztig’s Theorem. If \(W\) is dihedral, then Theorem L is easily checked using \([Lu3, \S17 \text{ and Lemma } 22.2]\). If \(W\) is of type \(F_4\), then Theorem L follows from inspection of \([Ge, \text{ Table } 2]\). Therefore, this shows that we may, and we will, assume that \(W\) is of type \(B_n\), with \(n \geq 2\). Write \(S = \{t, s_1, s_2, \ldots, s_{n-1}\}\) in such a way that the Dynkin diagram of \((W, S)\) is

\[
(\#) \quad t \quad s_1 \quad s_2 \quad \ldots \quad s_{n-1}
\]

Write \(b = \varphi(t)\) and \(a = \varphi(s_1) = \varphi(s_2) = \cdots = \varphi(s_{n-1})\). If \(b \notin a \mathbb{N}^*\), then \(\text{Cons}^\varphi_\psi(W) = \text{Irr}(W)\) (see \([Lu3, \text{ Proposition } 22.25]\)) and Theorem L becomes obvious. So we may assume that \(b = ra\) with \(r \in \mathbb{N}^*\), and since the notions are unchanged by multiplying \(\varphi\) by a positive real number, we may also assume that \(a = 1\). Therefore:

**Hypothesis and notation.** From now on, and until the end of this section, we assume that the Coxeter system \((W, S)\) is of type \(B_n\), with \(n \geq 2\), that \(S = \{t, s_1, s_2, \ldots, s_{n-1}\}\) is such that the Dynkin diagram of \((W, S)\) is given by \((\#)\) and that \(\varphi(t) = r \varphi(s_1) = r \varphi(s_2) = \cdots = r \varphi(s_{n-1}) = r\) with \(r \in \mathbb{N}^*\).

We will now review the combinatorics introduced by Lusztig (symbols, admissible involutions,...) in order to compute families and constructible characters in type \(B_n\) (see \([Lu3, \text{ \S22}]\) for further details).

1.B. Admissible involutions. — Let \(l \geq 0\) and let \(Z\) be a totally ordered set of size \(2l + r\). We will define by induction on \(l\) what is an \(r\)-admissible involution of \(Z\). Let \(\iota : Z \to Z\) be an involution. Then \(\iota\) is said \(r\)-admissible if it has \(r\) fixed points and, if \(l \geq 1\), there exist two consecutive elements \(b\) and \(c\) of \(Z\) such that \(\iota(b) = c\) and the restriction of \(\iota\) to \(Z \setminus \{b, c\}\) is \(r\)-admissible.

Note that, if \(\iota\) is an \(r\)-admissible involution and if \(\iota(b) = c > b\) and \(\iota(z) = z\), then \(z < b\) or \(z > c\) (this is easily proved by induction on \(|Z|\)).
1. C. Symbols. — We will denote by $\text{Sym}_k(r)$ the set of symbols $\Lambda = \left( \begin{array}{c} \beta \\ \gamma \end{array} \right)$ where $\beta = (\beta_1 < \beta_2 < \cdots < \beta_{k+r})$ and $\gamma = (\gamma_1 < \gamma_2 < \cdots < \gamma_k)$ are increasing sequences of non-zero natural numbers. We set

$$|\Lambda| = \sum_{i=1}^{k+r} (\beta_i - i) + \sum_{j=1}^{k} (\gamma_j - j)$$

and

$$b(\Lambda) = \sum_{i=1}^{k+r} (2k + 2r - 2i)(\beta_i - i) + \sum_{j=1}^{k} (2k + 1 - 2j)(\gamma_j - j).$$

The number $b(\Lambda)$ will be called the $b$-invariant of $\Lambda$. For simplifying our arguments, we will define

$$\nabla_{k,r} = \sum_{i=1}^{k+r} (2k + 2r - 2i)i + \sum_{j=1}^{k} (2k + 1 - 2j)j$$

so that

$$b(\Lambda) = \sum_{i=1}^{k+r} (2k + 2r - 2i)\beta_i + \sum_{j=1}^{k} (2k + 1 - 2j)\gamma_j - \nabla_{k,r}. $$

By abuse of notation, we denote by $\beta \cap \gamma$ the set $\{\beta_1, \beta_2, \ldots, \beta_{k+r}\} \cap \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ and by $\beta \cup \gamma$ the set $\{\beta_1, \beta_2, \ldots, \beta_{k+r}\} \cup \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$. We also set $\beta + \gamma = (\beta \cup \gamma) \setminus (\beta \cap \gamma)$.

We now define

$$z'(\Lambda) = (\beta_1, \beta_2, \ldots, \beta_r, \gamma_1, \beta_{r+1}, \gamma_2, \beta_{r+2}, \ldots, \gamma_k, \beta_{r+k})$$

and we will write

$$z'(\Lambda) = (z'_1(\Lambda), z'_2(\Lambda), \ldots, z'_{2k+r}(\Lambda)),$$

so that

$$b(\Lambda) = \sum_{i=1}^{r} (2k + 2r - 2i)z'_i(\Lambda) + \sum_{i=r+1}^{2k+r} (2k + r - i)z'_i(\Lambda) - \nabla_{k,r}$$

(1)

$$= \sum_{i=1}^{r} (r - i)z'_i(\Lambda) + \sum_{i=r+1}^{2k+r} (2k + r - i)z'_i(\Lambda) - \nabla_{k,r}$$

$$= \sum_{i=1}^{r} \left( \sum_{j=1}^{i} z'_j(\Lambda) \right) + \sum_{i=r+1}^{2k+r} \left( \sum_{j=i}^{2k+r} z'_j(\Lambda) \right) - \nabla_{k,r}.$$ (1.1)

1. D. Families of symbols. — We denote by $z(\Lambda)$ the sequence $z_1 \leq z_2 \leq \cdots \leq z_{2k+r}$ obtained after rewriting the sequence $(\beta_1, \beta_2, \ldots, \beta_{k+r}, \gamma_1, \gamma_2, \ldots, \gamma_k)$ in non-decreasing order.

Remark 1 - Note that the sequence $z'(\Lambda)$ determines the symbol $\Lambda$, contrarily to the sequence $z(\Lambda)$. However, $z(\Lambda)$ determines completely $|\Lambda|$ thanks to the formula $|\Lambda| = \sum_{z \in z(\Lambda)} z - r(r+1)/2 - (k+r)(k+r+1)/2$. □
We say that two symbols $\Lambda = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$ and $\Lambda' = \begin{pmatrix} \beta' \\ \gamma' \end{pmatrix}$ in $\text{Sym}_k(r)$ are in the same family if $z(\Lambda) = z(\Lambda')$. Note that this is equivalent to say that $\beta \cap \gamma = \beta' \cap \gamma'$ and $\beta \cup \gamma = \beta' \cup \gamma'$. If $\mathcal{F}$ is the family of $\Lambda$, we set $X_{\mathcal{F}} = \beta \cap \gamma$ and $Z_{\mathcal{F}} = \beta \cup \gamma$: note that $X_{\mathcal{F}}$ and $Z_{\mathcal{F}}$ depend only on $\mathcal{F}$ (and not on the particular choice of $\Lambda \in \mathcal{F}$).

If $\iota$ is an $r$-admissible involution of $Z_{\mathcal{F}}$, we denote by $\mathcal{F}_{\iota}$ the set of symbols $\Lambda = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$ in $\mathcal{F}$ such that $|\beta \cap \omega| = 1$ for all $\iota$-orbits $\omega$.

1.E. Lusztig families, constructible characters. — Let $\Lambda \in \text{Sym}_k(r)$ be such that $|\Lambda| = n$. Let $\text{Bip}(n)$ be the set of bipartitions of $n$. We set

$$\lambda_1(\Lambda) = (\beta_{k+r} - (k + r) \geq \cdots \geq \beta_2 - 2 \geq \beta_1 - 1),$$

$$\lambda_2(\Lambda) = (\gamma_{k} - k \geq \cdots \geq \gamma_2 - 2 \geq \gamma_1 - 1)$$

and

$$\lambda(\Lambda) = (\lambda_1(\Lambda), \lambda_2(\Lambda)).$$

Then $\lambda(\Lambda)$ is a bipartition of $n$. We denote by $\chi_{\lambda}$ the irreducible character of $W$ denoted by $\chi_{\lambda(\Lambda)}$ in [Lu3, §22] or in [GePf, §5.5.3]. Then [GePf, §5.5.3]

(♦)

$$b_{\chi_{\lambda}} = b(\Lambda).$$

With these notations, Lusztig described the $\varphi$-constructible characters in [Lu3, Proposition 22.24], from which the description of Lusztig $\varphi$-families follows by using [Lu3, Lemma 22.22]:

**Theorem 2 (Lusztig).** Let $\mathcal{F}_{\text{Lus}}$ be a Lusztig $\varphi$-family and let $\gamma \in \text{Cons}_{\varphi}^{\text{Lus}}(\mathcal{F}_{\text{Lus}})$. If we choose $k$ sufficiently large, then:

(a) There exists a family $\mathcal{F}$ of symbols in $\text{Sym}_k(r)$ such that

$$\mathcal{F}_{\text{Lus}} = \{ \chi_{\Lambda} \mid \Lambda \in \mathcal{F} \}.$$ 

(b) There exists an $r$-admissible involution $\iota$ of $Z_{\mathcal{F}}$ such that

$$\gamma = \sum_{\Lambda \in \mathcal{F}_{\iota}} \chi_{\Lambda}.$$ 

If $\Lambda = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$, we set $\Lambda' = \begin{pmatrix} \beta \setminus (\beta \cap \gamma) \\ \gamma \setminus (\beta \cap \gamma) \end{pmatrix}$.

**Definition 3.** The symbol $\Lambda$ is said special if $z(\Lambda') = z(\Lambda')$.

**Remark 4.** According to Remark 1, there is a unique special symbol in each family. It will be denoted by $\Lambda_{\mathcal{F}}$. Finally, note that, if $\Lambda, \Lambda'$ belong to the same family, then $|\Lambda| = |\Lambda'|$. □
Now, Theorem L follows from Theorem 2, Formula (◇) and the following next Theorem:

**Theorem 5.** Let $\mathcal{F}$ be a family of symbols in $\text{Sym}_k(r)$, let $\iota$ be an $r$-admissible involution of $Z_\mathcal{F}$ and let $\Lambda \in \mathcal{F}$. Then:

(a) $b(\Lambda) \geq b(\Lambda_{\mathcal{F}})$ with equality if and only if $\Lambda = \Lambda_{\mathcal{F}}$.

(b) There is a unique symbol $\Lambda_{\mathcal{F},i}$ in $\mathcal{F}_i$ such that, if $\Lambda \in \mathcal{F}_i$, then $b(\Lambda) \geq b(\Lambda_{\mathcal{F},i})$, with equality if and only if $\Lambda = \Lambda_{\mathcal{F},i}$.

The rest of this section is devoted to the proof of Theorem 5.

**1.F. First reduction.** — First, assume that $X_\mathcal{F} \neq \emptyset$. Let $b \in X_\mathcal{F}$ and let $\Lambda = \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \in \mathcal{F}$. Then $b \in \beta \cap \gamma = X_\mathcal{F}$ and we denote by $\beta(b)$ the sequence obtained by removing $b$ to $\beta$. Similarly, let $\Lambda[b] = \begin{pmatrix} \beta(b) \\ \gamma[b] \end{pmatrix}$.

Then $\Lambda[b] \in \text{Sym}_{k-1}(r)$ and

\[
\text{(◇)} \quad b(\Lambda) = b(\Lambda[b]) + \nabla_{k,r} - \nabla_{k-1,r} + b\left(4k + 2r + 1 - \sum_{z \in \mathcal{Z}(\Lambda), z \neq b} 2\right) + \sum_{z \in \mathcal{Z}(\Lambda), z < b} z.
\]

Proof of (◇). Let $i_0$ and $j_0$ be such that $\beta_{i_0} = b$ and $\gamma_{j_0} = b$. Then

\[
b(\Lambda) - b(\Lambda[b]) = \nabla_{k,r} - \nabla_{k-1,r} + (2k + 2r - 2i_0)b + \sum_{i=1}^{i_0-1} 2\beta_i + (2k + 1 - 2j_0)b + \sum_{j=1}^{j_0-1} 2\gamma_j.
\]

But the numbers $\beta_1, \beta_2, \ldots, \beta_{i_0}, \gamma_1, \gamma_2, \ldots, \gamma_{j_0}$ are exactly the elements of the sequence $\mathcal{Z}(\Lambda)$ which are $\leq b$. So

\[
i_0 + j_0 = \sum_{z \in \mathcal{Z}(\Lambda), z \leq b} 1
\]

and

\[
\sum_{i=1}^{i_0-1} \beta_i + \sum_{j=1}^{j_0-1} \gamma_j = \sum_{z \in \mathcal{Z}(\Lambda), z < b} z.
\]

This shows (◇). ■

Now, the family of $\Lambda[b]$ depends only on the family of $\Lambda$ (and not on $\Lambda$ itself): indeed, $\mathcal{Z}(\Lambda[b])$ is obtained from $\mathcal{Z}(\Lambda)$ by removing the two entries equal to $b$. We will denote by $\mathcal{F}[b]$ the family of $\Lambda[b]$. Moreover, $Z_{\mathcal{F}[b]} = Z_\mathcal{F}$ and the map $\Lambda \mapsto \Lambda[b]$ induces a bijection between $\mathcal{F}$ and $\mathcal{F}[b]$, and also induces a bijection between $\mathcal{F}_i$ and $\mathcal{F}[b]_i$. 

On the other hand, the formula (\(\triangledown\)) shows that the difference between \(b(\Lambda)\) and \(b(\Lambda[b])\) depends only on \(b\) and \(\mathcal{F}\), so proving Theorem 5 for the pair \((\mathcal{F}, \iota)\) is equivalent to proving Theorem 5 for the pair \((\mathcal{F}[b], \iota)\). By applying several times this principle if necessary, this means that we may, and we will, assume that \(X_{\mathcal{F}} = \emptyset\).

1.G. Proof of Theorem 5(a). — First, note that \(\mathbf{z}(\Lambda) = \mathbf{z}(\Lambda_{\mathcal{F}}) = \mathbf{z}'(\Lambda_{\mathcal{F}})\) (the last equality follows from the fact that \(\Lambda_{\mathcal{F}}\) is special and \(X_{\mathcal{F}} = \emptyset\)). As \(\mathbf{z}(\Lambda)\) is a permutation of the non-decreasing sequence \(\mathbf{z}'(\Lambda_{\mathcal{F}})\), we have

\[
\sum_{j=1}^{i} z'_j(\Lambda) \geq \sum_{j=1}^{i} z'_j(\Lambda_{\mathcal{F}})
\]

for all \(i \in \{1, 2, \cdots, 2k + r\}\). So, it follows from (\(\spadesuit\)) that

\[
b(\Lambda) - b(\Lambda_{\mathcal{F}}) = \sum_{i=1}^{r-1} \left( \sum_{j=1}^{i} (z'_j(\Lambda) - z'_j(\Lambda_{\mathcal{F}})) \right) + \sum_{i=1}^{2k+r-1} \left( \sum_{j=1}^{i} (z'_j(\Lambda) - z'_j(\Lambda_{\mathcal{F}})) \right).
\]

So \(b(\Lambda) \geq b(\Lambda_{\mathcal{F}})\) with equality only whenever \(\sum_{j=1}^{i} z'_j(\Lambda) = \sum_{j=1}^{i} z'_j(\Lambda_{\mathcal{F}})\) for all \(i \in \{1, 2, \cdots, 2k + r\}\). The proof of Theorem 5(a) is complete.

1.H. Proof of Theorem 5(b). — We denote by \(f_1 < \cdots < f_r\) the elements of \(Z_{\mathcal{F}}\) which are fixed by \(\iota\). We also set \(f_{r+1} = 0\) and \(f_0 = \infty\). As \(\iota\) is \(r\)-admissible, the set \(Z_{\mathcal{F}}^{(d)} = \{z \in Z_{\mathcal{F}} \mid f_{d+1} < z < f_d\}\) is \(\iota\)-stable and contains no \(\iota\)-fixed point (for \(d \in \{0, 1, \cdots, r\}\)). Let \(k_d = |Z_{\mathcal{F}}^{(d)}|/2\) and let \(\iota_d\) be the restriction of \(\iota\) to \(Z_{\mathcal{F}}^{(d)}\). Then \(\iota_d\) is a 0-admissible involution of \(Z_{\mathcal{F}}^{(d)}\).

If \(\Lambda = \left(\begin{array}{c} \beta \\ \gamma \end{array}\right) \in \mathcal{F}_r\), we set \(\beta^{(d)} = \beta \cap Z_{\mathcal{F}}^{(d)}, \gamma^{(d)} = \gamma \cap Z_{\mathcal{F}}^{(d)}\) and \(\Lambda^{(d)} = \left(\begin{array}{c} \beta^{(d)} \\ \gamma^{(d)} \end{array}\right)\). Then \(\Lambda^{(d)} \in \text{Sym}_{k_d}(0)\) and, if \(\mathcal{F}^{(d)}\) denotes the family of \(\Lambda^{(d)}\), then \(\Lambda^{(d)} \in \mathcal{F}^{(d)}_d\).

Now, if \(\Lambda' = \left(\begin{array}{c} \beta' \\ \gamma' \end{array}\right) \in \text{Sym}_{k_d}(0)\), we set

\[
b_d(\Lambda') = \sum_{i=1}^{k'-r} (2k' + 2d - 2i) \beta'_i + \sum_{j=1}^{k'-r} (2k' + 1 - 2j) \gamma'_j.
\]

The number \(b_d(\Lambda')\) is called the \(b_d\)-invariant of \(\Lambda'\). It then follows from the definition of \(b\) and \(\nabla_{k,r}\) that

\[(\spadesuit) \quad b(\Lambda) = \sum_{d=0}^{r} b_d(\Lambda^{(d)}) - \nabla_{k,r} + \sum_{d=1}^{r} 2(k_0 + k_1 + \cdots + k_{d-1})(f_d + \sum_{z \in Z^{(d)}} z).
\]
Since the map
\[ \mathcal{F}_t \longrightarrow \prod_{d=0}^r \mathcal{F}_{t_d}^{(d)} \]
\[ \Lambda \longrightarrow (\Lambda^{(0)}, \Lambda^{(1)}, \ldots, \Lambda^{(d)}) \]
is bijective and since \( b(\Lambda) - \sum_{d=0}^r b(\Lambda^{(d)}) \) depends only on \((\mathcal{F}, t)\) and not on \( \Lambda \) (as shown by the formula (♠)), Theorem 5(b) will follow from the following lemma:

**Lemma 6.** There exists a unique symbol in \( \mathcal{F}_{t_d}^{(d)} \) with minimal \( b_d \)-invariant.

The proof of Lemma 6 will be given in the next section.

### 2. Minimal \( b_d \)-invariant

For simplifying notation, we set \( Z = Z^{(d)}_\mathcal{F} \), \( l = k_d \), \( \mathcal{G} = \mathcal{F}^{(d)} \) and \( j = t_d \). Let us write \( Z = \{z_1, z_2, \ldots, z_{2l}\} \) with \( z_1 < z_2 < \cdots < z_{2l} \). Recall from the previous section that \( j \) is a 0-admissible involution of \( Z \).

#### 2.A. Construction. — We will define by induction on \( l \geq 0 \) a symbol \( \Lambda_j^{(d)}(Z) \in \mathcal{G}_j \). If \( l = 0 \), then \( \Lambda_j^{(d)}(Z) \) is obviously empty. So assume now that, for any set of non-zero integers \( Z' \) of order \( 2(l-1) \), for any 0-admissible involution \( j' \) of \( Z' \) and any \( d' \geq 0 \), we have defined a symbol \( \Lambda_{j'}^{(d')}(Z') \). Then \( \Lambda_j^{(d)}(Z) = \left( \beta_j^{(d)}(Z) \right) \right) \) is defined as follows: let \( Z' = Z \setminus \{z_1, i(z_1)\} \), \( j' \) the restriction of \( j \) to \( Z' \) and let

\[
\begin{align*}
    d' &= \begin{cases} 
        d - 1 & \text{if } d \geq 1, \\
        1 & \text{if } d = 0.
    \end{cases}
\end{align*}
\]

Then \( |Z'| = 2(l - 1) \) and \( j' \) is 0-admissible. So \( \Lambda_{j'}^{(d')}(Z') = \left( \beta_{j'}^{(d')}(Z') \right) \) is well-defined by the induction hypothesis. We then set

\[
\begin{align*}
    \beta_j^{(d)}(Z) &= \begin{cases} 
        \beta_{j'}^{(d')}(Z') \cup \{z_1\} & \text{if } d \geq 1, \\
        \beta_{j'}^{(d')}(Z') \cup j(z_1) & \text{if } d = 0,
    \end{cases}
\end{align*}
\]

and

\[
\begin{align*}
    \gamma_j^{(d)}(Z) &= \begin{cases} 
        \gamma_{j'}^{(d')}(Z') \cup j(z_1) & \text{if } d \geq 1, \\
        \gamma_{j'}^{(d')}(Z') \cup \{z_1\} & \text{if } d = 0.
    \end{cases}
\end{align*}
\]

Then Lemma 6 is implied by the next lemma:

**Lemma 6+.** Let \( \Lambda \in \mathcal{G}_j \). Then \( b_d(\Lambda) \geq b_d(\Lambda_j^{(d)}(Z)) \) with equality if and only if \( \Lambda = \Lambda_j^{(d)}(Z) \).
The rest of this section is devoted to the proof of Lemma $6^+$. We will first prove Lemma $6^+$ whenever $d \in \{0, 1\}$ using Lusztig's Theorem. We will then turn to the general case, which will be handled by induction on $l = |Z|/2$. We fix $\Lambda = \left( \begin{array}{c} \beta \\ \gamma \end{array} \right) \in \mathfrak{g}_j$.

2.B. **Proof of Lemma $6^+$ whenever $d = 1$.** — Let $z$ be a natural number strictly bigger than all the elements of $Z$. Let $\tilde{\Lambda} = \left( \begin{array}{c} \beta \cup \{z\} \\ \gamma \end{array} \right) \in \text{Sym}_k(1)$. Then $b_1(\Lambda) = b(\tilde{\Lambda}) + C$, where $C$ depends only on $Z$. Let $\tilde{\Lambda}_0 = \left( \begin{array}{c} z_1, z_3, \ldots, z_{2l-1}, z \\ z_2, \ldots, z_{2l} \end{array} \right)$. Since $l$ is 0-admissible, it is easily seen that, if $j(z_i) = z_j$, then $j - i$ is odd. So $\tilde{\Lambda}_0 \in \mathfrak{g}_j$. But, by [Lu1, §5], $b(\tilde{\Lambda}) \geq b(\tilde{\Lambda}_0)$ with equality if and only if $\tilde{\Lambda} = \tilde{\Lambda}_0$. So it is sufficient to notice that $\Lambda^{(1)}_j(Z) = \tilde{\Lambda}_0$, which is easily checked.

2.C. **Proof of Lemma $6^+$ whenever $d = 0$.** — Assume in this subsection, and only in this subsection, that $d = 0$ or $1$. We denote by $\Lambda^\text{op} = \left( \begin{array}{c} \gamma \\ \beta \end{array} \right) \in \mathfrak{g}_j$. It is readily seen from the construction that $\Lambda^{(0)}_j(Z)^\text{op} = \Lambda^{(1)}_j(Z)$ and that

$$b_1(\Lambda) = b_0(\Lambda^\text{op}) + \sum_{z} z.$$

So Lemma $6^+$ for $d = 0$ follows from Lemma $6^+$ for $d = 1$.

2.D. **Proof of Lemma $6^+$ whenever $d \geq 2$.** — Assume now, and until the end of this section, that $d \geq 2$. We will prove Lemma $6^+$ by induction on $l = |Z|/2$. The result is obvious if $l = 0$, as well as if $l = 1$. So we assume that $l \geq 2$ and that Lemma $6^+$ holds for $l' \leq l - 1$. Write $j(z) = z_{2m}$, where $m \leq l$ (note that $j(z_1) \notin \{z_1, z_3, z_5, \ldots, z_{2l-1}\}$ since $j$ is 0-admissible).

Assume first that $m < l$. Then $Z$ can be written as the union $Z = Z^+ \cup Z^-$, where $Z^+ = \{z_1, z_3, \ldots, z_{2m}\}$ and $Z^- = \{z_{2m+1}, z_{2m+3}, \ldots, z_{2l}\}$ are $j$-stable (since $j$ is 0-admissible). If $\varepsilon \in \{+,-\}$, let $j^\varepsilon$ denote the restriction of $j$ to $Z^\varepsilon$, let $\beta^\varepsilon = \beta \cap Z^\varepsilon$, $\gamma^\varepsilon = \gamma \cap Z^\varepsilon$ and $\Lambda^\varepsilon = \left( \begin{array}{c} \beta^\varepsilon \\ \gamma^\varepsilon \end{array} \right)$, and let $\mathfrak{g}_j^\varepsilon$ denote the family of $\Lambda^\varepsilon$. Then it is easily seen that $\Lambda^\varepsilon \in \mathfrak{g}_j^\varepsilon$, that $b_d(\Lambda) - \left( b_d(\Lambda^+) + b_d(\Lambda^-) \right)$ depends only on $(\mathfrak{g}_j^\varepsilon, j)$ and that $\Lambda^{(d)}_j(Z)^\varepsilon = \Lambda^{(d)}_j(Z^\varepsilon)$. By the induction hypothesis, $b_d(\Lambda^\varepsilon) \geq b_d(\Lambda^{(d)}_j(Z^\varepsilon))$ with equality if and only if $\Lambda^\varepsilon = \Lambda^{(d)}_j(Z^\varepsilon)$. So the result follows in this case. This means that we may, and we will, work under the following hypothesis:
Hypothesis. From now on, and until the end of this section, we assume that \( j(z_1) = z_{2l} \).

As in the construction of \( \Lambda^{(d)}_j(Z) \), let \( Z' = Z \setminus \{z_1, z_{2l}\} = \{z_2, z_3, \ldots, z_{2l-1}\} \), let \( j' \) denote the restriction of \( j \) to \( Z' \) and let

\[
d' = \begin{cases} 
  d - 1 & \text{if } d \geq 1, \\
  1 & \text{if } d = 0.
\end{cases}
\]

Then \( |Z'| = 2(l-1) \) and \( j' \) is 0-admissible. Let \( \Lambda' = \left( \beta', \gamma' \right) \) where \( \beta' = \beta \setminus \{z_1, z_{2l}\} \) and \( \gamma' = \gamma \setminus \{z_1, z_{2l}\} \). Since \( d \geq 2 \), we have \( z_1 \in \beta_j^{(d)}(Z) \) and \( z_{2l} \in \gamma_j^{(d)}(Z) \). This implies that

\[
(\star) \quad \mathbf{b}_d(\Lambda_j^{(d)}(Z)) = \mathbf{b}_{d-1}(\Lambda_{j'}^{(d-1)}(Z')) + z_{2l} + 2(l+d)z_1 + 2 \sum_{z \in Z'} z.
\]

If \( z_1 \in \beta \), then \( \Lambda = \Lambda_j^{(d)}(Z) \) if and only if \( \Lambda' = \Lambda_{j'}^{(d')}(Z') \) and again

\[
\mathbf{b}_d(\Lambda) = \mathbf{b}_{d-1}(\Lambda') + z_{2l} + 2(l+d)z_1 + 2 \sum_{z \in Z'} z.
\]

So the result follows from (\star) and from the induction hypothesis.

This means that we may, and we will, assume that \( z_1 \in \gamma \). In this case,

\[
\mathbf{b}_d(\Lambda) = \mathbf{b}_{d+1}(\Lambda') + 2d z_{2l} + (2l + 1)z_1.
\]

Then it follows from (\star) that

\[
\mathbf{b}_d(\Lambda) - \mathbf{b}_d(\Lambda_j^{(d)}(Z)) = \mathbf{b}_{d+1}(\Lambda') - \mathbf{b}_{d-1}(\Lambda_{j'}^{(d-1)}(Z')) + (2d-1)(z_{2l} - z_1) - 2 \sum_{z \in Z'} z.
\]

So, by the induction hypothesis,

\[
\mathbf{b}_d(\Lambda) - \mathbf{b}_d(\Lambda_j^{(d)}(Z)) \geq \mathbf{b}_{d+1}(\Lambda_{j'}^{(d+1)}(Z')) - \mathbf{b}_{d-1}(\Lambda_{j'}^{(d-1)}(Z')) + (2d-1)(z_{2l} - z_1) - 2 \sum_{z \in Z'} z.
\]

Since \( z_{2l} - z_1 > z_{2l-1} - z_2 \), it is sufficient to show that

\[
(?) \quad \mathbf{b}_{d+1}(\Lambda_{j'}^{(d+1)}(Z')) - \mathbf{b}_{d-1}(\Lambda_{j'}^{(d-1)}(Z')) \geq -(2d-1)(z_{2l-1} - z_2) + 2 \sum_{z \in Z'} z.
\]

This will be proved by induction on the size of \( Z' \). First, if \( j(z_2) < z_{2l-1} \), then we can separate \( Z' \) into two \( j' \)-stable subsets and a similar argument as before allows to conclude thanks to the induction hypothesis.
So we assume that $f'(z_2) = z_{2l-1}$. Let $Z'' = Z' \setminus \{z_2, z_{2l-1}\}$ and let $f''$ denote the restriction of $f'$ to $Z''$. Since $z_2 \in \beta_{j'}(Z')$, we can apply $(\star)$ one step further to get

$$b_{d+1}(\Lambda_{j'}^{d+1}(Z')) - b_{d-1}(\Lambda_{j'}^{d-1}(Z')) = b_d(\Lambda_{j'}^{d}(Z'')) + z_{2l-1} + 2(l + d)z_2 + 2 \sum_{z \in Z''} z \phantom{\sum_{z \in Z''}} - (b_{d-2}(\Lambda_{j''}^{d-2}(Z'')) + z_{2l-1} + 2(l + d - 2)z_2 + 2 \sum_{z \in Z''} z)$$

$$= b_d(\Lambda_{j'}^{d}(Z'')) - b_{d-2}(\Lambda_{j''}^{d-2}(Z'')) + 4z_2.$$

So, by the induction hypothesis,

$$b_{d+1}(\Lambda_{j'}^{d+1}(Z')) - b_{d-1}(\Lambda_{j'}^{d-1}(Z')) \geq -(2d - 3)(z_{2l-2} - z_3) + 2 \sum_{z \in Z''} z + 4z_2$$

$$\geq -(2d - 3)(z_{2l-1} - z_2) + 2 \sum_{z \in Z''} z + 2z_2 - 2z_{2l-1}$$

$$= -(2d - 1)(z_{2l-1} - z_2) + 2 \sum_{z \in Z''} z,$$

as desired. This shows $(\star)$ and completes the proof of Lemma 6$^*$. 

### 3. Complex reflection groups

If $\mathcal{W}$ is a complex reflection group, then R. Rouquier and the author have also defined Calogero-Moser cellular characters and Calogero-Moser families (see [BoRo1] or [BoRo2]). If $\mathcal{W}$ is of type $G(l, 1, n)$ (in Shephard-Todd classification), then Leclerc and Miyachi [LeMi, §6.3] proposed, in link with canonical bases of $U_q(sl_\infty)$-modules, a family of characters that could be a good analogue of constructible characters: let us call them the Leclerc-Miyachi constructible characters of $G(l, 1, n)$. If $l = 2$, then they coincide with constructible characters [LeMi, Theorem 10].

Of course, it would be interesting to know if Calogero-Moser cellular characters coincide with the Leclerc-Miyachi ones: this seems rather complicated but it should be at least possible to check if the Leclerc-Miyachi constructible characters satisfy the analogous properties with respect to the $b$-invariant.

### References


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