On the conservativity of cell centered Galerkin methods

Daniele A. Di Pietro

University of Montpellier 2, I3M, place Eugène Bataillon 34057 Montpellier CEDEX 5, France

Abstract

In this work we investigate the conservativity of the cell centered Galerkin method of [5] and provide an analytical expression for the conservative flux. The relation with the SUSHI method of [10] and with discontinuous Galerkin methods is also explored. The theoretical results are assessed on a numerical example using standard as well as general polygonal grids.

Résumé

Sur la conservativité des méthodes de Galerkine centrées aux mailles Dans cette note on étudie la conservativité de la méthode de Galerkine centrée aux mailles de [5] et on fournit une expression analytique pour le flux numérique. Le lien avec la méthode SUSHI de [10] et avec les méthodes de Galerkine discontinues est aussi détaillé. Les résultats théoriques sont validés la fois sur des maillages standards et polygonaux.

1. Introduction

Cell centered Galerkin (ccG) methods have been recently introduced in [4,5] combining ideas from the SUSHI method of Eymard, Gallouët, and Herbin [10] and discontinuous Galerkin methods, cf., e.g., [6]. A complete convergence analysis for a pure diffusion problem has been carried out in [5] using finite element techniques. The goal of this work is to revisit ccG methods in the spirit of finite volume methods and show that they enjoy a local conservation property analogous to the SUSHI method provided (i) interface unknowns are kept rather than eliminated by the local procedure described in [5, Section 2.3]; (ii) only the lowest-order part of the jumps is penalized and cell unknowns are used in the right-hand side. Both of these modifications can be interpreted as using reduced quadratures. As is the case for the SUSHI method, we show that face unknowns can be interpreted as the Lagrange multipliers of the flux continuity constraint. For the sake of simplicity, the discussion is based on the homogeneous Poisson problem,

$$-\triangle u = f \qquad \text{in } \Omega, \tag{1a}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{1b}$$

Email address: daniele.di-pietro@univ-montp2.fr (Daniele A. Di Pietro).

where $\Omega \in \mathbb{R}^d$, $d \geq 2$, denotes an open bounded connected polygonal or polyhedral domain and $f \in L^2(\Omega)$. The arguments below can be easily extended to anisotropic heterogeneous problems with more general boundary conditions.

Let $\mathcal{T}_h = \{T\}$ denote a family of disjoint open polygonal or polyhedral elements such that $\bigcup_{T\in\mathcal{T}_h} \overline{T} = \overline{\Omega}$. The planar faces of the elements in $\mathcal{T}_h$ that lie on the boundary of Ω are collected in the set $\mathcal{F}_h^{\mathrm{b}}$, while we denote by $\mathcal{F}_h^{\mathrm{i}}$ the set of interfaces, i.e., connected portions of planar element faces F such that there exist T_1, $T_2 \in \mathcal{T}_h$ with $F \subset \partial T_1 \cap \partial T_2$. Mesh faces are collected in the set $\mathcal{F}_h := \mathcal{F}_h^{\mathrm{i}} \cup \mathcal{F}_h^{\mathrm{b}}$ and, for all $T \in \mathcal{T}_h$, we let $\mathcal{F}_T := \{F \in \mathcal{F}_h \mid F \subset \partial T\}$. We also define $N_\partial := \max_{T\in\mathcal{T}_h} \mathrm{card}(\mathcal{F}_T)$. For all $F \in \mathcal{F}_h$, we define the set $\mathcal{T}_F := \{T \in \mathcal{T}_h \mid F \subset \partial T\}$. By definition, $\mathrm{card}(\mathcal{T}_F) = 2$ if $F \in \mathcal{F}_h^{\mathrm{i}}$ while $\mathrm{card}(\mathcal{T}_F) = 1$ if $F \in \mathcal{F}_h^{\mathrm{b}}$. For all $F \in \mathcal{F}_h^{\mathrm{i}}$ we choose an arbitrary but fixed orientation of the unit normal vector $\boldsymbol{n}_F$, while, for all $F \in \mathcal{F}_h^{\mathrm{b}}$, $\boldsymbol{n}_F$ is taken outward to Ω. For all $T \in \mathcal{T}_h$ and all $F \in \mathcal{F}_T$ we denote by $\boldsymbol{n}_{T,F}$ the unit normal vector to F outward to T. Finally, for all $F \in \mathcal{F}_h$, we let $\boldsymbol{x}_F := \int_F \boldsymbol{x}/|F|$. The mesh $\mathcal{T}_h$ is assumed to satisfy the regularity requirements of [5, Section 2.1], which are not detailed here for the sake of conciseness. For the purposes of the present work it suffices to recall that this assumption implies the existence of a family of cell centers $(\boldsymbol{x}_T)_{T\in\mathcal{T}_h}$ such that every cell $T \in \mathcal{T}_h$ is star-shaped with respect to $\boldsymbol{x}_T$ and, for all $F \in \mathcal{F}_T$, the F-based pyramid of apex $\boldsymbol{x}_T$ is non-degenerated.

In their hybrid versions, both the SUSHI and the ccG methods are based on cell and face unknowns collected in the vector space of degrees of freedom (DOFs) $\mathbb{V}_h := \mathbb{R}^{\mathcal{T}_h} \times \mathbb{R}^{\mathcal{F}_h}$. To identify the components of a generic DOF vector $\mathbb{v}_h \in \mathbb{V}_h$ we note $\mathbb{v}_h = ((v_T)_{T\in\mathcal{T}_h}, (v_F)_{F\in\mathcal{F}_h})$. The homogeneous Dirichlet condition (1b) is strongly enforced by defining the subspace $\mathbb{V}_{h0} := \{\mathbb{v}_h \in \mathbb{V}_h \mid v_F = 0, \forall F \in \mathcal{F}_h^{\mathrm{b}}\}$. Following [10, Section 2.4], the SUSHI bilinear form can be expressed as follows:

$$\forall \mathbb{w}_h,\, \mathbb{v}_h \in \mathbb{V}_{h0}, \qquad a_h^{\mathrm{sushi}}(\mathbb{w}_h, \mathbb{v}_h) = \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} |F| \Phi_{T,F}^{\mathrm{sushi}}(\mathbb{w}_h)(v_T - v_F). \tag{2}$$

We refer to the cited work for an explicit expression for $\Phi_{T,F}^{\mathrm{sushi}}$ depending on (local) geometric quantities and on the diffusion field when it is does not coincide with the unit tensor. Consider the discrete problem:

$$\text{Find } \mathbb{u}_h \in \mathbb{V}_{h0} \text{ such that } a_h^{\mathrm{sushi}}(\mathbb{u}_h, \mathbb{v}_h) = \sum_{T\in\mathcal{T}_h} |T| \langle f \rangle_T v_T \text{ for all } \mathbb{v}_h \in \mathbb{V}_h, \tag{3}$$

with $\langle f \rangle_T := \int_T f/|T|$. Setting $v_F = 1$ for $F \in \mathcal{F}_h^{\mathrm{i}}$ with $F \subset \partial T_1 \cap \partial T_2$ and $v_T = 0$ for all $T \in \mathcal{T}_h$ in (3) one can infer the following flux continuity relation:

$$\Phi_{T_1,F}^{\mathrm{sushi}}(\mathbb{u}_h) = -\Phi_{T_2,F}^{\mathrm{sushi}}(\mathbb{u}_h). \tag{4}$$

In addition, for all $T \in \mathcal{T}_h$, taking $v_{T'} = \delta_{TT'}$ for $T' \in \mathcal{T}_h$ and $v_F = 0$ for all $F \in \mathcal{F}_h$, there holds,

$$\forall T \in \mathcal{T}_h, \qquad \sum_{F\in\mathcal{F}_T} |F| \Phi_{T,F}^{\mathrm{sushi}}(\mathbb{u}_h) = |T| \langle f \rangle_T. \tag{5}$$

We emphasize that a key point to obtain (4) is that only cell unknowns appear in the right-hand side of (3). The goal of this work is precisely to show that results analogous to (3), (4), and (5) hold for the hybrid version of the ccG method of [5] and variants thereof. In proving these properties, we relate the conservative flux for the ccG method with the numerical flux of the corresponding dG formulation based on full piecewise polynomial spaces. We show, in particular, that the two numerical fluxes differ by an asymptotically consistent perturbation expressed in terms of discrete gradients and jump liftings.

2. Local conservation

The ccG space We briefly recall the construction of the ccG space. For all $T \in \mathcal{T}_h$ we consider the following gradient reconstruction inspired by Green's formula:

$$\boldsymbol{G}_T(\mathbb{v}_h) := \frac{1}{|T|}\sum_{F\in\mathcal{F}_T}|F|v_F\boldsymbol{n}_{T,F} = \frac{1}{|T|}\sum_{F\in\mathcal{F}_T}|F|(v_F - v_T)\boldsymbol{n}_{T,F}. \tag{6}$$

For an integer $k \geq 0$ we let $\mathbb{P}_d^k(\mathcal{T}_h) := \{v_h \in L^2(\Omega) \mid v_{h|T} \in \mathbb{P}_d^k(T), \forall T \in \mathcal{T}_h\}$ with $\mathbb{P}_d^k(T)$ spanned by the restriction to T of polynomial functions of degree $\leq k$. We also need the following average and jump trace operators defined for all $F \in \mathcal{F}_h^{\mathrm{i}}$ such that $F \subset \partial T_1 \cap \partial T_2$ and $\boldsymbol{n}_F$ points out of T_1:

$$\{\varphi\} := \tfrac{1}{2}(\varphi_{|T_1} + \varphi_{|T_2}), \qquad [\![\varphi]\!] := \varphi_{|T_1} - \varphi_{|T_2}.$$

On boundary faces $F \in \mathcal{F}_h^{\mathrm{b}}$ we conventionally set $\{\varphi\} = [\![\varphi]\!] = \varphi$. Let $\mathfrak{R}_h : \mathbb{V}_h \to \mathbb{P}_d^1(\mathcal{T}_h)$ be such that

$$\forall \mathbb{v}_h \in \mathbb{V}_h, \qquad \mathfrak{R}_h(\mathbb{v}_h)_{|T}(\boldsymbol{x}) = v_T + \boldsymbol{G}_T(\mathbb{v}_h)\cdot(\boldsymbol{x} - \boldsymbol{x}_T) \qquad \forall \boldsymbol{x} \in T. \tag{7}$$

The ccG space V_h is defined as the image of the DOF space $\mathbb{V}_{h0}$ through $\mathfrak{R}_h$, i.e., we set $V_h := \mathfrak{R}_h(\mathbb{V}_{h0})$.

IIP-ccG method It is instructive to first consider the following bilinear form on $V_h \times V_h$ inspired by the IIP-dG method of Dawson, Sun and Wheeler [3]:

$$a_h^{\mathrm{iip}}(u_h, v_h) := \int_\Omega \boldsymbol{\nabla}_h u_h \cdot \boldsymbol{\nabla}_h v_h + \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}\int_F \phi_{T,F}(u_h)\langle v_{h|T}\rangle_F, \tag{8}$$

where, for all φ regular enough, $\langle\varphi\rangle_F := \int_F \varphi/|F|$ and, for all $v_h \in V_h$,

$$\phi_{T,F}(v_h) := -\{\boldsymbol{\nabla}_h v_h\}\cdot\boldsymbol{n}_{T,F} + \frac{\eta}{h_F}\langle[\![v_h]\!]\rangle_F\epsilon_{T,F}, \qquad \frac{1}{h_F} := \sum_{T\in\mathcal{T}_F}\frac{|F|}{\mathrm{card}(\mathcal{T}_F)^2|T|}, \qquad \epsilon_{T,F} := \boldsymbol{n}_{T,F}\cdot\boldsymbol{n}_F. \tag{9}$$

Using the above expression for the linear face dimension h_F, the user dependent parameter η for the IIP bilinear form (8) (resp. SIP bilinear form (17)) should be taken strictly larger than $N_\partial/2$ (resp. N_∂) to ensure stability; cf. the discussion in [8, Section 3.2.2]. Observe that $\phi_{T,F}(v_h)$ is constant over F since we are only penalizing the lowest-order part of the jumps as in [2]. Moreover, by definition there holds

$$\phi_{T_1,F}(v_h) = -\phi_{T_2,F}(v_h). \tag{10}$$

The IIP-ccG method reads

$$\text{Find } u_h \in V_h \text{ such that } a_h^{\mathrm{iip}}(u_h, v_h) = \int_\Omega f v_h \text{ for all } v_h \in V_h. \tag{11}$$

Letting $k \geq 1$ and replacing V_h by $\mathbb{P}_d^k(\mathcal{T}_h)$ in (11) we obtain the IIP-dG method for which there holds, for all $T \in \mathcal{T}_h$, $\sum_{F\in\mathcal{F}_T}\phi_{T,F}(u_h) = \langle f\rangle_T$. This local conservation property is proved taking as a test function $v_h = \chi_T$ with χ_T characteristic function of T and using (10); see [6, Section 5.3.1.1] for the details. Unfortunately, this argument breaks down for the IIP-ccG method (11) since, in general, $\chi_T \notin V_h$. We will show, however, that the bilinear form a_h^{iip} on $V_h \times V_h$ admits a flux formulation analogous to (2) with a numerical flux $\Phi_{T,F}^{\mathrm{iip}}$ which is a perturbation of (9). Provided the right-hand side is approximated as in (3), this allows to prove a conservation property analogous to (5). We start by observing that, for all $T \in \mathcal{T}_h$, all $F \in \mathcal{F}_T$, and all $v_h = \mathfrak{R}_h(\mathbb{v}_h) \in V_h$, (7) yields

$$\langle v_{h|T}\rangle_F = v_h(\boldsymbol{x}_F) = v_T + \boldsymbol{G}_T(\mathbb{v}_h)\cdot(\boldsymbol{x}_F - \boldsymbol{x}_T).$$

Plugging this expression into the second term in the right-hand side of (8), subtracting the quantity $\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}|F|\phi_{T,F}(u_h)v_F=0$ (this equality is a consequence of (10)), it is inferred

$$\sum_{F\in\mathcal{F}_h}\sum_{T\in\mathcal{T}_F}|F|\phi_{T,F}(u_h)\langle v_{h|T}\rangle_F=\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}|F|\phi_{T,F}(u_h)(v_T-v_F)-\sum_{T\in\mathcal{T}_h}|T|\boldsymbol{\mathcal{G}}_T(\mathfrak{u}_h)\cdot\boldsymbol{G}_T(\mathfrak{v}_h),\tag{12}$$

where, for all $v_h=\mathfrak{R}_h(\mathfrak{v}_h)\in V_h$, we have introduced the following flux-based gradient reconstruction:

$$\boldsymbol{\mathcal{G}}_T(\mathfrak{v}_h):=\sum_{F\in\mathcal{F}_T}\frac{|F|}{|T|}\phi_{T,F}(v_h)(\boldsymbol{x}_T-\boldsymbol{x}_F).\tag{13}$$

It is worth noting that formula (13) has a strong analogy with the gradient reconstruction used by Eymard and Droniou [9]; cf. also [1, eq. (2.15)]. Plugging (12) into (8), letting $\boldsymbol{\Delta}_T^{\mathrm{iip}}(\mathfrak{u}_h):=\boldsymbol{\mathcal{G}}_T(\mathfrak{u}_h)-\boldsymbol{G}_T(\mathfrak{u}_h)$, and replacing $\boldsymbol{G}_T(\mathfrak{v}_h)$ by its definition (6) in the first term we obtain

$$\begin{aligned}a_h^{\mathrm{iip}}(u_h,v_h)&=-\sum_{T\in\mathcal{T}_h}|T|\boldsymbol{\Delta}_T^{\mathrm{iip}}(\mathfrak{u}_h)\cdot\boldsymbol{G}_T(\mathfrak{v}_h)+\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}|F|\phi_{T,F}(u_h)(v_T-v_F)\\&=\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}|F|\boldsymbol{\Delta}_T^{\mathrm{iip}}(\mathfrak{u}_h)\cdot\boldsymbol{n}_{T,F}(v_T-v_F)+\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}|F|\phi_{T,F}(u_h)(v_T-v_F)\\&=\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}|F|\Phi_{T,F}^{\mathrm{iip}}(\mathfrak{u}_h)(v_T-v_F),\end{aligned}$$

where, for all $v_h=\mathfrak{R}_h(\mathfrak{v}_h)\in V_h$,

$$\Phi_{T,F}^{\mathrm{iip}}(\mathfrak{v}_h):=\phi_{T,F}(v_h)+\boldsymbol{\Delta}_T^{\mathrm{iip}}(\mathfrak{v}_h)\cdot\boldsymbol{n}_{T,F}.\tag{14}$$

It is thus clear that the numerical flux for the IIP-ccG method is equal to the dG flux (9) plus a perturbation proportional to the difference between the gradient reconstruction (6) based on face unknowns and the gradient reconstruction (13) based on dG fluxes, both of which are consistent. Consider now the following variation of (11), where the sole difference lies in the approximation of the right-hand side:

$$\text{Find } u_h=\mathfrak{R}_h(\mathfrak{u}_h)\in V_h \text{ such that } a_h^{\mathrm{iip}}(u_h,v_h)=\sum_{T\in\mathcal{T}_h}|T|\langle f\rangle_T v_T \text{ for all } v_h=\mathfrak{R}_h(\mathfrak{v}_h)\in V_h.\tag{15}$$

Setting $v_F=1$ for $F\in\mathcal{F}_h^{\mathrm{i}}$ with $F\subset\partial T_1\cap\partial T_2$ and $v_T=0$ for all $T\in\mathcal{T}_h$ in (15) it is inferred

$$\Phi_{T_1,F}^{\mathrm{iip}}(\mathfrak{u}_h)=-\Phi_{T_2,F}^{\mathrm{iip}}(\mathfrak{u}_h),\tag{16}$$

which shows that the flux defined by (14) is continuous. Moreover, a conservation property analogous to (5) can be proved by a similar argument.

In the left panel of Figure 1 we show a numerical example where the convergence of the perturbation norm $\mathcal{N}_{\boldsymbol{\Delta}}^{\mathrm{iip}}:=(\sum_{T\in\mathcal{T}_h}\|\boldsymbol{\Delta}_T^{\mathrm{iip}}(\mathfrak{u}_h)\|_{L^2(T)^d}^2)^{1/2}$ is numerically evaluated on three successively refined mesh families. For every mesh type and refinement property (16) is verified up to machine precision.

SIP-ccG method The bilinear form a_h^{sip} corresponding to the SIP-ccG method of [5] contains an additional symmetry term, namely

$$a_h^{\mathrm{sip}}(u_h,v_h):=a_h^{\mathrm{iip}}(u_h,v_h)-\sum_{F\in\mathcal{F}_h}\int_F[\![u_h]\!]\{\boldsymbol{\nabla}_h v_h\}\cdot\boldsymbol{n}_F.\tag{17}$$

For all $v_h=\mathfrak{R}_h(\mathfrak{v}_h)\in V_h$, the symmetry term can be rewritten as follows:

$$-\sum_{F\in\mathcal{F}_h}\int_F[\![u_h]\!]\{\boldsymbol{\nabla}_h v_h\}\cdot\boldsymbol{n}_F=-\sum_{T\in\mathcal{T}_h}|T|\cdot\boldsymbol{L}_T(\mathfrak{u}_h)\boldsymbol{G}_T(\mathfrak{v}_h),\qquad\boldsymbol{L}_T(\mathfrak{u}_h):=\sum_{F\in\mathcal{F}_T}\frac{1}{\mathfrak{h}_{T,F}}\langle[\![u_h]\!]\rangle_F\boldsymbol{n}_F,$$

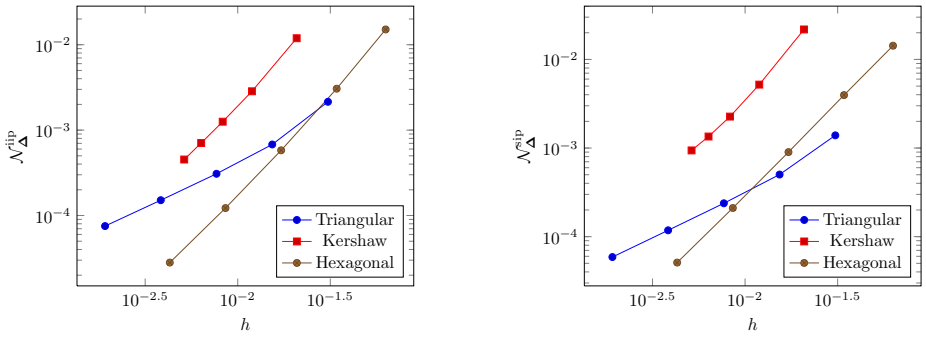


Figure 1. Convergence of the flux perturbation. The triangular and Kershaw mesh families correspond to the mesh families 1 and 4.1 of the FVCA5 benchmark [11] respectively, while the hexagonal-dominant mesh family coincides with the one proposed in [7]

where $\mathfrak{h}_{T,F} := \frac{|T|\,\mathrm{card}(\mathcal{T}_F)}{|F|}$. The operator $\boldsymbol{L}_T$ is analogous to the jump lifting defined in [8, Section 3.2.2]. Proceeding as for the IIP-ccG method it is inferred

$$a_h^{\mathrm{sip}}(u_h, v_h) = \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T} |F|\Phi_{T,F}^{\mathrm{sip}}(\mathfrak{u}_h)(v_T - v_F),$$

where, for all $v_h = \mathfrak{R}_h(\mathfrak{v}_h) \in V_h$,

$$\Phi_{T,F}^{\mathrm{sip}}(\mathfrak{v}_h) := \phi_{T,F}(v_h) + \boldsymbol{\Delta}_T^{\mathrm{sip}}(\mathfrak{v}_h)\cdot\boldsymbol{n}_{T,F}, \qquad \boldsymbol{\Delta}_T^{\mathrm{sip}}(\mathfrak{v}_h) := \boldsymbol{\Delta}_T^{\mathrm{iip}}(\mathfrak{v}_h) + \boldsymbol{L}_T(\mathfrak{v}_h).$$

The numerical convergence of the perturbation norm $\mathcal{N}_{\boldsymbol{\Delta}}^{\mathrm{sip}}$ defined by replacing $\boldsymbol{\Delta}_T^{\mathrm{iip}}$ by $\boldsymbol{\Delta}_T^{\mathrm{sip}}$ in the expression of $\mathcal{N}_{\boldsymbol{\Delta}}^{\mathrm{iip}}$ is shown in the right panel of Figure 1. Also in this case the flux continuity property $\Phi_{T_1,F}^{\mathrm{sip}}(\mathfrak{u}_h) = -\Phi_{T_2,F}^{\mathrm{sip}}(\mathfrak{u}_h)$ for all $F \in \mathcal{F}_h^{\mathrm{i}}$ such that $F \subset \partial T_1 \cap \partial T_2$ is verified to machine precision.

References

[1] L. Agélas, D. A. Di Pietro, and J. Droniou. The G method for heterogeneous anisotropic diffusion on general meshes. *M2AN Math. Model. Numer. Anal.*, 44(4):597–625, 2010.

[2] E. Burman and B. Stamm. Local discontinuous Galerkin method with reduced stabilization for diffusion equations. *Communications in Computational Physics*, 5:498–524, 2009.

[3] C. Dawson, S. Sun, and M. F. Wheeler. Compatible algorithms for coupled flow and transport. *Comp. Meth. Appl. Mech. Eng.*, 193:2565–2580, 2004.

[4] D. A. Di Pietro. Cell centered Galerkin methods. *C. R. Acad. Sci. Paris, Ser. I*, 348:31–34, 2010.

[5] D. A. Di Pietro. Cell centered Galerkin methods for diffusive problems. *M2AN Math. Model. Numer. Anal.*, 46(1):111–144, 2012.

[6] D. A. Di Pietro and A. Ern. *Mathematical aspects of discontinuous Galerkin methods*, volume 69 of *Mathématiques & Applications*. Springer-Verlag, Berlin, 2011.

[7] D. A. Di Pietro and S. Lemaire. An extension of the Crouzeix–Raviart space to general meshes with application to quasi-incompressible linear elasticity and Stokes flow. Preprint hal-00753660, 2012.

[8] D. A. Di Pietro and S. Nicaise. A locking-free discontinuous Galerkin method for linear elasticity in locally nearly incompressible heterogeneous media. *App. Num. Math.*, 63:105–116, 2013.

[9] J. Droniou and R. Eymard. A mixed finite volume scheme for anisotropic diffusion problems on any grid. *Numer. Math.*, 105(1):35–71, 2006.

[10] R. Eymard, T. Gallouët, and R. Herbin. Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes. SUSHI: a scheme using stabilization and hybrid interfaces. *IMA J. Num. Anal.*, 30(4):1009–1043, 2010.

[11] R. Herbin and F. Hubert. Benchmark on discretization schemes for anisotropic diffusion problems on general grids. In R. Eymard and J.-M. Hérard, editors, *Finite Volumes for Complex Applications V*, pages 659–692. John Wiley & Sons, 2008.