A toolbox on the distribution of the maximum of Gaussian process

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Abstract

In this paper we are interested in the distribution of the maximum, or the maximum of the absolute value, of certain Gaussian processes for which the result is exactly known.

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1 Introduction

This paper is devoted to a unified presentation of some classical results on the distribution of the maximum of stochastic processes. These results have a rather simple form and can be put in a toolbox.
Apart from some general methods that give bound or asymptotic expansions for some classes of stochastic process like, for example, the Rice method [AW09]. There exist a short list of cases where the distribution is known exactly. The simplest and the most famous case is the Brownian motion (see below). A full list is given in [AW09] p. 3–4. Among it, some results are very complicated and are not included in the present presentation, some are very simple and some other need clarification see, for example, Section 2.4.
After recalling these classical results, adding some new ones, we give in Section 4 a description of the toolbox. Our intention is to put a set of dispersed results at the disposal of the community.
Throughout the paper, $X((t)$ (or $U(t)$,$W(t)$) is a random process with real values and

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parameters (except for the Brownian motion in dimension n, see Section 2.4). Set

$$M_t = \sup_{0<s<t} X(s),$$
$$M_t^* = \sup_{0<s<t} |X(s)|.$$

The aim of the paper is to compute $\mathbb{P}\{M_t \leq u\}$ and $\mathbb{P}\{M_t^* \leq u\}$ or similar expression.

2 Some results on Gaussian processes

2.1 The Brownian motion

Most of the result of this section and the next one are an application of the reflection principle and a good reference among many others is [Dud04] for all the results cited, see also [BS96].
Let $W(t)$ be a standard Brownian motion, we mean the 1-dimensional Brownian motion starting from zero (also called Wiener process).

Unilateral case: It is well-known that for $u > 0$,

$$\mathbb{P}\{M_t \leq u\} = 2\Phi\left(\frac{u}{\sqrt{t}}\right) - 1,$$

where Φ stands for the standard normal cumulative distribution function [Dud04, p.459].

We turn now to the bilateral case. The distribution of maximum absolute value of the Brownian motion is given by

$$\begin{aligned}\mathbb{P}\{M_t^* \leq u\} &= \sum_{k=-\infty}^{\infty} (-1)^k \left[\Phi\left(\frac{u(2k+1)}{\sqrt{t}}\right) - \Phi\left(\frac{u(2k-1)}{\sqrt{t}}\right)\right] \\ &= \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left(-\frac{(2k+1)^2\pi^2 t}{8u^2}\right).\end{aligned}$$

The second expression is less known, we refer the reader to [Fel70, p. 342]. A detailed proof can be found in [Chu74, p. 223]. This infinite sum has two advantages respect to the first one. Firstly, it involves only the exponential function. It avoids the computation of the cumulative distribution function of the normal distribution. Secondly, it converges very quickly for large values of t.

Extensions : The most simple is the case of a Brownian motion with a drift μ. In such a case

$$\mathbb{P}\{\sup_{0<s<t} W(s) + \mu s \geq u\} = \Phi\Big(\frac{u-\mu t}{\sqrt{t}}\Big) + \exp(2\mu u)\overline{\Phi}\Big(\frac{u-\mu t}{\sqrt{t}}\Big)$$

where $\overline{\Phi} = 1 - \Phi$. A proof can be found in [BS96]. A generalization of this result to the case of two different drifts is given in [Bou90].

Extensions are possible considering the case of the maximum of a Brownian motion $W(s)$ on an interval $[0,t]$, subject to the condition $W(T)=\eta$ where t and T are distinct and η is non-zero. We consider here the case $t<T$ and refer to [BO99] for the other case. Regression formulas imply that the conditional process can be represented as

$$X(s) = W(s) - \frac{s}{T}(W(T)-\eta).$$

This formula implies, in particular, that a linear drift μt can be added to the Brownian motion without modifying the expression of $X(s)$. The main results of [BO99] are

$$\mathbb{P}\left\{\sup_{0\le s\le t} X(s) \le u\right\} = \exp\left(-2\frac{u(u-\eta)}{T}\right)\Phi\left(\frac{u(2t-T)-\eta t}{\sqrt{Tt(T-t)}}\right) + \overline{\Phi}\left(\frac{uT-\eta t}{\sqrt{Tt(T-t)}}\right),$$

and for the bilateral case

$$\mathbb{P}\left\{\sup_{0\le s\le t} |X(s)| \le u\right\} = \sum_{k=-\infty}^{+\infty}(-1)^k \exp\Big(-2\frac{ku(ku-\eta)}{T}\Big)\times$$
$$\left[\Phi\left(\frac{uT-2ku(T-t)-\eta t}{\sqrt{Tt(T-t)}}\right) - \Phi\left(\frac{-uT-2ku(T-t)-\eta t}{\sqrt{Tt(T-t)}}\right)\right].$$

2.2 The Brownian Bridge

The simplest way of defining of the Brownian bridge is considering the distribution of $W(s), 0\le s\le 1$ conditional to $W(1)=0$. It can be written as $X(s)=W(s)-sW(1), 0\le s\le 1$.

For the unilateral case it holds:

$$\mathbb{P}\{M_1 \le u\} = 1-\exp(-2u^2),$$

which corresponds to the Rayleigh distribution. By a direct scaling argument we can extend the result to an interval $[0,t]$,

$$\mathbb{P}\left\{\sup_{0\le s\le t} W(s) - \frac{s}{t}W(t) \le u\right\} = 1-\exp\left(-2\frac{u^2}{t}\right).$$

Bilateral case: the distribution of the maximum of the absolute value is given by the infinite sum

$$\mathbb{P}\left\{\sup_{0\le s\le t}\left|W(s)-\frac{s}{t}W(t)\right| \le u\right\} = \sum_{k=-\infty}^{\infty}(-1)^k\exp\left(-2\frac{k^2u^2}{t}\right).$$

2.3 The Sine-Cosine process

Let $X(t)$ be the process defined by

$$X(t) = \xi_1 \cos(\omega t) + \xi_2 \sin(\omega t), \tag{2.1}$$

where ω is a positive parameter and ξ_1, ξ_2 are two standard normal variables. This process can be written equivalently as

$$X(t) = Z \cos(\omega t + \theta),$$

where Z and θ are independent random variables with Rayleigh and uniform (on $[0, 2\pi]$) distribution respectively.
First it is clear that for $t > 2\pi/\omega$, $M_t = M_t^* = Z$. For $u > 0$ and $0 < t \leq \pi/\omega$, the distribution of M_t was given by Berman [Ber71]. Delmas [Del03] gives it for $\pi/\omega < t < 2\pi/\omega$. Putting these results together, we obtain the following general statement. For $u > 0$,

$$\mathbb{P}\{M_t \leq u\} = \begin{cases} \Phi(u) - \frac{t\omega}{2\pi} e^{-u^2/2}, & 0 < t \leq \pi/\omega, \\ \Phi(u) - \dfrac{t\omega}{2\pi} e^{-\frac{u^2}{2}} + \dfrac{1}{2\pi} \displaystyle\int_\pi^{t\omega} \exp\left[- \frac{u^2\big(1 - \cos(s)\big)}{\sin^2(s)} \right] ds, & \pi/\omega < t < 2\pi/\omega, \\ 1 - e^{-\frac{u^2}{2}}, & t \geq 2\pi/\omega. \end{cases}$$

Observe that in fact the second case is the most general and it can be extended, with some precautions, to the case $u < 0$ which is anyway not very relevant.
For M_t^* we can establish a similar result using the method of Delmas. As far as we know, this result is new. See detailed proof in Section 3.

Proposition 2.1. *For $u > 0$*

$$\mathbb{P}\{M_t^* \leq u\} = \begin{cases} 2\Phi(u) - 1 - \dfrac{t\omega}{\pi} e^{-\frac{u^2}{2}} + \dfrac{1}{\pi} \displaystyle\int_0^{t\omega} \exp\left[- \frac{u^2\big(1 + \cos(s)\big)}{\sin^2(s)} \right] ds, & 0 \leq t \leq \pi/\omega \\ 1 - e^{-\frac{u^2}{2}}, & t > \pi/\omega. \end{cases}$$

2.4 The normalized Bessel process

Let $W_n(t)$ be a n-dimensional Brownian motion. The normalized Bessel $U_n(t)$ process of order n is defined as

$$U_n(t) = \frac{\|W_n(t)\|}{\sqrt{t}},$$

where $\| \cdot \|$ stands for the corresponding Euclidean norm. Set

$$G_t^n(u) = \mathbb{P}\left\{ \sup_{1<s<t} U_n(s) \leq u \right\}.$$

DeLong [DeL81] gives a formula for $G_t^n(u)$. The explicit expression, where we have corrected a little sign error in (2.4) as pointed out also by Estrella [Est03], is

$$G_t^n(u) = \sum_{i=1}^{\infty} \alpha_i(u) t^{-\beta_i(u)} \tag{2.2}$$

where $\beta_i(u)$ is the i^{th} root as a function of x of

$$M\left(x + \frac{n}{2}, \frac{n}{2}, -\frac{u^2}{2}\right) = 0, \tag{2.3}$$

M represents the confluent hypergeometric function (see [AS72, §13]) and

$$\alpha_i(u) = -\frac{e^{-u^2/2}(u^2/2)^{n/2} M(\beta_i + n/2, n/2+1, -u^2/2)}{\Gamma(n/2+1)\beta_i(u)\frac{\partial}{\partial E} M(E + n/2, n/2, -u^2/2)\big|_{E=\beta_i(u)}}. \tag{2.4}$$

For $n = 1$ it is possible to consider the one-sided problem. More precisely, now we focus on calculating

$$\mathbb{P}\left\{\max_{1<t<t} \frac{W(s)}{s^{1/2}} \leq u\right\} \tag{2.5}$$

where $W(t)$ is the 1-dimensional standard Brownian motion.
Tables of values computed from (2.5) were presented by Delong [DeL81] but without giving the explicit formula he used. Following the DeLong's procedure, Estrella and Rodrigues [ER05, formula (19)] obtain

$$\mathbb{P}\left\{\max_{1<s<t} \frac{W(s)}{s^{1/2}} \leq u\right\} = \sum_{k=1}^{\infty} -t^{-v_k/2} \frac{\phi(u) D_{v_k-1}(-u)}{v_k \frac{\partial}{\partial v} D_v(-u)\big|_{v=v_k}} \tag{2.6}$$

where D is the parabolic cylinder function (see [AS72, §19]), v_i is i-th root of $D_v(-u) = 0$ as a function of v and ϕ stands for the standard normal density.

2.5 The Ornstein-Uhlenbeck process

In this paper the Ornstein-Uhlenbeck(OU) process $X(t)$ will be defined as the centered Gaussian stationary process with covariance $r(t) = \exp(-|t|)$. This process and the Bessel process are related by the following. When $n = 1$, the following representation holds

$$|X(t)| \stackrel{D}{=} U_1(e^{2t}).$$

Moreover if $X_1(t), ..., X_n(t)$ are n independent copies of the OU process,

$$X_1^2(t) + \cdots + X_n^2(t) \stackrel{D}{=} U_n^2(e^{2t}).$$

Let define

$$F_t(u) = \mathbb{P}\left\{ \sup_{s\in[0,t]} X(t) \leq u \right\},$$
$$F_t^*(u) = \mathbb{P}\left\{ \sup_{s\in[0,t]} |X(t)| \leq u \right\}.$$

We have in particular

$$F_t^*(u) = G^1_{\exp(2t)}(u) = \sum_{i=1}^{\infty} \alpha_i(u) \exp\big(- \beta_i(u)t\big) \tag{2.7}$$

where G is defined in (2.2). The first results in the literature concern the unilateral case and dates back to Slepian [Sle62]. A direct application of the reflection principle for the level $u = 0$ shows that

$$F_t(0) = \frac{1}{2} - 2\mathbb{P}\Big\{X(0) < 0, X(t) > 0\Big\} = \frac{2}{\pi} \arcsin\left(e^{-t}\right).$$

Another interesting and rather unknown result is due to Newell and Rosenblatt [NR62]. They give an expression of $F_t(u)$ under the form of a series with non very explicit terms. In fact Formula (2.6) implies that

$$F_t(u) = -\sum_{k=1}^{\infty} \frac{\phi(u) D_{v_k-1}(-u)}{v_k \frac{\partial}{\partial v} D_v(-u)\big|_{v=v_k}} \exp(-tv_k).$$

3 Proof of proposition 2.1

Without loss of generality we can assume that $\omega = 1$. We have, since we are considering at most a half period

$$\begin{aligned}\mathbb{P}\{M_t^* > u\} &= \mathbb{P}\{|X(0)| > u\} + \mathbb{P}\{|X(0)| < u, \mathbf{u}_u > 0\} + \mathbb{P}\{|X(0)| < u, \mathbf{d}_{-u} > 0\} \\ &= \mathbb{P}\{|X(0)| > u\} + 2\mathbb{P}\{|X(0)| < u, \mathbf{u}_u > 0\}\end{aligned}$$

where $\mathbf{d}_{-u}$ and $\mathbf{u}_u$ are the numbers of up-crossings of u. The variable $\mathbf{u}_u \ \mathbb{1}_{\{|X(0)|\leq u\}}$ takes the value 0 or 1. Thus by the Rice formula

$$\begin{aligned}\mathbb{P}\{|X(0)| < u, \mathbf{u}_u > 0\} &= \mathbb{E}\left(\mathbf{u}_u \ \mathbb{1}_{|X(0)|\leq u}\right) \\ &= \int_0^t ds \int_{-u}^{u} \ \mathbb{E}\big((X'(s))^+ \big| X(0) = x, X(s) = u\big) p_{X(0),X(s)}(x,u)dx \end{aligned} \tag{3.1}$$

Using Formula (2.1), it is easy to check that under the condition $X(0) = x$, $X(s) = u$,

$$X'(s) = \frac{1}{\sin(s)}[u\cos(s) - x].$$

On the other hand

$$p_{X(0),X(s)}(x,u) = \frac{1}{2\pi\sin(s)}\exp\Big(-\frac{x^2+u^2-2\cos(s)xu}{2\sin^2(s)}\Big).$$

The right hand side of (3.1) is equal to

$$\begin{aligned}\int_0^t ds \int_{-u}^{u\cos(s)} \frac{1}{2\pi}e^{-u^2/2}\frac{[u\cos(s)-x]^2}{\sin^2(s)}\exp\Big(-\frac{1}{2}\frac{u\cos(s)-x}{\sin^2(s)}\Big)dx \\ = \frac{1}{2\pi}e^{-u^2/2}\int_0^t 1-\exp\Big(-\frac{u^2(1+\cos(s))^2}{2\sin^2(s)}\Big)ds \\ = \frac{t}{2\pi}e^{-u^2/2} - \frac{1}{2\pi}\int_0^t \exp\Big(-\frac{u^2(1+\cos(s))}{\sin^2(s)}\Big)ds.\end{aligned}$$

4 Writing a toolbox

Since most of the special functions are available in Matlab, the second author made a toolbox `MAXGPBOX` that is available at
`http://www.math.univ-toulouse.fr/~azais/softwares.php`
and that permits to make all these computations. The more complex formulae we work with in this paper are those concerning the Bessel and OU process. Therefore, before describe the toolbox, it is worth discussing some few aspects about the numerical computations involved.

4.1 Computing (2.2) and (2.6)

In the computation of (2.2) there is involved the finding of the roots of equation (2.3). This part is the more complex and computational time demanding. Following Estrella (see [Est03] for details) we compute the roots using the Matlab standard function. The initial zero approximation is giving by steps of length 1. Propositions of [Est03] prove that this choice is optimal to capture all the roots. On the other hand, the partial sums in the RHS of (2.2) converge very quickly. Thus only a few roots are necessary to obtain a accurate approximation of the infinite sum, and consequently of the computation of G_t^n. Typical with 5 roots an approximation of order 10^{-5} is obtained.

To compute (2.6), once again, the challenging part is the calculation of the roots. Although, there is not similar Propositions to those of [Est03] to localize the roots, in [ER05] the authors use a similar grid strategy to find the zeros with good results. In our experience, we need to use a grid of large 0.5 to getting all the roots. As pointed by Estrella and Rodrigues we notice that the computation of (2.6) is more complex than (2.2). See [ER05] for the reasons of this.
To evaluate our implementation of algorithms we compute the values of G_t^n to compare with the ones in [And93], [DeL81], [Est03] and [ER05]. We also ran simulations to calculated the values. In general, for the two-sided problem our results matched Estrella's.

In the case of DeLong's the differences correspond to those found and explained by Estrella, i.e. $n = 1$, $u = 1$. In the case of the one-sided problem, our computations permit to recover the critical values in Table 1 of [ER05] and the values in DeLong's Table 1 [DeL81, p. 2210].

4.2 Toolbox description

Throughout the section the suffix `xx` in the name of routines represents any of the suffixes in the previous table.

suffix	stands for
bb	Brownian Bridge
bm	Brownian Motion
bp	normalized Bessel Process
ou	Ornstein-Uhlenbeck process
sc	Sine-Cosine process

Routines:

`pxx`: Gives the value of the distribution function at u of the maximum the process.

`qxx`: Gives the inverse of the distribution function at p of the maximum the process.

`pnxx`: Gives the value of the distribution function at u of the maximum norm of the process.

`qnxx`: Gives the inverse of the distribution function at u of the maximum norm of the process.

For details, about the routine call, and parameters see the comments inside each program.

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