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Modeling and simulation of sutured biomembranes

A. Atai^a, D.J. Steigmann^{b,*}

^a Department of Mechanical Engineering, University of Tehran, Tehran, Iran

^b Department of Mechanical Engineering, University of California, Berkeley, CA 94720, USA

ABSTRACT

The method of dynamic relaxation is applied to the analysis of wrinkling and slackening that accompany suturing in biological membranes composed of isotropic nonlinearly elastic materials.

1. Introduction

A number of investigations of the elasticity of biological tissues have been based on nonlinear membrane theory (see, for example, Davies et al., 1999; Lubarda, 2010). The version of membrane theory known as *tension-field theory* has also been applied to the study of wrinkling occurring in biological membranes when the edges of an internal boundary are sutured together (Danielson and Natarajan, 1975; Massabò and Gambarotta, 2007; Cavichi et al., 2009). Tension-field theory has the feature that it admits only tensile states of stress, extending the range of applicability of membrane theory to states of strain that would otherwise generate destabilizing compressive stresses.

An extended membrane theory that incorporates tension-field theory automatically may be obtained from conventional membrane theory via the process of *relaxation*, which entails the construction of an energy minimizing sequence of deformations containing ever-more finely spaced wrinkles (Pipkin, 1986). The limit of the sequence is a smooth deformation, and the strain-energy function associated with the limiting deformation is such that compressive stresses are automatically excluded. The resulting equilibrium stress distribution is called a tension field. The same model emerges from three-dimensional nonlinear elasticity via the method of gamma convergence (Le Dret and Raoult, 1995), a technique for extracting the leading-order variational problem in the small-thickness limit. These works are concerned with the form of the strain-energy function – the *relaxed energy* – associated with energy-minimizing membrane deformations. Further

work, in which the associated equilibrium theory is developed and analytical solutions are obtained, is given by Steigmann (1990, 2005), Steigmann and Pipkin (1989), Li and Steigmann (1993, 1995a,b), and Nadler and Steigmann (2006), while numerical solutions are discussed in Haseganu and Steigmann (1994), Haseganu and Steigmann (1996) and Atai and Steigmann (1998).

The obvious alternative to relaxation is to regularize the equations by including higher-order gradient (curvature) terms in the strain-energy function. This is tantamount to using plate theory in place of membrane theory. The advantage is that a length scale is then built directly into the model, which effectively determines the wavelength and amplitude of the wrinkles. This arises from the fact that the energetic penalty associated with curvature sets a lower bound to the wavelength. This is clearly more realistic than the idealizations inherent in relaxed membrane theory, where the absence of bending stiffness implies the absence of a lower bound and thus a continuous distribution of wrinkles over the membrane surface. Fortunately, the associated equilibrium equations for the tension field are statically determinate (Steigmann, 1990); the stress in a wrinkled region is thus insensitive to the details of the deformation and hence to the errors incurred in its description by adopting relaxed membrane theory. In this way the stress in a wrinkled region may be described using the relatively simple membrane theory, suitably relaxed to accommodate wrinkling.

However, offsetting the advantages of relaxed membrane theory is the fact that the absence of stress in certain connected regions of strain space implies an absence of stiffness also, so that conventional stiffness-based solution strategies for the computation of equilibria are either ineffective or require special measures. To overcome this difficulty we embed the equilibrium equations into an artificial damped dynamical system, constructed in such a way that the desired equilibria are globally asymptotically stable with

* Corresponding author.

E-mail address: steigman@newton.berkeley.edu (D.J. Steigmann).

respect to arbitrary initial data. We then discretize this system using a finite-difference method based on Green's theorem, and forward integrate in time using a simple difference scheme. Temporal accuracy is not an issue, as it is only the asymptotic states that are of interest. This allows for the use of simple explicit finite difference operators with mass-proportional damping to achieve an efficient vectorized system for computations, which is used here to simulate the state of stress in sutured bio-tissues.

The application of this technique to the analysis of suturing problems in bioelasticity constitutes the main contribution of this work. Its principal advantage *vis à vis* more conventional finite element analyses of such problems is the automatic generation of equilibria that are global energy minimizers of the energy without the special measures to accommodate wrinkling required by stiffness-based equilibrium solution procedures (Massabò and Gambarotta, 2007; Cavichi et al., 2009).

The basic theory and associated numerical analysis that form the basis of this work are well developed, and discussed in detail in the references cited. For this reason we present them in outline, while describing in detail their application to suturing of bioelastic membranes described by two strain-energy functions that have been developed to fit experimental data (Demiray, 1972; Demiray and Vito, 1976).

2. Theoretical background

The equilibrium equation for membranes has a simple divergence structure identical to that for conventional bulk continua. For convenience we use a plane Ω as reference configuration, taken here to be unstressed. Let $\mathbf{e}_\alpha$; $\alpha = 1, 2$, be orthonormal vectors spanning (the translation space of) Ω , and let $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$. We use the usual summation convention with Greek subscripts taking values in $\{1, 2\}$ and Latin indices in $\{1, 2, 3\}$, while Greek subscripts preceded by commas are used to denote partial derivatives with respect to the initial coordinates x_α . The equilibrium equation in the absence of lateral loads is

$$\text{div} \mathbf{T} = 0, \quad \text{or} \quad T_{i\alpha,\alpha} = 0 \quad (1)$$

where

$$\mathbf{T} = T_{i\alpha} \mathbf{e}_i \otimes \mathbf{e}_\alpha \quad (2)$$

is the Piola stress (resultant) and div is the two-dimensional divergence operator on Ω . In the case of elasticity the stress is determined by the deformation gradient

$$\mathbf{F} = F_{i\alpha} \mathbf{e}_i \otimes \mathbf{e}_\alpha, \quad \text{where} \quad F_{i\alpha} = r_{i,\alpha} \equiv \frac{\partial r_i}{\partial x_\alpha} \quad (3)$$

in which x_α and r_i are the Cartesian coordinates of a material point before and after deformation, respectively. The relationship is Haseganu and Steigmann (1994) and Steigmann (2005)

$$\mathbf{T} = W_{\mathbf{F}}, \quad \text{or} \quad T_{i\alpha} = \frac{\partial W}{\partial F_{i\alpha}}, \quad (4)$$

where W is the strain energy per unit area of Ω .

These equations are augmented by traction data $t_i = T_{i\alpha} \nu_\alpha$ or position data $r_i = R_i$ on complementary parts of the boundary, where ν_α are the components of the exterior unit normal to an edge in the reference configuration.

Isotropy of space, or *frame invariance*, requires that W depend on $\mathbf{F}$ through $\mathbf{C} = \mathbf{F}^t \mathbf{F}$. If the membrane material is isotropic relative to Ω , then this dependence occurs through the *principal stretches* λ and μ , the positive square roots of the eigenvalues of $\mathbf{C}$. This yields

$$\mathbf{T} = w_\lambda \mathbf{I} \otimes \mathbf{L} + w_\mu \mathbf{m} \otimes \mathbf{M}, \quad (5)$$

where w_λ and w_μ are the partial derivatives of the symmetric function

$$w(\lambda, \mu) = w(\mu, \lambda) = W(\mathbf{F}), \quad (6)$$

and $\{\mathbf{L}, \mathbf{m}\}$, $\{\mathbf{L}, \mathbf{M}\}$ are orthonormal principal strain axes. These generate a useful formula for the deformation gradient which is identical in form to (5):

$$\mathbf{F} = \lambda \mathbf{I} \otimes \mathbf{L} + \mu \mathbf{m} \otimes \mathbf{M}. \quad (7)$$

Relaxed membrane theory is based on a composite strain-energy function $w_r(\lambda, \mu)$ defined by Pipkin (1986)

$$w_r(\lambda, \mu) = \begin{cases} w(\lambda, \mu); & \lambda > \nu(\mu) \quad \text{and} \quad \mu > \nu(\lambda) \\ w(\lambda, \nu(\lambda)); & \lambda > 1 \quad \text{and} \quad \mu \leq \nu(\lambda) \\ w(\nu(\mu), \mu); & \mu > 1 \quad \text{and} \quad \lambda \leq \nu(\mu) \\ 0; & \lambda \leq 1 \quad \text{and} \quad \mu \leq 1 \end{cases}, \quad (8)$$

where $\nu(x)$ – the *natural width* in simple tension – is the (nominally unique) solution to the implicit equation $w_y(x, y) = 0$ and w is normalized by requiring that $w(1, 1) = 0$.

The second and third branches of (8) correspond to states of uniaxial tension along the principal strain axes. These are operative when the stretches λ or μ are less than the corresponding value of the natural width, the difference being attributed to fine-scale wrinkling without further change of energy and without compressive stress. The first branch is associated with biaxial tension and is operative in the corresponding region of the plane of principal stretches. The fourth branch corresponds to double wrinkling without stress, this being operative in slack, or stress-free, regions of the membrane. In general the determination of the regions on the material surface Ω which correspond to the various branches of (8) in equilibrium states constitutes a difficult free-boundary problem. In the present work these branches are selected automatically in the course of the numerical solution procedure.

The relaxed energy satisfies further convexity conditions associated with energy minimizers. These are discussed in Section 4 below in the context of particular formulations for bio-elastic materials.

3. Discussion of numerical method

We use a discrete version of the Green–Stokes theorem to discretize the equations directly on the computational plane Ω . In this method the plane is sub-divided into quadrilateral cells. These need not be rectangular, and so it is possible to cover a domain with curved or irregular boundaries with high precision by using a sufficient number of cells. This method, incorporating position and traction boundary conditions, is described comprehensively by Haseganu and Steigmann (1994), to which reference may be made for a detailed discussion. The associated nodal points are distributed at the intersections of a curvilinear grid of boundary-fitted coordinates computed using the grid-mapping procedure discussed in Wang and Steigmann (1997) and so again we omit the details.

This scheme is used to discretize the *artificial* dynamical system

$$\text{div} \mathbf{T} = \rho \ddot{\mathbf{r}} + \mathbf{c} \dot{\mathbf{r}}, \quad (9)$$

obtained by appending an explicit viscous damping term to the inertial term in the actual equations of motion. In turn, the spatially discretized system is used together with an explicit central-difference time integration scheme designed for efficient vectorization of the equations. This of course is *not* the actual dynamical equation for membrane motions. Rather, it is an artificial system introduced solely to expedite the computation of equilibria. The basic method, known as *dynamic relaxation*, is a powerful

tool for generating equilibria in a wide variety of nonlinear problems. It is used here in conjunction with quiescent initial conditions (assigned position, zero initial velocity).

In particular, (9) may be used to show that the total mechanical energy E , defined by

$$E = \int_{\Omega} \left(\frac{1}{2} \rho |\dot{\mathbf{r}}|^2 + W \right) da \quad (10)$$

for the null traction/assigned position boundary-value problem, satisfies (Atai and Steigmann, 1998)

$$\frac{d}{dt} E = - \int_{\Omega} c |\dot{\mathbf{r}}|^2 da \quad (11)$$

and hence decays on solution trajectories of (9), reaching its minimum, assuming one exists, in a static configuration. We show below that the relaxed energy defined by (8), obtained from the particular strain-energy functions considered, is such that any static configuration is in fact a minimizer of the total strain energy. Accordingly, the total energy furnishes a Lyapunov function for the dynamical system. The method is implemented using quiescent initial conditions to achieve equilibria in the long-time limit of the artificial dynamical system. Thus temporal accuracy is not an issue, whereas temporal stability is ensured by repeating the computations, as necessary, using successively smaller time steps.

Finally, the examples to be discussed entail suturing of parts of the membrane boundary to form a curve, the configuration of which is obtained as part of the solution. This relies on an algorithm developed in Atai and Steigmann (1998) which ensures that sutured parts of the boundary generate equal and opposite mutual interaction forces.

4. Two relaxed strain-energy functions for bio-elasticity

The two strain-energy functions w_{α} , where

$$w_{\alpha} = \frac{\beta}{2\gamma} \{ \exp[\gamma(I_{\alpha} - 3)] - 1 \}; \quad \alpha = 1, 2, \quad (12)$$

with β and γ positive constants,

$$I_1 = \lambda + \mu + (\lambda\mu)^{-1} \quad \text{and} \quad I_2 = \lambda^2 + \mu^2 + (\lambda^2\mu^2)^{-1}, \quad (13)$$

have been proposed in Demiray (1972) and Demiray and Vito (1976) and shown therein to furnish quantitative agreement with mechanical data on isotropic biological tissues that are incompressible in bulk. We construct the relaxed strain energies associated with these functions and use them to obtain solutions to equilibrium boundary-value problems for sutured bio-tissues.

The equilibrium Eq. (2) is the Euler–Lagrange equation for the potential energy

$$U[\mathbf{r}] = \int_{\Omega} W(\mathbf{F}) da \quad (14)$$

in a mixed position/zero-traction boundary-value problem. It is a necessary condition for a sufficiently smooth deformation $\mathbf{r}(\mathbf{x})$ to be an energy minimizer. When combined with the convexity condition

$$W(\mathbf{F} + \mathbf{A}) - W(\mathbf{F}) \geq \mathbf{A} \cdot \mathbf{W}_{\mathbf{F}}(\mathbf{F}), \quad \text{for all } \mathbf{A}, \quad (15)$$

it may be used to demonstrate that equilibria minimize the potential energy, i.e.

$$U[\mathbf{r}] \leq U[\bar{\mathbf{r}}], \quad (16)$$

where $\bar{\mathbf{r}}$ is any kinematically admissible position field (Haseganu and Steigmann, 1994).

For a given strain-energy function the convexity condition (15) is usually not easy to check. However, it implies the weaker, local, convexity condition

$$\mathbf{A} \cdot \mathbf{W}_{\mathbf{FF}}(\mathbf{F})[\mathbf{A}] \geq 0, \quad \text{or} \quad A_{i\alpha} A_{j\beta} \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}} \geq 0, \quad (17)$$

which is equivalent, for isotropic materials, to the inequalities (Pipkin, 1986)

$$w_{\lambda} \geq 0, \quad w_{\mu} \geq 0, \quad w_{\lambda\lambda} \geq 0, \quad w_{\mu\mu} \geq 0, \quad a \geq 0, \quad (18)$$

and

$$\frac{(w_{\lambda} - w_{\mu})}{(\lambda - \mu)} \geq 0, \quad w_{\lambda\lambda} w_{\mu\mu} \geq (w_{\lambda\mu})^2, \quad (19)$$

where

$$a = \frac{(\lambda w_{\lambda} - \mu w_{\mu})}{(\lambda^2 - \mu^2)}. \quad (20)$$

For the strain-energy functions (13) studied in the present work, we show that these inequalities are satisfied automatically by the relaxed energy (8). Because the domain of the relaxed energy is the quadrant in the λ, μ - plane defined by $\lambda \geq 0$ and $\mu \geq 0$, including the half-lines $\lambda = 0$ and $\mu = 0$, the function $W(\mathbf{F}) = w_r(\lambda, \mu)$ is defined on the entire linear space of tensors. This is a convex set, and so the local inequality (17) may be integrated along straight lines in this space to prove that the global convexity condition (15) is satisfied. It follows that equilibria obtained by using the relaxed energy automatically furnish global minimizers of the potential energy.

The natural width associated with the relaxed energy is easily shown to be

$$v(x) = x^{-1/2} \quad (21)$$

for either of the energies (13). Moreover, on the second branch of (8) the latter reduce to

$$\bar{w}_{\alpha}(\lambda) = w_{\alpha}(\lambda, v(\lambda)) = \frac{\beta}{2\gamma} \{ \exp[\gamma(\bar{I}_{\alpha}(\lambda) - 3)] - 1 \}, \quad (22)$$

where

$$\bar{I}_1(\lambda) = \lambda + 2\lambda^{-1/2} \quad \text{and} \quad \bar{I}_2(\lambda) = \lambda^2 + 2\lambda^{-1}, \quad (23)$$

and the same functions apply on the third branch of (8), with λ replaced by μ .

We have

$$\bar{w}'_{\alpha}(\lambda) = \frac{\beta}{2} \bar{I}'_{\alpha}(\lambda) \exp[\gamma(\bar{I}_{\alpha}(\lambda) - 3)], \quad (24)$$

where

$$\bar{I}'_1(\lambda) = 1 - \lambda^{-3/2} \quad \text{and} \quad \bar{I}'_2(\lambda) = 2(\lambda - \lambda^{-2}). \quad (25)$$

Further,

$$\bar{w}''_{\alpha}(\lambda) = \frac{\beta}{2} \{ \bar{I}''_{\alpha}(\lambda) + \gamma [\bar{I}'_{\alpha}(\lambda)]^2 \} \exp[\gamma(\bar{I}_{\alpha}(\lambda) - 3)], \quad (26)$$

where

$$\bar{I}''_1(\lambda) = \frac{3}{2} \lambda^{-5/2} \quad \text{and} \quad \bar{I}''_2(\lambda) = 2(1 + 2\lambda^{-3}). \quad (27)$$

These are positive for $\lambda > 1$, implying that inequalities (18)_{3,4} are satisfied on the second branch of (8), where w_r is independent of μ . In the same way, they are also satisfied on the third branch where w_r is independent of λ , and of course trivially on the fourth branch where the energy vanishes identically.

To show that inequalities (18)_{1,2} are satisfied on the first branch, we compute

$$(w_{\alpha})_{\lambda} = \frac{\beta}{2} (I_{\alpha})_{\lambda} \exp[\gamma(I_{\alpha} - 3)], \quad \text{where} \quad (I_1)_{\lambda} = 1 - \lambda^{-2} \mu^{-1} \quad (28)$$

$$\text{and} \quad (I_2)_{\lambda} = 2(\lambda - \lambda^{-3} \mu^{-2}),$$

and

$$(w_\alpha)_\mu = \frac{\beta}{2}(I_\alpha)_\mu \exp[\gamma(I_\alpha - 3)], \quad \text{where} \quad (I_1)_\mu = 1 - \mu^{-2}\lambda^{-1} \quad (29)$$

and $(I_2)_\mu = 2(\mu - \mu^{-3}\lambda^{-2}).$

These are easily seen to be positive on the first branch, where $\mu > \lambda^{-1/2}$ and $\lambda > \mu^{-1/2}$.

On the first branch we also compute

$$(w_\alpha)_\lambda - (w_\alpha)_\mu = \frac{\beta}{2}[(I_\alpha)_\lambda - (I_\alpha)_\mu] \exp[\gamma(I_\alpha - 3)], \quad (30)$$

where

$$(I_1)_\lambda - (I_1)_\mu = \lambda^{-2}\mu^{-2}(\lambda - \mu) \quad \text{and} \quad (I_2)_\lambda - (I_2)_\mu = 2(1 + \lambda^{-3}\mu^{-3})(\lambda - \mu) \quad (31)$$

and thus conclude that $(19)_1$ is satisfied. Verification of $(19)_5$ proceeds similarly; we obtain

$$\lambda(w_\alpha)_\lambda - \mu(w_\alpha)_\mu = \frac{\beta}{2}[\lambda(I_\alpha)_\lambda - \mu(I_\alpha)_\mu] \exp[\gamma(I_\alpha - 3)], \quad (32)$$

where

$$\lambda(I_1)_\lambda - \mu(I_1)_\mu = \lambda - \mu \quad \text{and} \quad \lambda(I_2)_\lambda - \mu(I_2)_\mu = 2(\lambda^2 - \mu^2). \quad (33)$$

Lastly, we observe that inequalities $(18)_{3,4}$ and $(19)_2$ are equivalent to the convexity of the strain energy as a function of λ and μ . For this it is sufficient that $w_\alpha(I_\alpha)$ be convex, increasing functions, and that I_α be convex functions of λ and μ . Verification of the first requirement is trivial, since

$$w'_\alpha(I_\alpha) = \frac{\beta}{2} \exp[\gamma(I_\alpha - 3)] \quad \text{and} \quad w''_\alpha(I_\alpha) = \gamma w'_\alpha(I_\alpha) \quad (34)$$

are both positive. To verify the second requirement we start with I_1 , obtaining

$$(I_1)_{\lambda\lambda} = 2\lambda^{-3}\mu^{-1}, \quad (I_1)_{\mu\mu} = 2\mu^{-3}\lambda^{-1} \quad \text{and} \quad (I_1)_{\lambda\mu} = \lambda^{-2}\mu^{-2}. \quad (35)$$

Therefore,

$$[(I_1)_{\lambda\lambda}][(I_1)_{\mu\mu}] - [(I_1)_{\lambda\mu}]^2 = 3\lambda^{-4}\mu^{-4}, \quad (36)$$

and in the same way we find

$$[(I_2)_{\lambda\lambda}][(I_2)_{\mu\mu}] - [(I_2)_{\lambda\mu}]^2 = 4\lambda\mu + 12(\lambda^{-4}\mu^{-1} + \lambda^{-1}\mu^{-4}) + 20\lambda^{-6}\mu^{-6}. \quad (37)$$

Both are positive, implying that the w_α are convex functions of λ and μ and hence, together with the remaining inequalities in (18) and (19), that they are convex functions of $\mathbf{F}$, yielding the conclusion that equilibria computed using the relaxed formulation furnish global minima of the energy. Accordingly, any equilibrium state obtained via dynamic relaxation is energetically optimal.

We remark that the restrictions $(18)_{1,2}$ have counterparts in the convexity condition for three-dimensional elasticity (Ciarlet, 1988), which is known to imply that the pairwise sums of the principal stresses are non-negative. For this and other reasons convexity is regarded as an overly restrictive condition in the three-dimensional theory. However, in membrane theory the same restrictions on stress are necessary conditions for energy minimizers. This follows from the Legendre–Hadamard necessary condition (Pipkin, 1986), which is given by (17) but with $\mathbf{A}$ restricted to have rank one; i.e. $\mathbf{A} = \mathbf{a} \otimes \mathbf{b}$ ($A_{i\alpha} = a_i b_\alpha$), for any three-vector $\mathbf{a}$ and two-vector $\mathbf{b} \in \Omega$. Accordingly, convexity of $W(\mathbf{F})$ is compatible with physically realistic membrane behavior. Indeed, the necessity of $(18)_{1,2}$ for energy minimizers furnishes the impetus for the introduction of the relaxed strain-energy function in the study of equilibria.

Fig. 1. Initial configuration of an annular membrane. Letters identify locations of maximum principal stretch in the examples.

5. Examples

In all examples the equations are non-dimensionalized by the positive modulus β and the solution is advanced in (artificial) time until the maximum value of local norm $|\beta^{-1} \text{div} \mathbf{T}|$ falls below a specified tolerance, on the order of 10^{-6} , corresponding to the approximate attainment of mechanical equilibrium. Here the divergence operator is non-dimensionalized by a problem-dependent length scale. We also equate the material parameter γ to unity in all examples. We are not concerned here with the assessment of specific suturing techniques or geometries that may arise in practice, but rather with the development of a general and versatile capability to model membrane suturing.

To this end we investigate a small number of simple representative problems. Each of these consists of two sub-examples, corresponding to the material model index $\alpha = 1$ or 2 (see (13)). In the first example we take the initial, unstressed configuration of the membrane to be a plane annulus bounded by concentric circles with a ratio of outer radius to inner radius equal to 4 and a standard radial/azimuthal mesh is used (Fig. 1). The inner radius is used to non-dimensionalize all length scales.

The outer boundary is fixed and the upper and lower edges of the inner boundary are sutured together using the algorithm described in Atai and Steigmann (1998). This generates a deformed suture that lies on the horizontal symmetry axis of the membrane. The deformation is nearly the same in both material models. The local distributions of tense, wrinkled and slack regions near the right-hand end of the suture line are depicted in Fig. 2a and b for material models $\alpha = 1, 2$, respectively. Here and elsewhere the short line segments represent the tension trajectories in wrinkled regions, whereas open circles represent fully slack regions with no stress; the membrane is under biaxial tension in regions lacking either segments or circles. The slack regions correspond, in the real membrane, to excess material that carries no load. The maximum principal stretches are 1.608 (at the inner edge on vertical axis labelled A), in the first example, and 1.543 (at point B) in the second, indicating that the deformation is well into the finite-strain range. In the second example the boundary to be sutured is a centered

Fig. 2. (a) Local pattern of slack and wrinkled regions in example 1 ($\alpha = 1$) and (b) local pattern of slack and wrinkled regions in example 1 ($\alpha = 2$).

Fig. 3. Annular membrane with elliptical hole. Letters indicate locations of maximum principal stretch in various cases.

ellipse, with semi-major and -minor axes equal to $1/2$ and $1/10$ of the outer radius. The mesh is generated by the grid-mapping procedure referred to in Section 3. Again the outer boundary is fixed while the ellipse is sutured along the semi-major (horizontal) axis.

The local details of the wrinkled, slack and biaxially stressed states near the right-hand end of the suture line are shown in Fig. 4a and b for the two material models. The maximum principal stretches are 1.138 (at point A in Fig. 3) in the first case ($\alpha = 1$), and 1.139 (at point B) in the second ($\alpha = 2$). Some minor qualitative differences due to the relative stiffness characteristics of the two models are evident.

The third example concerns an unstressed configuration (not shown) in which the inner boundary is the same ellipse as in the previous example, while the circular outer boundary is replaced by an ellipse with a horizontal major axis and vertical minor axis equal to 1.1 and 1.0 times the diameter of the previous circle, respectively. The outer edge is fixed while the inner ellipse is sutured as in the previous example. The minor adjustment to the initial membrane geometry is seen, in Fig. 5, to have a significant effect on the qualitative aspects of the solution. In particular, the extent of wrinkling is substantially diminished and the slack regions have disappeared altogether. The maximum principal stretch is now 1.161 (at point C in Fig. 3) in the first case ($\alpha = 1$), and 1.185 (at point D) in the second ($\alpha = 2$), indicating a substantial drop in strain (Figs. 4 and 5).

In the fourth and final example, the initial configuration is a region bounded by the same inner ellipse but now the exterior

Fig. 4. (a) Local pattern of slack and wrinkled regions in example 2 ($\alpha = 1$) and (b) local pattern of the slack and wrinkled regions in example 2 ($\alpha = 2$).

Fig. 5. (a) Local pattern of wrinkled region in example 3 ($\alpha = 1$) and (b) local pattern of wrinkled region in example 3 ($\alpha = 2$).

Fig. 6. (a) Local pattern of wrinkled and slack regions in example 4 ($\alpha = 1$) and (b) local pattern of wrinkled and slack regions in example 4 ($\alpha = 2$).

ellipse has vertical and horizontal major and minor axes, respectively, with lengths again equal to 1.1 and 1.0 times the diameter of the original circle. The substantial alteration of the pattern of wrinkled and slack states induced by this small change is shown in Fig. 6. The maximum stretch is 1.274 (at point E) in the first case ($\alpha = 1$) and 1.284 at (at point F) in the second ($\alpha = 2$).

We remark that the deformation in the interior of a slack region is essentially arbitrary and hence highly non-unique. The associated nodal configurations shown in the figures are simply the asymptotic limits of those generated by the present solution procedure on the basis of certain initial conditions. Further, wrinkling and slackening result in substantial redistribution of stress relative to the unrelaxed formulation in which compressive stresses are allowed. This is illustrated in the work of Massabò and Gambarotta (2007) but was not realized in the analytical treatment of suturing presented by Danielson and Natarajan (1975); in the latter work the regions of (unstable) compressive stress predicted by the conventional theory were simply removed and replaced by a tension field. In the present work such issues do not arise. Instead, we simply track the state of strain as the solution evolves in (artificial) time and select the appropriate branch of the relaxed strain-energy function accordingly.

On the basis of the foregoing numerical experiments it is appropriate to conclude that the details of the local distributions of strain in sutured biomembranes, as reflected by the distributions of

wrinkled, tense and slack regions, are relatively insensitive to the material model but highly sensitive to the initial geometry of the membrane prior to suturing. This underscores the importance of a robust computational platform upon which to investigate optimal incision geometries in applications.

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