Theory of elastic solids reinforced with fibers resistant to extension, flexure and twist

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A B S T R A C T

A model of non linearly elastic solids reinforced by continuously distributed embedded fibers is formulated in which elastic resistance of the fibers to extension, bending and twist is taken into account explicitly. This generalizes the conventional theory in which the solid is modeled as a transversely isotropic simple material.

1. Introduction

The mechanics of fiber reinforced solids is a well established subject with a long history [1,2] that has significantly enriched and advanced continuum mechanics in general. It has been based almost entirely on the concept of a simple anisotropic material in which the response functions depend on the conventional defor mation gradient, possibly augmented by constraints such as bulk incompressibility or fiber inextensibility. In the latter case the deformation is often so constrained as to be essentially kinema tically determinate. The associated theory also exhibits a number of novel features such as the hyperbolicity of the equilibrium equations wherein the fiber trajectories emerge as characteristic curves of the associated differential equations [1]. The continuum theory presumes the fibers to be so densely distributed as to render meaningful the idealization of a continuous distribution, and purports to describe homogenized fiber matrix composites.

Recently a significant advance in the continuum theory of fiber reinforced solids was achieved by introducing the bending resistance of the fibers explicitly [3]. This is framed in the setting of the non linear strain gradient theory [4 6] of anisotropic elasticity in which elastic resistance is assigned to changes in curvature (flexure) of the fibers. The latter is calculated from the second gradient of the continuum deformation in which the fibers are regarded as convected curves. The model also accounts for additional effects associated with the gradients of the fiber stretches.

In the present work we develop a model in which the fibers offer elastic resistance to twist in addition to flexure and stretch. Thus the fibers are regarded as continuously distributed spatial rods of the Kirchhoff type in which the kinematics are based on a position field and an orthonormal triad field [7 9]. Variation of the triad along the length of a fiber accounts for flexure and twist, while the position field generates the fiber stretch. We seek a model that accounts for these effects but which does not attribute elastic resistance to the gradient of fiber stretch. This restriction is in accord with established rod theories.

The present model is a special case of the Cosserat theory of non linear elasticity [4, 10, Section 98, 11,12], which has received renewed attention in recent years. Current applications of the general theory are treated in [13 16], for example, and mathe matical aspects of the subject are addressed in [17]. This frame work may be combined with strain gradient theory, which is also in a period of active development [18,19], to obtain a model in which bending and twisting effects are combined with fiber stretch gradients. We do not consider such a possibility here, however, as we are concerned with the simplest generalization of the conventional theory. For the same reason we suppress transverse shearing of the fiber cross sections and thus work in the setting of the constrained Cosserat theory in which the directors are described by a finite rotation tensor field. This is decoupled, at the local level, from the deformation of the homo genized continuum.

The basic elements of Kirchhoff rod theory are summarized in Section 2. This is followed, in Section 3, by a survey of non linear Cosserat elasticity [11,12,17]. There we also introduce a kinematic constraint on the rotation and deformation fields to ensure that fibers are convected as material curves. We focus on a special case

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of the general theory in which Cosserat effects are attributed exclusively to the fibers. The resulting model is similar in structure to the Kirchhoff theory, with the effects of fiber matrix interaction manifesting themselves as forces and couples distrib uted along the lengths of the embedded fibers. The theory of material symmetry is developed in Section 4, and specialized to the case of transverse isotropy in which the fibers are normal to the planes of isotropy in a reference configuration. This discussion leads to a non standard problem in representation theory which may be of independent interest. Rather than pursue the general solution to this problem, we simply record some example solu tions in the form of scalars that automatically satisfy the relevant invariance requirement. An example is used, in Section 5, to solve the problem of finite torsion of an elastic cylinder in which the fibers are aligned with the generators of the cylinder prior to deformation.

We use standard notation such as $\mathbf{A}^t$, $\mathbf{A}^{-1}$, $\mathbf{A}^*$, *Sym* $\mathbf{A}$, *Skw* $\mathbf{A}$ and tr $\mathbf{A}$. These are, respectively, the transpose, the inverse, the cofactor, the symmetric part, the skew part and the trace of a tensor $\mathbf{A}$, regarded as a linear transformation from a three dimensional vector space to itself. We also use *Sym* and *Skw* to denote the linear subspaces of symmetric and skew tensors and $Orth^+$ to identify the group of rotation tensors. The tensor product of three vectors is indicated by interposing the symbol $\otimes$, and the Euclidean inner product of tensors $\mathbf{A}$,$\mathbf{B}$ is denoted and defined by $\mathbf{A}\cdot\mathbf{B}=\mathrm{tr}(\mathbf{A}\mathbf{B}^t)$; the associated norm is $|\mathbf{A}|=\sqrt{\mathbf{A}\cdot\mathbf{A}}$. The symbol $|\cdot|$ is also used to denote the usual Euclidean norm of three vectors. Latin and Greek indices take values in {1,2,3} and {2,3}, respectively, and, when repeated, are summed over their ranges. Finally, the notation $F_{\mathbf{A}}$ stands for the tensor valued derivative of a scalar valued function $F(\mathbf{A})$.

2. Kirchhoff rod theory

In the present theory we regard the embedded fibers as continuously distributed spatial Kirchhoff rods [7 9,20,21]. Con figurations of a spatial rod are described by a position field $\mathbf{r}(S)$, where S measures arclength along the rod in a reference place ment, and a field $\{\mathbf{d}_i(S)\}$ of orthonormal vectors in which $\mathbf{d}_1$ is everywhere tangent to the space curve defined by $\mathbf{r}(S)$; $\mathbf{d}_\alpha$; $\alpha=2,3$ are vectors embedded in the rod cross section. Thus,

$$\mathbf{r}'(S)=\lambda\mathbf{d}_1 \quad \text{where } \lambda=|\mathbf{r}'(S)| \tag{1}$$

is the local stretch of the rod. Here and elsewhere the notation $(\cdot)'$ stands for $d(\cdot)/dS$. We use the vernacular of rod theory in referring to the *cross section*, but it should be borne in mind that the model to be developed does not identify this explicitly. Rather, the latter should be regarded as a microstructural feature that entails the introduction of a local length scale; this is accounted for indirectly via constitutive equations for the three dimensional continuum. We also use the term *matrix* to refer to material properties that are not attributable to the fibers.

The $\mathbf{d}_i$ are presumed to remain orthonormal in all configura tions of the rod. Accordingly, if $\mathbf{D}_i(S)$ are their values in the reference placement, then

$$\mathbf{d}_i=\mathbf{A}(S)\mathbf{D}_i \tag{2}$$

for some rotation field $\mathbf{A}(S)\in Orth^+$. The rates of change $\mathbf{d}_i'(S)$ describe the curvature and twist of the rod. Because the $\mathbf{d}_i(S)$ are orthonormal for all S in the domain (the length of the rod in the reference placement), if the rod is initially straight and untwisted ($\mathbf{D}_i'=\mathbf{0}$) then the curvature and twist are given by the axial vector $\boldsymbol{\alpha}(S)$ of the skew tensor $\mathbf{A}'\mathbf{A}^t$, i.e.

$$\mathbf{d}_i'=\boldsymbol{\alpha}\times\mathbf{d}_i. \tag{3}$$

In the frame invariant formulation of the theory the response of an elastic rod is described by a strain energy function $w(\lambda,\boldsymbol{\kappa})$, where $\boldsymbol{\kappa}=\mathbf{A}^t\boldsymbol{\alpha}=\kappa_i\mathbf{D}_i$, with

$$\kappa_i=\tfrac{1}{2}e_{ijk}\mathbf{d}_k\cdot\mathbf{d}_j', \tag{4}$$

where e_{ijk} is the usual permutation symbol ($e_{123}=+1$). The equations of equilibrium are [7 9,21]

$$\mathbf{m}'+\boldsymbol{\varpi}=\mathbf{f}\times\mathbf{r}' \quad \text{and} \quad \mathbf{f}'+\mathbf{g}=\mathbf{0}, \tag{5}$$

where

$$\mathbf{m}=(\partial w/\partial\kappa_i)\mathbf{d}_i \tag{6}$$

is the vector of bending and twisting moments exerted on the part $[0,S]$ of the rod by the part in the remainder,

$$\mathbf{f}=\lambda^{-1}(\partial w/\partial\lambda)\mathbf{r}'+f_\alpha\mathbf{d}_\alpha, \tag{7}$$

where f_α ($\alpha=2,3$) are constitutively undetermined, is the force on $[0,S]$ exerted on the cross section at arclength station S; and $\mathbf{g}(S)$, $\boldsymbol{\varpi}(S)$, respectively, are the distributed force and couple per unit reference length.

In the absence of axial extension, the strain energy function most commonly used for isotropic rods of circular section is [7,8]

$$w(1,\boldsymbol{\kappa})=\tfrac{1}{2}GJ\kappa_1^2+\tfrac{1}{2}EI\kappa_\alpha\kappa_\alpha, \tag{8}$$

where GJ, in which G is the shear modulus and J is the polar moment of the section, is the torsional stiffness; and EI, in which E is Young's modulus and I is the second moment, is the flexural stiffness. This yields [7]

$$\mathbf{m}=GJ\kappa_1\mathbf{d}_1+EI\kappa_\alpha\mathbf{d}_\alpha=GJ\kappa_1\mathbf{d}_1+EI\mathbf{d}_1\times\mathbf{d}_1'. \tag{9}$$

The same model applies to isotropic rods of non circular section provided that J is adjusted to account for warping of the cross section [7,8].

3. Cosserat elasticity theory

3.1. Kinematics

To model the kinematics of the embedded fibers, we assume the body, regarded as a homogenized continuum consisting of matrix material and fibers together, to be endowed with a rotation field $\mathbf{R}(\mathbf{X})$ in addition to the usual deformation $\boldsymbol{\chi}(\mathbf{X})$. To exhibit the main ideas as simply and clearly as possible, we confine attention here to materials that are reinforced by a single family of fibers.

Drawing on the structure of rod theory with axial extension, we further assume the existence of a referential energy density $U(\mathbf{F},\mathbf{R},\mathbf{S};\mathbf{X})$, where $\mathbf{F}$ is the usual deformation gradient and $\mathbf{S}$ is the rotation gradient; thus,

$$\mathbf{F}=F_{iA}\mathbf{e}_i\otimes\mathbf{E}_A, \quad \mathbf{R}=R_{iA}\mathbf{e}_i\otimes\mathbf{E}_A \quad \text{and} \quad \mathbf{S}=S_{iAB}\mathbf{e}_i\otimes\mathbf{E}_A\otimes\mathbf{E}_B \tag{10}$$

with

$$F_{iA}=\chi_{i,A} \quad \text{and} \quad S_{iAB}=R_{iA,B}, \tag{11}$$

where $(\cdot)_{,A}=\partial(\cdot)/\partial X_A$ and we use an older Cartesian index nota tion that emphasizes the two point character of the deformation gradient and rotation fields. Here $\{\mathbf{e}_i\}$ and $\{\mathbf{E}_A\}$ are fixed ortho normal bases associated with the coordinates x_i and X_A, where $x_i=\chi_i(X_A)$. Such notation delivers formulae which, though per haps unwieldy, are at least explicit and unambiguous.

The rotation field acts on the orthonormal triad field $\{\mathbf{D}_i(\mathbf{X})\}$ associated with the unit tangents and cross sections of embedded fibers. To make the roles of these vectors explicit, we write $\{\mathbf{D}_i\}=\{\mathbf{D},\mathbf{D}_\alpha\}$; $\alpha=2,3$, where $\mathbf{D}(=\mathbf{D}_1)$ is the unit tangent to a fiber in the reference configuration, denoted by ξ, and $\mathbf{D}_\alpha$ are cross sectional vectors embedded in the fiber, but *not* in the matrix. Thus,

$$\mathbf{d}_i = \mathbf{R}\mathbf{D}_i \tag{12}$$

is the (orthonormal) fiber triad in the current configuration, where $\mathbf{d}(=\mathbf{d}_1)$ is the unit tangent to a fiber.

We regard the fiber as an embedded curve, and hence the tangent field $\mathbf{D}(\mathbf{X})$ as being convected by the deformation $\boldsymbol{\chi}(\mathbf{X})$. This generates the connection (cf. (1))

$$\mathbf{F}\mathbf{D} = \lambda\mathbf{d} \quad \text{where } \mathbf{d} = \mathbf{R}\mathbf{D} \text{ and } \lambda = |\mathbf{F}\mathbf{D}|. \tag{13}$$

The cross sectional vectors $\mathbf{D}_\alpha$ are not embedded in the matrix, and so in general their images $\mathbf{d}_\alpha$ in the current configuration are not directly connected to the deformation of the matrix. Rather, they are free to shear relative to the matrix while remaining mutually unsheared. This effectively extends the kinematics of the ideal theory of elastic materials with embedded inextensible fibers [1,2] to allow for fiber extension, flexure and twist. Eqs. (12) and (13) generate the two constraints

$$\mathbf{R}\mathbf{D}_\alpha \cdot \mathbf{F}\mathbf{D} = 0, \quad \alpha = 2,3, \tag{14}$$

between the fiber rotation and matrix deformation, and thus yield the interpretation of the present model as a constrained variant of the Cosserat theory of non linear elasticity [4,11,12].

3.2. Strain energy function

We observe that the constraints (14) are invariant under the transformations $\mathbf{F}\to\mathbf{Q}\mathbf{F}$ and $\mathbf{R}\to\mathbf{Q}\mathbf{R}$, where $\mathbf{Q}$ is the spatially uniform rotation associated with an arbitrary superposed rigid body motion. We assume the energy density function to be similarly invariant, and thus impose the requirement

$$U(\mathbf{F},\mathbf{R},\mathbf{S};\mathbf{X}) = U(\mathbf{Q}\mathbf{F},\mathbf{Q}\mathbf{R},\mathbf{Q}\mathbf{S};\mathbf{X}), \tag{15}$$

where $(\mathbf{Q}\mathbf{S})_{iAB} = (Q_{ij}R_{jA})_{,B} = Q_{ij}S_{jAB}$. The restriction

$$U(\mathbf{F},\mathbf{R},\mathbf{S};\mathbf{X}) = W(\mathbf{E},\boldsymbol{\Gamma};\mathbf{X}), \tag{16}$$

where [11,12]

$$\mathbf{E} = \mathbf{R}^t\mathbf{F} = E_{AB}\mathbf{E}_A \otimes \mathbf{E}_B, \quad E_{AB} = R_{iA}F_{iB}, \tag{17}$$

$$\boldsymbol{\Gamma} = \Gamma_{DC}\mathbf{E}_D \otimes \mathbf{E}_C, \quad \Gamma_{DC} = \tfrac{1}{2}e_{BAD}R_{iA}R_{iB,C}, \tag{18}$$

with W a suitable function and e_{ABC} the permutation symbol, is both necessary and sufficient for the stated invariance. Sufficiency is nearly obvious, while necessity follows by choosing $\mathbf{Q} = \mathbf{R}^t_{|X}$, where X is the material point in question, and making use of the fact that for each fixed $C \in \{1,2,3\}$ the matrix $R_{iA}R_{iB,C}$ is skew; this of course follows by differentiating $R_{iA}R_{iB} = \delta_{AB}$, the usual Kronecker delta. The associated axial vectors γ_C have components

$$\gamma_{D(C)} = \tfrac{1}{2}e_{BAD}R_{iA}R_{iB,C}, \tag{19}$$

yielding [12]

$$\boldsymbol{\Gamma} = \gamma_C \otimes \mathbf{E}_C, \tag{20}$$

and so $\boldsymbol{\Gamma}$ is equivalent to $\mathbf{R}^t\mathbf{S}$.

3.3. Stationary energy and equilibrium

In the case of conservative loading equilibria may be interpreted as states that render stationary the potential energy

$$E = \int_\xi W\, dv - L, \tag{21}$$

where L is a suitable load potential. Among the numerous possibilities, we emphasize the dead load problem under negligible body forces and couples in which traction $\mathbf{t}$ is fixed on a part $\partial\xi_t$ of the boundary and couples $\mathbf{m}_i$ are fixed on a part $\partial\xi_c$, such that

$$L = \int_{\partial\xi_t} \mathbf{t}\cdot\boldsymbol{\chi}\, da + \int_{\partial\xi_c} \mathbf{m}_i \cdot \mathbf{d}_i\, da. \tag{22}$$

The virtual work of the force and couples is

$$\dot{L} = \int_{\partial\xi_t} \mathbf{t}\cdot\dot{\boldsymbol{\chi}}\, da + \int_{\partial\xi_c} \mathbf{c}\cdot\boldsymbol{\omega}\, da, \tag{23}$$

where the superposed dots refer to derivatives with respect to a parameter of a one parameter family $\{\mathbf{F}(\mathbf{X};\epsilon),\mathbf{R}(\mathbf{X};\epsilon)\}$ of deformation and rotation fields, $\boldsymbol{\omega} = ax(\boldsymbol{\Omega})$ is the axial vector of the skew tensor $\boldsymbol{\Omega} = \dot{\mathbf{R}}^t\mathbf{R}$ ($\boldsymbol{\Omega}\mathbf{v} = \boldsymbol{\omega}\times\mathbf{v}$ for all $\mathbf{v}$) and

$$\mathbf{c} = ax[(\mathbf{D}_i \otimes \mathbf{m}_i)\mathbf{R} - \mathbf{R}^t(\mathbf{m}_i \otimes \mathbf{D}_i)] \tag{24}$$

is the (configuration dependent) couple traction. This follows from $\mathbf{m}_i \cdot \dot{\mathbf{d}}_i = \mathbf{m}_i \cdot \dot{\mathbf{R}}\mathbf{D}_i$ with $\dot{\mathbf{R}} = -\mathbf{R}\boldsymbol{\Omega}$ and $\mathbf{m}_i \cdot \mathbf{R}\boldsymbol{\Omega}\mathbf{D}_i = \mathbf{R}^t\mathbf{m}_i \cdot \boldsymbol{\Omega}\mathbf{D}_i = \mathbf{R}^t\mathbf{m}_i \otimes \mathbf{D}_i \cdot \boldsymbol{\Omega}$.

Stationarity of the energy is subsumed under the general virtual work statement [11]

$$\int_\xi \dot{W}\, dv = \int_{\partial\xi_t} \mathbf{t}\cdot\dot{\boldsymbol{\chi}}\, da + \int_{\partial\xi_c} \mathbf{c}\cdot\boldsymbol{\omega}\, da, \tag{25}$$

where the derivatives are evaluated at equilibrium, corresponding to $\epsilon = 0$, say. We regard the values of $\boldsymbol{\chi}$ as being assigned on $\partial\xi\backslash\partial\xi_t$, and those of $\mathbf{R}$ as being assigned on $\partial\xi\backslash\partial\xi_c$. We emphasize the fact that this holds whenever the virtual work of the loads is expressible as a linear form in $\dot{\boldsymbol{\chi}}$ and $\boldsymbol{\omega}$, including the case of non conservative loads or conservative loads other than those discussed above.

Global balance statements may be derived from (25) by assuming that $\partial\xi_t = \partial\xi_c = \partial\xi$ and considering a rigid body motion

$$\boldsymbol{\chi}(\mathbf{X};\epsilon) = \mathbf{Q}(\epsilon)\boldsymbol{\chi}_0(\mathbf{X}) + \mathbf{b}(\epsilon), \quad \mathbf{R}(\mathbf{X};\epsilon) = \mathbf{Q}(\epsilon)\mathbf{R}_0(\mathbf{X}), \tag{26}$$

superposed on a configuration described by the fixed position and rotation fields $\boldsymbol{\chi}_0(\mathbf{X})$ and $\mathbf{R}_0(\mathbf{X})$, respectively, where $\mathbf{Q}(\epsilon) \in Orth^+$ with $\mathbf{Q}(0) = \mathbf{I}$. Because the strain energy remains invariant in such motions we have $\dot{W} = 0$; evaluating (25) at $\epsilon = 0$ then gives

$$\int_{\partial\xi} (\mathbf{t}\cdot\dot{\boldsymbol{\chi}} + \mathbf{c}\cdot\boldsymbol{\omega})\, da = 0, \tag{27}$$

where

$$\dot{\boldsymbol{\chi}} = \mathbf{a}\times(\boldsymbol{\chi}_0 - \mathbf{b}_0) + \dot{\mathbf{b}} \quad \text{and} \quad \boldsymbol{\omega} = -\mathbf{R}_0^t\mathbf{a} \tag{28}$$

in which $\mathbf{a} = ax(\dot{\mathbf{R}}\mathbf{R}^t)$ is evaluated at $\epsilon = 0$. To obtain the second result we use

$$(\mathbf{R}^t\dot{\mathbf{R}})\mathbf{R}^t\mathbf{v} = \mathbf{R}^t(\mathbf{a}\times\mathbf{v}) = \mathbf{R}^t\mathbf{a}\times\mathbf{R}^t\mathbf{v} \tag{29}$$

for any $\mathbf{v}$ in which the second equality follows from the fact that the rotation $\mathbf{R}^t$ is equal to its own cofactor. Using $\mathbf{R}^t\dot{\mathbf{R}} = -\boldsymbol{\Omega}$ we conclude that $\mathbf{a} = -\mathbf{R}\boldsymbol{\omega}$, and the stated result follows immediately. Dropping the subscripts, we then have

$$(\dot{\mathbf{b}} - \mathbf{a}\times\mathbf{b})\cdot\int_{\partial\xi} \mathbf{t}\, da + \mathbf{a}\cdot\int_{\partial\xi} (\boldsymbol{\chi}\times\mathbf{t} - \mathbf{R}\mathbf{c})\, da = 0, \tag{30}$$

and for this to hold for all $\dot{\mathbf{b}}$ and $\mathbf{a}$ it is necessary and sufficient that the global force and moment balances,

$$\int_{\partial\xi} \mathbf{t}\, da = \mathbf{0} \quad \text{and} \quad \int_{\partial\xi} (\boldsymbol{\chi}\times\mathbf{t} - \mathbf{R}\mathbf{c})\, da = \mathbf{0}, \tag{31}$$

respectively, be satisfied.

In the course of deriving further consequences of equations such as (25) in the general case it is conventional to use the Lagrange multiplier rule to accommodate any constraints that may be operative. However, for multiple integral problems of the kind considered here it is not a trivial matter to establish the existence of Lagrange multipliers [22]. Such matters are beyond the scope of this work. The issue is examined in detail in [23,24] in connection with the constraints of incompressibility and inextensibility in conventional finite elasticity theory. These difficulties are circumvented here by replacing the constrained problem by an unconstrained problem for the functional

$$\overline{E} = \int_{\xi} \overline{W}\, dv \;\; L \tag{32}$$

in which

$$\overline{W} = W + \Lambda_\alpha \mathbf{D}_\alpha \cdot \mathbf{ED}, \tag{33}$$

where Λ_α are Lagrange multipliers associated with (14). Here $\overline{E}$ is to be regarded as an unconstrained functional of $\boldsymbol{\chi}$, $\mathbf{R}$ and Λ_α. Thus E and $\overline{E}$ coincide when the constraints are in effect, whereas the latter effectively extends the former to states in which the constraints are relaxed. Variation of the multipliers simply returns the constraints, ensuring that states which render $\overline{E}$ stationary are also stationary states for E. This follows from the fact that stationarity in the class of unrestricted variations of $\boldsymbol{\chi}$ and $\mathbf{R}$ implies stationarity in the restricted class defined by the constraints. In this way equilibrium equations may be derived by requiring that $\overline{E}$ be stationary for unconstrained variations.

The results of the Appendix may be used to reduce the statement $(\overline{E})^{\cdot} = 0$ to

$$\int_{\xi} \{\dot{\Lambda}_\alpha \mathbf{D}_\alpha \cdot \mathbf{ED} + \boldsymbol{\omega} \cdot (Div\; \boldsymbol{\mu} + 2ax\; Skw[(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \mathbf{D})\mathbf{E}^t + \boldsymbol{\mu}\boldsymbol{\Gamma}^t])$$
$$\dot{\boldsymbol{\chi}} \cdot Div(\mathbf{R}\boldsymbol{\sigma} + \boldsymbol{\lambda} \otimes \mathbf{D})\}\, dv$$
$$= \int_{\partial\xi_t} \dot{\boldsymbol{\chi}} \cdot [\mathbf{t} \;\; (\mathbf{R}\boldsymbol{\sigma} + \boldsymbol{\lambda} \otimes \mathbf{D})\mathbf{n}]\, da + \int_{\partial\xi_c} \boldsymbol{\omega} \cdot (\mathbf{c} + \boldsymbol{\mu}\mathbf{n})\, da, \tag{34}$$

where $\mathbf{n}$ is the exterior unit normal to $\partial\xi$,

$$\boldsymbol{\Lambda} = \Lambda_\alpha \mathbf{D}_\alpha, \quad \boldsymbol{\lambda} = \mathbf{R}\boldsymbol{\Lambda}, \quad \boldsymbol{\sigma} = W_{\mathbf{E}} \quad \text{and} \quad \boldsymbol{\mu} = W_{\boldsymbol{\Gamma}} \tag{35}$$

and the variations of the multipliers have been made explicit. Hence the equilibrium equations

$$Div(\mathbf{R}\boldsymbol{\sigma} + \boldsymbol{\lambda} \otimes \mathbf{D}) = \mathbf{0},$$
$$Div\; \boldsymbol{\mu} + ax\{2\; Skw[(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \mathbf{D})\mathbf{E}^t + \boldsymbol{\mu}\boldsymbol{\Gamma}^t]\} = \mathbf{0} \quad \text{in } \xi, \tag{36}$$

and boundary conditions

$$\mathbf{t} = (\mathbf{R}\boldsymbol{\sigma} + \boldsymbol{\lambda} \otimes \mathbf{D})\mathbf{n} \quad \text{on } \partial\xi_t \quad \text{and} \quad \mathbf{c} + \boldsymbol{\mu}\mathbf{n} = \mathbf{0} \quad \text{on } \partial\xi_c. \tag{37}$$

The traction condition, with $\mathbf{n} = \mathbf{D}$, yields the interpretation of $\boldsymbol{\lambda} = \Lambda_\alpha \mathbf{d}_\alpha$ as a density of kinematically undetermined transverse shear force acting on the fiber cross sections.

Remarks. 1. Fiber inextensibility is accommodated by appending the constraint $\mathbf{RD} \cdot \mathbf{FD} = 1$. This affects the theory to the extent that $\boldsymbol{\Lambda}$ and $\boldsymbol{\lambda}$ are now 3 vectors given, respectively, by $\Lambda_i \mathbf{D}_i$ and $\Lambda_i \mathbf{d}_i$ in which Λ_1 is a kinematically undetermined density of axial force exerted on the fibers.

2. Incompressibility entails the constraint $\det \mathbf{F}(= \det \mathbf{E}) = 1$, which may be accommodated by using

$$\overline{W} = W + \Lambda_\alpha \mathbf{D}_\alpha \cdot \mathbf{ED} \;\; p(\det \mathbf{E} \;\; 1) \tag{38}$$

in place of (33), where p is the associated Lagrange multiplier. With reference to the Appendix, this affects only Eqs. $(36)_1$ and $(37)_1$, which are replaced by

$$Div(\mathbf{R}\boldsymbol{\sigma} \;\; p\mathbf{F}^* + \boldsymbol{\lambda} \otimes \mathbf{D}) = \mathbf{0} \quad \text{and} \quad \mathbf{t} = (\mathbf{R}\boldsymbol{\sigma} \;\; p\mathbf{F}^* + \boldsymbol{\lambda} \otimes \mathbf{D})\mathbf{n}, \tag{39}$$

respectively, augmented by the identity $Div\; \mathbf{F}^* = \mathbf{0}$.

3. The conventional theory of elasticity may be regarded as a special case of the Cosserat theory in which $\mathbf{c}$ vanishes, W is independent of $\boldsymbol{\Gamma}$ and $\mathbf{R}$ is constrained to be the rotation in the polar factorization of $\mathbf{F}$. Then, $\mathbf{E} = \mathbf{U}$ (the symmetric right stretch tensor) and the chain rule may be used, together with the symmetry of the second Piola Kirchhoff stress $\boldsymbol{\Pi}$ (which is necessary and sufficient for the rotation invariance of the strain energy) to obtain $\boldsymbol{\sigma} \cdot \dot{\mathbf{U}} = \dot{W} = \mathbf{R}^t\mathbf{P} \cdot \dot{\mathbf{U}}$, where $\mathbf{P}(= \mathbf{F}\boldsymbol{\Pi}) = W_{\mathbf{F}}$ is the usual Piola stress [25, p. 159]. This yields $\boldsymbol{\sigma} = Sym(\mathbf{R}^t\mathbf{P})$ and hence the identification of $\boldsymbol{\sigma}$ as the Biot stress of the conventional theory.

The connection between $\mathbf{R}$ and $\mathbf{F}$ means that the equilibrium equations *may not* be obtained simply by specializing (36) and (37). One way to accommodate this is to replace (14) by the constraint $Skw(\mathbf{R}^t\mathbf{F}) = \mathbf{0}$ and to replace (33) by $\overline{W} = W + \mathbf{W} \cdot \mathbf{R}^t\mathbf{F}$ with $\mathbf{W} \in Skw$. We obtain

$$(\overline{W})^{\cdot} = (\mathbf{P} + \mathbf{RW}) \cdot \dot{\mathbf{F}} + \mathbf{WU} \cdot \boldsymbol{\Omega} + \dot{\mathbf{W}} \cdot \mathbf{R}^t\mathbf{F}, \tag{40}$$

and the associated Euler equations, replacing (36), are

$$Div(\mathbf{P} + \mathbf{RW}) = 0 \quad \text{and} \quad Skw(\mathbf{WU}) = \mathbf{0}. \tag{41}$$

Because $\mathbf{U}$ is symmetric and positive definite the second of these yields $\mathbf{W} = \mathbf{0}$ [26, Lemma 1] and the equations, including boundary conditions, reduce to those of the conventional theory.

3.4. A simple model for fiber reinforced material

The kinematics of embedded fibers may be described in this framework by using (12) to write (cf. (4))

$$\kappa_i = \tfrac{1}{2} e_{ijk} \mathbf{D}_k \cdot \mathbf{R}^t\mathbf{R}'\mathbf{D}_j, \tag{42}$$

where $(\cdot)'$ is the directional, or *fiber* derivative along the fiber axis $\mathbf{D}$ and we have assumed, with minor loss of generality, that the fibers are straight and untwisted in ξ; i.e., that $\mathbf{D}'_j = \mathbf{0}$. Here we use $R'_{iA} = R_{iA,B} D_B$ to derive (cf. (A.9))

$$\mathbf{R}^t\mathbf{R}' = R_{iC} S_{iAB} D_B \mathbf{E}_C \otimes \mathbf{E}_A = e_{ACD} \Gamma_{DB} D_B \mathbf{E}_C \otimes \mathbf{E}_A, \tag{43}$$

which implies that $\boldsymbol{\kappa} = \kappa_i \mathbf{D}_i$ is determined by $\boldsymbol{\Gamma}$.

In view of the structure of the rod theory described in Section 2, we assume the constitutive response of the fiber reinforced material to depend on $\boldsymbol{\Gamma}$ via $\boldsymbol{\kappa}$; thus the strain energy is described by a (different) constitutive function $W(\mathbf{E}, \boldsymbol{\kappa})$. For the sake of illustration we further assume the material to be uniform, and thus that W does not depend explicitly on $\mathbf{X}$.

To determine the associated response function $\boldsymbol{\mu}$ for use in (36), we proceed indirectly, using [21]

$$\dot{\kappa}_i = \mathbf{d}_i \cdot \mathbf{a}' \quad \text{where } \mathbf{a} = ax(\dot{\mathbf{R}}\mathbf{R}^t) \tag{44}$$

(see (28)) in which the superposed dot refers, as in (25), to the derivative with respect to the parameter in a one parameter family of configurations. Accordingly, $\dot{\kappa}_i = \mathbf{D}_i \cdot \mathbf{R}^t\mathbf{a}' = \;\; \mathbf{D}_i \cdot \mathbf{R}^t (\mathbf{R}\boldsymbol{\omega})'$, yielding

$$\dot{\kappa}_i = (\mathbf{R}^t\mathbf{R}')\mathbf{D}_i \cdot \boldsymbol{\omega} \;\; \omega'_i \quad \text{where } \omega_i = \boldsymbol{\omega} \cdot \mathbf{D}_i. \tag{45}$$

Combining this with $\omega'_i = \omega_{i,A} D_A$ and fixing $\mathbf{E}$ ($\dot{\mathbf{E}} = \mathbf{0}$) we derive $\dot{W} = \mathbf{M} \cdot \dot{\boldsymbol{\kappa}}$, where

$$\mathbf{M} = M_i \mathbf{D}_i \quad \text{with } M_i = \partial W / \partial \kappa_i \tag{46}$$

and

$$\dot{W} = \boldsymbol{\omega} \cdot [Div(\mathbf{M} \otimes \mathbf{D}) + (\mathbf{R}^t\mathbf{R}')\mathbf{M}] \;\; Div[(\mathbf{M} \otimes \mathbf{D})^t \boldsymbol{\omega}]. \tag{47}$$

By equating this to the expression (A.10) for $\boldsymbol{\mu} \cdot \dot{\boldsymbol{\Gamma}}$ we conclude, on taking (A.9) into account, that

$$\boldsymbol{\mu} = \mathbf{M} \otimes \mathbf{D}, \tag{48}$$

and Eq. $(36)_2$ specializes, for uniform $\mathbf{D}(\mathbf{x})$, to

$$\mathbf{M}' + (\mathbf{R}^t\mathbf{R}')\mathbf{M} + ax\{2\; Skw[(\boldsymbol{\sigma} + \boldsymbol{\Lambda} \otimes \mathbf{D})\mathbf{E}^t]\} = \mathbf{0}$$
$$\text{where } \mathbf{M}' = (\nabla \mathbf{M})\mathbf{D}, \tag{49}$$

while the boundary condition $(37)_2$ becomes

$$\mathbf{c} = \;\; (\mathbf{D} \cdot \mathbf{n})\mathbf{M}, \tag{50}$$

implying that $\mathbf{c}$ vanishes at points where the fibers lie parallel to the boundary.

The model may be recast in a form more easily recognizable from rod theory by introducing the field

$$\mathbf{m} = M_i \mathbf{d}_i = \mathbf{R}\mathbf{M}. \tag{51}$$

This yields $\mathbf{M}' + (\mathbf{R}^t\mathbf{R}')\mathbf{M} = \mathbf{R}^t\mathbf{m}'$. Further, from (17) and $(35)_2$ we observe that

$$ax\{2\,Skw[(\boldsymbol{\Lambda} \otimes \mathbf{D})\mathbf{E}^t]\} = ax[2\,Skw(\mathbf{R}^t\boldsymbol{\lambda} \otimes \mathbf{R}^t\boldsymbol{\chi}')]$$
$$\text{where } \boldsymbol{\chi}' = \mathbf{F}\mathbf{D}. \tag{52}$$

Using the easily derived rule

$$ax(\mathbf{a} \otimes \mathbf{b} \quad \mathbf{b} \otimes \mathbf{a}) = \quad \mathbf{a} \times \mathbf{b} \tag{53}$$

we obtain

$$ax[2\,Skw(\mathbf{R}^t\boldsymbol{\lambda} \otimes \mathbf{R}^t\boldsymbol{\chi}')] = \mathbf{R}^t\boldsymbol{\chi}' \times \mathbf{R}^t\boldsymbol{\lambda} = \mathbf{R}^t(\boldsymbol{\chi}' \times \boldsymbol{\lambda}) \tag{54}$$

and substitute into (49), thereby reducing it to

$$\mathbf{m}' + \boldsymbol{\chi}' \times \boldsymbol{\lambda} + \mathbf{R}\{ax(\boldsymbol{\sigma}\mathbf{E}^t \quad \mathbf{E}\boldsymbol{\sigma}^t)\} = \mathbf{0} \quad \text{where } \mathbf{m}' = (\nabla\mathbf{m})\mathbf{D}, \tag{55}$$

whereas $(39)_1$ may be recast as

$$\boldsymbol{\lambda}' + Div(\mathbf{R}\boldsymbol{\sigma} \quad p\mathbf{F}^*) = \mathbf{0} \quad \text{where } \boldsymbol{\lambda}' = (\nabla\boldsymbol{\lambda})\mathbf{D} \tag{56}$$

in which the constraint of incompressibility has been incorpo rated. These may be regarded as the equilibrium equations for the reinforced solid.

Comparison with (5) and (46) furnishes the interpretation of $\mathbf{m}$ as a density of moment transmitted by a fiber, and confirms our earlier interpretation of $\boldsymbol{\lambda}$ as a density of force acting on a fiber. It also identifies the third term in (55) and the second term in (56) (which incorporates the effects of the axial force on a fiber (cf. (7))), respectively, as a density of distributed couples exerted by the matrix on a fiber and a distributed force exerted on the fiber. Further, the contribution to the net moment (cf. (31), (50) and (51)) from the embedded fibers reduces to

$$\mathbf{R}\mathbf{c} = \mathbf{m}(\mathbf{D} \cdot \mathbf{n}). \tag{57}$$

The dependence of the strain energy function on $\boldsymbol{\kappa}$ (or $\boldsymbol{\Gamma}$) introduces a natural length scale, l say, into the constitutive behavior which is on the order of that of the microstructure and hence of the diameter of a fiber cross section. Using this to define the dimensionless curvature twist vector $\overline{\boldsymbol{\kappa}} = l\boldsymbol{\kappa}$, supposing the latter to be small in typical applications and assuming that the fibers carry no bending or twisting moments when straight and untwisted, we find that W is given to leading order in $\overline{\boldsymbol{\kappa}}$ by

$$W(\mathbf{E},\boldsymbol{\kappa}) = W(\mathbf{E},\mathbf{0}) + \tfrac{1}{2}\boldsymbol{\kappa} \cdot \mathbf{K}(\mathbf{E})\boldsymbol{\kappa}, \tag{58}$$

where $\mathbf{K}(\mathbf{E}) = W_{\boldsymbol{\kappa}\boldsymbol{\kappa}}(\mathbf{E},\mathbf{0})$.

4. Material symmetry and transverse isotropy

4.1. General considerations

In this section we develop the theory of material symmetry for elastic Cosserat materials subject to the constraint (14). Our development borrows from that of Noll for conventional elasticity [27]. We first describe the manner in which the constitutive function for the strain energy may be computed for any choice of reference when that pertaining to any particular choice is given. We then derive a restriction on the constitutive function pertain ing to any given choice of reference following from the presumed existence of alternative choices that are related to the first by symmetry transformations.

Suppose, then, that ξ and μ are two references, and let $\mathbf{Y}(\mathbf{X})$ be the (invertible) map that takes points in ξ to points in μ. The deformation gradients relative to ξ and μ, denoted by $\mathbf{F}_\xi$ and $\mathbf{F}_\mu$, respectively, are related by

$$\mathbf{F}_\xi = \mathbf{F}_\mu\mathbf{H} \quad \text{where } \mathbf{H} = \nabla\mathbf{Y}. \tag{59}$$

We restrict attention to transformations $\mathbf{Y}$ with $\det\mathbf{H} = 1$, for reasons that are well known in conventional elasticity [27, p. 192], and impose $\mathbf{Y}(\mathbf{X}_0) = \mathbf{X}_0$. The specification of such a *pivot* removes an inessential translational degree of freedom from the discussion of symmetry that follows.

We have seen that the presumed rigidity of the director triad leads to the existence of a rotation $\mathbf{R}$ such that (12) is satisfied; here we write $\mathbf{d}_i = \mathbf{R}_\xi\mathbf{D}_i$. In the same way there is a rotation $\mathbf{R}_\mu$ such that $\mathbf{d}_i = \mathbf{R}_\mu\mathbf{G}_i$, where $\{\mathbf{G}_i(\mathbf{Y})\}$ is the positively oriented orthonormal director field defined in μ. Thus,

$$\mathbf{R}_\xi = \mathbf{R}_\mu\mathbf{L}, \tag{60}$$

where $\mathbf{L} = \mathbf{G}_i \otimes \mathbf{D}_i$ is the rotation field that maps the directors in ξ to their images in μ. We have $\mathbf{d} = \mathbf{R}_\xi\mathbf{D} = \mathbf{R}_\mu\mathbf{G}$, where $\mathbf{G}(=\mathbf{G}_1)$ is the unit tangent field to fibers in μ, so that $\mathbf{G} = \mathbf{L}\mathbf{D}$. To ensure that $\mathbf{D}$ is a material vector (cf. (13)), it is thus necessary to impose

$$\mathbf{H}\mathbf{D} = |\mathbf{H}\mathbf{D}|\mathbf{L}\mathbf{D}. \tag{61}$$

The rotation gradient fields $\mathbf{S}_\xi$ and $\mathbf{S}_\mu$, relative to ξ and μ, respectively, are related by

$$S^{(\xi)}_{iAB} = [S^{(\mu)}_{iCD}L_{CA} + R^{(\mu)}_{iC}L_{CA,D}]H_{DB} \quad \text{where } L_{CA,D} = \partial L_{CA}/\partial Y_D. \tag{62}$$

Given the constitutive function $U_\xi(\mathbf{F}_\xi,\mathbf{R}_\xi,\mathbf{S}_\xi;\mathbf{X}_0)$ pertaining to the reference ξ that pertaining to μ is given by

$$U_\mu(F^{(\mu)}_{iA},R^{(\mu)}_{iA},S^{(\mu)}_{iAB};X^0_A) = U_\xi(F^{(\mu)}_{iB}H_{BA},R^{(\mu)}_{iB}L_{BA},[S^{(\mu)}_{iCD}L_{CA} + R^{(\mu)}_{iC}L_{CA,D}]H_{DB};X^0_A). \tag{63}$$

Suppose now that ξ and μ respond identically to given deforma tion and director rotation fields; that is, suppose they are related by symmetry. Their constitutive functions then satisfy

$$U_\xi(\mathbf{F},\mathbf{R},\mathbf{S};\mathbf{X}_0) = U_\mu(\mathbf{F},\mathbf{R},\mathbf{S};\mathbf{X}_0), \tag{64}$$

and therefore

$$U_\xi(F_{iA},R_{iA},S_{iAB};X_A) = U_\xi(F_{iB}H_{BA},R_{iB}L_{BA},[S_{iCD}L_{CA} + R_{iC}L_{CA,D}]H_{DB};X_A). \tag{65}$$

Following Noll's characterization of solids (see [27]) we assume the existence of an *undistorted* reference and suppose ξ to be one of these. Thus we confine attention to proper orthogonal $\mathbf{H}$. Further, we remove an inessential orientational degree of freedom in the local change of reference by requiring that it preserve the *pivotal axis* $\mathbf{D}$; thus,

$$\mathbf{D} = \mathbf{H}\mathbf{D} = \mathbf{L}\mathbf{D}, \tag{66}$$

in place of (61).

To proceed further it is necessary to express the restriction (65) in terms of the reduced energy $W(\mathbf{E},\boldsymbol{\Gamma};\mathbf{X})$. Rather than pursue this in the general case, however, we proceed instead to the special case described in Section 3.4.

4.2. Application to the present model and specialization to transverse isotropy

For the simple model discussed in Section 3.4, the strain energy depends on $\mathbf{S}$ via $\boldsymbol{\kappa} = \kappa_i\mathbf{D}_i$, where κ_i is given by (4) in which the prime refers to the fiber derivative in ξ. In particular, for any function f we have $f' = (\nabla f)_\xi \cdot \mathbf{D} = (\nabla f)_\mu\mathbf{H} \cdot \mathbf{D}$, where the subscripts ξ and μ identify gradients with respect to $\mathbf{X} \in \xi$ and $\mathbf{Y} \in \mu$, respectively. Thus, from (66), $f' = (\nabla f)_\mu \cdot \mathbf{H}^t\mathbf{D} = (\nabla f)_\mu \cdot \mathbf{D}$, implying that the fiber derivative is invariant under transforma tions of the reference configuration that preserve the fiber axis. Accordingly, the κ_i are also invariant, and the curvature twist vectors $\boldsymbol{\kappa}_\xi$ and $\boldsymbol{\kappa}_\mu$ relative to the reference placements ξ and μ are related by

$$\boldsymbol{\kappa}_\xi = \mathbf{L}^t\boldsymbol{\kappa}_\mu, \tag{67}$$

whereas (cf. (17), (59), (60))

$$\mathbf{E}_\xi = \mathbf{L}^t\mathbf{E}_\mu\mathbf{H}. \tag{68}$$

These hold whether or not the fibers are straight or untwisted in μ, i.e. whether or not the $\mathbf{G}_i'$ vanish.

If ξ and μ are related by symmetry, then the associated strain energy functions satisfy (cf. (64))

$$W_\xi(\mathbf{E},\boldsymbol{\kappa}) = W_\mu(\mathbf{E},\boldsymbol{\kappa}) \tag{69}$$

at the pivot point X_0, where $W_\mu(\mathbf{E}_\mu,\boldsymbol{\kappa}_\mu) = W_\xi(\mathbf{E}_\xi,\boldsymbol{\kappa}_\xi)$; combining this with (67) and (68) yields the restriction

$$W_\xi(\mathbf{E},\boldsymbol{\kappa}) = W_\xi(\mathbf{L}^t\mathbf{E}\mathbf{H},\mathbf{L}^t\boldsymbol{\kappa}), \tag{70}$$

where the rotations $\mathbf{H}$ and $\mathbf{L}$ are connected by (66) but otherwise independent. This replaces (65) in the present circumstances.

If the reinforced material is transversely isotropic, with the fibers perpendicular to the planes of isotropy, then (70) holds without further restrictions on $\mathbf{H}$ or $\mathbf{L}$; that is, for *all* rotations $\mathbf{H},\mathbf{L} \in S$, where

$$S = \{\mathbf{Q} \in Orth^+ \text{ with } \mathbf{QD} = \mathbf{D}\}. \tag{71}$$

For example, strain energy functions of the type

$$W(\mathbf{E},\boldsymbol{\kappa}) = W_1(\mathbf{E}) + W_2(\mathbf{E})(\boldsymbol{\kappa}\cdot\mathbf{D})^2 + W_3(\mathbf{E})|\mathbf{1}\boldsymbol{\kappa}|^2$$
$$\text{with } \mathbf{1} = \mathbf{I} \quad \mathbf{D}\otimes\mathbf{D} \tag{72}$$

are suggested by (8) and furnish examples of (58). It is straight forward to verify that

$$\boldsymbol{\kappa}\cdot\mathbf{D} = \mathbf{L}^t\boldsymbol{\kappa}\cdot\mathbf{D} \quad \text{and} \quad |\mathbf{1}\boldsymbol{\kappa}| = |\mathbf{1}\mathbf{L}^t\boldsymbol{\kappa}| \tag{73}$$

for all $\mathbf{L} \in S$, and (70) is then satisfied for all $\boldsymbol{\kappa}$ provided that

$$W_i(\mathbf{E}) = W_i(\mathbf{L}^t\mathbf{E}\mathbf{H}), \quad i = 1,2,3. \tag{74}$$

To address this non standard representation problem we may derive necessary conditions by setting $\mathbf{L} = \mathbf{H} \in S$, writing the functions of $\mathbf{E}$ in terms of *Sym* $\mathbf{E}$ and *Skw* $\mathbf{E}$, and finally appealing to established theorems in representation theory (see [28] or [29]; Theorem 4.5.1 and Tables 4.1 4.7). This procedure yields the W_i in terms of a (possibly reducible) list of scalar invariants. Then, we may eliminate those scalars that do not remain invariant when $\mathbf{L},\mathbf{H} \in S$ are allowed to differ. For example, $\mathbf{L} = \mathbf{I}$ is permis sible and yields $W_i(\mathbf{E}) = W_i(\mathbf{EH})$ for all $\mathbf{H} \in S$. We would then eliminate the scalars that fail to remain invariant when $\mathbf{E}$ is replaced by $\mathbf{EH}$ for all such $\mathbf{H}$. This laborious process must then be repeated for all other choices of $\mathbf{L} \in S$. Indeed, the standard framework for deriving representations appears not to be well suited to the present theory.

Rather than pursue this procedure here, we simply record a list I of functionally independent scalars that are easily shown to satisfy (74) individually, for all $\mathbf{L},\mathbf{H} \in S$; namely,

$$I = \{I_1,\ldots,I_9\}, \tag{75}$$

where

$$I_1 = \mathrm{tr}(\mathbf{E}^t\mathbf{E}), \quad I_2 = \mathrm{tr}[(\mathbf{E}^t\mathbf{E})^2], \quad I_3 = \det\mathbf{E},$$
$$I_4 = \mathbf{D}\cdot\mathbf{ED}, \quad I_5 = \mathbf{D}\cdot(\mathbf{E}^t\mathbf{E})\mathbf{D},$$
$$I_6 = \mathbf{D}\cdot(\mathbf{E}\mathbf{E}^t)\mathbf{D}, \quad I_7 = \mathbf{D}\cdot\mathbf{E}^*\mathbf{D},$$
$$I_8 = \mathbf{D}\cdot(\mathbf{E}^t\mathbf{E})^2\mathbf{D}, \quad I_9 = \mathbf{D}\cdot(\mathbf{E}\mathbf{E}^t)^2\mathbf{D}, \tag{76}$$

and $\mathbf{E}^* = (\det\mathbf{E})\mathbf{E}^{-t}$ is the cofactor of $\mathbf{E}$. Thus any function of the elements of I automatically satisfies (74), but of course we have not shown that I is a function basis for transverse isotropy. It is included here mainly to establish that the representation problem defined by (74) is not vacuous. We observe that $\det\mathbf{E} = \det\mathbf{F}$, $\mathbf{E}^t\mathbf{E} = \mathbf{C}$ and $\mathbf{E}\mathbf{E}^t = \mathbf{R}^t\mathbf{B}\mathbf{R}$, where $\mathbf{C} = \mathbf{F}^t\mathbf{F}$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^t$ are the right and left Cauchy Green deformation tensors, respectively.

The response function $\boldsymbol{\sigma}$ derived from (72) and (75) is given by

$$\boldsymbol{\sigma} = W_{\mathbf{E}} = (W_1)_{\mathbf{E}} + (\boldsymbol{\kappa}\cdot\mathbf{D})^2(W_2)_{\mathbf{E}} + |\mathbf{1}\boldsymbol{\kappa}|^2(W_3)_{\mathbf{E}}, \tag{77}$$

with

$$(W_i)_{\mathbf{E}} = \sum_j W_{ij}(I_j)_{\mathbf{E}} \quad \text{where } W_{ij} = \partial W_i/\partial I_j \tag{78}$$

and $(I_j)_{\mathbf{E}}$ are the gradients of the invariants with respect to $\mathbf{E}$. To compute these we use the chain rule in the form

$$(I_j)_{\mathbf{E}}\cdot\dot{\mathbf{E}} = \dot{I}_j, \tag{79}$$

where the superposed dot is the derivative with respect to a parameter in a parametrized path $\mathbf{E}(\cdot)$. The procedure consists in expressing the right hand side as a linear form in $\dot{\mathbf{E}}$ and then using (79) to read off the associated gradient. To this end we use the identities $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA}) = \mathrm{tr}(\mathbf{B}^t\mathbf{A}^t)$ and $\mathbf{A}\cdot\mathbf{BC} = \mathbf{AC}^t\cdot\mathbf{B} = \mathbf{B}^t\mathbf{A}\cdot\mathbf{C}$ for arbitrary tensors $\mathbf{A},\mathbf{B},\mathbf{C}$, as needed, obtaining

$$(I_1)_{\mathbf{E}} = 2\mathbf{E}, \quad (I_2)_{\mathbf{E}} = 4\mathbf{EC}, \quad (I_3)_{\mathbf{E}} = \mathbf{E}^*,$$
$$(I_4)_{\mathbf{E}} = \mathbf{D}\otimes\mathbf{D}, \quad (I_5)_{\mathbf{E}} = 2\mathbf{E}(\mathbf{D}\otimes\mathbf{D}),$$
$$(I_6)_{\mathbf{E}} = 2(\mathbf{D}\otimes\mathbf{D})\mathbf{E}, \quad (I_7)_{\mathbf{E}} = I_7\mathbf{E}^{-t} \quad I_3\mathbf{E}^{-t}(\mathbf{D}\otimes\mathbf{D})\mathbf{E}^{-t},$$
$$(I_8)_{\mathbf{E}} = 2\mathbf{E}[(\mathbf{D}\otimes\mathbf{D})\mathbf{C} + \mathbf{C}(\mathbf{D}\otimes\mathbf{D})],$$
$$(I_9)_{\mathbf{E}} = 2[(\mathbf{D}\otimes\mathbf{D})\mathbf{EC} + \mathbf{E}\mathbf{E}^t(\mathbf{D}\otimes\mathbf{D})\mathbf{E}]. \tag{80}$$

To obtain the response function $\mathbf{M}$ (cf. (49)) we require the gradients

$$(\boldsymbol{\kappa}\cdot\mathbf{D})_{\boldsymbol{\kappa}} = \mathbf{D} \quad \text{and} \quad (|\mathbf{1}\boldsymbol{\kappa}|^2)_{\boldsymbol{\kappa}} = 2\mathbf{1}\boldsymbol{\kappa}. \tag{81}$$

Eq. (46) then delivers

$$\mathbf{M} = W_{\boldsymbol{\kappa}} = 2W_2(\mathbf{E})(\boldsymbol{\kappa}\cdot\mathbf{D})\mathbf{D} + 2W_3(\mathbf{E})\mathbf{1}\boldsymbol{\kappa} \tag{82}$$

and

$$\mathbf{m} = 2W_2(\mathbf{E})(\boldsymbol{\kappa}\cdot\mathbf{D})\mathbf{d} + 2W_3(\mathbf{E})\kappa_\alpha\mathbf{d}_\alpha,$$
$$\text{where } \kappa_\alpha\mathbf{d}_\alpha = \mathbf{d}\times\mathbf{d}' \quad \text{with } \mathbf{d}' = (\nabla\mathbf{d})\mathbf{D}, \tag{83}$$

which coincides with (9) provided that the torsional and flexural rigidities are replaced by the deformation dependent terms $2W_2(\mathbf{E})$ and $2W_3(\mathbf{E})$, respectively. Accordingly we impose $W_{2,3} > 0$, which in turn imply that the tensor $\mathbf{K}(\mathbf{E})$ of (58) is positive definite.

5. Example: torsion of a cylinder

We illustrate the theory by the simple example of finite torsion of a right circular cylinder. The reference placement ξ of the body is the region defined by $0 \le r \le a$, $0 \le \theta < 2\pi$, $0 \le z \le L$ in a cylindrical polar coordinate system (r,θ,z). Position of a material point in this region is given by

$$\mathbf{X} = r\mathbf{e}_r(\theta) + z\mathbf{k}, \tag{84}$$

where $\mathbf{e}_r$ is the radial unit vector at azimuth θ, directed away from the cylinder axis, $\mathbf{k}$ is the fixed unit vector along the axis and $\mathbf{e}_\theta = \mathbf{k}\times\mathbf{e}_r$. We pursue a standard semi inverse strategy and seek a deformation of the form

$$\boldsymbol{\chi}(\mathbf{X}) = r\mathbf{e}_r(\phi) + z\mathbf{k} \quad \text{where } \phi = \theta + \tau z \tag{85}$$

in which τ the twist per unit length is constant. The associated deformation gradient is [25]

$$\mathbf{F} = \mathbf{Q}[\mathbf{I} + r\tau\mathbf{e}_\theta(\theta)\otimes\mathbf{k}] \quad \text{where}$$
$$\mathbf{Q} = \mathbf{e}_r(\phi)\otimes\mathbf{e}_r(\theta) + \mathbf{e}_\theta(\phi)\otimes\mathbf{e}_r(\theta) + \mathbf{k}\otimes\mathbf{k} \in Orth^+. \tag{86}$$

This is isochoric and hence kinematically admissible in an incompressible material. Accordingly, we consider the incompres sibility constraint to be operative.

The fibers are assumed to be everywhere aligned with the axis of the cylinder in the reference placement; thus, $\mathbf{D}=\mathbf{k}$, the fiber derivative is $(\cdot)'=\partial(\cdot)/\partial z$, and the unit tangent $\mathbf{d}$ to a deformed fiber and the fiber stretch λ are given by

$$\lambda\mathbf{d}=\mathbf{F}\mathbf{k}=\mathbf{k}+r\tau\mathbf{e}_\theta(\phi),\quad \lambda=\sqrt{1+r^2\tau^2}. \tag{87}$$

This is sufficient to determine the action of the Cosserat rotation tensor on $\mathbf{k}$, i.e. $\mathbf{R}\mathbf{k}=\mathbf{d}$; we do not require the complete expression for $\mathbf{R}$. The trajectory of a fiber piercing a cross section at the point with coordinates (r,θ) is obtained by fixing the latter in the expression (85), yielding a circular helix of constant pitch. Such configurations are known to furnish equilibria for rods that are isolated in the sense that the distributed forces and moments exerted on them vanish identically [30]. We show below that this result is subsumed under the present theory.

Torsion is a standard problem in finite elasticity theory for isotropic incompressible materials [25]. To consider the simplest generalization of it to fiber reinforced solids, we suppose the torsional and flexural stiffnesses to be fixed and the leading term in (72) to be neo Hookean; thus,

$$W_1(\mathbf{E})=\tfrac{1}{2}\mu(I_1-3),\quad W_2(\mathbf{E})=\tfrac{1}{2}T\quad\text{and}\quad W_3(\mathbf{E})=\tfrac{1}{2}F \tag{88}$$

in which μ,T and F are positive constants. These generate the simple response functions

$$\boldsymbol{\sigma}=\mu\mathbf{E}\quad\text{and}\quad\mathbf{m}=T(\mathbf{k}\cdot\boldsymbol{\kappa})\mathbf{d}+F\mathbf{d}\times\mathbf{d}', \tag{89}$$

the first of which gives $\boldsymbol{\sigma}\mathbf{E}^t\in Sym$, implying (cf. (55)) that the matrix transmits no distributed couples to the fibers. Evidently this is not the case if the strain energy depends on the invariants I_4, I_6, I_7 or I_9. Using $\mathbf{R}\boldsymbol{\sigma}=\mu\mathbf{F}$ in the present circumstances, together with the rule $J\,div\,\mathbf{A}_1=Div\,\mathbf{A}_2$, with $\mathbf{A}_2=\mathbf{A}_1\mathbf{F}^*$ and $J=\det\mathbf{F}$, we find that the balance equations (55) and (56) reduce to

$$\mathbf{m}'+\lambda\mathbf{d}\times\boldsymbol{\lambda}=\mathbf{0}\quad\text{and}\quad\boldsymbol{\lambda}'+\mu\,div\,\mathbf{B}=grad\,p, \tag{90}$$

respectively, where *div* and *grad* are the divergence and gradient operations in the coordinate system (r,ϕ,z), and

$$\mathbf{B}=\mathbf{F}\mathbf{F}^t=\mathbf{I}+r\tau[\mathbf{e}_\theta(\phi)\otimes\mathbf{k}+\mathbf{k}\otimes\mathbf{e}_\theta(\phi)]+r^2\tau^2\mathbf{e}_\theta(\phi)\otimes\mathbf{e}_\theta(\phi) \tag{91}$$

is the left Cauchy Green deformation tensor with

$$div\,\mathbf{B}=-r\tau^2\mathbf{e}_r(\phi). \tag{92}$$

The standard finite elasticity problem corresponds to $(90)_2$ in which $\boldsymbol{\lambda}$ vanishes identically. In this case p reduces to a function of r only with $dp/dr=-\mu r\tau^2$, yielding

$$p(r)=p_0-\tfrac{1}{2}\mu\tau^2r^2, \tag{93}$$

where p_0 is a constant.

If the first of Eqs. (90) is scalar multiplied by $\mathbf{d}$ we find, using $(89)_2$, that $\kappa_1(=\mathbf{k}\cdot\boldsymbol{\kappa})$ is such that $\kappa_1'=0$, i.e. κ_1 is independent of z. Using the expression (87) for $\mathbf{d}$, together with

$$\mathbf{d}\times\mathbf{d}'=\lambda^{-2}r\tau^2[r\tau\mathbf{k}-\mathbf{e}_\theta(\phi)], \tag{94}$$

we then derive

$$\mathbf{m}'=\lambda^{-1}r\tau^2(\lambda^{-1}F\tau-T\kappa_1)\mathbf{e}_r(\phi). \tag{95}$$

This is sufficient to determine the fiber force $\boldsymbol{\lambda}$. For, $\boldsymbol{\lambda}\cdot\mathbf{d}$ vanishes identically (cf. $(35)_{1,2}$), whereas $(90)_1$ and (95) yield $\boldsymbol{\lambda}\times\mathbf{d}$. We obtain

$$\boldsymbol{\lambda}=(\boldsymbol{\lambda}\cdot\mathbf{d})\mathbf{d}+\mathbf{d}\times(\boldsymbol{\lambda}\times\mathbf{d})=-\lambda^{-3}r\tau^2(\lambda^{-1}F\tau-T\kappa_1)[r\tau\mathbf{k}-\mathbf{e}_\theta(\phi)]. \tag{96}$$

If (93) is to apply in the present setting then it is necessary that $\boldsymbol{\lambda}'=\mathbf{0}$. Because $\mathbf{e}_\theta'(\phi)=-\tau\mathbf{e}_r(\phi)$ is non zero, this in turn requires that

$$\kappa_1=\lambda^{-1}(F/T)\tau, \tag{97}$$

yielding the fiber twist as a function of r (cf. $(87)_2$) which is maximized on the axis of the cylinder. With this result we find that $\boldsymbol{\lambda}$ and $\mathbf{m}'$ vanish separately, so that $(90)_1$ is identically satisfied. With some algebra we also find, from (87), $(89)_2$ and (94), that

$$\mathbf{m}=F\tau\mathbf{k}, \tag{98}$$

implying that every fiber transmits the same moment. This result is interesting in light of the fact that the individual terms in $(89)_2$ associated with fiber twisting and bending are non uniform. In this solution the fibers are unforced and do not interact with the matrix.

To complete the solution we impose the traction condition (cf. $(39)_2$ with $\mathbf{D}=\mathbf{k}$)

$$(\mathbf{R}\boldsymbol{\sigma}-p\mathbf{F}^*)\mathbf{e}_r(\theta)=\mathbf{0}\quad\text{at } r=a. \tag{99}$$

This is equivalent to $(\mathbf{R}\boldsymbol{\sigma})\mathbf{F}^t\mathbf{e}_r(\phi)=p\mathbf{e}_r(\phi)$ and thus, in the present circumstances, to

$$\mu\mathbf{B}\mathbf{e}_r(\phi)=p\mathbf{e}_r(\phi)\quad\text{at } r=a, \tag{100}$$

yielding $p(a)=\mu$ and hence $p(r)=\frac{1}{2}\mu\tau^2(a^2-r^2)+\mu$. This furnishes

$$(\mathbf{R}\boldsymbol{\sigma})\mathbf{F}^t-p\mathbf{I}=\mu[\tfrac{1}{2}\tau^2(r^2-a^2)-1]\mathbf{I}+\mu\mathbf{B}. \tag{101}$$

We observe that $\mathbf{R}$ and $\boldsymbol{\sigma}$ never occur separately in the equili brium equations or boundary conditions. In fact, neither is determined by the analysis.

The overall response of the cylinder may be determined by computing the net force on a cross section and the net torque required to effect the torsion. These in turn require the traction

$$\mathbf{t}=[(\mathbf{R}\boldsymbol{\sigma})\mathbf{F}^t-p\mathbf{I}]\mathbf{k}=\tfrac{1}{2}\mu\tau^2(r^2-a^2)\mathbf{k}+\mu r\tau\mathbf{e}_\theta(\phi) \tag{102}$$

acting on a cross section. This is the same as the traction appearing in (25) because there is no change in cross sectional area in the course of the deformation. The resultant force is

$$\mathbf{f}=\int_0^{2\pi}\int_0^a\mathbf{t}r\,dr\,d\phi=f(\tau)\mathbf{k}, \tag{103}$$

where

$$f(\tau)=-\tfrac{1}{4}\pi a^4\mu\tau^2, \tag{104}$$

and is a manifestation of the well known normal stress effect in non linear elasticity theory.

Finally, the torque is (cf. (31), (50) and (51) with $\mathbf{D}=\mathbf{k}=\mathbf{n}$)

$$\boldsymbol{\rho}=\int_0^{2\pi}\int_0^a(\boldsymbol{\chi}\times\mathbf{t}+\mathbf{m})r\,dr\,d\phi=\rho(\tau)\mathbf{k}, \tag{105}$$

where

$$\rho(\tau)=\pi a^2\tau(F+\tfrac{1}{2}\mu a^2). \tag{106}$$

This problem may be cast in the framework of a conservative loading problem (cf. (21) and (22)), if desired, by taking $\partial\xi_t$ to be the lateral surface of the cylinder, where zero traction is assigned, and assigning position at $z=0$, L in accordance with (85). Thus the ends of the cylinder comprise $\partial\xi\backslash\partial\xi_t$. We note, from (87), that this also entails the assignment of $\mathbf{d}(=\mathbf{d}_1)$ at the ends of the cylinder. On the lateral surface we assume that no kinematical data are assigned and thus also identify it with $\partial\xi_c$. According to (22) and (23) the virtual work of the assigned couples $\mathbf{m}_i$ is $\mathbf{m}_i\cdot\dot{\mathbf{d}}_i=\mathbf{m}_i\cdot\dot{\mathbf{R}}\mathbf{R}^t\mathbf{d}_i=-\mathbf{m}_i\cdot\boldsymbol{\Omega}\mathbf{d}_i$; therefore,

$$\mathbf{c}\cdot\boldsymbol{\omega}=\mathbf{m}_i\times\mathbf{d}_i\cdot\boldsymbol{\omega}. \tag{107}$$

Because $\mathbf{c}$ vanishes on the lateral surface (cf. (50) with $\mathbf{D}=\mathbf{k}$ and $\mathbf{n}=\mathbf{e}_r(\theta)$), we assign $\mathbf{m}_i=\mathbf{0}$ there. At the ends $z=0$, L, where $\mathbf{d}$ is fixed, we may regard the $\mathbf{d}_\alpha=\mathbf{R}\mathbf{D}_\alpha$ as being fixed as well, where

$\{\mathbf{D}_\alpha\} = \{\mathbf{e}_r(\theta), \mathbf{e}_\theta(\theta)\}$, say. Then $\boldsymbol{\omega}$ vanishes there and the ends of the cylinder also comprise $\partial\xi \backslash \partial\xi_c$.

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Appendix A. Variational derivatives

Consider the one parameter families $\mathbf{F}(\mathbf{X};\epsilon)$ and $\mathbf{R}(\mathbf{X};\epsilon)$ of deformation and rotation fields, and let superposed dots stand for derivatives with respect to the parameter, evaluated at $\epsilon = 0$. Then,

$$\boldsymbol{\Omega} = \dot{\mathbf{R}}^t\mathbf{R} \tag{A.1}$$

is skew, and

$$\dot{\mathbf{E}} = \boldsymbol{\Omega}\mathbf{R}^t\mathbf{F} + \mathbf{R}^t\nabla\dot{\boldsymbol{\chi}}, \quad \dot{E}_{AB} = \Omega_{AC}E_{CB} + R_{iA}\dot{\chi}_{i,B}. \tag{A.2}$$

Further,

$$\dot{\boldsymbol{\Gamma}} = \dot{\Gamma}_{AB}\mathbf{E}_A \otimes \mathbf{E}_B, \quad \dot{\Gamma}_{AB} = \tfrac{1}{2}e_{CEA}(\dot{R}_{iC,B}R_{iD} + R_{iC,B}\dot{R}_{iD}) \tag{A.3}$$

in which

$$\dot{R}_{iA} = R_{iB}\Omega_{AB}. \tag{A.4}$$

The induced variation of the energy density is

$$\dot{W} = \sigma_{AB}\dot{E}_{AB} + \mu_{AB}\dot{\Gamma}_{AB},$$
$$\text{where } \sigma_{AB} = \partial W/\partial E_{AB} \text{ and } \mu_{AB} = \partial W/\partial \Gamma_{AB}. \tag{A.5}$$

We seek an expression for this in terms of $\dot{\chi}_i$ and Ω_{AB}. To this end we write

$$\begin{aligned}\mu_{AB}\dot{\Gamma}_{AB} &= \tfrac{1}{2}e_{BAE}\mu_{EC}(R_{iB,C}\dot{R}_{iA} + R_{iA}\dot{R}_{iB,C}) \\ &= \tfrac{1}{2}e_{BAE}[\mu_{EC}R_{iB,C}\dot{R}_{iA} - (\mu_{EC}R_{iA})_{,C}\dot{R}_{iB} + (\mu_{EC}R_{iA}\dot{R}_{iB})_{,C}] \\ &= \tfrac{1}{2}e_{BAE}[\mu_{EC}(R_{iB,C}\dot{R}_{iA} - R_{iA,C}\dot{R}_{iB})] \\ &\quad -\tfrac{1}{2}e_{BAE}\mu_{EC,C}R_{iA}\dot{R}_{iB} + \tfrac{1}{2}e_{BAE}(\mu_{EC}R_{iA}\dot{R}_{iB})_{,C}.\end{aligned} \tag{A.6}$$

Because the term in square brackets in the last line is skew in the subscripts B, A, we may simplify the expression to

$$\begin{aligned}\mu_{AB}\dot{\Gamma}_{AB} &= e_{BAE}\mu_{EC}R_{iB,C}R_{iD}\Omega_{AD} - \tfrac{1}{2}e_{BAE}\mu_{EC,C}\Omega_{BA} \\ &\quad + \tfrac{1}{2}e_{BAE}(\mu_{EC}\Omega_{BA})_{,C},\end{aligned} \tag{A.7}$$

where (A.4) has been used with $R_{iA}R_{iD} = \delta_{AD}$.

Let $\boldsymbol{\omega} = ax(\boldsymbol{\Omega})$ be the axial vector of $\boldsymbol{\Omega}$. Then, $e_{BAE}\Omega_{BA} = -2\omega_E$, $\Omega_{AD} = e_{DAF}\omega_F$ and $e_{BAE}e_{DAF}\omega_F = \omega_E\delta_{BD} - \omega_B\delta_{ED}$; the last of these following from one of the $e-\delta$ identities. We substitute into (A.7) and use $R_{iB,C}R_{iB} = (R_{iB}R_{iB})_{,C} - R_{iB}R_{iB,C}$ with $R_{iB}R_{iB} = 3$, obtaining

$$\mu_{AB}\dot{\Gamma}_{AB} = \omega_E(\mu_{EC,C} - \mu_{BC}R_{iE,C}R_{iB}) - (\omega_E\mu_{EC})_{,C}. \tag{A.8}$$

Using the inverse of $(18)_2$ in the form

$$R_{iB}R_{iE,C} = e_{EBD}\Gamma_{DC}, \tag{A.9}$$

we finally arrive at

$$\mu_{AB}\dot{\Gamma}_{AB} = \omega_E(\mu_{EC,C} + e_{EDB}\mu_{BC}\Gamma_{DC}) - (\omega_E\mu_{EC})_{,C}. \tag{A.10}$$

The first expression in $(A.5)_1$ yields much more easily; we use (A.2) to obtain

$$\begin{aligned}\sigma_{AB}\dot{E}_{AB} &= R_{iA}\sigma_{AB}\dot{\chi}_{i,B} + e_{CAD}\sigma_{AB}E_{CB}\omega_D \\ &= (R_{iA}\sigma_{AB}\dot{\chi}_i)_{,B} - \dot{\chi}_i(R_{iA}\sigma_{AB})_{,B} + e_{CAD}\sigma_{AB}E_{CB}\omega_D.\end{aligned} \tag{A.11}$$

The variation of the expression in (33) involving the constraint is

$$(\Lambda_\alpha\mathbf{D}_\alpha \cdot \mathbf{ED})^{\cdot} = \dot{\Lambda}_\alpha\mathbf{D}_\alpha \cdot \mathbf{ED} + \boldsymbol{\Lambda} \cdot \dot{\mathbf{E}}\mathbf{D}, \tag{A.12}$$

where $\boldsymbol{\Lambda} = \Lambda_\alpha\mathbf{D}_\alpha$. Using (A.2) we reduce the second term to

$$\boldsymbol{\Lambda} \cdot \dot{\mathbf{E}}\mathbf{D} = \boldsymbol{\lambda} \otimes \mathbf{D} \cdot \nabla\dot{\boldsymbol{\chi}} + ax(\boldsymbol{\Lambda} \otimes \mathbf{ED} - \mathbf{ED} \otimes \boldsymbol{\Lambda}) \cdot \boldsymbol{\omega}$$
$$\text{where } \boldsymbol{\lambda} = \mathbf{R}\boldsymbol{\Lambda} = \Lambda_\alpha\mathbf{d}_\alpha. \tag{A.13}$$

The variation of the term in (38) involving the constraint of incompressibility is

$$[p(\det\mathbf{E} - 1)]^{\cdot} = \dot{p}(\det\mathbf{E} - 1) + p\mathbf{E}^* \cdot \dot{\mathbf{E}}, \tag{A.14}$$

where

$$\mathbf{E}^* \cdot \dot{\mathbf{E}} = \mathbf{E}^*\mathbf{E}^t \cdot \boldsymbol{\Omega} + \mathbf{R}\mathbf{E}^* \cdot \nabla\dot{\boldsymbol{\chi}} \tag{A.15}$$

in which the first term on the right hand side vanishes identically and $\mathbf{RE}^* = \mathbf{RR}^t\mathbf{F}^* = \mathbf{F}^*$, yielding $\mathbf{E}^* \cdot \dot{\mathbf{E}} = \mathbf{F}^* \cdot \nabla\dot{\boldsymbol{\chi}}$.

With all constraints incorporated we then have

$$\begin{aligned}(\overline{W})^{\cdot} &= \boldsymbol{\omega} \cdot \{Div\,\boldsymbol{\mu} + 2ax[Skw(\boldsymbol{\sigma}\mathbf{E}^t + \boldsymbol{\mu}\boldsymbol{\Gamma}^t + \boldsymbol{\Lambda} \otimes \mathbf{ED})]\} \\ &\quad - \dot{\boldsymbol{\chi}} \cdot Div(\mathbf{R}\boldsymbol{\sigma} - p\mathbf{F}^* + \boldsymbol{\lambda} \otimes \mathbf{D}) \\ &\quad + Div[(\mathbf{R}\boldsymbol{\sigma} - p\mathbf{F}^* + \boldsymbol{\lambda} \otimes \mathbf{D})^t\dot{\boldsymbol{\chi}}] - Div(\boldsymbol{\mu}^t\boldsymbol{\omega}) \\ &\quad + \dot{\Lambda}_\alpha(\mathbf{D}_\alpha \cdot \mathbf{ED}) - \dot{p}(\det\mathbf{E} - 1).\end{aligned} \tag{A.16}$$

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