Analysis and control of quadrotor via a Normal Form approach

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ANALYSIS AND CONTROL OF QUADROTOR VIA A NORMAL FORM APPROACH

JING WANG*,† ISLAM BOUSSAADA*,† ARBEN CELA,‡ HUGUES MOUNIER* AND SILVIU-IULIAN NICULESCU*

Abstract. This paper focuses on the analysis and control of some mathematical models representing the dynamics of a quadrotor. By using a normal form approach, the highly coupled parts in the quadrotor system are eliminated, while all possible properties of the original system are not changed. The bifurcations of the system are then analyzed. A two dimensional system is deduced at the origin which can determine the stability and possible local bifurcations of the system. Based on the normal form and indirect method of Lyapunov, we propose a state feedback control method with computational simplicity as well as practical implementation facility. Comparing to a standard PID control, the proposed method has faster response time and less tracking errors especially with wind disturbance.

Key words. Normal forms, Quadrotor control, Center manifold.

AMS subject classifications. 93C10, 93C35, 93D15

1. Introduction. The quadrotor (see in Figure 1.1) is a mini unmanned aerial vehicle (UVA) with four rotors, which has been widely studied in the last decades [1, 2, 3, 4]. It is a system with four inputs, six outputs and highly coupled states. Due to its simplicity both in mechanical structure and maneuver, it is widely used in surveillance, search and rescue, mobile sensor networks [1]. Many methods have been proposed for controlling quadrotors. For example, Bouabdallah et al.[2] have proposed a backstepping control used separately in two subsystems. Besnard et al.[3] have proposed a sliding mode control driven by a disturbance observer. Wang et al.[4] have presented an event driven model free control which can avoid heavy computation. However, to the best of the authors' knowledge, the bifurcation of the dynamical system have never been studied.

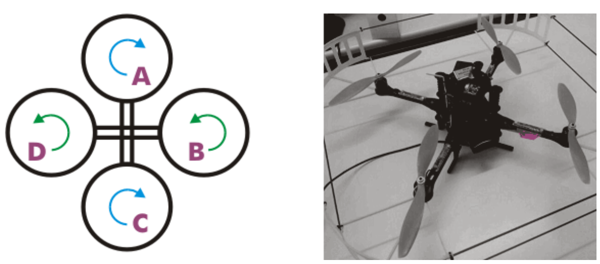

Fig. 1.1. *The quadrotor(right) from Ascending technologies(Available at ESIEE).*

The method of normal forms is an useful approach in studying the dynamical system properties [5]. Its purpose is employing successive coordinate transformations

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to construct the simplest form of the system. The normal form exhibits all possible properties of the original system. The normal forms of any degree with a single input were obtained by using change of coordinates and feedback [6]. For multi-input systems, the normal forms are deduced from the system with two inputs [7]. Based on the normal forms, the bifurcations and its control were studied by several authors [6, 8]. Center manifold is usually applied with the normal forms. It reduces the system to a center manifold associated with parts of the system with the eigenvalues with zero real parts at a bifurcation point [9].

To the best of our knowledge, the normal form and center manifold theories have never been used in the analysis and control of quadrotor. In this paper, the normal form of the quadrotor system is firstly calculated. By using such a methodology, the highly coupled parts in quadrotor system are eliminated. Under certain control laws, the normal form is reduced into a two dimensional system at the bifurcation point by using center manifold theory. Also, a simple control method based on the normal form using state feedback is proposed. The control laws are proposed to ensure the asymptotical stability of the system by moving all the eigenvalues of the system to the open left half plane. Comparing to a standard PID control, the proposed method has faster response time and less tracking errors especially when there is wind disturbance, as illustrated at the end of the paper. The interest of considering such control laws lies in the simplicity of the controller as well as in its practical implementation facility.

The paper is organized as follows: In Section 2, the model of quadrotor is given. In Section 3, the normal form of quadrotor is deduced. In Section 4, the bifurcation of the system under certain control laws is analyzed. In Section 5, simulations with and without wind disturbance using the proposed method and PID control are given.

2. The quadrotor model. The chosen model of quadrotor is depicted in equations (2.1). The rotation angles ϕ, θ and ψ are along the world axis x, y and z respectively, namely roll, pitch and yaw. $w_i (i = 1..4)$ are the accelerations caused by four rotors, which are the inputs of the system. ($g = 9.8\text{m/s}^2$ the gravity).

$$\begin{aligned} &\ddot{x} = -w_1 sin\theta, && \ddot{y} = w_1 cos\theta sin\phi, && \ddot{z} = w_1 cos\theta cos\phi - g, \\ &\ddot{\phi} = w_2, && \ddot{\theta} = w_3, && \ddot{\psi} = w_4. \end{aligned} \tag{2.1}$$

We introduce the variables as $x_1 = x$, $x_2 = \dot{x}$, $x_3 = y$, $x_4 = \dot{y}$, $x_5 = z$, $x_6 = \dot{z}$, $x_7 = \phi$, $x_8 = \dot{\phi}$, $x_9 = \theta$, $x_{10} = \dot{\theta}$, $x_{11} = \psi$, $x_{12} = \dot{\psi}$. Therefore, we can rewrite the system as:

$$\begin{aligned} &\dot{x}_1 = x_2, && \dot{x}_2 = -w_1 sin(x_9), && \dot{x}_3 = x_4, && \dot{x}_4 = w_1 cos(x_9) sin(x_7), \\ &\dot{x}_5 = x_6, && \dot{x}_6 = w_1 cos(x_9) cos(x_7) - g, && \dot{x}_7 = x_8, && \dot{x}_8 = w_2, \\ &\dot{x}_9 = x_{10}, && \dot{x}_{10} = w_3, && \dot{x}_{11} = x_{12}, && \dot{x}_{12} = w_4. \end{aligned} \tag{2.2}$$

3. Normal form of the system. It is easy to see that the equilibria of the system (2.2) are $x_e = (c_1, 0, c_2, 0, c_3, 0, k\pi, 0, k\pi, 0, c_4, 0)$, $w = (g, 0, 0, 0)$, where $k = 0, \pm1, \pm2, ...$, $c_i (i = 1..4) \in \mathbb{R}$ are constants and g is the gravity. Note in the real control system, $\phi, \theta \in (-\pi/2, \pi/2)$ and $\psi \in [0, \pi)$. Therefore, without losing generality, only the equilibrium $x_0 = (x, w) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, g, 0, 0, 0)$ is considered. We move x_0 to the origin by changing the coordinates of the inputs $w_1 = u_1 + g, w_2 = u_2, w_3 = u_3, w_4 = u_4$. Then, using the Taylor series of function $sin(x)$ and $cos(x)$ at $x = 0$. The system (2.2) can be written in polynomial form as follows. Here, O^5 are the polynomials with 5th and higher degree:

$$\dot{x}_1 = x_2, \;\; \dot{x}_2 = -gx_9 - u_1 x_9 + \frac{gx_9^3}{6} + \frac{u_1 x_9^3}{6} + O^5,$$

$$\dot{x}_3 = x_4, \quad \dot{x}_4 = gx_7 + u_1x_7 - \frac{gx_7^3}{6} - \frac{gx_9^2x_7}{2} - \frac{u_1x_7^3}{6} - \frac{u_1x_9^2x_7}{2} + O^5,$$
$$\dot{x}_5 = x_6, \quad \dot{x}_6 = u_1 - \frac{gx_7^2}{2} - \frac{gx_9^2}{2} - \frac{u_1x_7^2}{2} - \frac{u_1x_9^2}{2} + \frac{gx_7^4}{24} + \frac{gx_9^2x_7^2}{4} + \frac{gx_9^4}{24} + O^5,$$
$$\dot{x}_7 = x_8, \quad \dot{x}_8 = u_2, \quad \dot{x}_9 = x_{10}, \quad \dot{x}_{10} = u_3, \quad \dot{x}_{11} = x_{12}, \quad \dot{x}_{12} = u_4.$$

Using the state and input transformation $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_3$, $y_4 = x_4$, $y_5 = x_5$, $y_6 = x_6$, $y_7 = gx_7$, $y_8 = gx_8$, $y_9 = gx_9$, $y_{10} = gx_{10}$, $y_{11} = x_{11}$, $y_{12} = x_{12}$, $v_1 = u_1$, $v_2 = gu_2$, $v_3 = gu_3$, $v_4 = u_4$, we change the system (3.1) into Brunovsky form:

$$\begin{aligned}
&\dot{y}_1 = y_2, \quad \dot{y}_2 = -y_9 - \frac{v_1y_9}{g} + \frac{y_9^3}{6g^2} + \frac{v_1y_9^3}{6g^3} + O^5,\\
&\dot{y}_3 = y_4, \quad \dot{y}_4 = y_7 + \frac{v_1y_7}{g} - \frac{y_7y_9^2}{2g^2} - \frac{y_7^3}{6g^2} - \frac{v_1y_7^3}{6g^3} - \frac{v_1y_7y_9^2}{2g^3} + O^5,\\
&\dot{y}_5 = y_6, \quad \dot{y}_6 = v_1 - \frac{y_7^2}{2g} - \frac{y_9^2}{2g} - \frac{v_1y_7^2}{2g^2} - \frac{v_1y_9^2}{2g^2} + \frac{y_7^4}{24g^3} + \frac{y_7^2y_9^2}{4g^3} + \frac{y_9^4}{24g^3} + O^5,\\
&\dot{y}_7 = y_8, \quad \dot{y}_8 = v_2, \quad \dot{y}_9 = y_{10}, \quad \dot{y}_{10} = v_3, \quad \dot{y}_{11} = y_{12}, \quad \dot{y}_{12} = v_4.
\end{aligned} \tag{3.1}$$

The system (3.1) can be written as:

$$\dot{y} = f(y) + g(y)v = Ay + f^{(2)}(y) + f^{(3)}(y) + Bv + g^{(1)}(y)v + g^{(2)}(y)v + O^4 \tag{3.2}$$

where A, B are the coefficients of the linear parts, $f^{(2)}(y)$, $g^{(1)}(y)v$ are the second degree homogeneous polynomials of the system, $f^{(3)}(y)$, $g^{(2)}(y)v$ are the third degree homogeneous polynomials.

We take a third-degree homogeneous transformation for example [10]:

$$y = z + \phi^{(2)}(z) + \phi^{(3)}(z) \tag{3.3}$$

which z are the new states of the system. $\phi^{(2)}(z)$ is a second degree homogeneous polynomial and $\phi^{(3)}(z)$ is a third degree homogeneous polynomial of the states z, whose coefficients will be defined later.

We get the derivative of equation (3.3). Therefore, the derivative of the new states z are:

$$\dot{z} = (I + \frac{d\phi^{(2)}}{dz} + \frac{d\phi^{(3)}}{dz})^{-1}\dot{y} \tag{3.4}$$

where,

$$(I + \frac{d\phi^{(2)}}{dz} + \frac{d\phi^{(3)}}{dz})^{-1} = I - \frac{d\phi^{(2)}}{dz} - \frac{d\phi^{(3)}}{dz} + (\frac{d\phi^{(2)}}{dz})^2 + (\frac{d\phi^{(3)}}{dz})^2 + 2\frac{d\phi^{(2)}}{dz}\frac{d\phi^{(3)}}{dz}...$$

In (3.2), we rewrite the $f(y)$ and $g(y)$ using the new states z.

$$f(y) = A(y) + f^{(2)}(y) + f^{(3)}(y) + O^4 = Az + A\phi^{(2)}(z) + f^{(2)}(z) + A\phi^{(3)}(z) + f^{(3)}(z)...$$
$$g(y) = B + g^{(1)}(y) + g^{(2)}(y) + O^3 = B + g^{(1)}(z) + g^{(1)}(\phi^{(2)}(z)) + g^{(2)}(z)...$$

Therefore, with the help of the equations (3.2), (3.4), by now we have the new system:

$$\begin{aligned}
\dot{z} = {} & Az + Bv + A\phi^{(2)}(z) + f^{(2)}(z) + g^{(1)}(z)v - \frac{d\phi^{(2)}}{dz}Az - \frac{d\phi^{(2)}}{dz}Bv + A\phi^{(3)}(z)\\
& + f^{(3)}(z) + g^{(2)}(z)v + g^{(1)}(\phi^{(2)}(z))v - \frac{d\phi^{(2)}}{dz}(A\phi^{(2)}(z) + f^{(2)}(z) + g^{(1)}(z)v)\\
& - \frac{d\phi^{(3)}}{dz}(Az + Bv) + (\frac{d\phi^{(2)}}{dz})^2(Az + Bv) + O^4
\end{aligned}$$

For the simplicity of the system, the states z and the inputs v should be separated. In the third degree normal form, the polynomial $g^{(1)}(z)v$, $g^{(2)}(z)v$ should be canceled.

$$g^{(1)}(z) - \frac{d\phi^{(2)}}{dz}B = 0$$
$$g^{(2)}(z) + g^{(1)}(\phi^{(2)}(z)) - \frac{d\phi^{(2)}}{dz}g^{(1)}(z) - \frac{d\phi^{(3)}}{dz}B + (\frac{d\phi^{(2)}}{dz})^2 B = 0$$

Therefore, the transformation in equation (3.3) should be:

$$\phi^{(2)}(z) = (0, -\frac{z_6 z_9}{g}, 0, \frac{z_6 z_7}{g}, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$\phi^{(3)}(z) = (0, 0, 0, 0, 0, -\frac{z_6 z_7^2}{2g^2} - \frac{z_6 z_9^2}{2g^2}, 0, 0, 0, 0, 0, 0).$$

Using the same method, we can calculate the normal form of any degree. A Maple package 'QualitativeODE' [11] has been made for calculating the normal form of quadrotor. Using this programme, we get the third degree normal form of the system (3.1) as:

$$\begin{aligned}
&\dot{z}_1 = z_2 - \frac{z_6 z_9}{g}, && \dot{z}_2 = -z_9 + \frac{z_6 z_{10}}{g} - \frac{z_9^3}{3g^2} - \frac{z_7^2 z_9}{2g^2} + O^4,\\
&\dot{z}_3 = z_4 + \frac{z_6 z_7}{g}, && \dot{z}_4 = z_7 - \frac{z_6 z_8}{g} + \frac{z_7^3}{3g^2} + O^4,\\
&\dot{z}_5 = z_6 - \frac{z_6 z_7^2}{2g^2} - \frac{z_6 z_9^2}{2g^2}, && \dot{z}_6 = v_1 - \frac{z_7^2}{2g} - \frac{z_9^2}{2g} + \frac{z_6 z_7 z_8}{g^2} + \frac{z_6 z_9 z_{10}}{g^2} + O^4,\\
&\dot{z}_7 = z_8, \quad \dot{z}_8 = v_2, \quad \dot{z}_9 = z_{10}, && \dot{z}_{10} = v_3, \quad \dot{z}_{11} = z_{12}, \quad \dot{z}_{12} = v_4.
\end{aligned} \tag{3.5}$$

4. Bifurcation and simplification of the control system.

4.1. Bifurcation of the roots. It is easy to see that in the linear part of the equation (3.5), z_1 is related only to z_2, z_9, z_{10}, v_3; z_3 is related to z_4, z_7, z_8, v_2; z_5 is related to z_6, v_1; z_{11} is related to z_{12}, v_4. Therefore, the control laws can be defined as:

$$\begin{aligned}
&v_1 = K_{11} z_5 + K_{12} z_6, && v_2 = K_{21} z_3 + K_{22} z_4 + K_{23} z_7 + K_{24} z_8,\\
&v_4 = K_{41} z_{11} + K_{42} z_{12}, && v_3 = K_{31} z_1 + K_{32} z_2 + K_{33} z_9 + K_{34} z_{10}.
\end{aligned}$$

In this way, we can move the related eigenvalues in each group separately without changing the eigenvalues in other groups. Here, we define $v_i (i = 1..4)$ as:

$$\begin{aligned}
&v_1 = -256 z_5 + K_{12} z_6, && v_2 = -100 z_3 - 308 z_4 - 256 z_7 - 32 z_8,\\
&v_4 = -1024 z_{11} + K_{42} z_{12}, && v_3 = 100 z_1 + 308 z_2 - 256 z_9 - 32 z_{10}.
\end{aligned}$$

The system has three equilibria $P_1^e = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$, $P_2^e = (0, 0, 43.45, 0, -0.057, 0, -16.97, 0, 0, 0, 0, 0)$ and $P_3^e = (0, 0, -43.45, 0, -0.057, 0, 16.97, 0, 0, 0, 0, 0)$. However, only the origin P_1^e can be stable when K_{12}, K_{42} change.

At the equilibrium P_1^e, for simplicity $K_{12} = K_{42}$, when K_{12} changes, the real and imaginary parts of the eigenvalues are in Figure 4.1. When $K_{12} < 0$, the system has four eigenvalues with positive real parts, and the system becomes unstable. When $K_{12} > 0$, the system has all eigenvalues with negative real parts, and the system is asymptotically stable. When $K_{12} = 0$, the system has two pairs of pure imaginary eigenvalues $\pm 16i$ and $\pm 32i$, and all other eigenvalues have negative real parts, which is a four dimensional center manifold. The stability cannot be determined by the linear part of the system. It depends on the nonlinearity of the system.

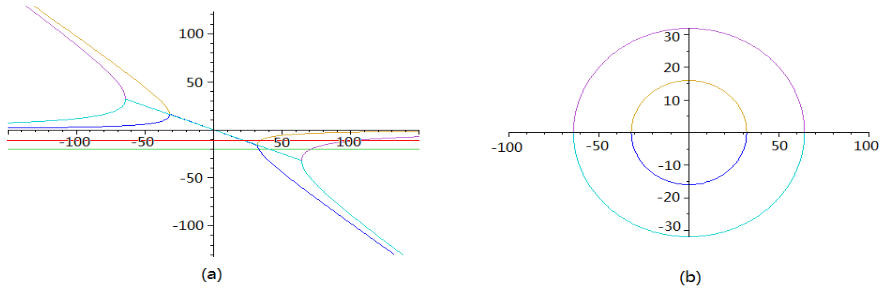

FIG. 4.1. *The eigenvalues when K_{12} changes from -150 to 150: (a) the real parts. (b) the imaginary parts.*

4.2. Center manifold. The aim of this part is to get the reduced system which can determine the stability and possible local bifurcations of the system at one bifurcation point [12]. A system can be written as:

$$\dot{x} = A(b)x + F(x), \qquad x \in R^n$$

where b is a free parameter, $b \in R$.

At its origin $x = [0, ..., 0]$, $J(b)$ is the Jordan form of the matrix $A(b)$ and Q is a matrix which enables $Q(b)J(b)Q^{-1}(b) = A(b)$. Therefore, we have:

$$\dot{x} = Q(b)J(b)Q^{-1}(b)x + F(x) \quad \Rightarrow \quad Q^{-1}(b)\dot{x} = J(b)Q^{-1}(b)x + Q^{-1}(b)F(x)$$

we define $y = Q^{-1}(b)x$, then

$$\dot{y} = J(b)y + Q^{-1}(b)F(Q(b)y) = J(b)y + \tilde{F}(y) \tag{4.1}$$

we can separate the Jordan matrix J as matrices B and C whose eigenvalues have zero real parts and negative real parts respectively. Therefore, we can rewrite the system (4.1) at the origin with $x = [0, ..., 0]$.

$$\dot{y}_0 = By_0 + f(y_0, y_-), \qquad \dot{y}_- = Cy_0 + g(y_0, y_-).$$

Since the center manifold is tangent to E^c(the $y_- = 0$ space), we define

$$y_- = h(y_0, b), \quad h(0,0) = Dh(0,0) = 0, \quad \dot{b} = 0. \tag{4.2}$$

We can calculate the function $h(y_0, b)$ by using

$$\dot{y}_- = Dh(y_0, b)\dot{y}_0 = Dh(y_0, b)[By_0 + f(y_0, h(y_0, b))] = Cy_0 + g(y_0, h(y_0, b))$$

Therefore, we can get the local evolution equations of y_0 which can determine the stability of the original system.

In quadrotor center manifold analysis, the control laws are defined as:

$$v_1 = -256z_5 - bz_6 - z_5^3, \qquad v_2 = -100z_3 - 308z_4 - 256z_7 - 32z_8,$$
$$v_4 = -10_{11} - 24z_{12}, \qquad v_3 = 100z_1 + 308z_2 - 256z_9 - 32z_{10}.$$

The bifurcation of the system is like in previous subsection. When $b < 0$, the system has two eigenvalues with positive real parts. When $b > 0$, the system has all eigenvalues with negative real parts. When $b = 0$, the system has two pure imaginary eigenvalues $\pm 16i$, and all other eigenvalues have negative real parts. The stability depends on the nonlinear parts of the system. We can use the center manifold theory to simplify the system, and further simplify the study of the bifurcation of the system. In this control system, $y_0 = [y_1, y_2]^T = [z_5, z_6]^T$ and $y_- = [y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}]^T = [z_1, z_2, z_3, z_4, z_7, z_8, z_9, z_{10}, z_{11}, z_{12}]^T$.

We seek a quadratic center manifold (a are parameters to be defined later):

$$y_i = a_{i200}y_1^2 + a_{i020}y_2^2 + a_{i002}b^2 + a_{i110}y_1y_2 + a_{i101}y_1b + a_{i011}y_2b, \qquad i = 3..12$$

Using the method mentioned before, we get $h(y_0, b) = [-0.62b^2, -0.62b^2, -0.76b^2, -0.76b^2, 0, 0, 0, 0, -0.42b^2, -23.58b^2]$ in equation (4.2).

Therefore, the reduced system on the center manifold can be written:

$$\begin{aligned} \dot{y}_1 &= 16y_2 - 0.41b^4 - 0.011b^8 - (b + 0.057b^4)y_1 + 0.00024y_2^3 \\ \dot{y}_2 &= -16y_1 + 0.67b^4y_1 \end{aligned} \tag{4.3}$$

In the reduced system, when b is positive (negative), the origin is a stable (unstable) focus. When $b = 0$, the origin is a center. The phase portrait of equation (4.3) when $b = -0.5$, $b = 0$ and $b = 0.5$ are depicted in Figure 4.2.

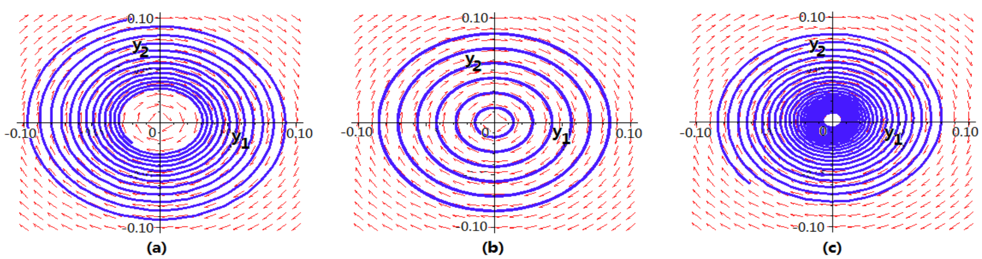

FIG. 4.2. *The phase portrait of the reduced system: (a) b=-0.5. (b) b=0. (c) b=0.5.*

5. Quadrotor control. Here we propose a control method based on the normal form and Lyapunov theory. In equation (3.5), the Jacobian matrix of the system can be easily found. If the system is time invariant, the indirect method of Lyapunov says that if the eigenvalues of Jacobian matrix of the system at the origin are in the open left half complex plane, then the origin is asymptotically stable. Therefore, we can define the state feedback as follows to move all the eigenvalues of the system to the open left half plane. x_r, y_r, z_r, ψ_r are the references.

$$\begin{aligned} v_1 &= -256(z_5 - z_r) - 32z_6, \qquad & v_2 &= -1700(z_3 - y_r) - 1000z_4 - 256z_7 - 32z_8 \\ v_4 &= -256(z_{11} - \psi_r) - 32z_{12}, \qquad & v_3 &= 1700(z_1 - x_r) + 1000z_2 - 256z_9 - 32z_{10} \end{aligned}$$

The simulation task is to let quadrotor follow a square path with the length of 2m while hovering at the altitude of 10m, which is given in Figure 5.1. The totally sample time is 20s. For comparison, the simulations using a standard PID control are also given.

5.1. Simulation without wind disturbance. The simulation results are given in Figure 5.2. The desired response time is 1s. We can see that the proposed method has better performance than a standard PID control.

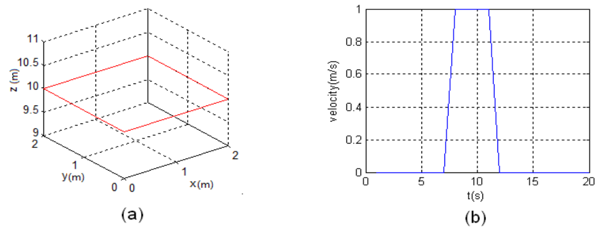

FIG. 5.1. *(a) Reference trajectory for the quadrotor. (b) Wind disturbance.*

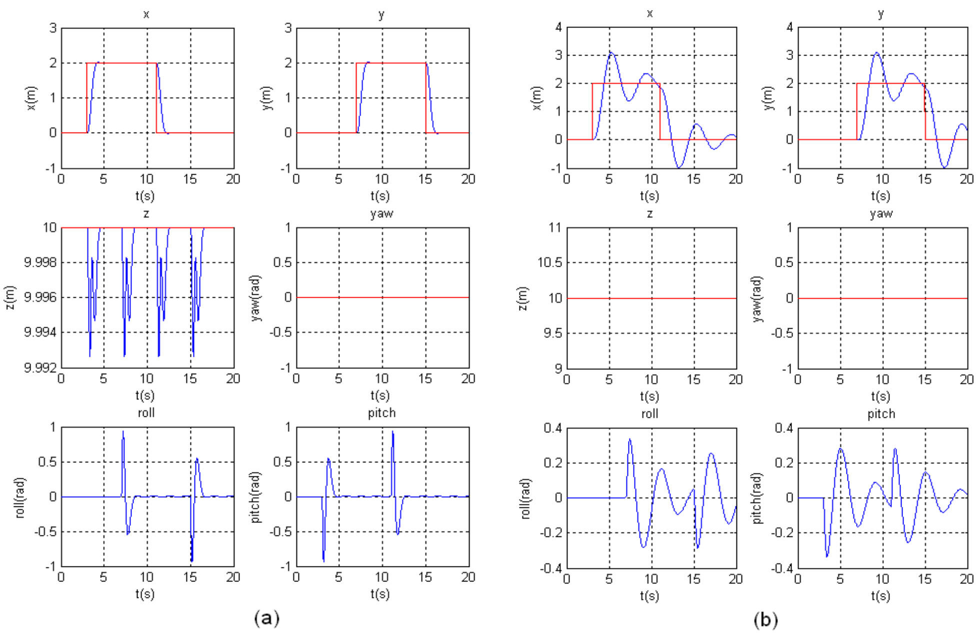

FIG. 5.2. *The simulation without wind disturbance: (a) the proposed method. (b) a standard PID.*

5.2. Simulation with wind disturbance. During the trajectory, there may have wind disturbance with velocity 1m/s as in Figure 5.1, which occurs in all x, y and z axis. The simulation results are given in Figure 5.3. The desired response time is 1s. The proposed method can keep the stability during the wind disturbance, and has better performance than a standard PID control.

6. Conclusion. In this paper, the normal form of quadrotor is deduced. A Maple package 'QualitativeODE' [11] has been written for calculating the normal form of any degree of the system. From equation (3.5), we can see that the highly coupled parts in quadrotor system are eliminated. This makes the analysis of the dynamical system easier. Under certain control laws, the system can be further deduced using center manifold theorem. A two dimensional system is deduced which can determine the stability and possible local bifurcations of the control system at the origin. Based on the normal form and indirect method of Lyapunov, we proposed a state feedback control method with computational simplicity as well as practical implementation facility. This method achieved good results. In the simulations, the system can remain stable with small tracking errors even if there is wind disturbance. Also, this

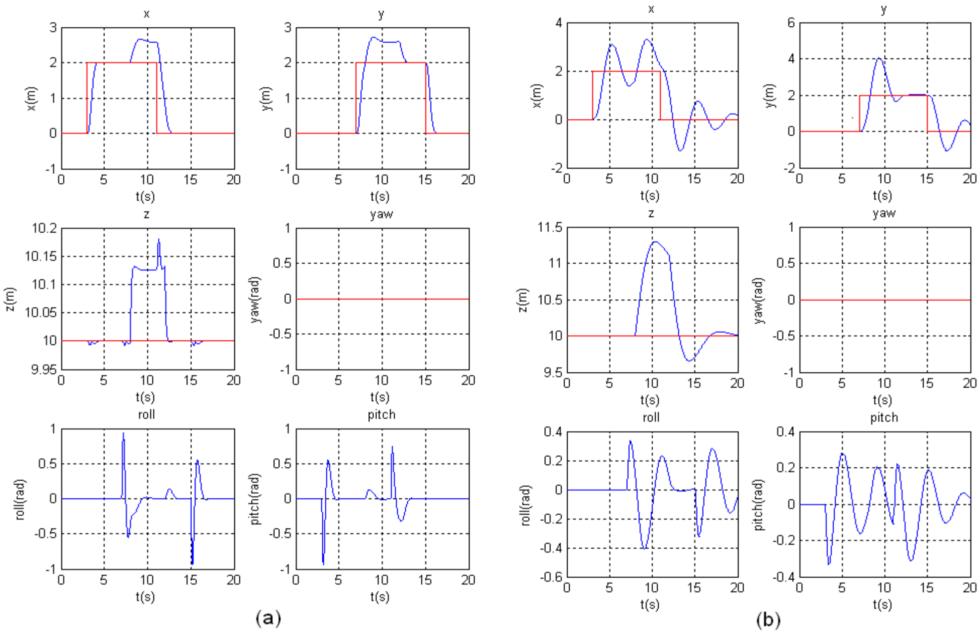

FIG. 5.3. *The simulation with wind disturbance: (a) the proposed method. (b) a standard PID.*

method has faster response time than a standard PID control.

REFERENCES

[1] G. M. HOFFMANN, S. L. WASLANDER, C. J. TOMLIN, *Distributed Cooperative Search using Information-Theoretic Costs for Particle Filters with Quadrotor Applications*, Proceedings of the AIAA Guidance, Navigation, and Control Conference, Keystone, CO, Aug 2006.

[2] S. BOUABDALLAH, R. SIEGWART, *Backstepping and Sliding-mode Techniques Applied to an Indoor Micro Quadrotor*, Proceedings of the 2005 IEEE International Conference onrobotics and Automation, Apr 2005.

[3] L. BESNARD, Y. SHTESSEL, B. LANDRUM, *Control of a quadrotor vehicle using sliding mode disturbance observer*, Proceedings of the American Control Conference, pp. 5230-5235, 2007.

[4] J. WANG, H. MOUNIER, A. CELA, S.I. NICULESCU, *Event driven intelligent PID controllers with applications to motion control*, Proceedings of 18th IFAC World Congress, Milan, Aug 2011.

[5] Y. KUZNETSOV, *Elements of applied bifurcation theory*, New york, Springer Verlog, 1998.

[6] W. KANG, *Extended controller form and invariants of nonlinear control systems with a single input*, Journal of Mathematical Systems Estimation and control,6, 27-51, 1996.

[7] I.A. TALL, W. RESPONDEK, *Normal forms for two-inputs nonlinear control systems*, Proceedings of the 41st IEEE Conference on Decision and Control, Dec 2002.

[8] A.J. KRENER, W. KANG, *Control bifurcations*, IEEE Transactions on Automatic Control, pp. 1231-1246,Aug 2004 .

[9] J. CARR, *Applications of center manifold theory*, New york, Springer Verlog, 1981.

[10] A.J. KRENER, S. KARAHAN, M. HUBBARD, R. FREZZA, *Higher order linear approximations to nonlinear control systems*, Proceedings of the 26th IEEE Conference on Decision and Control, 1987.

[11] I. BOUSSAADA, *Analysis and control of quadrotor (French, in preparation, available upon request)*, Technical report IPSA, 2012.

[12] J. GUCKENHEIMER, P. HOLMES, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, New york, Springer Verlog, 1983.