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Dynamics of Coupled Oscillators Excited by Dry Friction

Madeleine Pascal

In this paper, we present an analytical method to investigate the behavior of a two-degree-of-freedom oscillator excited by dry friction. The system consists of two masses connected by linear springs. These two masses are in contact with a driving belt moving at a constant velocity. The contact forces between the masses and the belt are obtained assuming Coulomb’s friction law. Two families of periodic motions are found in closed form. The first one includes stick-slip oscillations with two switches per period, the second one is also composed of stick-slip motion, but includes three switches per period. In both cases, the initial conditions and the time duration of each kind of motions (stick or slip phases) are obtained in analytical form.

Keywords: coupled oscillators, dry friction, periodic motions, analytical method, stick-slip motion

1 Introduction

Vibrating systems excited by dry friction are frequently encountered in technical applications, including turbine blade joints, robot joints, electric motor drives, wheel rail coupling of mass transit systems, and brake systems. These systems are strongly nonlinear and they are usually modeled as spring-mass oscillators. They have been the subject of several investigations [1,2]. In the simplest case of a one-degree-of-freedom system [3], an analytical investigation of the system’s behavior, including periodic motions, chatter, and chaos, have been performed. In the case of multidegrees-of-freedom systems [4], only the numerical approaches have been used. In this work, our attention is focused on the analysis of an undamped self-excited two-degree-of-freedom oscillator with dry friction. We assume Coulomb’s friction law and, by using the method already presented in Ref. [5] for a coupled oscillator with impact, periodic motions with stick-slip oscillations are obtained by analytical methods.

2 Model

The system (Fig. 1) consists of two masses $m_1, m_2$, connected by linear springs $k_1, k_2$. These two masses are in contact with a driving belt moving at a constant velocity $v_0$. Friction forces $F_1, F_2$ act between the masses $m_1, m_2$ and the belt.

This two-degree-of-freedom oscillator is governed by the following differential system:

$$MX'' + KX = R, \quad X = (x_1, x_2)' \quad R = (F_1, F_2)'$$

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad K = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}$$

$$X = X(t'), \quad \ddot{X} = \frac{d^2X}{dt'^2}$$

Here, $F_1, F_2$ are the contact friction forces obtained from Coulomb’s laws:

$$v_0 - \dot{x}_i \neq 0, \quad F_i = F_{si} \text{ sign}(v_0 - \dot{x}_i) \quad (i = 1, 2)$$

$$v_0 - \dot{x}_1 = 0$$

$$F_1 = \begin{cases} (k_1 + k_2)x_1 - k_2x_2 & \text{if } |(k_1 + k_2)x_1 - k_2x_2| < F_{s1} \\ eF_{s1} & \text{if } e[(k_1 + k_2)x_1 - k_2x_2] > F_{s1} \quad (e = \pm 1) \end{cases}$$

$$v_0 - \dot{x}_2 = 0$$

$$F_2 = \begin{cases} k_2(x_2 - x_1) & \text{if } |k_2(x_2 - x_1)| < F_{s2} \\ eF_{s2} & \text{if } e(k_2(x_2 - x_1) > F_{s2} \quad (e = \pm 1) \end{cases}$$

$$F_{s1}, F_{s2}$$ are the friction forces when slip motion occurs, while $F_{s1}, F_{s2}$ are the static friction forces. The systems (1) and (2) are normalized using

$$t = \omega_0 t', \quad \omega_0 = \sqrt{\frac{k_1 + k_2}{m_1}}, \quad \eta = \frac{m_1}{m_2}$$

$$x_1 + X - x_2 = u_1, \quad \chi = \frac{k_2}{k_1 + k_2}, \quad \eta = \frac{m_1}{m_2}$$

$$x_2 + \chi x_2 - x_1 = \eta u_2, \quad u_i = \frac{F_i}{k_1 + k_2} \quad (i = 1, 2)$$

3 Prediction of the Oscillations Exhibited by the System

The coupled oscillator system excited by dry friction is modeled as a piecewise-linear system. The dynamical behavior is very complex and includes slip or stick oscillations of the masses. All possible motions of the system are composed of several phases of stick motion for both masses, stick motion of $m_1$, and slip motion of $m_2$ or vice versa and at last, stick motion for the two masses. For each kind of motions, the close form solution can be obtained.

3.1 Slip Motion of the Two Masses. This motion is related to the following initial conditions: $x_{i0}, \dot{x}_{i0} < V \quad (i = 1, 2), \quad (V = v_0/\omega_0)$. The motion of the oscillators is given by the solution of Eq. (3) with $u_i = u_{i0} \quad (i = 1, 2)$:

$$Z'(t) = H(t)Z_0, \quad Z' = \begin{pmatrix} X - d_0 \\ \chi \dot{X} \end{pmatrix}, \quad Z_0 = \begin{pmatrix} X'(0) \end{pmatrix}, \quad X = (x_1, x_2)'$$

$$H = \begin{pmatrix} H_1 \bar{H}_2 \\ \bar{H}_3 H_1 \end{pmatrix}, \quad H_i = \Lambda B \Lambda^{-1} \quad (i = 1, 2, 3)$$
The motion equations are given by

\[ \begin{align*}
\Sigma = \begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} \cos t & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} \sin t & 0 \\ 0 & 1 \end{pmatrix}, \\
\gamma_3 = \gamma_2', \\
\delta_0 = (u_{i1},0)' 
\end{align*} \tag{14} \]

For the \( \Gamma_i \) matrices, the following properties hold:

\[ \Gamma_1^2 - \Gamma_2 \Gamma_3 = I, \quad \Gamma_i \Gamma_j = \Gamma_j \Gamma_i \quad (i,j = 1,2,3) \tag{15} \]

The solution can be also formulated as

\[ Z'(t) = \Gamma(t)Z_0, \quad Z^* = \begin{pmatrix} X - d_0 \\ X' \end{pmatrix} \tag{16} \]

One variant of this solution is related to the initial conditions:

\[ x'_{10} > V, \quad x'_{20} = V, \quad \chi|x_{20} - x_{10}| < u_{i2} \]

### 3.3 Stick Motion of the First Mass, Slip Motion of the Second Mass

This motion is related to the following initial conditions:

\[ x'_{10} = V, \quad x'_{20} < V, \quad |x_{10} - \chi x_{20}| < u_{i1} \]

and to the following motion equations:

\[ x_i'' = 0 \tag{7} \]

The solution is given by

\[ Z'(t) = C(t)Z_0, \quad Z^* = \begin{pmatrix} X - e_0 \\ X' \end{pmatrix}, \quad Z_0 = Z'(0), \quad C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} \tag{8} \]

\[ C_i = \Pi A_i \Pi^{-1} \quad (i = 1,2,3), \quad \Pi = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \]

\[ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\sin(\omega t)}{\omega} \end{pmatrix}, \quad A_2 = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}, \quad A_3 = A_1' \tag{10} \]

As for the \( \Gamma_i \) matrices, the following properties hold for the \( C_i \) matrices:

\[ C_i^2 - C_i C_j = I \quad (i,j = 1,2,3) \tag{11} \]

\[ (C_i - I)(e_0 - d_0) = 0, \quad \Pi C_i(e_0 - d_0) = 0 \tag{12} \]

Eq. (18) can be formulated as

\[ Z'(t) = C(t)Z_0, \quad Z^* = \begin{pmatrix} X - d_0 \\ X' \end{pmatrix} \tag{20} \]

One variant of this solution is related to the initial conditions

\[ x'_{10} = V, \quad x'_{20} > V, \quad |x_{10} - \chi x_{20}| < u_{i1} \]

### 3.4 Stick Motion of the Two Masses

This last case is related to the initial conditions:

\[ x'_{10} = x'_{20} = V, \quad |x_{10} - \chi x_{20}| < u_{i1}, \quad \chi|x_{20} - x_{10}| < u_{i2} \]

The two masses perform a stick motion given by:

\[ X = X_0 + tX'_0, \quad X' = X'_0 \tag{21} \]
4 Periodic Motions With Transitions Between Slip and Stick Motions

A transition between slip and stick motions occurs if at least one of the velocities of the masses reaches the velocity of the belt. In this case, the eventual transition depends on some criteria related to the restoring force applied to the mass (i.e., on Coulomb’s laws (2)). A detailed description of the analytical predictions of the stick and nonstick motions for a one-degree-of-freedom damped and forced oscillator with dry friction can be found in Ref. [6]. In the following, periodic solutions including two or three transitions between each kind of motions are obtained in analytical form.

4.1 Solution I: Periodic Motion With Two Transitions per Period. Let us assume that at $t=0$,

$$
\chi(x_{20} - x_{10}) = u_{22} \text{ (22)}
$$

From Eq. (3), we deduce $x'_3(0) = \eta(u_{22} - u_{22})$.

A motion with $u_2 = u_2$ gives $x'_3(0) < 0$; hence for $t > 0$, $x'_3$ is a decreasing function of $t$, and for some period of time $0 < t < \tau$ $x'_3 < V$. The motion of the oscillators is given by Eq. (4).

This motion finishes at time $t = \tau$, if $x'_3 = \chi(V) = V$. Let us assume that $X_i = X(\tau), X'_i = X'(\tau)$ are given by

$$
X_i - d_0 = -(X_0 - d_0), \quad X'_i = X'_0 \quad (i = 1, 2, 3) \text{ (23)}
$$

From Eq. (4), it follows that $X_0, X'_1$ are deduced from the conditions:

$$
(H_1 + I)(X_0 - d_0) + H_2 X'_0 = 0
$$

$$
H_3(X_0 - d_0) + (H_1 - I)X'_0 = 0
$$

$$
H_i = H_i(\tau) \quad (i = 1, 2, 3) \text{ (24)}
$$

This system provides four scalar equations for the determination of the three unknown parameters $x_{10}, x_{10}, \tau$ ($x_{20}, x_{20}$ are deduced from conditions (22)). Taking into account the properties (8) of the $H_i$ matrices, these equations are not independent and reduce only to two scalar equations. From Eq. (23), we deduce

$$
\chi(x_{20} - x_{10}) = -\chi(x_{20} - x_{10}) + 2\chi(d_0 - d_0) = -u_{22} + 2u_{22} < u_{22} \quad \text{ (25)}
$$

It results that for $t > \tau$, $\chi(x_{20} - x_{10}) < u_{22}$, and the first mass performs a slip motion while the second one performs a stick motion. From Eq. (16), the solution is written in the following form:

$$
Z'(t) = \Gamma(t - \tau)Z'_0, \quad Z'_0 = EZ'_0, \quad E = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \text{ (26)}
$$

This motion ends at $t = \tau + T$, when $\chi(x_{20} - x_{10})$ reaches the limiting value $u_{22}$. This condition is fulfilled if we assume that $x_i(\tau + T) = x_{i0}, x'_i(\tau + T) = x'_{i0}, (i = 1, 2, 3)$.

It results to the following conditions:

$$
(\Gamma_1 + I)(X_0 - d_0) - \Gamma_2 X'_0 = 0
$$

$$
-\Gamma_2(X_0 - d_0) + (H_1 - I)X'_0 = 0
$$

$$
\Gamma_i = \Gamma_i(\tau) \quad (i = 1, 2, 3) \text{ (27)}
$$

Taking into account the properties (15) of the $\Gamma_i$ matrices, these four scalar equations are not independent and reduce only to two scalar equations. It results that Eqs. (24) and (27) provide four scalar equations for the determination of the four parameters $x_{10}, x'_{10}, \tau, T$. To each solution of these equations, a periodic motion of the coupled oscillators of period $\tau + T$, with two transitions per period, is obtained.

In conclusion, the initial conditions and the duration times of the slip motion $\tau$ and of the stick motion $T$ related to a periodic solution (with one slip-slip mode and one stick-slip mode per period) are obtained from the system

$$
x_{20} = VT - d_0, \quad x_{10} = x_{20} - u_{22} \times x_{10} = \chi V + \cot \left( \frac{1}{2} \right) \left( \frac{u_{22} - u_{22}}{x_{10}} + \frac{VT}{x_{10}} (1 - \chi) \right) \text{ (28)}
$$

$$
(H_1 + 1)\left( \frac{u_{22} - u_{22}}{x_{10}} + \frac{VT}{x_{10}} \right) + H_2 \left( \frac{VT}{x_{10}} \right) + H_3 x_{10} + H_4 V = 0
$$

$$
H_2 \left( \frac{u_{22} - u_{22}}{x_{10}} + \frac{VT}{x_{10}} \right) + (H_2 + 1) \left( \frac{VT}{x_{10}} \right) + H_2 x_{10} + H_2 V = 0
$$

$$
H_1 = H_i(\tau, i, j = 1, 2), \quad H_2 = H_i(\tau, i, j = 3, 4) \text{ (28)}
$$

However, to obtain a realistic solution, the results deduced from system (28) must fulfill the conditions

$$
\tau > 0, \quad T > 0, \quad x'_1 - V < 0, \quad x'_2 - V < 0 \quad (0 < t < \tau)
$$

$$
z_1 = \chi(x_2 - x_1) - u_{22} < 0
$$

$$
z_2 = \chi(x_2 - x_1) + u_{22} > 0 \quad (\tau < t < \tau + T) \text{ (29)}
$$

An interesting property of symmetry can be found for this solution (see Appendix and Ref. [7]):

$$
Z(t) = EZ(\tau - t), \quad 0 < t < \tau/2
$$

$$
Z(t) = EZ(2\tau + T - t), \quad T < t < \tau + T/2
$$

It seems that a similar periodic solution without this property cannot be found for any set of data. It is not difficult to show that dissymmetrical periodic solutions such as Solution I can be found only for some particular values of the data.

4.2 Solution II: Periodic Solution With Three Transitions per Period. Let us assume that at $t = 0$,

$$
x'_{30} = x'_2 = V, \quad x_{10} - \chi(x_{20} - x_{10}) < u_{22} \quad \text{ (30)}
$$

For some period of time $0 < t < \tau$, the motion is composed of a slip motion for $m_1(u_{11} = u_{11}, x'_i \neq V)$ and a stick motion for $m_2$:

$$
Z'(t) = \Gamma(t)Z'_0 \text{ (31)}
$$

A transition occurs at $t = \tau$ if

$$
\chi(x_{20} - x_{10}) = u_{22}, \quad x_{10} = x_{10}(\tau) \quad (i = 1, 2) \text{ (32)}
$$

For a new period of time $(\tau < t < \tau + \tau_1)$, the two masses undergo a slip motion:

$$
Z'(t) = H(t - \tau)Z'_0, \quad Z'_0 = Z'(\tau) \text{ (33)}
$$

Again, a switch occurs at $t = \tau + \tau_1$ if

$$
x'_{1d} = x'_1(\tau + T) = V, \quad x'_{2d} = x'_2(\tau + T) = x'_1(\tau + T) \quad (i = 1, 2) \text{ (34)}
$$

For $\tau + \tau_1 < t < \tau + \tau_1 + T$, the first mass undergoes a stick motion, while the second one undergoes a slip motion:

$$
Z'(t) = C(t - \tau - \tau_1)Z'_0, \quad Z'_0 = Z'(\tau + \tau_1) \text{ (35)}
$$

A periodic solution of period $\tau + \tau_1 + T$ is obtained under the conditions

$$
Z'(\tau + \tau_1 + T) = C(T)Z'_0 \text{ (36)}
$$

This condition includes the relation $x'_{1d} = V$.

The obtained periodic solution depends on the four parameters: $x_{20}$ (the other initial conditions are obtained from Eq. (30)), and...
\( \tau, \tau_1, T \) (time duration of each part of the periodic solution). On the other hand, the conditions (32) and (36) provide five scalar equations for the determination of these parameters. It results that in contrast to the periodic Solution I, this new solution is obtained only for some particular set of the data, which includes \( \chi, \eta, V, u_{i1}, u_{i2}, (i=1,2) \). The periodicity condition (36) gives the following equations:
\[
N_1(X_0 - d_0) + N_2 X_0' = 0 \\
N_3(X_0 - d_0) + N_4 X_0' = 0 \\
N = C(T)H(\tau_1)\Gamma(\tau) - I_4 = \begin{pmatrix} N_1 & N_2 \noalign{
\vskip 1ex}
N_3 & N_4 \end{pmatrix} \\
I_4 = \begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix}, \quad X_0' = \begin{pmatrix} V \\
V \end{pmatrix}
\]
(37)
From Eq. (37), we deduce two compatibility conditions:
\[
M_{11} + M_{12} = 0, \quad M_{21} + M_{22} = 0 \\
M = \begin{pmatrix} M_{11} & M_{12} \\
M_{21} & M_{22} \end{pmatrix} = N^{-1}_1 N_2 - N^{-1}_1 N_4
\]
(38)
The relations (38) provide two equations linking the time duration \( (\tau, \tau_1, T) \) of each part of the motion. From Eq. (37), we deduce the value of \( X_0 - d_0 \):
\[
X_0 - d_0 = -N^{-1}_1 N_2 X_0' 
\]
(39)
This last result, together with Eq. (38), gives four scalar equations for the determination of the parameters \( x_0, \tau_1, \tau, T \) in terms of the data. Inserting these values in the condition (32), a compatibility condition for the existence of such a periodic solution is obtained. This compatibility condition provides, for example, the value of \( u_{i2} \), in terms of the data \( (\chi, \eta, V, u_{i1}, u_{i2}, (i=1,2)) \).

5 Numerical Applications

Numerical validations will be made by using the MATLAB program for several values of the data \( \chi, \eta, V, u_{i1}, u_{i2}, (i=1,2) \) and for each kind of periodic motions.

Solution I. From Eq. (28), the time duration \( \tau \) of the slip motion is obtained from the condition
\[
F(\tau) = a_{11} + a_{12} - \chi(a_{21} + a_{22}) + B\varphi_1 \varphi_2 \\
+ [1 - \chi + B(a_{22} + \chi a_{21})] \cot\left(\frac{B\varphi_2 - 1}{a_{21} + a_{22}}\right) \\
= 0 \quad (40)
\]
\[
a_{11} = d(\lambda_1 \varphi_2 - \lambda_2 \varphi_1), \quad a_{12} = d(\varphi_1 - \varphi_2) \\
a_{21} = d(\lambda_1 \varphi_2 - \varphi_1), \quad a_{22} = d(\lambda_1 \varphi_1 - \lambda_2 \varphi_2) \\
d = \frac{1}{\lambda_2 - \lambda_1}, \quad B = \frac{u_{i2} - u_{i1}}{V}, \quad \varphi_i = \omega_i \cot(\omega_i \tau/2) \quad (i=1,2)
\]
(41)
For the following values of the parameters:
\[
\chi = 0.4001, \quad \eta = 4, \quad V = 0.6773, \quad u_{i1} = 0.0127 \\
u_{i2} = 0.5, \quad B = -2.4994
\]
we obtain \( \tau = 2.65, \quad T = 2.2407 \).

The corresponding behaviors of the two masses are shown on Figs. 2 and 3.

The constraints
\[
x_0' - V < 0 \quad (0 < t < \tau)
\]
two kinds of periodic solutions for a two-degree-of-freedom oscillator excited by dry friction have been obtained in analytical form. These two solutions differ by the number of switches occurring during one period. In Ref. [4], similar solutions have been obtained by numerical methods. However, in the work performed in Ref. [4], the friction model is not Coulomb’s friction law.

The investigation of other kinds of periodic motions will be the subject of future researches, together with the analysis of the stability of the obtained periodic motions.

Appendix: Symmetry of the Solution

For 0 < t < \tau/2,

\[ Z(t) = H(t)Z_0 \]

From the identities

\[ H(t) = H(-t)H(\tau), \quad Z(t) = H(\tau)Z_0 = EZ_0, \quad H(-t)E = EH(t) \]

it results to

\[ Z(t) = EH(t)Z_0 = EZ(t) \]

In particular, for t = \tau/2, this relation gives

\[ Z(\tau/2) = EZ(\tau/2) \]

Hence, the following relation holds:
$X(\tau/2) = d_0$

A similar property of symmetry is obtained for the other part of the solution $(\tau < t < \tau + T)$. In particular, $X(\tau + T/2) = d_0$.

References


