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Energy Estimates for Low Regularity Bilayer Schrödinger Equations

Nabille Boussaïd * Marco Caponigro ** Thomas Chambrion ***

* Laboratoire de mathématiques, Université de Franche-Comté, 25030 Besançon, France (e-mail: Nabille.Boussaid@univ-fcomte.fr)
** Département Ingénierie Mathématiques, Conservatoire National des Arts et Métiers, 75003 Paris, France (e-mail: marco.caponigro@cnam.fr)
*** Université de Lorraine, Institut Élie Cartan de Lorraine, Vandœuvre-lès-Nancy, F-54506, France, CNRS, IECU, Vandœuvre-lès-Nancy, F-54506, France, Inria, CORIDA, Villers-lès-Nancy, F-54600, France (e-mail: Thomas.Chambrion@univ-lorraine.fr)

Abstract: This paper presents an energy estimate in terms of the total variation of the control for bilinear infinite dimensional quantum systems with unbounded potentials. These estimates allow a rigorous construction of propagators associated with controls of bounded variation. Moreover, upper bounds of the error made when replacing the infinite dimensional system by its finite dimensional Galerkin approximations is presented.

Keywords: Bilinear systems, quantum systems, well-posedness, approximation.

1. INTRODUCTION

1.1 Physical context

The state of a quantum system evolving in a Riemannian manifold Ω is described by its wave function, a point $\psi$ in $L^2(\Omega, \mathbb{C})$. When the system is submitted to an electric field (e.g., a laser), the time evolution of the wave function is given, under the dipolar approximation and neglecting decoherence, by the Schrödinger bilinear equation:

$$i\frac{\partial \psi}{\partial t} = (-\Delta + V(x))\psi(x, t) + u(t)W(x)\psi(x, t)$$

where $\Delta$ is the Laplace-Beltrami operator on $\Omega$, $V$ and $W$ are real potential accounting for the properties of the free system and the control field respectively, while the real function of the time $u$ accounts for the intensity of the laser.

In view of applications (for instance in NMR), it is important to know whether and how it is possible to chose a suitable control $u : [0, T] \rightarrow \mathbb{R}$ in order to steer (1) from a given initial state to a given target. This question has raised considerable interest in the community in the last decade. After the negative results of Ball et al. (1982) and Turinici (2000) excluding exact controllability on the natural domain of the operator $-\Delta + V$ when $W$ is bounded, the first, and at this day the only one, description of the attainable set for an example of bilinear quantum system was obtained by (Beauchard (2005); Beauchard and Coron (2006)). Further investigations of the approximate controllability of (1) were conducted using Lyapunov techniques (Nersesyan (2010, 2009); Beauchard and Nersesyan (2010); Beauchard et al. (2007); Mirrahimi et al. (2005); Mirrahimi (2006)) and geometric techniques (Chambrion et al. (2009); Bosan et al. (2012)).

In most of the references cited above, the potentials $V$ and $W$ in (1) are bounded. The very general (and irregular) systems considered by Bosan et al. (2012) allow to define the solutions of (1) for piecewise constant controls only. The aim of this paper is to present a coherent framework to deal with unbounded potentials in (1). This includes a rigorous definition of the solution of (1) for control that are not necessarily piecewise constant and the extension of some quantitative energy estimates.

1.2 Abstract framework and notations

We reformulate the control problem in more abstract framework, in such a way that we can use some of the powerful tools of functional analysis. In a separable Hilbert space $H$, we consider a pair $(A, B)$ of (possibly unbounded) linear operators that satisfy Hypothesis 1

**Hypothesis 1.** $(A, B)$ is a pair of linear operators such that

1. $A$ is skew-adjoint on its domain $D(A)$;
2. $iA$ is bounded from below;
3. $B$ is skew-symmetric;
4. there exists $a, b \geq 0$ such that $\|B\psi\| \leq a\|A\psi\| + b\|\psi\|$ for any $\psi$ in $D(A)$.

Following Kato (1953), Hypothesis 1 is the minimal framework for our developments. For many examples encoun-
tered in the physics literature, $A$ has a discrete spectrum and we will consider the more restrictive Hypothesis 2.

**Hypothesis 2.** $(A, B, (\phi_j)_{j \in \mathbb{N}}, \alpha)$ is a quadruple such that

1. $(A, B)$ satisfies Hypothesis 1;
2. $(\phi_j)_{j \in \mathbb{N}}$ is a Hilbert basis of $H$;
3. $0 \leq \alpha \leq 1$;
4. $A$ has discrete spectrum $(-i\lambda_j)_{j \in \mathbb{N}}$ with $\lambda_j \to +\infty$ as $j \to \infty$;
5. for any $j \in \mathbb{N}$, $A\phi_j = -i\lambda_j \phi_j$;
6. there exists $d \geq 0$ such that $\|B\psi\| \leq d\|A^{1/2}\psi\|$ for any $\psi$ in $D(A^{1/2})$.

Thanks to the Kato-Rellich theorem (see Kato (1995)), Hypotheses 1.1, 1.3 and 1.4 imply that, for any $u$ in $(-1/\alpha, 1/\alpha)$, $A + uB$ is skew-adjoint with domain $D(A)$ and generates a unitary propagator $t \mapsto e^{t(A+uB)}$. In particular, this allows to define by concatenation the propagator $\Upsilon^u : t \mapsto \Upsilon^u_t$ for the control system

$$\frac{d\psi}{dt} = (A + u(t)B)\psi$$

for $u$ piecewise constant $u$ taking value in $(-1/\alpha, 1/\alpha)$.

Recall that a function $u : [0, T] \to \mathbb{R}$ has bounded variation (or is BV) if there exists a constant $C$ such that, for any partition $0 = a_0 < a_1 < \ldots < a_n = T$ of $[0, T]$, $\sum_{k=1}^n |u(a_k) - u(a_{k-1})| \leq C$. The smallest $C$ satisfying this property for any partition of $[0, T]$ is the total variation of $u$, denoted $TV(u)$.

We define the set $U$ of the functions $u : \mathbb{R} \to \mathbb{R}$ with bounded variation such that $u(t) = 0$ for $t \leq 0$. In $U$, the sequence $(u_n)_{n \in \mathbb{N}}$ converges to $u$ if $\sup_{n} TV_{\mathbb{R}}(u_n) \leq TV_{\mathbb{R}}(u)$ and $u_n(t)$ tends to $u(t)$ as $n$ goes to infinity for almost any $t$ in $\mathbb{R}$.

### 1.3 Contribution of this paper

This paper presents a rigorous yet elementary construction of the solutions of (2) associated with controls of bounded variation, inspired from Kato (1953). Among other byproducts of our energy estimates, we give a lower bound for the number of switches needed to steer (2) from a given source to a given target using controls with value in $\{0, 1\}$ and we give an upper bound of the error made when one replaces the original infinite dimensional system (2) by one of its finite dimensional Galerkin approximation. Such estimates are instrumental in practice, both for theoretical analysis, design of control laws and numerical simulations.

The strength of our results is the relative generality of our assumptions. In this sense, this paper may be seen as an extension of the results of Boussaïd et al. (2012b) to systems that are not weakly-coupled (according to (Boussaïd et al., 2013, Definition 1)).

### 1.4 Content of the paper

The first part of the paper (Section 2) is concerned with the construction of the solutions of (2) for controls with bounded variation. The key point of this construction is an energy estimate in terms of the total variation of the control (see Proposition 3). The second part of the paper (Section 3) presents some consequences of this energy estimate in terms of approximation of the original infinite dimensional system by its finite dimensional dynamics. Finally, we apply our results to various types of quantum oscillators encountered in the physics literature (Section 4).

## 2. CONSTRUCTION OF THE PROPAGATORS

To begin with, we consider the simple case where $\|B\psi\| \leq a\|A\psi\|$ for any $\psi$ in $D(A)$. The general case of operators $B$ relatively bounded with respect to $A$ satisfying Hypothesis 1.4 will be treated in Subsection 2.3.

### 2.1 Estimates on the $A$ norm

For any $\psi$ in $D(A)$, for any $u$ in $\mathbb{R}$ such that $a|u| < 1$,

$$\|B\psi\| \leq a\|A\psi\|$$

$$\leq a\|((A + uB)\psi) + |u|\|B\psi\|$$

Hence,

$$\|B\psi\| \leq \frac{a}{1 - |u|a}\|A + uB\|\psi\|$$

For any $u_1, u_2$ in $(-1/\alpha, 1/\alpha)$, $t$ in $\mathbb{R}$ and $\psi$ in $D(A)$, $\psi$ is in $D(A + u_2B)$ by Hypothesis 1.4. Hence, $e^{t(A+u_2B)}\psi$ belongs to $D(A + u_2B) = D(A) = D(A + u_1B)$. Moreover,

$$\|(A + u_1B)e^{t(A+u_2B)}\psi\| \leq \|(A + u_2B)e^{t(A+u_2B)}\psi\| + \|(u_1 - u_2)B e^{t(A+u_2B)}\psi\|$$

and hence

$$\|(A + u_1B)e^{t(A+u_2B)}\psi\| \leq \left(1 + \frac{|u_1 - u_2|\|B\|}{1 - |u_1|a}\right)\|A + u_2B\|\psi\|$$

Let $u^* > 0$ be given such that $|u^*| < 1/a$. For any $t \geq 0$, for any $u_1, u_2$ in $(-u^*, u^*)$ one has, with $\Gamma = \frac{1}{1 - |u^*|a}$,

$$\|(A + u_1B)e^{t(A+u_2B)}\psi\| \leq \exp(\Gamma|u_2 - u_1|)\|(A + u_2B)\psi\|.$$

Consider now a piecewise constant control $u : [0, T] \to (-1/\alpha, 1/\alpha)$ taking value $u_j$ for time $t_j$, $t_j \geq 0$ $1 \leq j \leq p$, $p \in \mathbb{N}$. We get by concatenation, for any $\psi$ in $D(A)$,
\[\|A T_{t,0}^{u} \psi\| \leq \exp(\Gamma |u|) \times \prod_{j=2}^{\infty} \exp(\Gamma (|u_j - u_{j-1}|) \|A \psi\| \leq \exp(2T T V_{0,T}(u)) \|A \psi\|.\]

We obtain, similarly to Kato (1953), the following result.

Proposition 3. For any \( \delta \in (0,1) \), let \((A,B)\) satisfy Hypothesis 1. Then, for any piecewise constant \( u \in \mathbb{U} \) with \( 0, T \) converging \( \lambda \) large enough, \( \|A T_{t,0}^{u}\psi\| = \|A T_{t,0}^{u}\psi\| \) for every \( \psi \) in \( D(A) \).

Below we write \( \|A + \lambda T_{t,0}^{u}\psi\| \leq e^{-\delta t T V_{0,T}(u)} \|(A + \lambda)\psi\| \).

3. GOOD GALERKIN APPROXIMATIONS

For applications (design of control laws or numerical simulations), it is common to replace the original infinite dimensional system (2) by a suitable finite dimensional approximation. It is often possible to bound the error due to this approximation. Under Hypothesis 2, we derive in this section an explicit upper bound of this error that depends only on the \( L^1 \) norm and the total variation of the control. The results presented here extend the results of Boussaid et al. (2012b).

3.1 Notion of Good Galerkin Approximations

Let \( \Phi = (\phi_j)_{j \in \mathbb{N}} \) be a Hilbert basis of \( H \). For any \( N \) in \( \mathbb{N} \), we define the orthogonal projection

\[\pi_N^\Phi \psi \in H \mapsto \sum_{j \leq N} \langle \phi_j, \psi \rangle \phi_j \in H.\]

Definition 7. Let \((A,B,\Phi,1)\) satisfy Hypothesis 2 and \( N \in \mathbb{N} \). The Galerkin approximation of (2) of order \( N \) is the system in \( H \)

\[\dot{x} = (A(\Phi,N) + u(t)B(\Phi,N))x \quad (7)\]

where \( A(\Phi,N) = \pi_N^\Phi A_{1 \leq n \leq N} \) and \( B(\Phi,N) = \pi_N^\Phi B_{1 \leq n \leq N} \) are the compressions of \( A \) and \( B \) (respectively).

We denote by \( X_N^\Phi(t,s) \) the propagator of (7) associated with a \( L^1 \) function \( u \).

Remark 8. The operators \( A(\Phi,N) \) and \( B(\Phi,N) \) are defined on the infinite dimensional space \( H \). However, they have finite rank and the dynamics of \( (\Sigma_N) \) leaves invariant the \( N \)-dimensional space \( \mathcal{L}_N \). Thus, \( (\Sigma_N) \) can be seen as a finite dimensional bilinear system in \( \mathcal{L}_N \).
for a functional norm $N(\cdot)$ on a functional space $U$ in a subspace $D$ (with norm $\|\cdot\|_D$) of $H$ if, for any $K, \varepsilon > 0$, for any $\psi$ in $D$, there exists $N \in \mathbb{N}$ such that, for any $u$ in $U, N(u) \leq K$ implies $\|X^\lambda_{(\Phi,N)}(t,0) - \mathbf{Y}^\lambda_{(\Phi,N)}(0)\|_D < \varepsilon$ for any $t < \varepsilon$.

3.2 GGA for BV controls

**Proposition 9.** Let $(A, B, \Phi)$ satisfy Hypotheses 1, 2, 2.4 and 2.5. Then, for any $\delta \in (0, 1)$, for any $r \in (0, 1)$ for any $n \in \mathbb{N}, N \in \mathbb{N}, (\psi_j)_{1 \leq j \leq n}$ in $D(|A|)^n$, and for any function $u$ in $B_{\delta,a}$, there exists $N \in \mathbb{N}$ such that for any $t \geq 0$ and $j = 1, \ldots, n$.

\[
\|(|A| - \pi_N^\lambda)\mathbf{Y}^\lambda_{(\Phi,N)}(t,0)\|_r \leq \frac{\varepsilon^r_{\delta,T_{\mathbb{N}}(\psi)}\|\psi\|_1}{\inf_{j > N} \lambda_j^{-r}}. \tag{8}
\]

for any $t \geq 0$ and $j = 1, \ldots, n$.

**Proof.** Fix $j \in \{1, \ldots, n\}$. For any $N > 1$, one has

\[
\|(|A| - \pi_N^\lambda)\mathbf{Y}^\lambda_{(\Phi,N)}(t,0)\|_r = \sum_{n = N+1}^{\infty} \lambda_n^r \|\psi\|_1 \leq \inf_{j > N} \lambda_j^{-r} \|\psi\|_1.
\]

By Proposition 3, for any $t > 0$.

\[
\|\mathbf{Y}^\lambda_{(\Phi,N)}(t,0)\|_r \leq \varepsilon^r_{\delta,T_{\mathbb{N}}(\psi)}\|\psi\|_1.
\]

**Proposition 10.** (Good Galerkin Approximation). Let $\delta \in (0, 1), \alpha \in (0, 1)$ and $(A, B, \Phi, \alpha)$ satisfy Hypothesis 2. Then for any $\varepsilon > 0, K \geq 0, n \in \mathbb{N}$, and $(\psi_j)_{1 \leq j \leq n}$ in $D(|A|)u_n$ there exists $N \in \mathbb{N}$ such that for any $L^1$ function $u$ in $U_{\delta,a}$.

\[
\|u\|_L^1 + T_{\mathbb{N}}(u) < K \Rightarrow \|\mathbf{Y}^\lambda_{(\Phi,N)}(t,0)\|_r \pi_N \|\psi\|_1 < \varepsilon,
\]

for any $t \geq 0$ and $j = 1, \ldots, n$.

**Proof.** Fix $j \in \{1, \ldots, n\}$ and consider the map $t \mapsto \pi_N \mathbf{Y}^\lambda_{(\Phi,N)}(t,0)$ that is absolutely continuous and satisfies, for almost any $t \geq 0$.

\[
\frac{d}{dt} \pi_N \mathbf{Y}^\lambda_{(\Phi,N)}(t,0) = (A^\lambda_{(\Phi,N)} + u(t)B^\lambda_{(\Phi,N)})\pi_N \mathbf{Y}^\lambda_{(\Phi,N)}(t,0).
\]

Hence, by variation of constants, for any $t \geq 0$.

\[
\pi_N \mathbf{Y}^\lambda_{(\Phi,N)}(t,0) = X^\lambda_{(\Phi,N)}(t,0)\pi_N \mathbf{Y}^\lambda_{(\Phi,N)}(t,0) + \int_0^t X^\lambda_{(\Phi,N)}(t,s)\pi_N B(\mathbf{Y}^\lambda_{(\Phi,N)}(s,0)u(t)\sigma(t)ds.
\]

By Proposition 9, the norm of $t \mapsto B(\mathbf{Y}^\lambda_{(\Phi,N)}(t,0)$ is less than $\frac{d}{dt} \pi_N \mathbf{Y}^\lambda_{(\Phi,N)}(t,0)$ is unitary.

\[
\|\mathbf{Y}^\lambda_{(\Phi,N)}(t,0) - X^\lambda_{(\Phi,N)}(t,0)\|_r \|\psi\|_1 \leq \|u\|_L^1 \inf_{j > N} \lambda_j^{-1} e^{(A,B)K} \|\psi\|_1.
\]

Then

\[
\|\mathbf{Y}^\lambda_{(\Phi,N)}(t,0) - X^\lambda_{(\Phi,N)}(t,0)\|_r \|\psi\|_1 \leq \|(|A| - \pi_N^\lambda)\mathbf{Y}^\lambda_{(\Phi,N)}(t,0)\|_r \pi_N \|\psi\|_1 \leq \|u\|_L^1 \inf_{j > N} \lambda_j^{-1} e^{(A,B)K} \|\psi\|_1.
\]

This completes the proof since $\lambda_n$ tends to infinity as $n$ goes to infinity.

4. EXAMPLES

4.1 Tri-diagonal systems

**Definition 11.** A system $(A, B, \Phi)$ is tri-diagonal if $(A, B)$ satisfies Hypotheses 1.1, 1.2, 1.3, 2.2, 2.4 and 2.5 and if, for any $k \in \mathbb{N}, |j - k| > 1$ implies $\langle \phi_j, B\phi_k \rangle = 0$.

In the following, we denote $b_{j,k} = \langle \phi_j, B\phi_k \rangle$.

**Proposition 12.** Let $(A, B, \Phi)$ be a tri-diagonal system and let $r$ be a positive number. Assume that the sequences $(\frac{b_{n+1}}{\lambda_n})_{n \in \mathbb{N}}, (\frac{b_{n+2}}{\lambda_n})_{n \in \mathbb{N}}$ and $(\frac{b_{n+3}}{\lambda_n})_{n \in \mathbb{N}}$ are bounded by $C$.

Then, for any $\psi$ in $D(|A|)u$, $\|B\psi\| \leq \sqrt{6}C\|\psi\|$.

In particular, if $r \leq 1$ (resp. $r < 1$), then $(A, B)$ satisfies Hypothesis 1 (resp. $(A, B, \Phi, r)$ satisfies Hypothesis 2).

**Proof.** For any $\psi$ in $D(|A|)u$.

\[
\|B\psi\|^2 = \sum_{k \in \mathbb{N}} \langle \phi_k, B\psi \rangle \|\phi_k\|^2 \leq \sum_{k \in \mathbb{N}} \langle b_{k-1}, \phi_k \rangle \|\phi_k\|^2 + \langle b_{k}, \phi_k \rangle \|\phi_k\|^2 + \langle b_{k+1}, \phi_k \rangle \|\phi_k\|^2 \leq 2C^2 \sum_{k \in \mathbb{N}} \langle b_{k-1}, \phi_k \rangle \|\phi_k\|^2 + \langle b_{k}, \phi_k \rangle \|\phi_k\|^2 + \langle b_{k+1}, \phi_k \rangle \|\phi_k\|^2 \leq 6C^2 \|\psi\|^2\]

4.2 A toy model: the anharmonic oscillator

Consider the system.

\[
\frac{\partial \psi}{\partial x} = ([-\Delta + x^2] \alpha + u(t)x^3)\|\psi\|_1,
\]

with $x$ in $\mathbb{R}$, $\psi$ in $L^2(\mathbb{R}, C)$, $\alpha, \beta \in \mathbb{N}$.

Given $\alpha$ and $\beta$ satisfying Hypothesis 1.1, $\alpha$ is the most important quantum system, it is the standard quantum harmonic oscillator submitted to a uniform electric field. For $\beta = 1$ the system is tri-diagonal.

With our notations, $H = L^2(\mathbb{R}, C)$, $A : \psi \mapsto -i(-\Delta + x^2)\psi$ and $B : \psi \mapsto -ix\psi$. Operator $A$ is skew-adjoint on its domain $D(A), B$ is skew-symmetric. A Hilbert basis $\Phi$ of $L^2(\mathbb{R}, C)$ made of eigenvectors of $A$ is given by the sequence $(\phi_k)_{k \in \mathbb{N}}$ of the normalized Hermite functions.

\[
\phi_k : x \mapsto (-1)^k(2k!)\sqrt{\pi}^{-1/2}e^{-x^2/2} \frac{d^k}{dx^k} e^{-x^2}.
\]

For any $k \in \mathbb{N}$, the eigenvector $\phi_k$ is associated with the eigenvalue $-\lambda_k = -i(k + 1)^2$.

**Proposition 13.** If $2\alpha > \beta$ then system (11) satisfies Hypothesis 1. If $2\alpha > \beta$ then system (11) satisfies Hypothesis 2.

**Proof.** We show that the system satisfies Hypothesis 1.4 if $2\alpha > \beta$ and Hypothesis 2.6 if $2\alpha > \beta$. The system clearly fulfills all other hypotheses. For any $k \in \mathbb{N}$,

\[
x_k(x) = \sqrt{\frac{k}{2}} \phi_{k-1}(x) + \sqrt{\frac{k + 1}{2}} \phi_{k+1}(x).
\]
Iterating $\beta$ times this equality, one gets for any $k,$

$$|\langle x^\beta \phi_k, \psi \rangle| \leq \frac{C}{(k + \beta)^{3/2}} \sum_{j = -\beta}^\beta |\langle \phi_{k+j}, \psi \rangle|$$

hence, following the idea of the chain of inequalities (10) one has

$$\|B\psi\|^2 = \sum_{k \in \mathbb{N}} |\langle B\phi_k, \psi \rangle|^2 \leq C\|\psi\|^2 + 2^{-\beta} \sum_{k > \beta} (k + \beta)^{3/2} \sum_{j = -\beta}^\beta |\langle \phi_{k+j}, \psi \rangle|^2 \leq C\|\psi\|^2 + 2^{-\beta} \sum_{j = -\beta}^\beta (2k + 1)^{3/2} |\langle \phi_{k+j}, \psi \rangle|^2 \leq C\|\psi\|^2 + 2^{-\beta} (2\beta + 1)\|A(\beta/2\alpha)\| \psi\|^2,$$

which concludes the proof.

Thanks to Proposition 13, we can apply Proposition 6 and prove the well-posedness of (11)

**Proposition 14.** If $2\alpha = \beta,$ then (11) is well-posed for any control $u_t$ with bounded variation and $L^\infty$ norm smaller than $\sqrt{(2\beta + 1)^2 - \beta}.$ If $2\alpha > \beta,$ then (11) is well-posed for any control $u_t$ of bounded variation.

Notice that Proposition 10 applies also to systems that are not weakly-coupled, see (Boussaid et al., 2013, Definition 1). For instance, using the set $\{(k, k + 1), k \in \mathbb{N}\}$ as a non-resonant chain of connectedness, see (Boscain et al., 2012, Definition 2.5), and the fact that $\{(b_k, k + 1\})_{k \in \mathbb{N}}$ is in $l^1,$ we get the following.

**Proposition 15.** Assume that $\beta \geq 3$ odd and $\alpha > \beta/2.$ Then, there exists $K = \sum_{k \in \mathbb{N}} 2^{k+1} \geq 0$ such that, for any even functions $\psi_0, \psi_1$ in the unit sphere of $L^2(\mathcal{C}, \mathbf{C}),$ for any $\varepsilon > 0,$ there exists a control $u : [0, T_\varepsilon) \to \{0, +\infty\}$ such that $\|Y_{T_\varepsilon, 0}^{\psi_0} \psi_0 - \psi_1\|_{L^2} \leq \varepsilon$ and $\|u\|_{L^1([0, T_\varepsilon])} < K.$

In other words, if $2\alpha > \beta \geq 3$ and $\beta$ is odd, there is no Galerkin approximation for (11) in $L^2(\mathbb{R}, \mathbf{C})$ in terms of the $L^1$ norm of the control. However, from Proposition 10, system (11) admits a sequence of Good Galerkin approximations in $L^2(\mathbb{R}, \mathbf{C})$ in terms of the $(L^1 + TV)$ norm of the control.

### 4.3 Rotation of a 2D molecule

We consider a linear molecule whose only degree of freedom is the planar rotation, in a fixed plane, about its fixed center of mass. In this model, the Schrödinger equation reads

$$i\frac{\partial \psi}{\partial t} = -\Delta \psi + \cos\theta \psi, \quad \theta \in \Omega,$$

where $\Omega = \mathbf{R}/2\pi\mathbf{Z}$ is the unit circle endowed with the Riemanian structure inherited from $\mathbf{R},$ $H$ is the space of odd functions of $L^2(\Omega, \mathbf{C}),$ $A = i\Delta(\Delta$ is the restriction to $H$ of the Laplace-Beltrami operator of $\Omega$) and $B : \psi \mapsto (\theta \mapsto \cos(\theta)\psi(\theta))$ is the multiplication by cosine.

In the Hilbert basis $\Phi = (\theta \mapsto \sin(k\theta))_{k \in \mathbb{N}}$ of $H,$ $A$ is diagonal with diagonal $-i k^2, k = 1, \ldots, \infty$ and $B$ is tri-diagonal with $b_{k, k} = 0, b_{k, k+1} = -i/2$ for any $k \in \mathbb{N}.$

System 12 is both tri-diagonal and weakly-coupled and it has been thoroughly studied (see for instance Boscani et al. (2009) and Boscain et al. (2012)). For instance, it was known that (12) admits a sequence of Good Galerkin Approximations in terms of $L^1$ norm of the control. More preciselyly (Boussaid et al., 2013, Section IV.C) for every $\phi$ with norm 1 in span($\phi_1, \phi_2$),

$$\|X^u_{\{\phi_1, \phi_2\}}(t, 0) - \pi_{\mathbb{N}, T}(\phi)\| \leq \frac{K_{N-1}}{(N - 2)!}.$$

Approximate controllability of (12) was established in Boscain et al. (2012). In Chambrion (2012) is given an explicit control law to steer (12) from $\phi_1$ to any neighborhood of $\phi_2$ using periodic functions with frequency $2\pi/3.$ Defining $u_n := t \mapsto \cos(3t)/n$ and $T^* = 2\pi,$ we have $|\langle \phi_2, Y_{nT^*}^{\phi_1} \phi_1 \rangle| \leq \frac{9}{n}.$ Since $|\|B\psi\| \leq \sqrt{2|A\psi|}$ for every $\psi$ in $D(A),$ Proposition 3 implies that every control $u : [0, T] \to \{0, 1\} with bounded variation satisfying $|\langle \phi_2, Y_{nT^*}^{\phi_1} \phi_1 \rangle| > 1 - \varepsilon$ has total variation larger than $\log(2(1 - \varepsilon))/4.$ This lower bound is rather conservative, and we will give better estimates using the boundedness of $B.$

For every $u_1, u_2, t_1, t_2$ in $\mathbf{R},$ for every $\psi$ in $H,$ one has

$$\|\langle (A + u_1 B)e^{t_1(A+u_2 B)} \psi \rangle\| = \|\langle (A + u_2 B)e^{t_1(A+u_2 B)} \psi + (u_2 - u_1) Be^{t_1(A+u_2 B)} \psi \rangle\| \leq \|\langle (A + u_2 B)e^{t_1(A+u_2 B)} \psi \rangle\| + \|\langle (u_2 - u_1) Be^{t_1(A+u_2 B)} \psi \rangle\| \leq \|\langle (A + u_2 B) \psi \rangle\| + \|u_2 - u_1\| \|B\psi\|.$$
For any even function $\psi \in L^2(\mathbb{R}, \mathbb{C})$, for any $T > 0$, for any $\alpha > 0$, for any $\omega : [0, T] \to (\alpha, +\infty)$ with bounded variation, $\alpha > 0$, the inequality (13) admits a unique solution $t \mapsto \Upsilon_t^\omega \psi_0$ satisfying $\Upsilon_T^\omega \psi_0 = \psi_0$.

5. CONCLUSION

We obtained an elementary proof of the well-posedness of bilinear Schrödinger equations by adapting classical tools developed by Kato to the simple structure of bilinear conservative systems. The key ingredient of our construction is an a priori upper bound on the growth of some energy functional in terms of the total variation of the control.

As a consequence we prove a general method to obtain explicit bounds on the number of switches of a control steering the system from a given source to a given target, in the case in which the control takes value in a discrete set. These bounds are of importance when considering quantum systems for which the dipolar approximation (leading to a bilinear modeling as in the present paper) is not valid anymore, see Morancey (2011) and Boussaïd et al. (2012a).

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