On the Relationship between the Cosserat and Kirchhoff-Love Theories of Elastic Shells

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Abstract: An attempt is made to clarify the interpretation of Kirchhoff-Love shells as constrained one-director Cosserat surfaces.

1. INTRODUCTION

Our purpose in this note is to clarify the relationship of the Kirchhoff-Love theory of elastic shells to the more general Cosserat theory of deformable surfaces with a single director. The latter has as its kinematical basis two vector fields defined on the surface, one that defines the particle position, and the other, the director, that is intended to account for finite-thickness effects.

Specifically, we obtain the Kirchhoff-Love theory by imposing constraints on the director field and deriving the general forms of the associated response functions through a careful application of the rigorous Lagrange multiplier rule. Although this rule is standard, the constraints considered are of an unusual type, and the development thus includes more detail than one usually finds in the literature on constrained elasticity.

In Naghdi's [1, secs. 10, 15] treatment of this subject, the Kirchhoff-Love theory is not derived from the Cosserat theory but is considered separately on the basis of distinct balance and invariance postulates. This contrasts with our view that the Cosserat theory should reduce to the Kirchhoff-Love theory upon the introduction of appropriate constraints. There are additional differences between Naghdi's treatment and ours. For example, the theory put forth in [1] relies on the moment-of-momentum equation to determine the skew part of a tensor whose symmetric part alone is deemed to be constitutively determinate. However, in conventional finite elasticity theory, the moment-of-momentum equation is identically satisfied if the strain energy is invariant under superposed rigid motions. In the present work, the response function in question is shown to be fully determinate and the moment-of-momentum equation reduces to an identity for invariant strain energies. Furthermore, in [1] the constitutively indeterminate skew part of another tensor that furnishes the moments transmitted across material curves is arbitrarily set to zero to obtain a determinate system of equations. We show this to be unnecessary. Nevertheless, the resulting model is substantially

equivalent to Naghdi's, and thus the differences between the two approaches are primarily conceptual, the main distinction being that the logical framework of the present development conforms to that of finite elasticity theory.

2. COSSERAT THEORY

We recall here the balance laws for the purely mechanical Cosserat theory of elastic shells with a single director. An exhaustive account is given in [1], and a concise summary may be found in [2].

Let $\mathbf{r}(\theta^\alpha, t)$ be the position in 3-space at time t of a particle with convected coordinates θ^α . Here and elsewhere, Greek indices take values in $\{1,2\}$, Latin indices take values in $\{1,2,3\}$, and standard tensor notation is used. The position function and the director field $\mathbf{d}(\theta^\alpha, t)$ constitute the basic kinematical variables of the theory. Let ω be an open, simply connected region of the surface in a given configuration with piecewise smooth boundary $\partial\omega$. The global balance postulates consist of the conservations of mass

$$\frac{d}{dt} \int_{\omega} \rho da = 0, \quad (2.1)$$

momentum

$$\frac{d}{dt} \int_{\omega} \rho (\mathbf{v} + \alpha \mathbf{w}) da = \int_{\omega} \rho \mathbf{f} da + \int_{\partial\omega} \mathbf{N} ds, \quad (2.2)$$

director momentum

$$\frac{d}{dt} \int_{\omega} \rho (\alpha \mathbf{v} + \beta \mathbf{w}) da = \int_{\omega} (\rho \mathbf{l} - \mathbf{k}) da + \int_{\partial\omega} \mathbf{M} ds, \quad (2.3)$$

and moment of momentum

$$\frac{d}{dt} \int_{\omega} \rho \mathbf{A} da = \int_{\omega} \rho (\mathbf{r} \times \mathbf{f} + \mathbf{d} \times \mathbf{l}) da + \int_{\partial\omega} (\mathbf{r} \times \mathbf{N} + \mathbf{d} \times \mathbf{M}) ds, \quad (2.4)$$

where

$$\mathbf{A} = \mathbf{r} \times (\mathbf{v} + \alpha \mathbf{w}) + \mathbf{d} \times (\alpha \mathbf{v} + \beta \mathbf{w}). \quad (2.5)$$

Our notation and terminology are similar to that used in [1]. Thus, ρ is the mass per unit area; $\mathbf{v} = \dot{\mathbf{r}}$ and $\mathbf{w} = \dot{\mathbf{d}}$ are the particle and director velocities, respectively, where the superposed dot is used to denote the time derivative at fixed values of θ^α ; $\mathbf{f}$ and $\mathbf{l}$ are the distributed force and distributed director force, respectively; $\mathbf{k}$ is the intrinsic director force; $\mathbf{N}$ and $\mathbf{M}$ are the traction and the director traction, respectively; and α, β are prescribed functions of θ^α . The specific forms of the inertia terms are not of primary importance to our development of the Kirchhoff-Love theory. Depending on the application, other forms may be more appropriate, particularly in the presence of finite strains [3].

As remarked by [1], the foregoing postulates may be motivated by appealing to conventional shell theory in which the position function in a thin three-dimensional body is assumed to be an affine function of a third convected coordinate associated with thickness. However, the mathematical status of the latter theory vis-à-vis three-dimensional elasticity is uncertain, and thus (2.1)-(2.4) are regarded here as being fundamental.

As in conventional continuum mechanics, the traction and director traction are assumed to depend on the curve $\partial\omega$ through its local orientation. This leads to the usual argument, outlined in [1] and [3], and developed in detail in [4] and [5], for the existence of stress vectors $\mathbf{N}^a$ and $\mathbf{M}^a$ such that

$$\mathbf{N} = \mathbf{N}^a v_a, \quad \mathbf{M} = \mathbf{M}^a v_a, \quad (2.6)$$

where $v_a = \varepsilon_{a\beta} d\theta^\beta / ds$ are the covariant components of the unit normal to the curve, $\theta^a(s)$ is the arclength parametrization of the curve, and $\varepsilon_{a\beta}$ are the covariant components of the permutation tensor density on the closure of ω .

Assuming sufficient smoothness, and recalling that ω is simply connected, the global balances (2.2), (2.3) together with (2.6) furnish the local equations [1]:

$$\mathbf{N}_{;a}^a + \rho \bar{\mathbf{f}} = 0 \quad \mathbf{M}_{;a}^a - \mathbf{k} + \rho \bar{\mathbf{l}} = 0, \quad (2.7)$$

where

$$\bar{\mathbf{f}} = \mathbf{f} - (\dot{\mathbf{v}} + \alpha \dot{\mathbf{w}}) \quad \bar{\mathbf{l}} = \mathbf{l} - (\alpha \dot{\mathbf{v}} + \beta \dot{\mathbf{w}}), \quad (2.8)$$

and the semicolon is used to denote covariant differentiation on ω . With these results, the local form of (2.4) may be written [1]:

$$\mathbf{a}_a \times \mathbf{N}^a + \mathbf{g}_a \times \mathbf{M}^a + \mathbf{d} \times \mathbf{k} = 0, \quad (2.9)$$

where

$$\mathbf{a}_a = \mathbf{r}_{,a}, \quad \mathbf{g}_a = \mathbf{d}_{,a} \quad (2.10)$$

and $(\cdot)_{,a} = \partial(\cdot)/\partial\theta^a$. The local form of the conservation of mass is

$$(\rho J)' = 0, \quad J = (a/A)^{1/2}, \quad (2.11)$$

where $a(>0)$ is the determinant of the symmetric, positive definite matrix

$$C_{a\beta} = \mathbf{a}_a \cdot \mathbf{a}_\beta \quad (2.12)$$

and A is the value of a in a fixed reference configuration.

Following [1], we impose the condition

$$\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{d} > 0, \quad (2.13)$$

which implies that $\mathbf{a}_\alpha$ are not collinear and that the director is nowhere tangent to the surface ω . This is the surface analog of the local impenetrability inequality of conventional continuum mechanics. We further assume that under arbitrary rigid-body motions, the transformation rules

$$\mathbf{a}_\alpha \rightarrow \mathbf{Q}\mathbf{a}_\alpha, \quad \mathbf{d} \rightarrow \mathbf{Q}\mathbf{d}, \quad \mathbf{g}_\alpha \rightarrow \mathbf{Q}\mathbf{g}_\alpha \quad (2.14)$$

apply, with $\mathbf{Q}$ a rotation.

A mechanical power identity may be constructed by scalar multiplying the first and second of equations (2.7) with $\mathbf{v}$ and $\mathbf{w}$, respectively, adding the resulting equations, and integrating over the simply connected region ω . This yields

$$\frac{dK}{dt} + S = P, \quad (2.15)$$

where

$$K = \frac{1}{2} \int_{\omega} \rho (\mathbf{v} \cdot \mathbf{v} + 2\alpha \mathbf{v} \cdot \mathbf{w} + \beta \mathbf{w} \cdot \mathbf{w}) da \quad (2.16)$$

is the kinetic energy of the material occupying the closure of ω ,

$$P = \int_{\omega} \rho (\mathbf{f} \cdot \mathbf{v} + \mathbf{l} \cdot \mathbf{w}) da + \int_{\partial\omega} (\mathbf{N} \cdot \mathbf{v} + \mathbf{M} \cdot \mathbf{w}) ds \quad (2.17)$$

is the power of the forces and director forces acting on the material, and

$$S = \int_{\omega} (\mathbf{N}^\alpha \cdot \dot{\mathbf{a}}_\alpha + \mathbf{M}^\alpha \cdot \dot{\mathbf{g}}_\alpha + \mathbf{k} \cdot \dot{\mathbf{d}}) da \quad (2.18)$$

is the stress power.

We introduce elasticity by assuming the existence of a strain energy Ψ per unit mass that, together with $\mathbf{N}^\alpha$, $\mathbf{M}^\alpha$, and $\mathbf{k}$, depends on $\mathbf{a}_\alpha$, $\mathbf{g}_\alpha$, and $\mathbf{d}$ and is such that $S = dU/dt$ for any subregion ω of the surface, where

$$U = \int_{\omega} \rho \Psi da. \quad (2.19)$$

This yields

$$\frac{dH}{dt} = P, \quad (2.20)$$

where $H = K + U$ is the mechanical energy. The same assumption, combined with conservation of mass, also implies that

$$\dot{W} = J\mathbf{N}^\alpha \cdot \dot{\mathbf{a}}_\alpha + J\mathbf{M}^\alpha \cdot \dot{\mathbf{g}}_\alpha + J\mathbf{k} \cdot \dot{\mathbf{d}} \quad (2.21)$$

pointwise, where $W = J\rho\Psi$ is the strain energy per unit area of the fixed surface used in the definition of J . Here, W is regarded as a function of the coefficient matrices in the expressions

$$\mathbf{a}_\alpha = F_{i\alpha} \mathbf{e}_i, \quad \mathbf{g}_\alpha = G_{i\alpha} \mathbf{e}_i, \quad \mathbf{d} = d_i \mathbf{e}_i, \quad (2.22)$$

where $\mathbf{e}_i$ is a set of fixed orthonormal vectors in 3-space. The values of the first pair of variables depend on the choice of coordinates θ^α , but the strain energy is required to be an absolute scalar field with values that are unaffected by this choice. Thus, the *form* of the function W must depend on coordinates.

On substituting (2.22) into (2.21), and assuming that $\dot{\mathbf{a}}_\alpha$, $\dot{\mathbf{g}}_\alpha$, and $\dot{\mathbf{d}}$ may be assigned independently and arbitrarily, we obtain the constitutive equations for unconstrained elastic Cosserat shells [1]:

$$J\mathbf{N}^\alpha = \frac{\partial W}{\partial F_{i\alpha}} \mathbf{e}_i = \frac{\partial W}{\partial \mathbf{a}_\alpha}, \quad J\mathbf{M}^\alpha = \frac{\partial W}{\partial G_{i\alpha}} \mathbf{e}_i = \frac{\partial W}{\partial \mathbf{g}_\alpha}, \quad J\mathbf{k} = \frac{\partial W}{\partial d_i} \mathbf{e}_i = \frac{\partial W}{\partial \mathbf{d}}, \quad (2.23)$$

in which the second equality in each expression is to be regarded as a definition.

3. CONSTRAINTS

In the Kirchhoff-Love theory, the strain energy is determined by the local deformation through the values of $\mathbf{a}_\alpha$ and $\mathbf{n}_{,\alpha}$, where $\mathbf{n} = \mathbf{a}_1 \times \mathbf{a}_2 / |\mathbf{a}_1 \times \mathbf{a}_2|$ is the local orientation field of the surface evaluated at the particle in question. It is also assumed that the gradient of $\mathbf{n}$ satisfies the Weingarten equation $\mathbf{n}_{,\alpha} = -b_{\beta\alpha} \mathbf{a}^\beta$, where $b_{\alpha\beta} = \mathbf{n} \cdot \mathbf{a}_{\alpha,\beta} = b_{\beta\alpha}$ are the symmetric coefficients of the second fundamental form of the surface. Here, $\mathbf{a}^\alpha$ are the unique vectors on the tangent plane such that $\mathbf{a}^\alpha \cdot \mathbf{a}_\beta = \delta_\beta^\alpha$ and $\mathbf{n} \cdot \mathbf{a}^1 \times \mathbf{a}^2 > 0$, where δ_β^α is the Kronecker delta.

A model with these features, regarded as being distinct from the Cosserat theory, is described in [1, secs. 10, 15], where it is referred to as the restricted direct theory. In particular, Naghdi's treatment involves dynamical variables formally similar to those appearing in (2.1)-(2.4) above, but with certain components set to zero a priori. In addition, the director momentum balance (2.3) is dropped from the list of postulates. However, in view of the fact that the constitutive assumptions of the Kirchhoff-Love theory are subsumed under those of the Cosserat theory, it seems more natural to consider the former theory within the framework of the latter rather than as a distinct model. This is the viewpoint adopted by Antman [5, ch. 14, secs. 10, 13], who incorporated the constraint $\mathbf{d} = \mathbf{n}$ in a global virtual work statement but omitted the director momentum balance. Thus, it appears that the status of the Kirchhoff-Love theory vis-à-vis the full Cosserat theory as presented in [1] has not been discussed in the literature.

In this work, we recover the Kirchhoff-Love theory by imposing the constraints

$$\mathbf{d} = \mathbf{n} \quad \text{and} \quad \mathbf{g}_\alpha = \mathbf{n}_{,\alpha} = -b_{\beta\alpha} \mathbf{a}^\beta \quad (3.1)$$

at the constitutive level. Constraints on $\mathbf{d}$ and its derivatives $\mathbf{g}_\alpha$ are needed, since the constitutive equations of the Cosserat theory involve the independent values of $\mathbf{d}$ and $\mathbf{d}_{,\alpha}$

at a material point. Moreover, $b_{\alpha\beta}$ are symmetric but otherwise arbitrary. This requires that $\mathbf{n} \cdot \mathbf{a}_{\alpha,\beta} = \mathbf{n} \cdot \mathbf{a}_{\beta,\alpha}$, but $\mathbf{a}_{\alpha,\beta}$ and $\mathbf{a}_{\beta,\alpha}$ need not be equal. Thus, there is no need to restrict attention to holonomic $\mathbf{a}_\alpha$ in the development of the constitutive theory. This allows for the possibility, not considered here, of a non-Riemannian material manifold [6].

To obtain constraints involving only the variables that appear in the Cosserat Theory, we note that in view of (2.13), equation (3.1)₁ is equivalent to the conditions $\phi = 0$ and $\phi_\alpha = 0$, where

$$\phi = \mathbf{d} \cdot \mathbf{d} - 1, \quad \phi_\alpha = \mathbf{d} \cdot \mathbf{a}_\alpha. \quad (3.2)$$

If these are satisfied, then (3.1)₂ is equivalent to the conditions $\psi = 0$ and $\psi_\alpha = 0$, where

$$\psi = e^{\alpha\beta} \mathbf{a}_\beta \cdot \mathbf{g}_\alpha, \quad \psi_\alpha = \mathbf{d} \cdot \mathbf{g}_\alpha \quad (3.3)$$

and $e^{\alpha\beta}$ is the unit alternator ($e^{12} = +1$). We note that the constraints are invariant under the transformations (2.14).

Naghdi [2] discussed the application of (3.2)₂ and (3.3)₁ to a model for a special Cosserat shell with a deformable director $\mathbf{d}$ constrained to be aligned with $\mathbf{n}$.

We use the standard Lagrange multiplier rule [7, 8] to derive representation formulas for the various dynamical variables in the presence of the constraints. To apply this rule, we must verify that the gradients of (3.2) and (3.3) are linearly independent vectors in a vector space E , with elements of the type $\mathbf{x} = (\mathbf{a}_\alpha, \mathbf{g}_\alpha, \mathbf{d})$. In view of (2.22), we may regard E as $\mathbb{R}^{15}$ endowed with the usual vector space structure and the Euclidean inner product. We also need to show that the constraint equations are solvable in terms of variables that locally parametrize a constraint manifold $\mathcal{M} \subset E$ obtained by restricting $\mathbf{x}$ such that all the constraints are satisfied simultaneously. Verification of the latter requirement is trivial: The general solution of the constraint equations (3.2) and (3.3) is $\mathbf{d} = \mathbf{a}_1 \times \mathbf{a}_2 / |\mathbf{a}_1 \times \mathbf{a}_2|$ and $\mathbf{g}_\alpha = K_{\beta\alpha} \mathbf{a}^\beta$, where

$$2K_{\beta\alpha} = \mathbf{g}_\alpha \cdot \mathbf{a}_\beta + \mathbf{g}_\beta \cdot \mathbf{a}_\alpha. \quad (3.4)$$

Thus, $\mathcal{M}$ is a nine-dimensional manifold parametrized by the variables $F_{i\alpha}, K_{\beta\alpha}$ with typical point $\mathbf{y} = (\mathbf{a}_\alpha, K_{\beta\alpha} \mathbf{a}^\beta, \mathbf{n})$. In view of the remarks following (3.1), we identify $-K_{\beta\alpha}$ with the coefficients of the second fundamental form of ω at the particle in question.

Linear independence is verified by regarding the constraints as functions of $\mathbf{x} \in E$ and using the Euclidean inner product to calculate their gradients. Thus, $d\phi(\mathbf{x}) = \nabla\phi(\mathbf{x}) \cdot d\mathbf{x}$ where $\nabla\phi \in E$, and so on, and we obtain

$$\begin{aligned} \nabla\phi(\mathbf{x}) &= (0, 0, 0, 0, 2\mathbf{d}) & \nabla\psi(\mathbf{x}) &= (-\mathbf{g}_2, \mathbf{g}_1, \mathbf{a}_2, -\mathbf{a}_1, 0) \\ \nabla\phi_1(\mathbf{x}) &= (\mathbf{d}, 0, 0, 0, \mathbf{a}_1) & \nabla\phi_2(\mathbf{x}) &= (0, \mathbf{d}, 0, 0, \mathbf{a}_2) \\ \nabla\psi_1(\mathbf{x}) &= (0, 0, \mathbf{d}, 0, \mathbf{g}_1) & \nabla\psi_2(\mathbf{x}) &= (0, 0, 0, \mathbf{d}, \mathbf{g}_2), \end{aligned} \quad (3.5)$$

in which the first and second pairs of entries are the counterparts of the components $\mathbf{a}_\alpha$ and $\mathbf{g}_\alpha$, respectively, of $\mathbf{x}$. An arbitrary linear combination of these gradients is a vector of the form

$$(c_1 \mathbf{d} - c_3 \mathbf{g}_2, c_2 \mathbf{d} + c_3 \mathbf{g}_1, c_3 \mathbf{a}_2 + c_5 \mathbf{d}, c_6 \mathbf{d} - c_3 \mathbf{a}_1, c_4 \mathbf{d} + c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2 + c_5 \mathbf{g}_1 + c_6 \mathbf{g}_2),$$

which vanishes for any $\mathbf{g}_\alpha$ and for all $\mathbf{a}_\alpha$ and $\mathbf{d}$ subject to (2.13) if and only if c_1 through c_6 are zero.

Let $\mathbf{x}(t)$ be a curve on $\mathcal{M}$ obtained by specifying $F_{i\alpha}(t)$ and $K_{\beta\alpha}(t)$ for t in an open interval. By differentiating (3.2) and (3.3) with respect to t , we verify that $\nabla\phi(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t)$, and so on, vanish for all $\dot{\mathbf{x}}(t) \in T(\mathbf{x}(t))$, where, by virtue of the continuity and linear independence of the gradients (3.5), $T(\mathbf{y})$ is the unique nine-dimensional vector subspace of E tangent to $\mathcal{M}$ at $\mathbf{y}$. Thus, E is the direct sum of the orthogonal subspaces $T(\mathbf{y})$ and $N(\mathbf{y}) = \text{Span}\{\nabla\phi(\mathbf{y}), \nabla\psi(\mathbf{y}), \nabla\phi_\alpha(\mathbf{y}), \nabla\psi_\alpha(\mathbf{y})\}$.

In (2.21), the strain energy is regarded as a function defined on $\mathcal{M}$, which we denote by $\bar{W}(\mathbf{y})$. To obtain $\bar{W}$, we introduce a smooth extension $W'(\mathbf{x})$ of W to E such that $W' = W$ when $\mathbf{x} \in \mathcal{M}$. Then, for t in an open interval and $\mathbf{x}(t) \in \mathcal{M}$, the chain rule [7] yields $\dot{W} = \dot{W}' = \nabla W'(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t)$, with $\dot{\mathbf{x}}(t) \in T(\mathbf{x}(t))$ and $\nabla W'(\mathbf{x}) = (\frac{\partial W'}{\partial \mathbf{a}_\alpha}, \frac{\partial W'}{\partial \mathbf{g}_\alpha}, \frac{\partial W'}{\partial \mathbf{d}})$. Here, the entries are defined as in (2.23) so that $\nabla W'(\mathbf{x}) \in E$. We note that, given W , there is a degree of arbitrariness in the extension W' . To see this, let $\bar{W}$ be another extension, let t belong to an open interval containing zero, and consider an arbitrary curve $\mathbf{x}(t) \in \mathcal{M}$ with $\mathbf{x}(0) = \mathbf{x}_0$. Both extensions are equal to W when restricted to $\mathcal{M}$, and the difference of their time derivatives vanishes on $\mathbf{x}(t)$. It follows that $\nabla \bar{W}(\mathbf{x}_0) - \nabla W'(\mathbf{x}_0) \in N(\mathbf{x}_0)$. We exploit this result in the next section.

The remaining response functions in (2.21) are also defined on $\mathcal{M}$. By decomposing them as in (2.22), we easily verify that $\mathbf{S}(\mathbf{y}) \in E$, where $\mathbf{S}(\mathbf{y}) = J(\mathbf{N}^\alpha, \mathbf{M}^\alpha, \mathbf{k})$ is the response vector.

We now write (2.21) as $\mathbf{B}(\mathbf{x}_0) \cdot \dot{\mathbf{x}} = 0$ for $\mathbf{x}_0 \in \mathcal{M}$ and $\dot{\mathbf{x}}$ an arbitrary element of $T(\mathbf{x}_0)$, where $\mathbf{B}(\mathbf{x}_0) = \mathbf{S}(\mathbf{x}_0) - \nabla \bar{W}(\mathbf{x}_0)$. Since $\mathbf{S}$ and $\nabla \bar{W}$ belong to E , we then have $\mathbf{B}(\mathbf{x}_0) \in N(\mathbf{x}_0)$, and $\mathbf{B}(\mathbf{x}_0)$ is therefore expressible as a linear combination of the gradients (3.5) evaluated at $\mathbf{x}_0 \in \mathcal{M}$. This is the Lagrange multiplier rule in the present context. Of course, the same conclusion applies with $\bar{W}$ replaced by W' .

To use this result, we recall our notational convention for (3.5) and obtain

$$\begin{aligned} \mathbf{N}^\alpha - \bar{\mathbf{N}}^\alpha &= q^\alpha \mathbf{n} + q \varepsilon^{\beta\alpha} \mathbf{n}_{,\beta} \\ \mathbf{M}^\alpha - \bar{\mathbf{M}}^\alpha &= q \varepsilon^{\alpha\beta} \mathbf{a}_\beta + r^\alpha \mathbf{n} \\ \mathbf{k} - \bar{\mathbf{k}} &= q^\alpha \mathbf{a}_\alpha + r \mathbf{n} + r^\alpha \mathbf{n}_{,\alpha}, \end{aligned} \tag{3.6}$$

where, in view of the physical context, the Lagrange multipliers (q, r) , (q^α, r^α) , respectively, are the values at θ^α and t of arbitrary scalar fields and contravariant vector fields on the closure of ω , $\varepsilon^{\alpha\beta} = a^{-1/2} e^{\alpha\beta}$ is the associated permutation tensor density, and

$$J\bar{\mathbf{N}}^\alpha = \frac{\partial \bar{W}}{\partial \mathbf{a}_\alpha}, \quad J\bar{\mathbf{M}}^\alpha = \frac{\partial \bar{W}}{\partial \mathbf{g}_\alpha}, \quad J\bar{\mathbf{k}} = \frac{\partial \bar{W}}{\partial \mathbf{d}}, \tag{3.7}$$

where the derivatives are evaluated on $\mathcal{M}$.

4. INVARIANCE AND THE REDUCED CONSTITUTIVE EQUATIONS

Since the constraint manifold is the domain of the constitutive functions, the latter are determined by the variables that parametrize $\mathcal{M}$; namely, $\mathbf{a}_\alpha$ and $K_{\alpha\beta} = -b_{\alpha\beta}$. In particular, the strain energy W at a particle is the value of a function $F(\mathbf{a}_\alpha, -b_{\alpha\beta})$.

We assume W to be unaltered by superposed rigid motions. In view of (2.14)₁, we thus require that $F(\mathbf{a}_\alpha, -b_{\alpha\beta}) = F(\mathbf{Q}\mathbf{a}_\alpha, -b_{\alpha\beta})$ for all rotations $\mathbf{Q}$, in which $b_{\alpha\beta}$, being invariant [1], play a passive role. From (2.12) and Cauchy's theorem on hemitropic functions [5], we obtain the necessary and sufficient condition

$$W = \bar{F}(C_{\alpha\beta}, -b_{\alpha\beta}) \quad (4.1)$$

for some function $\bar{F}$ whose domain is the set consisting of all ordered pairs of symmetric matrices, the first members of which are positive definite. To ensure that W is a scalar field, the form of the function $\bar{F}$ must reflect the choice of surface coordinates θ^a . The condition on W is met by requiring that it be a scalar-valued function of the tensors induced by the arguments of $\bar{F}$ on the tangent plane of a reference surface at the particle in question.

An extension $\bar{W}(\mathbf{a}_\alpha, \mathbf{g}_\alpha, \mathbf{d})$ of W to E may be obtained by using (2.12) and (3.4) to define extensions of $C_{\alpha\beta}$ and $K_{\alpha\beta}$ for $\mathbf{x} \in E$. We then set

$$\bar{W} = \bar{F}(C_{\alpha\beta}, K_{\alpha\beta}). \quad (4.2)$$

Equations (2.12), (2.14), and (3.4) imply that $\bar{W}$ thus defined is invariant under superposed rigid motions. Since the restrictions to $\mathcal{M}$ of all extensions are given by (4.1), it follows that the former are necessarily invariant under such motions. We satisfy this requirement by adopting as an additional assumption the invariance of all extensions of W for all $\mathbf{x} \in E$.

For $\mathbf{x}_0 \in \mathcal{M}$, let $\mathbf{S}(\mathbf{x}_0) \in E$ be the response vector obtained by using the extension (4.2). From the analysis of the previous section, we conclude that another invariant extension generates a response vector that differs from $\mathbf{S}(\mathbf{x}_0)$ by an element of $N(\mathbf{x}_0)$. This difference may be absorbed into the Lagrange multipliers, and thus no generality is lost by using the extension $\bar{W}$ defined by (4.2).

To demonstrate this explicitly, we consider an arbitrary invariant extension $W'(\mathbf{a}_\alpha, \mathbf{g}_\alpha, \mathbf{d})$ defined on E . If (2.13) is satisfied, then W' is invariant under rigid motions if and only if it is expressible as a function F' of the variables¹

$$C_{\alpha\beta}, \phi, \phi_\alpha, \psi_\alpha \text{ and } \lambda_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{g}_\beta, \quad (4.3)$$

where ϕ , ϕ_α , and ψ_α are defined in (3.2) and (3.3) but need not vanish. Let $\Pi = \bar{W} - W'$. Then, as remarked in Section 3, for any $\mathbf{x}_0 \in \mathcal{M}$ and $\dot{\mathbf{x}} \in T(\mathbf{x}_0)$, we have

$$0 = \nabla \Pi(\mathbf{x}_0) \cdot \dot{\mathbf{x}} = \frac{\partial \Pi}{\partial \mathbf{a}_\alpha} \cdot \dot{\mathbf{a}}_\alpha + \frac{\partial \Pi}{\partial \mathbf{g}_\alpha} \cdot \dot{\mathbf{g}}_\alpha + \frac{\partial \Pi}{\partial \mathbf{d}} \cdot \dot{\mathbf{d}}, \quad (4.4)$$

in which all quantities are evaluated on $\mathcal{M}$. In terms of $\Gamma = \bar{F} - F'$, the elements of $\nabla \Pi$ are

$$\begin{aligned}\frac{\partial \Pi}{\partial \mathbf{a}_\alpha} &= \left(\frac{\partial \Gamma}{\partial C_{\alpha\beta}} + \frac{\partial \Gamma}{\partial C_{\beta\alpha}} \right) \mathbf{a}_\beta + \frac{\partial \Gamma}{\partial \lambda_{\alpha\beta}} \mathbf{g}_\beta + \frac{\partial \Gamma}{\partial \phi_\alpha} \mathbf{d}, \\ \frac{\partial \Pi}{\partial \mathbf{g}_\alpha} &= \frac{\partial \Gamma}{\partial \lambda_{\beta\alpha}} \mathbf{a}_\beta + \frac{\partial \Gamma}{\partial \psi_\alpha} \mathbf{d}, \quad \frac{\partial \Pi}{\partial \mathbf{d}} = 2 \frac{\partial \Gamma}{\partial \phi} \mathbf{d} + \frac{\partial \Gamma}{\partial \phi_\alpha} \mathbf{a}_\alpha + \frac{\partial \Gamma}{\partial \psi_\alpha} \mathbf{g}_\alpha,\end{aligned}\quad (4.5)$$

and, with the constraints in force, we find after some labor that (4.4) is equivalent to

$$\frac{1}{2} \left(\frac{\partial \Gamma}{\partial C_{\alpha\beta}} + \frac{\partial \Gamma}{\partial C_{\beta\alpha}} \right) \dot{C}_{\alpha\beta} + \frac{\partial \Gamma}{\partial \lambda_{\beta\alpha}} \dot{K}_{\beta\alpha} = 0, \quad (4.6)$$

where $\dot{C}_{\alpha\beta}$ and $\dot{K}_{\beta\alpha}$ are arbitrary symmetric matrices. This in turn is satisfied if and only if

$$\frac{\partial \Gamma}{\partial C_{\alpha\beta}} + \frac{\partial \Gamma}{\partial C_{\beta\alpha}} = 0 \quad \text{and} \quad \frac{\partial \Gamma}{\partial \lambda_{\alpha\beta}} + \frac{\partial \Gamma}{\partial \lambda_{\beta\alpha}} = 0 \quad (4.7)$$

on $\mathcal{M}$.

Next, we use (3.7) and (4.2) together with $2K_{\alpha\beta} = \lambda_{\alpha\beta} + \lambda_{\beta\alpha}$ and formulas similar to (4.5) to derive

$$\bar{\mathbf{N}}^\alpha = \bar{N}^{\beta\alpha} \mathbf{a}_\beta, \quad \bar{\mathbf{M}}^\alpha = \bar{M}^{\beta\alpha} \mathbf{a}_\beta, \quad \bar{\mathbf{k}} = \mathbf{0}, \quad (4.8)$$

where

$$\bar{N}^{\beta\alpha} = \bar{\sigma}^{\beta\alpha} - b_\lambda^\beta \bar{M}^{\lambda\alpha}, \quad J\bar{\sigma}^{\beta\alpha} = \frac{\partial \bar{F}}{\partial C_{\alpha\beta}} + \frac{\partial \bar{F}}{\partial C_{\beta\alpha}}, \quad J\bar{M}^{\beta\alpha} = \frac{1}{2} \left(\frac{\partial \bar{F}}{\partial K_{\alpha\beta}} + \frac{\partial \bar{F}}{\partial K_{\beta\alpha}} \right). \quad (4.9)$$

These are evaluated at $K_{\alpha\beta} = -b_{\alpha\beta}$, and (3.1)₂ has been used in the form $\mathbf{g}_\alpha = -b_\alpha^\beta \mathbf{a}_\beta$.

Let $\mathbf{N}'^\alpha$, $\mathbf{M}'^\alpha$, and $\mathbf{k}'$ be defined as in (3.7), with $\bar{W}$ replaced by W' . According to (3.1), (4.5), and (4.7), we have

$$\begin{aligned}J(\bar{\mathbf{N}}^\alpha - \mathbf{N}'^\alpha) &= \frac{\partial \Gamma}{\partial \phi_\alpha} \mathbf{n} + J\gamma \varepsilon^{\beta\alpha} \mathbf{n}_\beta \\ J(\bar{\mathbf{M}}^\alpha - \mathbf{M}'^\alpha) &= J\gamma \varepsilon^{\alpha\beta} \mathbf{a}_\beta + \frac{\partial \Gamma}{\partial \psi_\alpha} \mathbf{n}, \quad J(\bar{\mathbf{k}} - \mathbf{k}') = \frac{\partial \Pi}{\partial \mathbf{d}}\end{aligned}\quad (4.10)$$

on $\mathcal{M}$, where $J\gamma = \frac{1}{2} \varepsilon^{\alpha\beta} \frac{\partial \Gamma}{\partial \lambda_{\beta\alpha}}$. Comparison with (3.6) shows that $J^{-1} \left(\frac{\partial \Gamma}{\partial \phi_\alpha}, \frac{\partial \Gamma}{\partial \psi_\alpha}, 2 \frac{\partial \Gamma}{\partial \phi} \right)$ may be absorbed into (q^α, r^α, r) and γ into q , and thus the response functions are given, without loss of generality, by (3.6), (4.8), and (4.9). We note that although the difference between two sets of Lagrange multipliers obtained in this way is invariant under rigid motions, there is nothing in the theory that requires the multipliers themselves to be invariant. The significance of this point has been discussed in [9].

5. EQUATIONS OF MOTION AND THE KIRCHHOFF EDGE CONDITIONS

The equations of motion for a Kirchhoff-Love shell have a well-known divergence structure [10] that may be recovered by first eliminating certain combinations of the Lagrange multipliers from the director-momentum equation (2.7)₂. To this end, we use (3.6)₂, together with the Gauss equation $\mathbf{a}_{\beta;\alpha} = b_{\beta\alpha} \mathbf{n}$ [1], the identity $\varepsilon_{;\alpha}^{\alpha\beta} = 0$, and the skew symmetry of $\varepsilon^{\alpha\beta}$, to derive

$$\mathbf{M}_{;\alpha}^{\alpha} - \mathbf{k} = \bar{\mathbf{M}}_{;\alpha}^{\alpha} - s^{\alpha} \mathbf{a}_{\alpha} - p \mathbf{n}, \quad (5.1)$$

where s^{α} and p are defined by

$$s^{\alpha} = q^{\alpha} + \varepsilon^{\alpha\beta} q_{;\beta}, \quad p = r - r_{;\alpha}^{\alpha}. \quad (5.2)$$

Using (4.8)₂, (2.7)₂ may then be solved to obtain

$$s^{\alpha} = \bar{M}_{;\beta}^{\alpha\beta} + \rho \bar{l}^{\alpha} \quad p = \bar{M}^{\alpha\beta} b_{\alpha\beta} + \rho \bar{l}, \quad (5.3)$$

where $\bar{l}^{\alpha} = \bar{\mathbf{l}} \cdot \mathbf{a}^{\alpha}$ and $\bar{l} = \bar{\mathbf{l}} \cdot \mathbf{n}$, respectively, are the tangential and normal components of $\bar{\mathbf{l}}$. With the constraints in force, (2.8)₂ becomes $\mathbf{l} - \bar{\mathbf{l}} = \alpha \dot{\mathbf{v}} + \beta \dot{\mathbf{n}}$, which is determined by the position function $\mathbf{r}(\theta^{\alpha}, t)$.

Following [2, sec. 6], we use (3.6)₁, (5.2)₁, and the symmetry of $\mathbf{n}_{;\beta\alpha}$ to obtain

$$\mathbf{N}_{;\alpha}^{\alpha} = \bar{\mathbf{N}}_{;\alpha}^{\alpha} + q_{;\alpha}^{\alpha} \mathbf{n} + s^{\alpha} \mathbf{n}_{;\alpha}. \quad (5.4)$$

The symmetry of $q_{;\beta\alpha}$ further implies that $q_{;\alpha}^{\alpha} = s_{;\alpha}^{\alpha}$. Consequently,

$$\mathbf{N}_{;\alpha}^{\alpha} = \mathbf{T}_{;\alpha}^{\alpha}, \quad (5.5)$$

where $\mathbf{T}^{\alpha} = \bar{\mathbf{N}}^{\alpha} + s^{\alpha} \mathbf{n}$.

It is of interest to note, from (3.6)₁ and (5.2)₁, that

$$\mathbf{N}^{\alpha} - \mathbf{T}^{\alpha} = \varepsilon^{\beta\alpha} (q \mathbf{n})_{;\beta}, \quad (5.6)$$

with divergence $\varepsilon^{\beta\alpha} (q \mathbf{n})_{;\beta\alpha}$ that vanishes identically, in accordance with (5.5)₁. Thus, the divergence form of the final equations, obtained by combining the foregoing with (2.7)₁, is

$$\mathbf{T}_{;\alpha}^{\alpha} + \rho \bar{\mathbf{f}} = 0. \quad (5.7)$$

If the constraints are operative, then $\bar{\mathbf{f}}$ is expressible in terms of $\mathbf{r}(\theta^{\alpha}, t)$ and the assigned field $\mathbf{f}$. Equations (4.8), (4.9), (5.3)₁, (5.5)₂, and (5.7) then furnish a formally determinate system of three equations for the three components of the position function, granted suitable boundary and initial conditions. The sole function of (5.3)₂ is to evaluate the combination of Lagrange multipliers defined by (5.2)₂ after the fact. These multipliers do not appear elsewhere in the balance laws. Therefore, (5.3)₂ and (5.7) are decoupled. The former

equation, which may now be regarded as irrelevant to the problem of determining $\mathbf{r}(\theta^\alpha, t)$, has no counterpart in alternative treatments of the Kirchhoff-Love theory.

It is straightforward to use Stokes's theorem with (5.7) to generate a global balance law identical in form to the linear momentum balance (2.2) in which $\mathbf{N} = \mathbf{N}^\alpha \nu_\alpha$ is replaced by $\mathbf{T}^\alpha \nu_\alpha$. The two balance laws are equivalent, since the difference between $\mathbf{N}^\alpha$ and $\mathbf{T}^\alpha$ is divergence free.

Naghdi's form of the equations [1, sec. 10] may be recovered by setting the inertia coefficients α and β to zero and taking $\bar{\mathbf{l}} = \mathbf{l}$ to be a tangential vector field. Indeed, it is apparent from (5.3), (5.5), and (5.7) that in general only the tangential components of $\bar{\mathbf{l}}$ contribute to the final equations for $\mathbf{r}(\theta^\alpha, t)$. However, in the present framework, it is not necessary to suppress the normal component of $\bar{\mathbf{l}}$, since the latter merely affects the values of p . Furthermore, it is not possible to eliminate this component if the inertia coefficients are present. An argument for the retention of these coefficients on mathematical grounds is given in [5].

Naghdi [1, sec. 15; 2, sec. 6] stipulates that the skew part of the matrix of tangential components of $\mathbf{M}^\alpha$ is to be set to zero to obtain a determinate system of equations. This is tantamount to requiring that $q = 0$ (cf. (3.6), (4.8), and (4.9)). However, we have seen that the gradient of q appears only in the combination (5.2)₁, and that q itself appears only in a divergence-free term that does not affect the local or global forms of the linear momentum balance. We will show that it has no direct effect on the Kirchhoff edge tractions or couples either. Thus, it is not necessary to eliminate q , although one may do so without loss of generality insofar as the problem of calculating the mechanical response of the shell is concerned. Variables with this property are called null-Lagrangians in variational treatments of mechanics.

To obtain the Kirchhoff conditions, we consider the part of the mechanical power P in (2.15) arising from boundary terms. According to (2.17), this is P_b , where

$$P_b = \int_{\partial\omega} (\mathbf{N} \cdot \mathbf{v} + \mathbf{M} \cdot \dot{\mathbf{n}}) ds, \quad (5.8)$$

in which the constraint $\mathbf{w} = \dot{\mathbf{n}}$ ($\mathbf{d} = \mathbf{n}$) has been imposed. We now use

$$\dot{\mathbf{n}} = -(\mathbf{n} \cdot \mathbf{v}_{,\alpha}) \mathbf{a}^\alpha \quad (5.9)$$

[1], together with (2.6), (3.6)₂, (4.8)₂, and (5.6), to write

$$\mathbf{M} \cdot \dot{\mathbf{n}} = -(\bar{M}^{\beta\alpha} + q\varepsilon^{\alpha\beta}) \nu_\alpha \mathbf{n} \cdot \mathbf{v}_{,\beta}, \quad \mathbf{N} \cdot \mathbf{v} = [\mathbf{T}^\alpha + \varepsilon^{\beta\alpha} (q\mathbf{n})_{,\beta}] \nu_\alpha \cdot \mathbf{v}, \quad (5.10)$$

and combine these to obtain

$$\mathbf{N} \cdot \mathbf{v} + \mathbf{M} \cdot \dot{\mathbf{n}} = \mathbf{T}^\alpha \nu_\alpha \cdot \mathbf{v} - \bar{M}^{\beta\alpha} \nu_\alpha \mathbf{n} \cdot \mathbf{v}_{,\beta} + \varepsilon^{\beta\alpha} (q\mathbf{n} \cdot \mathbf{v})_{,\beta} \nu_\alpha. \quad (5.11)$$

The vector field contracted with ν_α in the last term has zero divergence and, thus, makes no contribution to (5.8). In the second term, we substitute

$$\mathbf{v}_{,\beta} = v_\beta \mathbf{v}_\nu + \tau_\beta \mathbf{v}', \quad (5.12)$$

where τ_β are the covariant components of the unit tangent $\boldsymbol{\tau} = \mathbf{n} \times \boldsymbol{\nu}$ to $\partial\omega$ and $\mathbf{v}'$ and $\mathbf{v}_\nu$, respectively, are the independent arclength and normal derivatives of $\mathbf{v}$ on $\partial\omega$ [1]. The part of the resulting expression involving τ_β is

$$\bar{M}^{\beta\alpha} v_\alpha \tau_\beta \mathbf{n} \cdot \mathbf{v}' = (\bar{M}^{\beta\alpha} v_\alpha \tau_\beta \mathbf{n} \cdot \mathbf{v})' - (\bar{M}^{\beta\alpha} v_\alpha \tau_\beta \mathbf{n})' \cdot \mathbf{v}, \quad (5.13)$$

in which the first term on the right is regarded as a distributional derivative if $\partial\omega$ is piecewise smooth with a finite number of corners where $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$ are discontinuous. It follows that

$$P_b = \int_{\partial\omega} (\mathbf{t} \cdot \mathbf{v} - \mathbf{m} \cdot \mathbf{v}_\nu) ds + \sum \mathbf{F}_i \cdot \mathbf{v}_i, \quad (5.14)$$

with

$$\mathbf{t} = \mathbf{T}^\alpha v_\alpha + (\bar{M}^{\beta\alpha} v_\alpha \tau_\beta \mathbf{n})', \quad \mathbf{m} = M \mathbf{n}, \quad M = M^{\beta\alpha} v_\beta v_\alpha, \quad (5.15)$$

and

$$\mathbf{F} = -\bar{M}^{\beta\alpha} [v_\alpha \tau_\beta] \mathbf{n}, \quad (5.16)$$

where $\mathbf{F}_i$ is the point force at the i th corner of $\partial\omega$ required to support the motion; $\mathbf{v}_i$ is the particle velocity at the corner, the sum ranges over all the corners; and the notation $[\cdot]$ is used to denote the forward jump of the enclosed quantity as a corner is traversed in the sense of increasing s .

A further reduction may be achieved by using (5.9) and (5.12) to derive

$$\dot{\mathbf{n}} = -(\mathbf{n} \cdot \mathbf{v}_\nu) \boldsymbol{\nu} - (\mathbf{n} \cdot \mathbf{v}') \boldsymbol{\tau}. \quad (5.17)$$

Therefore, $-\mathbf{m} \cdot \mathbf{v}_\nu$ may be replaced by $M \boldsymbol{\nu} \cdot \dot{\mathbf{n}}$ in (5.14). We conclude that M is power conjugate to $\boldsymbol{\nu} \cdot \dot{\mathbf{n}}$ and, thus, equal to the bending couple along the edge. Independent justification of this interpretation is provided in Section 6.

Finally, $\mathbf{t}$ is conjugate to $\mathbf{v}$ and, thus, provides the Kirchhoff edge traction. On substituting (4.8)₁ and (5.5)₂ into (5.15)₁, we find

$$\mathbf{n} \cdot \mathbf{t} = s^\alpha v_\alpha + (\bar{M}^{\beta\alpha} v_\alpha \tau_\beta)', \quad (5.18)$$

and conclude that s^α contribute to the transverse shear traction across material curves. The multiplier q does not appear explicitly and, therefore, does not directly affect the mechanical response of any subregion of the shell, as we have already noted. Thus, we may identify s^α with q^α (cf. (5.2)₁) without loss of generality. Using (5.15)₁ and (5.16), we may also write the global balance law associated with (5.7) in the form (2.2) with

$$\int_{\partial\omega} \mathbf{N} ds = \int_{\partial\omega} \mathbf{t} ds + \sum \mathbf{F}_i. \quad (5.19)$$

6. MOMENT OF MOMENTUM

In finite elasticity theory, it is well known that the moment-of-momentum equation is identically satisfied if the response functions are derived from a strain energy that is insensitive to superposed rigid motions. The corresponding result for unconstrained Cosserat surfaces follows easily by evaluating (2.21) on the motion described by (2.14), with $\mathbf{Q}$ being an arbitrary time dependent rotation. The resulting equality is equivalent to the moment of momentum equation (2.9) by virtue of the arbitrariness of the axial vector of $\dot{\mathbf{Q}}\mathbf{Q}^T$.

For the Kirchhoff-Love theory, it follows from the procedure used to arrive at (3.6) that (2.9) remains valid with the response functions replaced by the constitutive functions $\bar{\mathbf{N}}^\alpha$, $\bar{\mathbf{M}}^\alpha$, and $\bar{\mathbf{k}}$. The same result follows directly by substituting (3.6) into (2.9), and with (4.8) this ultimately yields

$$\varepsilon_{\alpha\beta}\bar{N}^{\alpha\beta} = \varepsilon_{\alpha\beta}b_\lambda^\beta\bar{M}^{\alpha\lambda}. \quad (6.1)$$

In [1, sec. 15], the counterpart of this equation is used to determine the skew part of the matrix corresponding to our $\bar{N}^{\alpha\beta}$. Here, we use (4.9)₁ and the symmetry of $\bar{\sigma}^{\alpha\beta}$ to obtain $\varepsilon_{\alpha\beta}\bar{N}^{\alpha\beta} = -\varepsilon_{\alpha\beta}b_\lambda^\alpha\bar{M}^{\lambda\beta}$, which is identically equal to the right-hand side of (6.1) by virtue of the symmetry of $\bar{M}^{\lambda\beta}$.

The global form of the moment of momentum identity for Kirchhoff-Love shells is given by (2.4) and (2.5), in which the boundary integral is

$$I = \int_{\partial\omega} (\mathbf{r} \times \mathbf{N}^\alpha \nu_\alpha + \mathbf{n} \times \mathbf{M}^\alpha \nu_\alpha) ds. \quad (6.2)$$

We combine (3.6)₂ with (5.6) to reduce this to

$$I = \int_{\partial\omega} (\mathbf{r} \times \mathbf{T}^\alpha \nu_\alpha + \mathbf{n} \times \bar{\mathbf{M}}^\alpha \nu_\alpha) ds, \quad (6.3)$$

in which we have used $\mathbf{r} \times (q\mathbf{n})_{,\beta} = (\mathbf{r} \times q\mathbf{n})_{,\beta} + q\mathbf{n} \times \mathbf{a}_\beta$ to write the term in (5.6) involving the multiplier q as the sum of a divergence-free vector field and another term that cancels its counterpart in (3.6)₂. We now invoke (5.15)₁ together with $\mathbf{r}' = \boldsymbol{\tau}$ and $\boldsymbol{\tau} \times \mathbf{n} = \boldsymbol{\nu}$ to write

$$\mathbf{r} \times \mathbf{T}^\alpha \nu_\alpha = \mathbf{r} \times \mathbf{t} + (\bar{M}^{\beta\alpha} \nu_\alpha \tau_\beta) \boldsymbol{\nu} - (\bar{M}^{\beta\alpha} \nu_\alpha \tau_\beta \mathbf{r} \times \mathbf{n})'. \quad (6.4)$$

Similarly, (4.8)₂ may be used with the identity $\mathbf{a}_\beta = \nu_\beta \boldsymbol{\nu} + \tau_\beta \boldsymbol{\tau}$ and (5.15)₃ to obtain

$$\mathbf{n} \times \bar{\mathbf{M}}^\alpha \nu_\alpha = \mathbf{c} - (\bar{M}^{\beta\alpha} \nu_\alpha \tau_\beta) \boldsymbol{\nu}, \quad (6.5)$$

where

$$\mathbf{c} = M\boldsymbol{\tau}. \quad (6.6)$$

Thus, for piecewise smooth $\partial\omega$, I takes the simple form

$$I = \int_{\partial\omega} (\mathbf{r} \times \mathbf{t} + \mathbf{c}) ds + \sum \mathbf{r}_i \times \mathbf{F}_i, \quad (6.7)$$

which implies that $\mathbf{c}$ is a bending couple along $\partial\omega$ in accordance with the interpretation of M given in Section 5.

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NOTE

1. Naghdi [1, sec. 13, eq. 13.36] proved that if (2.13) holds, then any function of the type considered is invariant under superposed rigid motions if and only if it is expressible as a function of $C_{\alpha\beta}$, $\lambda_{\alpha\beta}$, ϕ_α , σ , and σ_α , where $\sigma = (\mathbf{d} \cdot \mathbf{n})^2$ and $\sigma_\alpha = (\mathbf{n} \cdot \mathbf{d})(\mathbf{n} \cdot \mathbf{g}_\alpha)$. We decompose $\mathbf{d}$ in the basis $\{\mathbf{a}_\alpha, \mathbf{n}\}$ and use (2.13) in the form $\mathbf{d} \cdot \mathbf{n} > 0$ to obtain $\mathbf{n} = (\mathbf{d} - \phi^\alpha \mathbf{a}_\alpha) / (\mathbf{d} \cdot \mathbf{n})$, where $\mathbf{d} \cdot \mathbf{n} = |\mathbf{d} - \phi^\alpha \mathbf{a}_\alpha|$, $\phi^\alpha = a^{\alpha\beta} \phi_\beta$, and $(a^{\alpha\beta}) = (C_{\alpha\beta})^{-1}$. It follows that $\sigma = \phi + \phi^\alpha \phi^\beta C_{\alpha\beta} - 2\phi^\alpha \phi_\alpha$ and $\sigma_\alpha = \psi_\alpha - \phi^\beta \lambda_{\beta\alpha}$. Thus, a function of $\mathbf{a}_\alpha$, $\mathbf{g}_\alpha$, and $\mathbf{d}$ subject to (2.13) is invariant only if it is expressible as a function of the variables listed in (4.3). Sufficiency is an immediate consequence of (2.14).

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