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Henri Anciaux[*], Pascal Romon[†]

Abstract

It is a classical fact that the cotangent bundle $T^*\mathcal{M}$ of a differentiable manifold $\mathcal{M}$ enjoys a canonical symplectic form Ω^*. If $(\mathcal{M}, \mathrm{J}, g, \omega)$ is a pseudo-Kähler or para-Kähler $2n$-dimensional manifold, we prove that the tangent bundle $T\mathcal{M}$ also enjoys a natural pseudo-Kähler or para-Kähler structure $(\tilde{\mathrm{J}}, \tilde{g}, \Omega)$, where Ω is the pull-back by g of Ω^* and $\tilde{g}$ is a pseudo-Riemannian metric with neutral signature $(2n, 2n)$. We investigate the curvature properties of the pair $(\tilde{\mathrm{J}}, \tilde{g})$ and prove that: $\tilde{g}$ is scalar-flat, is not Einstein unless g is flat, has nonpositive (resp. nonnegative) Ricci curvature if and only if g has nonpositive (resp. nonnegative) Ricci curvature as well, and is locally conformally flat if and only if $n = 1$ and g has constant curvature, or $n > 2$ and g is flat. We also check that (i) the holomorphic sectional curvature of $(\tilde{\mathrm{J}}, \tilde{g})$ is not constant unless g is flat, and (ii) in $n = 1$ case, that $\tilde{g}$ is never anti-self-dual, unless conformally flat.

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Introduction

It is a classical fact that given any differentiable manifold $\mathcal{M}$, its cotangent bundle $T^*\mathcal{M}$ enjoys a canonical symplectic structure Ω^*.

Moreover, given a linear connection ∇ on a manifold $\mathcal{M}$, (e.g. the Levi-Civita connection of a Riemannian metric), the bundle $TT\mathcal{M}$ splits into a direct sum of two subbundles $H\mathcal{M}$ and $V\mathcal{M}$, both isomorphic to $T\mathcal{M}$. This allows to define an almost complex structure J by setting $J(X_h, X_v) := (-X_v, X_h)$, where, for $X \in TT\mathcal{M} = H\mathcal{M} \oplus V\mathcal{M}$, we write $X \simeq (X_h, X_v) \in T\mathcal{M} \times T\mathcal{M}$. Analogously, one may introduce a natural almost para-complex (or bi-Lagrangian) structure, setting $J'(X_h, X_v) := (X_v, X_h)$.

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It is also well known that the tangent bundle of a Riemannian manifold $(\mathcal{M}, g)$ can be given a natural Riemannian structure, called *Sasaki metric*. A simple way to understand this construction, which extends *verbatim* to the case of a pseudo-Riemannian metric g with signature $(p, m-p)$, is as follows: using the splitting $TT\mathcal{M} = H\mathcal{M} \oplus V\mathcal{M}$, we set:

$$G\big((X_h, X_v), (Y_h, Y_v)\big) := g(X_h, Y_h) + g(X_v, Y_v).$$

This metric has signature $(2p, 2(m-p))$ and is well behaved with respect to J in two ways: (i) G is compatible with J, i.e. $G(.,.) = G(J., J.)$, and (ii) the symplectic form $\Omega := G(J., .)$ is nothing but the pull-back of Ω^* by the musical isomorphism between $T\mathcal{M} \simeq_g T^*\mathcal{M}$. In other words, the triple (J, G, Ω) defines an "almost pseudo-Kähler" structure[1] on $T\mathcal{M}$.

Unfortunately, this construction suffers two flaws: J is not integrable unless ∇ is flat and the metric G is somewhat "rigid": for example, if G has constant scalar curvature, then g is flat (see [15]). We refer to [5, 16] and the survey [9] for more detail on the Sasaki metric.

Another construction can be made in the case where $\mathcal{M}$ is complex (resp. para-complex): in this case both $T\mathcal{M}$ and $T^*\mathcal{M}$ enjoy a canonical complex (resp. para-complex) structure which are defined as follows: given a family of holomorphic (resp. para-holomorphic[2]) local charts $\varphi : \mathcal{M} \to \mathcal{U} \subset \mathbb{R}^{2n}$ on $\mathcal{M}$, we define holomorphic (resp. para-holomorphic) local charts $\bar{\varphi} : T\mathcal{M} \to \mathcal{U} \times \mathbb{R}^{2n}$ by $\bar{\varphi}(x, V) = (\varphi(x), d\varphi_x(V)),\ \forall (x, V) \in T\mathcal{M}$ for the tangent bundle, and $\bar{\varphi} : T^*\mathcal{M} \to \mathcal{U} \times \mathbb{R}^{2n}$ by $\bar{\varphi}(x, \xi) = (\varphi(x), ((d\varphi_x)^t)^{-1}(\xi)),\ \forall (x, \xi) \in T^*\mathcal{M}$ for the cotangent bundle. In the first section, we shall see that if $\mathcal{M}$ is merely almost complex (resp. almost para-complex), then a more subtle argument allows to define again a canonical almost complex structure (resp. almost para-complex structure) on $T\mathcal{M}$. On the other hand, we shall prove in the second section that if $\mathcal{M}$ is pseudo- or para-Kähler, the corresponding structure on $T\mathcal{M}$ can also be constructed using the splitting $H\mathcal{M} \oplus V\mathcal{M}$ induced by the Levi-Civita connection of the Kählerian metric.

Combining the canonical symplectic structure Ω^* of $T^*\mathcal{M}$ with the canonical complex (resp. para-complex) structure $\tilde{\mathrm{J}}^*$ just defined, it is natural to introduce a 2-tensor $\tilde{g}^*$ by the formula

$$\tilde{g}^* := \Omega^*(., \tilde{\mathrm{J}}^*.).$$

However, it turns out that Ω^* is not compatible with $\tilde{\mathrm{J}}^*$, since it turns out that $\Omega^*(\tilde{\mathrm{J}}^*., \tilde{\mathrm{J}}^*.) = -\varepsilon\Omega^*$ instead of the required formula $\Omega^*(\tilde{\mathrm{J}}^*., \tilde{\mathrm{J}}^*.) = \varepsilon\Omega^*$ (here and in the following, in order to deal simultaneously with the complex

[1] We might also define an "almost para-Kähler" structure on $T\mathcal{M}$ by introducing the *para-Sasaki metric*

$$G'\big((X_h, X_v), (Y_h, Y_v)\big) := g(X_h, Y_h) - g(X_v, Y_v).$$

This metric has neutral signature (m, m) (m being the dimension of $\mathcal{M}$), is compatible with J' and verifies $\Omega := -G'(J'., .)$.

[2] The terminology *split-holomorphic* is sometimes used.

and para-complex cases, we define ε to be such that $(\tilde{\mathrm{J}}^*)^2 = -\varepsilon \mathrm{Id}$, i.e. $\varepsilon = 1$ in the complex case and $\varepsilon = -1$ in the para-complex case). It follows that the tensor $\tilde{g}^*$ is not symmetric and therefore we failed in constructing a canonical pseudo-Riemannian structure on $T^*\mathcal{M}$.

On the other hand, the same idea works well if one considers, instead of the cotangent bundle, the tangent bundle of a pseudo- or para-Kähler manifold $(\mathcal{M}, \mathrm{J}, g)$, thus obtaining a canonical pseudo- or para-Kähler structure. The purpose of this note is to investigate in detail this construction and to study its curvature properties. The results are summarized in the following:

Main Theorem *Let $(\mathcal{M}, \mathrm{J}, g, \omega)$ be a pseudo- or para-Kähler manifold. Then $T\mathcal{M}$ enjoys a natural pseudo- or para-Kähler structure $(\tilde{\mathrm{J}}, \tilde{g}, \Omega)$ with the following properties:*

— $\tilde{\mathrm{J}}$ *is the canonical complex or para-complex structure of $T\mathcal{M}$ induced from that of $\mathcal{M}$;*

— Ω *is the pull-back of Ω^* by the metric isomorphism $T\mathcal{M} \simeq_g T^*\mathcal{M}$;*

— *The pseudo-Riemannian metric $\tilde{g}$ can be recovered from $\tilde{\mathrm{J}}$ and Ω by the equation $\tilde{g}(.,.) := \Omega(., \tilde{\mathrm{J}}.)$;*

— *According to the splitting $TT\mathcal{M} = H\mathcal{M} \oplus V\mathcal{M}$ induced by the Levi-Civita connection of g, the triple $(\tilde{\mathrm{J}}, \tilde{g}, \Omega)$ takes the following expression:*

$$\begin{aligned}
\tilde{\mathrm{J}}(X_h, X_v) &:= (\mathrm{J}X_h, \mathrm{J}X_v) \\
\tilde{g}\big((X_h, X_v), (Y_h, Y_v)\big) &:= g(X_v, \mathrm{J}Y_h) - g(X_h, \mathrm{J}Y_v) \\
\Omega\big((X_h, X_v), (Y_h, Y_v)\big) &:= g(X_v, Y_h) - g(X_h, Y_v);
\end{aligned}$$

— *The pseudo-Riemannian metric $\tilde{g}$ has the following properties:*

(i) *$\tilde{g}$ has neutral signature neutral $(2n, 2n)$ and is scalar flat;*

(ii) *$(T\mathcal{M}, \tilde{g})$ is Einstein if and only if $(\mathcal{M}, g)$ is flat, and therefore $(T\mathcal{M}, \tilde{g})$ is flat as well;*

(iii) *the Ricci curvature $\widetilde{\mathrm{Ric}}$ of $\tilde{g}$ has the same sign as the Ricci curvature Ric of g;*

(iv) *$(T\mathcal{M}, \tilde{g})$ is locally conformally flat if and only if $n = 1$ and g has constant curvature, or $n > 2$ and g is flat; if $n = 1$, $\tilde{g}$ is always self-dual, so anti-self-duality is equivalent to conformal flatness;*

(v) *the pair $(\tilde{\mathrm{J}}, \tilde{g})$ has constant holomorphic curvature if and only if g is flat.*

Remark 1. *We use in (iv) the general property that four-dimensional neutral pseudo-Kähler or para-Kähler manifolds are self-dual if and only if their scalar curvature vanishes. This is analogous to the case of Kähler four-dimensional manifolds, except that self-duality is exchanged with anti-self-duality. A proof of this statement is given in Theorem A.2 in the appendix.*

This result is a generalization of previous work on the tangent bundle of a Riemannian surface (see [10], [11], [3]). The authors wish to thank Brendan Guilfoyle for his valuable suggestions and comments.

1 Almost complex and para-complex structures on the tangent bundle

Given a manifold $\mathcal{M}$ endowed with an almost complex or almost para-complex structure J, it is only natural to ask whether its tangent or cotangent bundle inherit such a structure. The answer is positive:

Proposition 1. *Let $(\mathcal{M}, \mathrm{J})$ be an almost complex (resp. para-complex) manifold. Then its tangent bundle admits a canonical almost complex (resp. para-complex) structure $\tilde{\mathrm{J}}$. Furthermore, if J is complex (resp. para-complex), so is $\tilde{\mathrm{J}}$.*

Remark 2. *Such a result has been proven already by Lempert & Szöke [14] for the tangent bundle in the almost complex case. Their construction uses the jets over $\mathcal{M}$ and is quite a bit more technical than our proof. However it gives an interesting interpretation of the meaning of $\tilde{\mathrm{J}}$. We shall see below in Proposition 2 a different and simpler way of defining and understanding $\tilde{\mathrm{J}}$, provided $\mathcal{M}$ is a pseudo- or para-Kähler manifold.*

Proof. We prove the result using coordinate charts, which amounts to showing that $\tilde{\mathrm{J}}$ can be defined independently of any change of variable. Let $y = \varphi(x)$ be a local change of coordinates on $\mathbb{R}^n$ and write ξ and η respectively for the tangent coordinates induced by the charts (i.e. $\sum_i \xi^i \partial/\partial x^i = \sum_i \eta^i \partial/\partial y^i$). The change of tangent coordinates at x is $\xi \mapsto \eta = d\varphi(x)\xi$, in other words φ induces a chart Φ on $\mathbb{R}^{2n}$, $\Phi : (x, \xi) \mapsto (\varphi(x), d\varphi(x)\xi)$. The tangent coordinates at (x, ξ) (resp. (y, η)) are denoted by (X, Ξ) (resp. (Y, H)) and the change of (doubly) tangent coordinates is

$$d\Phi(x, \xi) : (X, \Xi) \mapsto (Y, H) = (d\varphi(x)X, d^2\varphi(x)(X, \xi) + d\varphi(x)\Xi).$$

Assume moreover that we have a $(1, 1)$ tensor, which reads in the x coordinate as the matrix $\mathrm{J}(x)$ and in the y coordinate as the matrix $\mathrm{J}'(y) = \mathrm{J}'(\varphi(x)) = d\varphi(x) \circ \mathrm{J}(x) \circ (d\varphi(x))^{-1}$. Equivalently for any X and $Y = d\varphi(x)X$, we have $\mathrm{J}'(y)Y = \mathrm{J}'(\varphi(x))d\varphi(x)X = d\varphi(x)\mathrm{J}(x)X$. Differentiating this equality along ξ yields

$$\begin{aligned}(D_{d\varphi(x)\xi}\mathrm{J}')(\varphi(x))d\varphi(x)X + \mathrm{J}'(\varphi(x))d^2\varphi(x)(X, \xi)&\\ = d\varphi(x)(D_\xi \mathrm{J})(x)X + d^2\varphi(x)(\mathrm{J}(x)X, \xi),& \quad (1)\end{aligned}$$

where $(D_\xi \mathrm{J})(x)$ denotes in this proof the directional derivative of the matrix J at x in the direction ξ (not a covariant derivative).

We now define the $(1,1)$ tensor $\tilde{\mathrm{J}}$ in the (x,ξ) coordinate by

$$\tilde{\mathrm{J}}(x,\xi) : (X,\Xi) \mapsto (\mathrm{J}(x)X, \mathrm{J}(x)\Xi + D_\xi \mathrm{J}(x)X).$$

Let us prove that this definition is coordinate-independent (for greater readability we will often write J, J' for $\mathrm{J}(x), \mathrm{J}'(y)$). Using (1) and the symmetry of the second order differential $d^2\varphi(x)$,

$$\begin{aligned}
d\Phi(x,\xi)(\mathrm{J}(X,\Xi)) &= d\Phi(x,\xi)(\mathrm{J}X, \mathrm{J}\Xi + D_\xi \mathrm{J}(x)X) \\
&= (d\varphi(x)\mathrm{J}X, d^2\varphi(x)(\mathrm{J}X,\xi) + d\varphi(x)(\mathrm{J}\Xi + D_\xi \mathrm{J}(x)X)) \\
&= (\mathrm{J}'Y, \mathrm{J}'d\varphi(x)\Xi \\
&\quad + (D_{d\varphi(x)\xi}\mathrm{J}')(\varphi(x))d\varphi(x)X + \mathrm{J}'d^2\varphi(x)(X,\xi)) \\
&= (\mathrm{J}'Y, \mathrm{J}'(d\varphi(x)\Xi + d^2\varphi(x)(X,\xi)) \\
&\quad + (D_{d\varphi(x)\xi}\mathrm{J}')(\varphi(x))d\varphi(x)X) \\
&= (\mathrm{J}'Y, \mathrm{J}'H + D_\eta \mathrm{J}'(y)Y) = \tilde{\mathrm{J}}'(y,\eta)(Y,H),
\end{aligned}$$

where $\tilde{\mathrm{J}}'$ denotes the map corresponding to $\tilde{\mathrm{J}}$ in the (y,η) coordinates. Consequently the tensor on $\mathcal{M}$ extends naturally to $T\mathcal{M}$.

We have so far defined a $(1,1)$ tensor on $T\mathcal{M}$ without extra assumptions. Suppose now that J is an almost complex (resp. para-complex) structure, so that $\mathrm{J}^2 = -\varepsilon \mathrm{Id}$. Differentiating this property yields $\mathrm{J}\, D_\xi \mathrm{J} + (D_\xi \mathrm{J})\, \mathrm{J} = 0$. Then

$$\begin{aligned}
\tilde{\mathrm{J}}^2(X,\Xi) &= (\mathrm{J}^2X, \mathrm{J}(\mathrm{J}\Xi + D_\xi \mathrm{J}X) + D_\xi \mathrm{J}(\mathrm{J}X)) \\
&= (-\varepsilon X, -\varepsilon\Xi + \mathrm{J}(d\mathrm{J}\xi)X + (d\mathrm{J}\xi)(\mathrm{J}X) = -\varepsilon(X,\Xi)
\end{aligned}$$

so that $\tilde{\mathrm{J}}$ is also an almost complex (resp. para-complex) structure.

Finally if J is a complex (resp. para-complex) structure then we can use complex (resp. para-complex) coordinate charts, which amounts to saying that J is a constant matrix. Then $\tilde{\mathrm{J}}$ defined in the associated charts on $T\mathcal{M}$ takes a simpler expression, and is also constant:

$$\tilde{\mathrm{J}}(x,\xi) : (X,\Xi) \mapsto (\mathrm{J}X, \mathrm{J}\Xi)$$

and that characterizes a complex (resp. para-complex) structure. □

Remark 3. *Finding a similar almost-complex structure on $T^*\mathcal{M}$ is much more difficult, and may not be true in all generality. The Reader will note that, whenever $\mathcal{M}$ is endowed with a pseudo-Riemannian metric, we have a musical correspondence between $T\mathcal{M}$ and $T^*\mathcal{M}$, and $\tilde{\mathrm{J}}$ induces a corresponding structure $\tilde{\mathrm{J}}^*$ on $T^*\mathcal{M}$. However different metrics will yield different structures on $T^*\mathcal{M}$. There is one unambiguous case, which will be the setting in the remainder of this article, namely when J is integrable.*

2 The Kähler structure

Let $\mathcal{M}$ be a differentiable manifold. We denote by π and π^* the canonical projections $T\mathcal{M} \to \mathcal{M}$ and $T\mathcal{M}^* \to \mathcal{M}$. The subbundle $\ker(d\pi) := V\mathcal{M}$ of $TT\mathcal{M}$ (it is thus a bundle over $T\mathcal{M}$) will be called *the vertical bundle.*

Assume now that $\mathcal{M}$ is equipped with a linear connection ∇. The corresponding horizontal bundle is defined as follows: let $\bar{X}$ be a tangent vector to $T\mathcal{M}$ at some point (x_0, V_0). This implies that there exists a curve $\gamma(s) = (x(s), V(s))$ such that $(x(0), V(0)) = (x_0, V_0)$ and $\gamma'(0) = \bar{X}$. If $X \notin V\mathcal{M}$ (which implies $x'(0) \neq 0$), we define the connection map (see [7], [3]) $\mathrm{K} : TT\mathcal{M} \to T\mathcal{M}$ by $\mathrm{K}\bar{X} = \nabla_{x'(0)}V(0)$, where ∇ denotes the Levi-Civita connection of the metric g. If X is vertical, we may assume that the curve γ stays in a fiber so that $V(s)$ is a curve in a vector space. We then define $\mathrm{K}\bar{X}$ to be simply $V'(0)$. The horizontal bundle is then $Ker(\mathrm{K})$ and we have a direct sum

$$\begin{array}{rcl} TT\mathcal{M} = H\mathcal{M} \oplus V\mathcal{M} & \simeq & T\mathcal{M} \oplus T\mathcal{M} \\ \bar{X} & \simeq & (\Pi\bar{X}, \mathrm{K}\bar{X}). \end{array} \tag{2}$$

Here and in the following, Π is a shorthand notation for $d\pi$.

Lemma 1. [7] *Given a vector field X on $(\mathcal{M}, \nabla)$ there exists exactly one vector field X^h and one vector field X^v on $T\mathcal{M}$ such that $(\Pi X^h, \mathrm{K}X^h) = (X, 0)$ and $(\Pi X^v, \mathrm{K}X^v) = (0, X)$. Moreover, given two vector fields X and Y on $(\mathcal{M}, \nabla)$, we have, at the point (x, V):*

$$\begin{array}{rcl} [X^v, Y^v] & = & 0, \\ [X^h, Y^v] & = & (\nabla_X Y)^v \simeq (0, \nabla_X Y), \\ [X^h, Y^h] & \simeq & ([X, Y], -\mathrm{R}(X, Y)V), \end{array}$$

where R *denotes the curvature of* ∇ *and we use the direct sum notation* (2)*.*

The Reader should not confuse the horizontal *lift* X^h, which is a vector field on $T\mathcal{M}$ constructed from a vector field $X \in \mathfrak{X}(\mathcal{M})$, with the notation $\bar{X}_h = \Pi\bar{X}$ denoting the horizontal *part* of $\bar{X} \in \mathfrak{X}(T\mathcal{M})$. Similarly, the vertical lift X^v is *not* the vertical projection $\bar{X}_v = K\bar{X}$.

We say that a vector field $\bar{X}$ on $T\mathcal{M}$ is *projectable* if it is constant on the fibres, i.e. $(\Pi\bar{X}, \mathrm{K}\bar{X})(x, V) = (\Pi\bar{X}, \mathrm{K}\bar{X})(x, V')$. According to the lemma above, it is equivalent to the fact that there exists two vector fields X_1 and X_2 on $\mathcal{M}$ such that $\bar{X} = (X_1)^h + (X_2)^v$.

Assume now that $\mathcal{M}$ is equipped with a pseudo-Riemannian metric g, i.e. a non-degenerate bilinear form. By the non-degeneracy assumption, we can identify $T^*\mathcal{M}$ with $T\mathcal{M}$ by the following (musical) isomorphism:

$$\begin{array}{rccc} \iota : & T\mathcal{M} & \to & T\mathcal{M}^* \\ & (x, V) & \mapsto & (x, \xi), \end{array}$$

where ξ is defined by

$$\xi(W) = g(V, W), \quad \forall W \in T_x\mathcal{M}.$$

The *Liouville form* $\alpha \in \Omega^1(T^*\mathcal{M})$ is the 1-form defined by $\alpha_{(x,\xi)}(\bar{X}) = \xi_x(d\pi^*(\bar{X}))$, where $\bar{X}$ is a tangent vector at the point (x,ξ) of $T^*\mathcal{M}$. The canonical symplectic form on $T\mathcal{M}^*$ is defined to be $\Omega^* := -d\alpha$. There is an elegant, explicit formula for the symplectic form $\Omega := \iota^*(\Omega^*)$ in terms of the metric g and the splitting induced by the Levi-Civita connection ∇ (see [2], [13]):

Lemma 2. *Let $\bar{X}$ and $\bar{Y}$ be two tangent vectors to $T\mathcal{M}$; we have*

$$\Omega(\bar{X},\bar{Y}) = g(\mathrm{K}\bar{X},\Pi\bar{Y}) - g(\Pi\bar{X},\mathrm{K}\bar{Y}).$$

Proposition 2. *Let $(\mathcal{M},\mathrm{J},g)$ be a pseudo- or para-Kähler manifold. The canonical structure $\tilde{\mathrm{J}}$ satisfies*

$$\tilde{\mathrm{J}}\bar{X} \simeq \tilde{\mathrm{J}}(\Pi\bar{X},\mathrm{K}\bar{X}) = (\mathrm{J}\Pi\bar{X},\mathrm{J}\mathrm{K}\bar{X}).$$

Corollary 1. *Let $(\mathcal{M},\mathrm{J},g)$ be a pseudo- or para-Kähler manifold. The 2-tensor $\tilde{g}(.,.) := \Omega(.,\tilde{\mathrm{J}}.)$ satisfies*

$$\tilde{g}(\bar{X},\bar{Y}) = g(\mathrm{K}\bar{X},\mathrm{J}\Pi\bar{Y}) - g(\Pi\bar{X},\mathrm{J}\mathrm{K}\bar{Y}).$$

Moreover, $\tilde{g}$ is symmetric and therefore defines a pseudo-Riemannian metric on $T\mathcal{M}$.

Proof of Proposition 2. Let us write the splitting of $TT\mathcal{M}$ in a local coordinate x as in the proof of Proposition 1 ([3]). The Levi–Civita connection is expressed through its connection form μ: $\nabla_X Y = dY(X) + \mu(X)Y$. Consequently, if $(X,\Xi) \in T_{(x,\xi)}T\mathcal{M}$, $\Pi(X,\Xi) = X$ and $\mathrm{K}(X,\Xi) = \Xi + \mu(X)\xi$. Thus

$$\Pi(\tilde{\mathrm{J}}(X,\Xi)) = \mathrm{J}X \text{ and } \mathrm{K}(\tilde{\mathrm{J}}(X,\Xi)) = \mathrm{J}(x)\Xi + (d\mathrm{J}(x)\xi)X + \mu(\mathrm{J}(x)X)\xi.$$

Because J is integrable, we may choose x to be a complex coordinate, so that J is a constant endomorphism, and $d\mathrm{J}(x)\xi$ vanishes. Because $\mathcal{M}$ is Kähler, we know that $\mu(X)$ commutes with J. However, ∇ being without torsion, $\mu(X)Y = \mu(Y)X$, so

$$\mathrm{K}(\tilde{\mathrm{J}}(X,\Xi)) = \mathrm{J}\Xi + \mathrm{J}\mu(X)\xi = \mathrm{J}\mathrm{K}(X,\Xi).$$ □

Corollary 2. *The symplectic form Ω is compatible with the complex or para-complex structure $\tilde{\mathrm{J}}$.*

Proof. Using Lemma 2, we compute

$$\begin{aligned}
\Omega(\tilde{\mathrm{J}}\bar{X},\tilde{\mathrm{J}}\bar{Y}) &= g(\mathrm{K}\tilde{\mathrm{J}}\bar{X},\Pi\tilde{\mathrm{J}}\bar{Y}) - g(\Pi\tilde{\mathrm{J}}\bar{X},\mathrm{K}\tilde{\mathrm{J}}\bar{Y}) \\
&= g(\mathrm{J}\mathrm{K}\bar{X},\mathrm{J}\Pi\bar{Y}) - g(\mathrm{J}\Pi\bar{X},\mathrm{J}\mathrm{K}\bar{Y}) \\
&= \varepsilon g(\mathrm{K}\bar{X},\Pi\bar{Y}) - \varepsilon g(\Pi\bar{X},\mathrm{K}\bar{Y}) \\
&= \varepsilon\Omega(\bar{X},\bar{Y}).
\end{aligned}$$

□

[3] The Reader should be aware of the conflicting notation: the splitting of $TT\mathcal{M} \simeq \mathbb{R}^{4n}$ as $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$ induced by the coordinate charts (e.g. $\bar{X} \simeq ((x,\xi),(X,\Xi))$) differs a priori from the connection-induced splitting $\bar{X} \simeq (\Pi\bar{X},\mathrm{K}\bar{X})$.

3 The Levi-Civita connection of $\tilde{g}$

The following lemma describes the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}$ in terms of the direct decomposition of $TT\mathcal{M}$, the Levi-Civita connection ∇ of g and its curvature tensor R.

Lemma 3. *Let $\bar{X}$ and $\bar{Y}$ be two vector fields on $T\mathcal{M}$ and assume that $\bar{Y}$ is projectable, then at the point (x,V) we have*

$$(\tilde{\nabla}_{\bar{X}}\bar{Y})_{|V} = \left(\nabla_{\Pi\bar{X}}\Pi\bar{Y}, \nabla_{\Pi\bar{X}}\mathrm{K}\bar{Y} - \mathrm{T}_1(\Pi\bar{X},\Pi\bar{Y},V)\right),$$

where

$$\mathrm{T}_1(X,Y,V) \quad = \quad \frac{1}{2}\Big(\mathrm{R}(X,Y)V - \varepsilon\mathrm{R}(V,\mathrm{J}X)\mathrm{J}Y - \varepsilon\mathrm{R}(V,\mathrm{J}Y)\mathrm{J}X\Big)$$

Moreover, if $\mathcal{M}$ is a pseudo-Riemannian surface with Gaussian curvature c, we have

$$\mathrm{T}_1(X,Y,V) = \begin{cases} -2cg(V,X)Y & \textit{in the Kähler case,} \\ +2cg(V,Y)X & \textit{in the para-Kähler case.} \end{cases}$$

Proof. We use Lemma 1 together with the Koszul formula:

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_{\bar{X}}\bar{Y},\bar{Z}) \quad &= \quad \bar{X}\tilde{g}(\bar{Y},\bar{Z}) + \bar{Y}\tilde{g}(\bar{X},\bar{Z}) - \bar{Z}\tilde{g}(\bar{X},\bar{Y}) + \tilde{g}([\bar{X},\bar{Y}],\bar{Z}) \\ &\quad -\tilde{g}([\bar{X},\bar{Z}],\bar{Y}) - \tilde{g}([\bar{Y},\bar{Z}],\bar{X}), \end{aligned}$$

where X, Y and Z are three vector fields on $T\mathcal{M}$. From the fact that $[X^v,Y^v]$ and $\tilde{g}(X^v,Y^v)$ vanish we have:

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_{X^v}Y^v,Z^v) \quad &= \quad X^v\tilde{g}(Y^v,Z^v) + Y^v\tilde{g}(X^v,Z^v) - Z^v\tilde{g}(X^v,Y^v) \\ &\quad +\tilde{g}([X^v,Y^v],Z^v) - \tilde{g}([X^v,Z^v],Y^v) - \tilde{g}([Y^v,Z^v],X^v) \\ &= \quad 0. \end{aligned}$$

Moreover, taking into account that $\tilde{g}(Y^v,Z^h)$ and similar quantities are constant on the fibres, we obtain

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_{X^v}Y^v,Z^h) \quad &= \quad X^v\tilde{g}(Y^v,Z^h) + Y^v\tilde{g}(X^v,Z^h) - Z^h\tilde{g}(X^v,Y^v) \\ &\quad +\tilde{g}([X^v,Y^v],Z^h) - \tilde{g}([X^v,Z^h],Y^v) - \tilde{g}([Y^v,Z^h],X^v) \\ &= \quad -\tilde{g}(-(\nabla_Z X)^v,Y^v) - \tilde{g}(-(\nabla_Z Y)^v,X^v) \\ &= \quad 0. \end{aligned}$$

From these last two equations we deduce that $\tilde{\nabla}_{X^v}Y^v$ vanishes. Analogous computations show that $\tilde{\nabla}_{X^v}Y^h$ vanishes as well. From Lemma 1 and the formula $[\bar{X},\bar{Y}] = \tilde{\nabla}_{\bar{X}}\bar{Y} - \tilde{\nabla}_{\bar{Y}}\bar{X}$, we deduce that

$$\tilde{\nabla}_{X^h}Y^v \simeq (0,\nabla_X Y). \tag{3}$$

Finally, introducing

$$\mathrm{T}_1(X,Y,V) := \frac{1}{2}\Big(\mathrm{R}(X,Y)V - \varepsilon\mathrm{R}(V,\mathrm{J}Y)\mathrm{J}X - \varepsilon\mathrm{R}(V,\mathrm{J}X)\mathrm{J}Y\Big),$$

we compute that

$$
\begin{aligned}
2\tilde{g}(\tilde{\nabla}_{X^h}Y^h, Z^h) &= -g(\mathrm{R}(X,Y)V,\mathrm{J}Z) + g(\mathrm{R}(X,Z)V,\mathrm{J}Y) + g(\mathrm{R}(Y,Z)V,\mathrm{J}X)\\
&= -g(\mathrm{R}(X,Y)V,\mathrm{J}Z) + g(\mathrm{R}(V,\mathrm{J}Y)X,Z) + g(\mathrm{R}(V,\mathrm{J}X)Y,Z)\\
&= -g(\mathrm{R}(X,Y)V,\mathrm{J}Z) + \varepsilon g(\mathrm{R}(V,\mathrm{J}Y)\mathrm{J}X,\mathrm{J}Z) + \varepsilon g(\mathrm{R}(V,\mathrm{J}X)\mathrm{J}Y,\mathrm{J}Z)\\
&= -g(2\mathrm{T}_1(X,Y,V),\mathrm{J}Z)
\end{aligned}
$$

and

$$\tilde{g}(\tilde{\nabla}_{X^h}Y^h, Z^v) = -g(\nabla_X Y, \mathrm{J}Z),$$

from which we deduce that

$$\tilde{\nabla}_{X^h}Y^h(V) = (\nabla_X Y, -\mathrm{T}_1(X,Y,V)). \tag{4}$$

From (3) and (4) we deduce the required formula for $\tilde{\nabla}_{\bar{X}}\bar{Y}$.

If $n=1$, we have $\mathrm{R}(X,Y)Z = c(g(Y,Z)X - g(X,Z)Y)$, hence the tensor T_1 becomes:

$$
\begin{aligned}
2\mathrm{T}_1(X,Y,V) &= \mathrm{R}(X,Y)V + \varepsilon\mathrm{J}\mathrm{R}(V,\mathrm{J}X)Y + \varepsilon\mathrm{J}\mathrm{R}(V,\mathrm{J}Y)X\\
&= c\Big(g(Y,V)X - g(X,V)Y\\
&\quad -\varepsilon\mathrm{J}\big(g(\mathrm{J}X,Y)V - g(V,Y)\mathrm{J}X + g(\mathrm{J}Y,X)V - g(V,X)\mathrm{J}Y\big)\Big)\\
&= c\big(g(Y,V)X - g(X,V)Y\\
&\quad -\varepsilon\big(g(\mathrm{J}X,Y)\mathrm{J}V + g(V,Y)X + g(\mathrm{J}Y,X)\mathrm{J}V + g(V,X)Y\big)\big)\\
&= c\big((1-\varepsilon)g(V,Y)X - (1+\varepsilon)g(V,X)Y\big).
\end{aligned}
$$

□

Remark 4. *It should be noted that covariant derivatives with respect to a projectable vertical field X^v always vanish.*

Proposition 3. *The structure $\tilde{\mathrm{J}}$ is parallel with respect to $\tilde{\nabla}$.*

Proof. It can be seen as a trivial consequence of the fact that $\tilde{\mathrm{J}}$ is complex (resp. para-complex) and Ω is closed, but can also be checked directly, using the equivariance properties of J w.r.t. the connection ∇ and the curvature tensor R. Using the definition of $\tilde{\mathrm{J}}$ and Lemma 3, $\tilde{\nabla}_{\bar{X}}\tilde{\mathrm{J}}\bar{Y}$ is obvious provided $\mathrm{T}_1(X,\mathrm{J}Y,V) = \mathrm{J}\mathrm{T}_1(X,Y,V)$. That is indeed the case since

$$
\begin{aligned}
&2(\mathrm{T}_1(X,\mathrm{J}Y,V) - \mathrm{J}\mathrm{T}_1(X,Y,V))\\
&\quad =\mathrm{R}\,(X,\mathrm{J}Y)\,V + \mathrm{R}\,(V,\mathrm{J}X)\,Y + \mathrm{R}\,(V,Y)\,\mathrm{J}X\\
&\qquad - \mathrm{R}\,(X,Y)\,\mathrm{J}V - \mathrm{R}\,(V,\mathrm{J}X)\,Y - \mathrm{R}\,(V,\mathrm{J}Y)\,X\\
&\quad =\mathrm{R}\,(X,\mathrm{J}Y)\,V + \mathrm{R}\,(\mathrm{J}Y,V)\,X + \mathrm{J}\,(\mathrm{R}\,(V,Y)\,X + \mathrm{R}\,(Y,X)\,V)\\
&\quad =\mathrm{R}\,(V,X)\,\mathrm{J}Y + \mathrm{J}\mathrm{R}\,(X,V)\,Y = 0,
\end{aligned}
$$

where we have used Bianchi's identity. □

4 Curvature properties of $(\tilde{\mathrm{J}}, \tilde{g})$

4.1 The Riemannian curvature tensor of $\tilde{g}$

Proposition 4. *The curvature tensor* $\widetilde{\mathrm{Rm}} := -\tilde{g}(\tilde{R}.,.)$ *of* $\tilde{g}$ *at* (x, V) *is given by the formula*

$$\begin{aligned}
\widetilde{\mathrm{Rm}}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) \;\; = \;\; & g(T_2(\Pi\bar{X}, \Pi\bar{Y}, \Pi\bar{Z}, V), \mathrm{J}\Pi\bar{W}) \\
& -\mathrm{Rm}(\Pi\bar{X}, \Pi\bar{Y}, \Pi\bar{Z}, \mathrm{JK}\bar{W}) - \mathrm{Rm}(\Pi\bar{X}, \Pi\bar{Y}, \mathrm{JK}\bar{Z}, \Pi\bar{W}) \\
& -\mathrm{Rm}(\Pi\bar{X}, \mathrm{JK}\bar{Y}, \Pi\bar{Z}, \Pi\bar{W}) + \mathrm{Rm}(\mathrm{JK}\bar{X}, \Pi\bar{Y}, \Pi\bar{Z}, \Pi\bar{W}),
\end{aligned}$$

where

$$\mathrm{T}_2(X, Y, Z, V) := (\nabla_X \mathrm{T}_1)(Y, Z, V) - (\nabla_Y \mathrm{T}_1)(X, Z, V).$$

Moreover, $(T\mathcal{M}, \tilde{g})$ *is scalar flat and the Ricci tensor of* $\tilde{g}$ *is*

$$\widetilde{\mathrm{Ric}}(\bar{X}, \bar{Y}) = 2\mathrm{Ric}(\Pi\bar{X}, \Pi\bar{Y}).$$

Corollary 3. $(T\mathcal{M}, \tilde{g})$ *is Einstein if and only if* $(\mathcal{M}, g)$ *is flat. Moreover* $(T\mathcal{M}, \tilde{g})$ *has nonnegative (resp. nonpositive) Ricci curvature if and only if* $(\mathcal{M}, g)$ *has nonnegative (resp. nonpositive) Ricci curvature as well.*

Proof of Proposition 4. We will compute the curvature tensor for projectable vector fields, and need only do so for the following six cases, due to the symmetries of $\widetilde{\mathrm{Rm}}$. Remark 4 simplifies computations greatly, since most vertical derivatives vanish, except when the derived vector field is not projectable. In particular $\tilde{\mathrm{R}}(X^v, Y^v)$ vanishes as endomorphism, hence:

$$\begin{aligned}
\widetilde{\mathrm{Rm}}(X^v, Y^v, Z^v, W^v) &= 0 \\
\widetilde{\mathrm{Rm}}(X^v, Y^v, Z^v, W^h) &= 0 \\
\widetilde{\mathrm{Rm}}(X^v, Y^v, Z^h, W^v) &= 0
\end{aligned}$$

To obtain the last three combinations, let us first derive $\tilde{\mathrm{R}}(X^h, Y^h)Z^h$. This is more delicate since we have to covariantly differentiate non-projectable quantities. Indeed

$$\begin{aligned}
\tilde{\mathrm{R}}(X^h, Y^h)Z^h \;\; = \;\; & \tilde{\nabla}_{X^h}\tilde{\nabla}_{Y^h}Z^h - \tilde{\nabla}_{Y^h}\tilde{\nabla}_{X^h}Z^h - \tilde{\nabla}_{[X^h, Y^h]}Z^h \\
= \;\; & \tilde{\nabla}_{X^h}(\nabla_Y Z, -\mathrm{T}_1(Y, Z, V)) - \tilde{\nabla}_{Y^h}(\nabla_X Z, -\mathrm{T}_1(X, Z, V)) \\
& -\tilde{\nabla}_{([X,Y], -\mathrm{R}(X,Y)V)}Z^h \\
= \;\; & (\nabla_X\nabla_Y Z, -\mathrm{T}_1(X, \nabla_Y Z, V)) - D_{X^h}(0, \mathrm{T}_1(Y, Z, V)) \\
& -(\nabla_Y\nabla_X Z, -\mathrm{T}_1(Y, \nabla_X Z, V)) + D_{Y^h}(0, \mathrm{T}_1(X, Z, V)) \\
& -(\nabla_{[X,Y]}Z, -\mathrm{T}_1([X, Y], Z, V)) \\
= \;\; & (\mathrm{R(X, Y)Z}, 0) \\
& - (0, \mathrm{T}_1(X, \nabla_Y Z, V) - \mathrm{T}_1(\mathrm{Y}, \nabla_{\mathrm{X}}\mathrm{Z}, \mathrm{V}) - \mathrm{T}_1([\mathrm{X}, \mathrm{Y}], \mathrm{Z}, \mathrm{V})) \\
& -\tilde{\nabla}_{X^h}(0, \mathrm{T}_1(Y, Z, V)) + \tilde{\nabla}_{Y^h}(0, \mathrm{T}_1(X, Z, V))
\end{aligned}$$

Recalling the lemma[4] in [12], there exists a vector field U on M such that $U(x) = V$ and $(\nabla_X U)(x) = 0$. Then the vertical lift of $\mathrm{T}_1(Y, Z, U)$ is seen to agree to first order with

$$(x, V) \mapsto (0, \mathrm{T}_1(X(x), Z(x), V))$$

thus allowing us to use the formula in Lemma 3:

$$\begin{aligned}
\tilde{\nabla}_{X^h}(0, \mathrm{T}_1(Y, Z, \cdot)) &= \tilde{\nabla}_{X^h}(\mathrm{T}_1(Y, Z, U)^v) \\
&= (0, \nabla_X(\mathrm{T}_1(Y, Z, U))) \\
&= \big(0, (\nabla_X \mathrm{T}_1)(Y, Z, U) + \mathrm{T}_1(\nabla_X Y, Z, U) \\
&\qquad + \mathrm{T}_1(Y, \nabla_X Z, U) + \mathrm{T}_1(Y, Z, \nabla_X U)\big)
\end{aligned}$$

which, evaluated at (x, V), yields

$$\tilde{\nabla}_{X^h}(0, \mathrm{T}_1(Y, Z, \cdot))|_{(x,V)} = (0, (\nabla_X \mathrm{T}_1)(Y, Z, V) + \mathrm{T}_1(\nabla_X Y, Z, V) + \mathrm{T}_1(Y, \nabla_X Z, V)).$$

Summing up,

$$\begin{aligned}
\tilde{\mathrm{R}}(X^h, Y^h)Z^h|_{(x,V)} &= \Big(\mathrm{R(X, Y)Z}, \\
&\quad -\mathrm{T}_1(X, \nabla_Y Z, V) + \mathrm{T}_1(Y, \nabla_X Z, V) \\
&\quad +\mathrm{T}_1([X, Y], Z, V) - (\nabla_X \mathrm{T}_1)(Y, Z, V) \\
&\quad -\mathrm{T}_1(\nabla_X Y, Z, V) - \mathrm{T}_1(Y, \nabla_X Z, V) \\
&\quad +(\nabla_Y \mathrm{T}_1)(X, Z, V) + \mathrm{T}_1(\nabla_Y X, Z, V) \\
&\quad +\mathrm{T}_1(X, \nabla_Y Z, V)\Big) \\
&= \Big(\mathrm{R}(X, Y)Z, -(\nabla_X \mathrm{T}_1)(Y, Z, V) + (\nabla_Y \mathrm{T}_1)(X, Z, V)\Big) \\
&= \Big(\mathrm{R}(X, Y)Z, -\mathrm{T}_2(X, Y, Z, V)\Big).
\end{aligned}$$

From that we deduce directly

$$\begin{aligned}
\widetilde{\mathrm{Rm}}(X^h, Y^h, Z^h, W^v) &= -\mathrm{Rm}(X, Y, Z, \mathrm{J}W) \\
\widetilde{\mathrm{Rm}}(X^h, Y^h, Z^h, W^h)|_{(x,V)} &= g(\mathrm{T}_2(X, Y, Z, V), \mathrm{J}W).
\end{aligned}$$

On the other hand, using repeatedly Remark 4,

$$\begin{aligned}
\widetilde{\mathrm{Rm}}(X^h, Y^v, Z^h, W^v) &= \tilde{g}(\tilde{\nabla}_{X^h}\tilde{\nabla}_{Y^v}W^v - \tilde{\nabla}_{Y^v}\tilde{\nabla}_{X^h}W^v - \tilde{\nabla}_{[X^h, Y^v]}W^v, Z^h) \\
&= \tilde{g}(-\tilde{\nabla}_{Y^v}(0, \nabla_X W), Z^h) = \tilde{g}(0, Z^h) = 0.
\end{aligned}$$

The claimed formula is easily deduced using the symmetries of the curvature tensor.

[4] Note that computations in [12] are done for the Sasaki metric, hence direct results do not apply.

In order to calculate the Ricci curvature of $\tilde{g}$, we consider a Hermitian pseudo-orthonormal basis $(e_1, \ldots, e_{2n})$ of $T_x\mathcal{M}$, i.e. $g(e_a, e_b) = \varepsilon_a \delta_{ab}$, where $\varepsilon_a = \pm 1$, and $e_{n+a} = \mathrm{J}e_a$. In particular, $\varepsilon_{n+a} = \varepsilon\varepsilon_a$.This gives a (non-orthonormal) basis of $T_{(x,V)}T\mathcal{M}$:

$$\bar{e}_a := (e_a)_h \qquad \bar{e}_{2n+a} := (e_a)^v.$$

A calculation using Corollary 1 shows that the expression of $\tilde{g}$ in this basis is:

$$[\tilde{g}_{\mu\nu}]_{1\leq\mu,\nu\leq 4n} := \begin{pmatrix} 0 & 0 & 0 & \Delta \\ 0 & 0 & -\Delta & 0 \\ 0 & -\Delta & 0 & 0 \\ \Delta & 0 & 0 & 0 \end{pmatrix},$$

where $\Delta = \varepsilon\mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n) = \mathrm{diag}(\varepsilon_{n+1}, \ldots, \varepsilon_{2n})$. It follows that $\widetilde{\mathrm{Ric}}(X^v, Y^v)$ and $\widetilde{\mathrm{Ric}}(X^h, Y^v)$ vanish.

Moreover, noting that $\tilde{g}^{\mu\nu} = \tilde{g}_{\mu\nu}$,

$$\begin{aligned}
\widetilde{\mathrm{Ric}}(X^h, Y^h) &= \sum_{\mu,\nu=1}^{4n} \tilde{g}^{\mu\nu}\widetilde{\mathrm{Rm}}(X^h, \bar{e}_\mu, Y^h, \bar{e}_\nu) \\
&= \sum_{a=1}^{n} \varepsilon\varepsilon_a \Big(\widetilde{\mathrm{Rm}}(X^h, (e_a)^h, Y^h, (\mathrm{J}e_a)^v) - \widetilde{\mathrm{Rm}}(X^h, (\mathrm{J}e_a)^h, Y^h, (e_a)^v) \\
&\quad -\widetilde{\mathrm{Rm}}(X^h, (e_a)^v, Y^h, (\mathrm{J}e_a)^h) + \widetilde{\mathrm{Rm}}(X^h, (\mathrm{J}e_a)^v, Y^h, (e_a)^h)\Big) \\
&= \sum_{a=1}^{n} \varepsilon\varepsilon_a \Big(- \mathrm{Rm}(X, e_a, Y, \mathrm{J}^2 e_a) + \mathrm{Rm}(X, \mathrm{J}e_a, Y, \mathrm{J}e_a) \\
&\quad +\mathrm{Rm}(Y, \mathrm{J}e_a, X, \mathrm{J}e_a) - \mathrm{Rm}(Y, e_a, X, \mathrm{J}^2 e_a)\Big) \\
&= 2\sum_{a=1}^{n} \Big(\varepsilon_a \mathrm{Rm}(X, e_a, Y, e_a) + \varepsilon_{a+n}\mathrm{Rm}(X, e_{a+n}, Y, e_{a+n})\Big) \\
&= 2\sum_{k=1}^{2n} \varepsilon_k \mathrm{Rm}(X, e_k, Y, e_k) = 2\mathrm{Ric}(X, Y).
\end{aligned}$$

We see easily that $\widetilde{\mathrm{Ric}}$ vanishes whenever one of the vectors is along the vertical fiber, thus the expected formula.

Finally the scalar curvature

$$\widetilde{\mathrm{Scal}} = \sum_{\mu,\nu=1}^{4} \tilde{g}^{\mu\nu}\widetilde{\mathrm{Ric}}(\bar{e}_\mu, \bar{e}_\nu) = 0,$$

since $\tilde{g}^{\mu\nu}$ vanishes as soon as both $\bar{e}_\mu, \bar{e}_\nu$ are both horizontal. $\square$

4.2 The Weyl curvature tensor of $\tilde{g}$

Proposition 5. *The Weyl tensor $\widetilde{\mathrm{W}}$ at (x, V) is given by*

$$\begin{aligned}\widetilde{\mathrm{W}}(\bar{X},\bar{Y},\bar{Z},\bar{W}) &= \widetilde{\mathrm{Rm}}(\bar{X},\bar{Y},\bar{Z},\bar{W}) \\ &\quad -\tfrac{1}{2n-1}\Big(\mathrm{Ric}(\Pi\bar{X},\Pi\bar{Z})\tilde{g}(\bar{Y},\bar{W}) + \mathrm{Ric}(\Pi\bar{Y},\Pi\bar{W})\tilde{g}(\bar{Y},\bar{W}) \\ &\qquad -\mathrm{Ric}(\Pi\bar{X},\Pi\bar{W})\tilde{g}(\bar{Y},\bar{Z}) - \mathrm{Ric}(\Pi\bar{Y},\Pi\bar{Z})\tilde{g}(\bar{X},\bar{W})\Big).\end{aligned}$$

In particular, if $n = 1$,

$$\widetilde{\mathrm{W}}(\bar{X},\bar{Y},\bar{Z},\bar{W}) = g(\mathrm{T}_2(\Pi\bar{\mathrm{X}},\Pi\bar{\mathrm{Y}},\Pi\bar{\mathrm{Z}},\mathrm{V}),\mathrm{J}\Pi\bar{\mathrm{W}}).$$

Corollary 4. *$(T\mathcal{M},\tilde{g})$ is locally conformally flat if and only if $n = 1$ and g has constant curvature, or $n \geq 2$ and g is flat.*

Remark 5. *This result has been proved in the case $n = 1$ and $\varepsilon = 1$ in [10].*

Proof of Proposition 5. Since the scalar curvature vanishes, we have

$$\widetilde{\mathrm{W}} = \widetilde{\mathrm{Rm}} - \frac{1}{4n-2}\widetilde{\mathrm{Ric}} \owedge \tilde{g},$$

where $\owedge$ denotes the Kulkarni–Nomizu product. Recall that $\widetilde{\mathrm{Ric}}(\bar{X},\bar{Y}) = 0$ if one of the two vectors $\bar{X}$ and $\bar{Y}$ is vertical. Consequently

$$\begin{aligned}\widetilde{\mathrm{Ric}} \owedge \tilde{g}(\bar{X},\bar{Y},\bar{Z},\bar{W}) &= 2\left(\mathrm{Ric}(\Pi\bar{X},\Pi\bar{Z})\tilde{g}(\bar{Y},\bar{W}) + \mathrm{Ric}(\Pi\bar{Y},\Pi\bar{W})\tilde{g}(\bar{Y},\bar{W})\right. \\ &\qquad \left.-\mathrm{Ric}(\Pi\bar{X},\Pi\bar{W})\tilde{g}(\bar{Y},\bar{Z}) - \mathrm{Ric}(\Pi\bar{Y},\Pi\bar{Z})\tilde{g}(\bar{X},\bar{W})\right).\end{aligned}$$

The expression of the Weyl tensor follows easily.

In the case $n = 1$ of a surface with Gaussian curvature c, we have $\mathrm{Ric}(X,Y) = cg(X,Y)$ and $\mathrm{Rm}(X,Y,Z,W) = c\big(g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\big)$. Hence using Proposition 4, the expression of Weyl tensor simplifies and we get the claimed formula. □

Proof of Corollary 4. We first deal with the case $n = 1$. Lemma 3 implies that $\mathrm{T}_1(X,Y,Z) = -2cg(Z,X)Y$ when $\varepsilon = 1$ (resp. $2cg(Z,Y)X$ when $\varepsilon = -1$). Therefore, if $\varepsilon = 1$,

$$\begin{aligned}\mathrm{T}_2(X,Y,Z,W) &= \nabla_X\mathrm{T}_1(Y,Z,W) - \nabla_Y\mathrm{T}_1(X,Z,W) \\ &= -2(X.c)g(W,Y)Z + 2(Y.c)g(W,X)Z \\ &= 2g\Big((Y.c)X - (X.c)Y, W\Big)Z,\end{aligned}$$

which vanishes if and only if $(X.c)Y = (Y.c)X$ for all vectors X, Y, i.e. the curvature c is constant. Analogously, if $\varepsilon = -1$,

$$\begin{aligned}\mathrm{T}_2(X,Y,Z,W) &= \nabla_X\mathrm{T}_1(Y,Z,W) - \nabla_Y\mathrm{T}_1(X,Z,W) \\ &= 2(X.c)g(W,Z)Y - 2(Y.c)g(W,Z)X \\ &= 2\Big((X.c)Y - (Y.c)X\Big)g(W,Z),\end{aligned}$$

which again vanishes if and only if the curvature c is constant.

Assume now that $(T\mathcal{M}, \tilde{g})$ is conformally flat with $n \geq 2$. Thus in particular

$$\begin{aligned}\widetilde{\mathrm{W}}&(X^h, Y^h, Z^h, W^v) \\ &= -\operatorname{Rm}(X, Y, Z, \mathrm{J}W) \\ &\quad - \frac{1}{2n-1}\Big(-\operatorname{Ric}(X, Z)g(Y, \mathrm{J}W) + \operatorname{Ric}(Y, Z)g(X, \mathrm{J}W)\Big)\end{aligned}$$

vanishes, so

$$\operatorname{Rm}(X, Y, Z, \mathrm{J}W) = \frac{1}{2n-1}\Big(\operatorname{Ric}(X, Z)g(Y, \mathrm{J}W) - \operatorname{Ric}(Y, Z)g(X, \mathrm{J}W)\Big).$$

(Observe that this equation always holds if $\mathcal{M}$ is a surface.) Let us apply the symmetry property of the curvature tensor to this equation with $Z = X$ and $\mathrm{J}W = Y$, assuming furthermore that X and Y are two non-null vectors:

$$\begin{aligned} 0 &= (2n-1)\big(\operatorname{Rm}(X, Y, X, Y) - \operatorname{Rm}(Y, X, Y, X)\big) \\ &= \operatorname{Ric}(X, X)g(Y, Y) - \operatorname{Ric}(Y, X)g(X, Y) \\ &\quad -\operatorname{Ric}(Y, Y)g(X, X) + \operatorname{Ric}(X, Y)g(Y, X) \\ &= \operatorname{Ric}(X, X)g(Y, Y) - \operatorname{Ric}(Y, Y)g(X, X).\end{aligned}$$

Hence

$$\frac{\operatorname{Ric}(X, X)}{g(X, X)} = \frac{\operatorname{Ric}(Y, Y)}{g(Y, Y)}.$$

The set of non null vectors being dense in $T\mathcal{M}$, it follows by continuity that g is Einstein. We deduce that

$$\begin{aligned}\operatorname{Rm}(X, Y, X, Y) &= \frac{1}{2n-1}\Big(\operatorname{Ric}(X, X)g(Y, Y) - \operatorname{Ric}(Y, X)g(X, Y)\Big) \\ &= c\Big(g(X, X)g(Y, Y) - g(X, Y)g(X, Y)\Big),\end{aligned}$$

so g has constant curvature. But since $\mathcal{M}$ is Kähler and has dimension $2n \geq 4$, it must be flat. $\square$

Finally, we recall the general result linking the Weyl tensor to the scalar curvature in dimension four: for a neutral pseudo-Kähler or para-Kähler metric, self-duality is equivalent to scalar flatness (see Theorem A.2 in annex). We can therefore conclude

Corollary 5. *In dimension four* $(n = 1)$*, the metric* $\tilde{g}$ *is anti-self-dual if and only the curvature* c *of* g *is constant.*

Proof. Thanks to proposition 4, we know that $\tilde{g}$ is scalar flat, hence self-dual (W^- vanishes identically). In order for $\tilde{g}$ to be also anti-self-dual, the Weyl tensor has to vanish completely, which amounts, following corollary 4, to having constant (sectional) curvature c on $\mathcal{M}$. $\square$

4.3 The holomorphic sectional curvature of $(\tilde{\mathrm{J}}, \tilde{g})$

Proposition 6. *$(\tilde{\mathrm{J}}, \tilde{g})$ has constant holomorphic sectional curvature if and only if g is flat.*

Proof. Define the holomorphic sectional curvature tensor of $\tilde{g}$ by $\widetilde{\mathrm{Hol}}(\bar{X}) := \widetilde{\mathrm{Rm}}(\bar{X}, \tilde{\mathrm{J}}\bar{X}, \bar{X}, \tilde{\mathrm{J}}\bar{X})$. Writing any doubly tangent vector $\bar{X}$ as the sum of a horizontal and a vertical factor, we will compute $\widetilde{\mathrm{Hol}}(X^h + Y^v)$. We deduce from Proposition 4 that $\widetilde{\mathrm{Rm}}$ vanishes whenever two or more entries are vertical. Hence, using the antisymmetric properties of the Riemann tensor w.r.t. the complex or para-complex structure,

$$\begin{aligned}
\widetilde{\mathrm{Hol}}(X^h + Y^v) &= \widetilde{\mathrm{Rm}}(X^h, \mathrm{J}X^h, X^h, \mathrm{J}X^h) \\
&\quad + \widetilde{\mathrm{Rm}}(X^h, \mathrm{J}X^h, X^h, \mathrm{J}Y^v) + \widetilde{\mathrm{Rm}}(X^h, \mathrm{J}X^h, Y^v, \mathrm{J}X^h) \\
&\quad + \widetilde{\mathrm{Rm}}(X^h, \mathrm{J}Y^v, X^h, \mathrm{J}X^h) + \widetilde{\mathrm{Rm}}(Y^v, \mathrm{J}X^h, X^h, \mathrm{J}X^h) \\
&= \widetilde{\mathrm{Rm}}(X^h, \mathrm{J}X^h, X^h, \mathrm{J}X^h) + 4\widetilde{\mathrm{Rm}}(X^h, \mathrm{J}X^h, X^h, \mathrm{J}Y^v) \\
&= g(\mathrm{T}_2(X, \mathrm{J}X, X, V) - 4\varepsilon \mathrm{R}(X, Y)X, \mathrm{J}X).
\end{aligned}$$

In particular,

$$\begin{aligned}
\widetilde{\mathrm{Hol}}(X^v) &= 0 \\
\widetilde{\mathrm{Hol}}(X^h + X^v) &= g(\mathrm{T}_2(X, \mathrm{J}X, X, V), \mathrm{J}X) \\
\widetilde{\mathrm{Hol}}(X^h + (\mathrm{J}X)^v) &= g(\mathrm{T}_2(X, \mathrm{J}X, X, V), \mathrm{J}X) + 4\varepsilon \mathrm{Hol}(X).
\end{aligned}$$

It follows from the first equation that if $\widetilde{\mathrm{Hol}}$ is constant, it must be zero. Hence, from the second and third equation we deduce that Hol must vanish, i.e. g is flat. □

5 Examples

The simplest examples where we may apply the construction above is where $(\mathcal{M}, \mathrm{J}, g, \omega)$ is the plane $\mathbb{R}^2$ equipped with the flat metric $g := dq_1^2 + \varepsilon dq_2^2$ and the complex or para-complex structure J defined by $\mathrm{J}(\partial_{q_1}, \partial_{q_2}) = (-\varepsilon\partial_{q_2}, \partial_{q_1})$. In other words, $\mathbb{R}^2$ is identified with the complex plane $\mathbb{C}$ or the para-complex plane $\mathbb{D}$. We recall that $\mathbb{D}$, called the algebra of double numbers, is the two-dimensional real vector space $\mathbb{R}^2$ endowed with the commutative algebra structure whose product rule is given by

$$(u, v).(u', v') = (uu' + vv', uv' + u'v).$$

The number $(0, 1)$, whose square is $(1, 0)$ and not $(-1, 0)$, will be denoted by τ.

We claim that in the complex case $\varepsilon = 1$, the structure $(\tilde{\mathrm{J}}, \tilde{g}, \Omega)$ just constructed on $T\mathbb{C}$ is equivalent to that of the standard complex pseudo-Euclidean plane $(\mathbb{C}^2, \bar{\mathrm{J}}, \langle ., . \rangle_2, \omega_1)$, where $\bar{\mathrm{J}}$ is the canonical complex structure, $(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$ are the canonical coordinates and

$$\begin{array}{rcl} \langle ., . \rangle_2 & := & -dx_1^2 - dx_2^2 + dx_2^2 + dy_2^2 \\ \omega_1 & := & -dx_1 \wedge dy_1 + dx_2 \wedge dy_2. \end{array}$$

To see this, it is sufficient to consider the following complex change of coordinates

$$\left\{ \begin{array}{rcl} z_1 & := & \frac{\sqrt{2}}{2}((p_1 + ip_2) + i(q_1 + iq_2)) \\ z_2 & := & \frac{\sqrt{2}}{2}(p_1 + ip_2 - i(q_1 + iq_2)), \end{array} \right.$$

which preserves the symplectic form, since we have

$$\omega_1 := -dx_1 \wedge dy_1 + dx_2 \wedge dy_2 = dq_1 \wedge dp_1 + dq_2 \wedge dp_2 = \Omega,$$

where Ω is the canonical symplectic form of $T^*\mathbb{C} \simeq_g T\mathbb{C}$. The metric of a pseudo-Kähler structure being determined by the complex structure and the symplectic form through the formula $\tilde{g} = \Omega(., \tilde{\mathrm{J}}.)$, we have the required identification.

Analogously, in the para-complex case $\varepsilon = -1$, the structure $(\tilde{\mathrm{J}}, \tilde{g}, \Omega)$ constructed on $T\mathbb{D}$ is equivalent to that of the standard para-complex plane $(\mathbb{D}^2, \bar{\mathrm{J}}, \langle ., . \rangle_*, \omega_*)$, where $\bar{\mathrm{J}}$ is the canonical para-complex structure, $(w_1 = u_1 + \tau u_1, w_2 = u_2 + \tau y_2)$ are the canonical coordinates and

$$\begin{array}{rcl} \langle ., . \rangle_* & := & du_1^2 - dv_1^2 + du_2^2 - dv_2^2 \\ \omega_* & := & du_1 \wedge dv_1 + du_2 \wedge dv_2. \end{array}$$

Here we have to be careful with the identification of $T^*\mathbb{D}$ with $T\mathbb{D}$: since the metric g is $dq_1^2 - dq_2^2$, we have $q_1 := dp_1 \simeq_g \partial_{p_1}$ and $q_2 := dp_2 \simeq_g -\partial_{q_2}$. Hence $\Omega^* = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$ and $\Omega = dq_1 \wedge dp_1 - dq_2 \wedge dp_2$. Introducing the change of para-complex coordinates

$$\left\{ \begin{array}{rcl} w_1 & := & \frac{\sqrt{2}}{2}((p_1 + \tau p_2) - \tau(q_1 + \tau q_2)) \\ w_2 & := & \frac{\sqrt{2}}{2}(\tau(p_1 + \tau p_2) + (q_1 + \tau q_2)), \end{array} \right.$$

we check that

$$\omega_* = du_1 \wedge dv_1 + du_2 \wedge dv_2 = dq_1 \wedge dp_1 - dq_2 \wedge dp_2 = \Omega,$$

hence we obtain the identification between $(T\mathbb{D}, \tilde{\mathrm{J}}, \tilde{g}, \Omega)$ and $(\mathbb{D}^2, \bar{\mathrm{J}}, \langle ., . \rangle_*, \omega_*)$. Of course the metrics considered in these two examples are flat.

The next simplest examples of pseudo-Riemannian surfaces are the two-dimensional space forms, namely the sphere $\mathbb{S}^2$, the hyperbolic plane $\mathbb{H}^2 := \{x_1^2 + x_2^2 - x_3^2 = -1\}$ and the de Sitter surface $d\mathbb{S}^2 := \{x_1^2 + x_2^2 - x_3^2 = 1\}$. Their tangent bundles enjoy a interesting geometric interpretation (see [10]):

the tangent bundle $T\mathbb{S}^2$ is canonically identified with the set of oriented lines of Euclidean three-space:

$$L(\mathbb{R}^3) \ni \{V + tx|\, t \in \mathbb{R}\} \simeq (x, V - \langle V, x\rangle_0 x) \in T\mathbb{S}^2.$$

Analogously, the tangent bundle $T\mathbb{H}^2$ is canonically identified with the set of oriented negative (timelike) lines of three-space endowed with the metric $\langle ., .\rangle_1 := dx_1^2 + dx_2^2 - dx_3^2$:

$$\mathbb{L}^3_{1,-} \ni \{V + tx|\, t \in \mathbb{R}\} \simeq (x, V - \langle V, x\rangle_1 x) \in T\mathbb{H}^2,$$

Finally, the tangent bundle $Td\mathbb{S}^2$ is canonically identified with the set of oriented positive (spacelike) lines of three-space endowed with the metric $\langle ., .\rangle_1$:

$$\mathbb{L}^3_{1,+} \ni \{V + tx|\, t \in \mathbb{R}\} \simeq (x, V - \langle V, x\rangle_1 x) \in Td\mathbb{S}^2.$$

Observe that the metric constructed on $T\mathbb{S}^2$ (resp. $T\mathbb{H}^2$) has non-negative (resp. non-positive) Ricci curvature.

A The Weyl tensor in the pseudo-Kähler or para-Kähler cases

The Riemann curvature tensor Rm of a pseudo-Riemannian manifold $\mathcal{N}$ may be seen as a symmetric form R on bivectors of $\Lambda^2 T\mathcal{N}$ (see [4] for references). Splitting R along the eigenspaces $\Lambda^+ \oplus \Lambda^-$ of the Hodge operator $*$ on $\Lambda^2 T\mathcal{N}$, yields the following block decomposition

$$\mathrm{R} = \begin{pmatrix} \mathrm{W}^+ + \frac{\mathrm{Scal}}{12} I & \mathrm{Z} \\ \mathrm{Z}^* & \mathrm{W}^- + \frac{\mathrm{Scal}}{12} I \end{pmatrix}$$

where Z^* denotes the adjoint w.r.t. the induced metric on $\Lambda^2 T\mathcal{N}$, so that $\mathrm{W} = \mathrm{W}^+ \oplus \mathrm{W}^-$, the Weyl tensor seen as a 2-form on $\Lambda^2 T\mathcal{N}$, is the traceless, Hodge-commuting part of the Riemann curvature operator R. Hence the following formula

$$\mathrm{W} = \mathrm{Rm} - \frac{1}{2}\mathrm{Ric} \owedge g + \frac{\mathrm{Scal}}{12} g \owedge g\,.$$

If, additionally, $\mathcal{N}$ is a four dimensional Kähler manifold, then

Theorem A.1. W^+ *can be written as a multiple of the scalar curvature by a parallel non-trivial 2-form on* $\Lambda^2 T\mathcal{N}$*.*

See Prop. 2 in [6] for a proof and the explicit formula for the tensor involved. We do not need it explicitly since we are only interested in the following

Corollary 6. $(\mathcal{N}, g, \mathrm{J})$ *is* anti-self-dual ($\mathrm{W}^+ = 0$) *if and only if the scalar curvature vanishes.*

The result extends to the two cases considered in this article: (1) neutral pseudo-Kähler manifolds and (2) para-Kähler manifolds, with a slight twist: W^+ is replaced by W^-. Precisely:

Theorem A.2. *Let $(\mathcal{N}, g, \mathrm{J})$ be a four dimensional manifold endowed with a pseudo-Kähler neutral metric (respectively a para-Kähler metric, necessarily neutral). Then the Weyl tensor W commutes with the Hodge operator and $\mathcal{N}$ is* self-dual *($\mathrm{W}^- = 0$) if and only if the scalar curvature vanishes.*

The result for neutral pseudo-Kähler manifolds is probably known and relates to representation theory (see [4] for introduction and references), but since we could not find an explicit proof in the literature[5], we will give a simple one below. To our knowledge, the proof for the para-Kähler case is new (albeit similar).

A.1 The pseudo-Kähler case

We will write explicitly the Weyl tensor in a given positively oriented orthonormal frame, denoted by $(e_1, e_{1'}, e_2, e_{2'})$, where $e_{1'} = \mathrm{J}e_1$, $e_{2'} = \mathrm{J}e_2$, $g(e_1) = g(e_{1'}) = -1$ and $g(e_2) = g(e_{2'}) = +1$. (For brevity, $g(X)$ denotes the norm $g(X, X)$.) The pseudo-metric g extends to bivectors, has signature $(2, 4)$, and will be again denoted by g: $g(e_a \wedge e_b) = g(e_a)g(e_b) - g(e_a, e_b)^2 = g(e_a)g(e_b)$, so that $\mathcal{B} = (e_1 \wedge e_{1'}, e_1 \wedge e_2, e_1 \wedge e_{2'}, e_{1'} \wedge e_2, e_{1'} \wedge e_{2'}, e_2 \wedge e_{2'})$ is an orthonormal frame of Λ^2, with $g(e_a \wedge e_b) = -1$, except for $g(e_1 \wedge e_{1'}) = g(e_2 \wedge e_{2'}) = +1$. (Note that the other convention, taking $-g$ does not change the induced metric on Λ^2.)

Since the volume $e_1 \wedge e_{1'} \wedge e_2 \wedge e_{2'}$ is positively oriented, we construct an orthonormal eigenbasis for the Hodge star on $\Lambda^2 T\mathcal{N}$:

$$\left\{ \begin{array}{l} E_1^{\pm} = \frac{\sqrt{2}}{2}(e_1 \wedge e_{1'} \pm e_2 \wedge e_{2'}) \\ E_2^{\pm} = \frac{\sqrt{2}}{2}(e_1 \wedge e_2 \pm e_{1'} \wedge e_{2'}) \\ E_3^{\pm} = \frac{\sqrt{2}}{2}(e_1 \wedge e_{2'} \mp e_{1'} \wedge e_2) \end{array} \right.$$

so that $\Lambda^{\pm}$ is generated by $E_1^{\pm}, E_2^{\pm}, E_3^{\pm}$.

The Kähler condition implies

$$\mathrm{Rm}(\mathrm{J}X, \mathrm{J}Y, Z, T) = \mathrm{Rm}(X, Y, Z, T) = \mathrm{Rm}(X, Y, \mathrm{J}Z, \mathrm{J}T),$$

because J is isometric and parallel. The matrix of the symmetric 2-form R in

[5] On the contrary, some authors seem to imply that scalar flatness is equivalent to anti-self-duality, see [8]). However this contradiction could possibly come from a different choice of orientation, which would exchange self-dual with anti-self-dual.

the orthonormal frame $\mathcal{B}$ is

	$e_{11'}$	e_{12}	$e_{12'}$	$e_{1'2}$	$e_{1'2'}$	$e_{22'}$
$e_{11'}$	$\mathrm{R}_{11'11'}$	$\mathrm{R}_{11'12}$	$\mathrm{R}_{11'12'}$	$\mathrm{R}_{11'1'2} = -\mathrm{R}_{11'12'}$	$\mathrm{R}_{11'1'2'} = \mathrm{R}_{11'12}$	$\mathrm{R}_{11'22'}$
e_{12}		R_{1212}	$\mathrm{R}_{1212'}$	$\mathrm{R}_{121'2} = -\mathrm{R}_{1212'}$	$\mathrm{R}_{131'2'} = \mathrm{R}_{1212}$	$\mathrm{R}_{1222'}$
$e_{12'}$			$\mathrm{R}_{12'12'}$	$\mathrm{R}_{12'1'2} = -\mathrm{R}_{12'12'}$	$\mathrm{R}_{12'1'2'} = \mathrm{R}_{1212'}$	$\mathrm{R}_{12'22'}$
$e_{1'2}$				$\mathrm{R}_{1'21'2} = \mathrm{R}_{12'12'}$	$\mathrm{R}_{1'21'2'} = -\mathrm{R}_{1212'}$	$\mathrm{R}_{1'222'} = -\mathrm{R}_{12'22'}$
$e_{1'2'}$					$\mathrm{R}_{1'2'1'2'} = \mathrm{R}_{1212}$	$\mathrm{R}_{1'2'22'} = \mathrm{R}_{1222'}$
$e_{22'}$						$\mathrm{R}_{22'22'}$

where e_{ab} stands for $e_a \wedge e_b$, for greater legibility. We have written the matrix as a table for clarity and to make symmetries more obvious, and because R is symmetric we need only write half the matrix. We have used the internal symmetries of R, to choose among equivalent coefficients the ones lowest in the lexicographic order of the indices.

The Weyl tensor satisfies *some* of the J-symmetries of R: indeed

$$\begin{aligned}
\mathrm{Ric}(\mathrm{J}X, \mathrm{J}Y) &= \sum_{i=1}^{4} g(e_i)\mathrm{Rm}(\mathrm{J}X, e_i, \mathrm{J}Y, e_i) = \sum_{i=1}^{4} g(e_i)\mathrm{Rm}(X, \mathrm{J}e_i, Y, \mathrm{J}e_i) \\
&= \sum_{i=1}^{4} g(\mathrm{J}e_i)\mathrm{Rm}(X, \mathrm{J}e_i, Y, \mathrm{J}e_i) = \mathrm{Ric}(X, Y)
\end{aligned}$$

because $(\mathrm{J}e_i)$ is again an orthonormal basis. In particular, this invariance implies $r_{11'} = \mathrm{Ric}(e_1, e_{1'}) = r_{1'1} = -r_{11'}$, so $r_{11'}$ vanish (and so does $r_{22'}$). For the Kulkarni–Nomizu product,

$$\begin{aligned}
\mathrm{Ric} \owedge g(\mathrm{J}X, Y, Z, T) =& \mathrm{Ric}(\mathrm{J}X, Z)g(Y, T) + \mathrm{Ric}(Y, T)g(\mathrm{J}X, Z) \\
& - \mathrm{Ric}(\mathrm{J}X, T)g(Y, Z) - \mathrm{Ric}(Y, Z)g(\mathrm{J}X, T) \\
=& - \mathrm{Ric}(X, \mathrm{J}Z)g(\mathrm{J}Y, \mathrm{J}T) - \mathrm{Ric}(\mathrm{J}Y, \mathrm{J}T)g(X, \mathrm{J}Z) \\
& + \mathrm{Ric}(X, \mathrm{J}T)g(\mathrm{J}Y, \mathrm{J}Z) + \mathrm{Ric}(\mathrm{J}Y, \mathrm{J}Z)g(X, \mathrm{J}T) \\
=& - \mathrm{Ric} \owedge g(X, \mathrm{J}Y, \mathrm{J}Z, \mathrm{J}T)
\end{aligned}$$

so

$$\mathrm{Ric} \owedge g(\mathrm{J}X, \mathrm{J}Y, Z, T) = -\mathrm{Ric} \owedge g(X, \mathrm{J}^2 Y, \mathrm{J}Z, \mathrm{J}T) = \mathrm{Ric} \owedge g(X, Y, \mathrm{J}Z, \mathrm{J}T)\,.$$

Hence the following symmetries (fewer than for Rm) in the coefficients of $\mathrm{Ric} \owedge g$,

$g \owedge g$ and Rm, and therefore W:

	$e_{11'}$	$e_1 \wedge e_2$	$e_{12'}$	$e_{1'} \wedge e_2$	$e_{1'2'}$	$e_{22'}$
$e_{11'}$	$\mathrm{W}_{11'11'}$	$\mathrm{W}_{11'12}$	$\mathrm{W}_{11'12'}$	$\mathrm{W}_{11'1'2} = -\mathrm{W}_{11'12'}$	$\mathrm{W}_{11'1'2'} = \mathrm{W}_{11'12}$	$\mathrm{W}_{11'22'}$
e_{12}		W_{1212}	$\mathrm{W}_{1212'}$	$\mathrm{W}_{121'2}$	$\mathrm{W}_{121'2'}$	$\mathrm{W}_{1222'}$
$e_{12'}$			$\mathrm{W}_{12'12'}$	$\mathrm{W}_{12'1'2}$	$\mathrm{W}_{12'1'2'} = -\mathrm{W}_{121'2}$	$\mathrm{W}_{12'22'}$
$e_{1'2}$				$\mathrm{W}_{1'21'2} = \mathrm{W}_{12'12'}$	$\mathrm{W}_{1'21'2'} = -\mathrm{W}_{1212'}$	$\mathrm{W}_{1'222'} = -\mathrm{W}_{12'22'}$
$e_{1'2'}$					$\mathrm{W}_{1'2'1'2'} = \mathrm{W}_{1212}$	$\mathrm{W}_{1'2'22'} = \mathrm{W}_{1222'}$
$e_{22'}$						$\mathrm{W}_{22'22'}$

Expanding on the above eigenbasis of $\Lambda^+ \oplus \Lambda^-$ (which differs from the one in the positive definite case) yields the following Weyl tensor coefficients, which we have simplified using the symmetries above (up to a factor 1/2 due to normalization):

	E_1^+	E_2^+	E_3^+
E_1^+	$\mathrm{W}_{11'11'} + \mathrm{W}_{22'22'} + 2\mathrm{W}_{11'22'}$	$2(\mathrm{W}_{11'12} + \mathrm{W}_{1222'})$	$2(\mathrm{W}_{11'12'} + \mathrm{W}_{12'22'})$
E_2^+		$2(\mathrm{W}_{1212} + \mathrm{W}_{121'2'})$	$2(\mathrm{W}_{1212'} - \mathrm{W}_{121'2})$
E_3^+			$2(\mathrm{W}_{12'12'} - \mathrm{W}_{12'1'2})$
E_1^-			
E_2^-			
E_3^-			

	E_1^-	E_2^-	E_3^-
E_1^+	$\mathrm{W}_{11'11'} - \mathrm{W}_{22'22'}$	0	0
E_2^+	$2(\mathrm{W}_{11'12} - \mathrm{W}_{1222'})$	0	0
E_3^+	$2(\mathrm{W}_{11'12'} - \mathrm{W}_{12'22'})$	0	0
E_1^-	$\mathrm{W}_{11'11'} + \mathrm{W}_{22'22'} - 2\mathrm{W}_{11'22'}$	0	0
E_2^-		$2(\mathrm{W}_{1212} - \mathrm{W}_{121'2'})$	$2(\mathrm{W}_{1212'} + \mathrm{W}_{121'2})$
E_3^-			$2(\mathrm{W}_{12'12'} + \mathrm{W}_{12'1'2})$

(Again only half the coefficients are written down.) Further simplifications come from computing W, and using

$$
\begin{aligned}
\mathrm{Scal} &= -r_{11} - r_{1'1'} + r_{22} + r_{2'2'} = 2(r_{22} - r_{11}) \\
&= 2(-(-\mathrm{R}_{11'11'} + \mathrm{R}_{1212} + \mathrm{R}_{12'12'}) + (-\mathrm{R}_{1212} - \mathrm{R}_{1'21'2} + \mathrm{R}_{22'22'})) \\
&= 2(\mathrm{R}_{11'11'} - 2(\mathrm{R}_{1212} + \mathrm{R}_{12'12'}) + \mathrm{R}_{22'22'})\,.
\end{aligned}
$$

First prove that the Hodge star commutes with W by considering $\mathrm{W}(\Lambda^+, \Lambda^-)$:

$$
\begin{aligned}
\mathrm{W}_{11'11'} &= \mathrm{R}_{11'11'} + \frac{1}{2}(r_{11} + r_{1'1'}) + \frac{\mathrm{Scal}}{6} = \mathrm{R}_{11'11'} + r_{11} + \frac{\mathrm{Scal}}{6} \\
&= \mathrm{R}_{1212} + \mathrm{R}_{12'12'} + \frac{\mathrm{Scal}}{6}
\end{aligned}
$$

$$\begin{aligned}
\mathrm{W}_{22'22'} &= \mathrm{R}_{22'22'} - \frac{1}{2}(r_{22} + r_{2'2'}) + \frac{\mathrm{Scal}}{6} = \mathrm{R}_{22'22'} - r_{22} + \frac{\mathrm{Scal}}{6} \\
&= \mathrm{R}_{1212} + \mathrm{R}_{12'12'} + \frac{\mathrm{Scal}}{6}
\end{aligned}$$

so that $\mathrm{W}_{11'11'} - \mathrm{W}_{22'22'} = 0$. Similarly

$$\mathrm{W}_{11'12} = \mathrm{R}_{11'12} + \frac{r_{1'2}}{2}, \quad \mathrm{W}_{1222'} = \mathrm{R}_{1222'} + \frac{r_{12'}}{2} = \mathrm{R}_{1222'} - \frac{r_{1'2}}{2}$$

so

$$\mathrm{W}_{11'12} - \mathrm{W}_{1222'} = \mathrm{R}_{11'12} - \mathrm{R}_{1222'} + r_{1'2} = 0$$

$$\mathrm{W}_{11'12'} = \mathrm{R}_{11'12'} + \frac{r_{1'2'}}{2} = \mathrm{R}_{11'12'} + \frac{r_{12}}{2}, \quad \mathrm{W}_{12'22'} = \mathrm{R}_{12'22'} - \frac{r_{12}}{2},$$

$$\mathrm{W}_{11'12'} - \mathrm{W}_{12'22'} = \mathrm{R}_{11'12'} - \mathrm{R}_{12'22'} + r_{12} = 0.$$

That proves that W is block-diagonal.

The W^- term satisfies

$$\begin{aligned}
\mathrm{W}_{11'11'} + \mathrm{W}_{22'22'} - 2\mathrm{W}_{11'22'} &= \mathrm{R}_{11'11'} + r_{11} + \mathrm{R}_{22'22'} - r_{22} + \frac{\mathrm{Scal}}{3} \\
&\quad -2\mathrm{R}_{11'22'} \\
&= \mathrm{R}_{11'11'} + \mathrm{R}_{22'22'} - 2\mathrm{R}_{11'22'} - \frac{\mathrm{Scal}}{6} \\
&= \mathrm{R}_{11'11'} + \mathrm{R}_{22'22'} - 2(\mathrm{R}_{1212} + \mathrm{R}_{12'12'}) \\
&\quad -\frac{\mathrm{Scal}}{6} \\
&= \frac{\mathrm{Scal}}{2} - \frac{\mathrm{Scal}}{6} = \frac{\mathrm{Scal}}{3}
\end{aligned}$$

using the first Bianchi identity (and the invariance of Rm):

$$\mathrm{R}_{11'22'} = -\mathrm{R}_{1'212'} - \mathrm{R}_{211'2'} = \mathrm{R}_{12'12'} + \mathrm{R}_{1212}.$$

$$\begin{aligned}
\mathrm{W}_{1212} - \mathrm{W}_{121'2'} &= \mathrm{R}_{1212} + \frac{r_{22} - r_{11}}{2} - \frac{\mathrm{Scal}}{6} - \mathrm{R}_{121'2'} = \frac{\mathrm{Scal}}{4} - \frac{\mathrm{Scal}}{6} \\
&= \frac{\mathrm{Scal}}{12}
\end{aligned}$$

$$\mathrm{W}_{12'12'} + \mathrm{W}_{12'1'2} = \mathrm{R}_{12'12'} + \frac{\mathrm{Scal}}{4} - \frac{\mathrm{Scal}}{6} + \mathrm{R}_{12'1'2} = \frac{\mathrm{Scal}}{12}$$

$$\mathrm{W}_{1212'} + \mathrm{W}_{121'2} = \mathrm{R}_{1212'} + \frac{r_{22'}}{2} + \mathrm{R}_{121'2} - \frac{r_{11'}}{2} = \frac{1}{2}(r_{22'} - r_{11'}) = 0.$$

Finally,

$$\mathrm{W}^- = \mathrm{Scal}\begin{pmatrix} 1/3 & & \\ & 1/6 & \\ & & 1/6 \end{pmatrix} = \frac{\mathrm{Scal}}{6}\mathrm{Id} + \frac{\mathrm{Scal}}{6} E_1^- \otimes E_1^-$$

(and indeed this matrix is traceless w.r.t. the pseudo-metric g). One should note that the above expression differs from the Riemannian case, where

$$\mathrm{W}^+ = \mathrm{Scal}\begin{pmatrix} 1/3 & & \\ & -1/6 & \\ & & -1/6 \end{pmatrix} = -\frac{\mathrm{Scal}}{6}\mathrm{Id} + \frac{\mathrm{Scal}}{3}E_1^+ \otimes E_1^+ .$$

We let the Reader check that in the neutral case, the W^+ part is not a multiple of the scalar curvature, which completes the proof of Theorem A.2.

A.2 The para-Kähler case

The computations are almost identical, but the results differ from the pseudo-Kähler setup, because the para-complex structure J is now an anti-isometry: $\mathrm{R}(\mathrm{J}X, \mathrm{J}Y)Z = -\mathrm{R}(X,Y)Z$. We pick an orthonormal basis $(e_1, e_{1'}, e_2, e_{2'})$ with $e_{1'} = \mathrm{J}e_1$, $e_{2'} = \mathrm{J}e_2$, and $g(e_1) = g(e_2) = +1$, $g(e_{1'}) = g(e_{2'}) = -1$. The frame $\mathcal{B} = (e_1\wedge e_{1'}, e_1\wedge e_2, e_1\wedge e_{2'}, e_{1'}\wedge e_2, e_{1'}\wedge e_{2'}, e_2\wedge e_{2'})$ of $\Lambda^2 T\mathcal{N}$ is also orthonormal w.r.t. the induced metric on Λ^2, again denoted by g, which has signature $(2,4)$: $g(e_a \wedge e_b) = g(e_a)g(e_b) = -1$, except for $g(e_1 \wedge e_2) = g(e_{1'} \wedge e_{2'}) = +1$.

An orthonormal eigenbasis for the Hodge operator is the following:

$$\begin{cases} E_1^\pm = \frac{\sqrt{2}}{2}(e_1 \wedge e_{1'} \mp e_2 \wedge e_{2'}) \\ E_2^\pm = \frac{\sqrt{2}}{2}(e_1 \wedge e_2 \mp e_{1'} \wedge e_{2'}) \\ E_3^\pm = \frac{\sqrt{2}}{2}(e_1 \wedge e_{2'} \mp e_{1'} \wedge e_2) \end{cases}$$

where the E_a^+ (resp. E_a^-) span Λ^+ (resp. Λ^-). (Note the sign differences w.r.t. the pseudo-Kähler case.)

Since J is anti-isometric and parallel,

$$\mathrm{Rm}(\mathrm{J}X, \mathrm{J}Y, Z, T) = -\mathrm{Rm}(X, Y, Z, T) = \mathrm{Rm}(X, Y, \mathrm{J}Z, \mathrm{J}T)\,.$$

Hence the following symmetries of the riemannian curvature operator R, expressed in the frame $\mathcal{B}$ (for symmetry reasons and greater legibility, lower left coefficients are not written in this and the subsequent matrices):

	$e_{11'}$	e_{12}	$e_{12'}$	$e_{1'} \wedge e_2$	$e_{1'2'}$	$e_{22'}$
$e_{11'}$	$\mathrm{R}_{11'11'}$	$\mathrm{R}_{11'12}$	$\mathrm{R}_{11'12'}$	$\mathrm{R}_{11'1'2}$ $= -\mathrm{R}_{11'12'}$	$\mathrm{R}_{11'1'2'}$ $= -\mathrm{R}_{11'12}$	$\mathrm{R}_{11'22'}$
e_{12}		R_{1212}	$\mathrm{R}_{1212'}$	$\mathrm{R}_{121'2}$ $= -\mathrm{R}_{1212'}$	$\mathrm{R}_{121'2'}$ $= -\mathrm{R}_{1212}$	$\mathrm{R}_{1222'}$
$e_{12'}$			$\mathrm{R}_{12'12'}$	$\mathrm{R}_{12'1'2}$ $= -\mathrm{R}_{12'12'}$	$\mathrm{R}_{12'1'2'}$ $= -\mathrm{R}_{1212'}$	$\mathrm{R}_{12'22'}$
$e_{1'2}$				$\mathrm{R}_{1'21'2}$ $= \mathrm{R}_{12'12'}$	$\mathrm{R}_{1'21'2'}$ $= \mathrm{R}_{1212'}$	$\mathrm{R}_{1'222'}$ $= -\mathrm{R}_{12'22'}$
$e_{1'2'}$					$\mathrm{R}_{1'2'1'2'}$ $= \mathrm{R}_{1212}$	$\mathrm{R}_{1'2'22'}$ $= -\mathrm{R}_{1222'}$
$e_{22'}$						$\mathrm{R}_{22'22'}$

(Note again the similarity with the pseudo-Kähler case: only a few signs change.)

The Weyl tensor satisfies *some* of the J-symmetries of Rm since

$$\begin{aligned}\mathrm{Ric}(\mathrm{J}X,\mathrm{J}Y) &= \sum_{i=1}^{4} g(e_i)\mathrm{Rm}(\mathrm{J}X,e_i,\mathrm{J}Y,e_i) = \sum_{i=1}^{4} g(e_i)\mathrm{Rm}(X,\mathrm{J}e_i,Y,\mathrm{J}e_i) \\ &= -\sum_{i=1}^{4} g(\mathrm{J}e_i)\mathrm{Rm}(X,\mathrm{J}e_i,Y,\mathrm{J}e_i) = -\mathrm{Ric}(X,Y)\end{aligned}$$

since $(\mathrm{J}e_i)$ is also an orthonormal basis. In particular this invariance implies $r_{1'1} = r_{11'} = -r_{1'1}$, so $r_{11'}$ vanishes (and so does $r_{22'}$). Finally,

$$\frac{\mathrm{Scal}}{2} = r_{11} + r_{22} = -\mathrm{R}_{11'11'} + 2(\mathrm{R}_{1212} - \mathrm{R}_{12'12'}) - \mathrm{R}_{22'22'}.$$

The Kulkarni–Nomizu product $\mathrm{Ric} \owedge g$ satisfies

$$\begin{aligned}\mathrm{Ric} \owedge g(\mathrm{J}X,Y,Z,T) &= \mathrm{Ric}(\mathrm{J}X,Z)g(Y,T) + \mathrm{Ric}(Y,T)g(\mathrm{J}X,Z) \\ &\quad -\mathrm{Ric}(\mathrm{J}X,T)g(Y,Z) - \mathrm{Ric}(Y,Z)g(\mathrm{J}X,T) \\ &= \mathrm{Ric}(X,\mathrm{J}Z)g(\mathrm{J}Y,\mathrm{J}T) + \mathrm{Ric}(\mathrm{J}Y,\mathrm{J}T)g(X,\mathrm{J}Z) \\ &\quad -\mathrm{Ric}(X,\mathrm{J}T)g(\mathrm{J}Y,\mathrm{J}Z) - \mathrm{Ric}(\mathrm{J}Y,\mathrm{J}Z)g(X,\mathrm{J}T) \\ &= \mathrm{Ric} \owedge g(X,\mathrm{J}Y,\mathrm{J}Z,\mathrm{J}T)\end{aligned}$$

so

$$\mathrm{Ric} \owedge g(\mathrm{J}X,\mathrm{J}Y,Z,T) = \mathrm{Ric} \owedge g(X,\mathrm{J}^2Y,\mathrm{J}Z,\mathrm{J}T) = \mathrm{Ric} \owedge g(X,Y,\mathrm{J}Z,\mathrm{J}T)$$

and the same property holds for $g \owedge g$. Hence the following symmetries (fewer than for Rm) in the coefficients of $\mathrm{Ric} \owedge g$, $g \owedge g$ and Rm, and therefore W:

	$e_{11'}$	e_{12}	$e_{12'}$	$e_{1'2}$	$e_{1'2'}$	$e_{22'}$
$e_{11'}$	$\mathrm{W}_{11'11'}$	$\mathrm{W}_{11'12}$	$\mathrm{W}_{11'12'}$	$\mathrm{W}_{11'1'2} = -\mathrm{W}_{11'12'}$	$\mathrm{W}_{11'1'2'} = -\mathrm{W}_{11'12}$	$\mathrm{W}_{11'22'}$
e_{12}		W_{1212}	$\mathrm{W}_{1212'}$	$\mathrm{W}_{121'2}$	$\mathrm{W}_{121'2'}$	$\mathrm{W}_{1222'}$
$e_{12'}$			$\mathrm{W}_{12'12'}$	$\mathrm{W}_{12'1'2}$	$\mathrm{W}_{12'1'2'} = \mathrm{W}_{121'2}$	$\mathrm{W}_{12'22'}$
$e_{1'2}$				$\mathrm{W}_{1'21'2} = \mathrm{W}_{12'12'}$	$\mathrm{W}_{1'21'2'} = \mathrm{W}_{1212'}$	$\mathrm{W}_{1'222'} = -\mathrm{W}_{12'22'}$
$e_{1'2'}$					$\mathrm{W}_{1'2'1'2'} = \mathrm{W}_{1212}$	$\mathrm{W}_{1'2'22'} = -\mathrm{W}_{1222'}$
$e_{22'}$						$\mathrm{W}_{22'22'}$

Let us now express W in the Hodge basis defined earlier, using the above symmetries (up to a factor 1/2 due to normalization).

	E_1^+	E_2^+	E_3^+
E_1^+	$\mathrm{W}_{11'11'}+\mathrm{W}_{22'22'}-2\mathrm{W}_{11'22'}$	$2(\mathrm{W}_{11'12}-\mathrm{W}_{1222'})$	$2(\mathrm{W}_{11'12'}-\mathrm{W}_{12'22'})$
E_2^+		$2(\mathrm{W}_{1212}-\mathrm{W}_{121'2'})$	$2(\mathrm{W}_{1212'}-\mathrm{W}_{121'2})$
E_3^+			$2(\mathrm{W}_{12'12'}-\mathrm{W}_{12'1'2})$
E_1^-			
E_2^-			
E_3^-			

	E_1^-	E_2^-	E_3^-
E_1^+	$\mathrm{W}_{11'11'}-\mathrm{W}_{22'22'}$	0	0
E_2^+	$2(\mathrm{W}_{11'12}+\mathrm{W}_{1222'})$	0	0
E_3^+	$2(\mathrm{W}_{11'12'}+\mathrm{W}_{12'22'})$	0	0
E_1^-	$\mathrm{W}_{11'11'}+\mathrm{W}_{22'22'}$ $+2\mathrm{W}_{11'22'}$	0	0
E_2^-		$2(\mathrm{W}_{1212}+\mathrm{W}_{121'2'})$	$2(\mathrm{W}_{1212'}+\mathrm{W}_{121'2})$
E_3^-			$2(\mathrm{W}_{12'12'}+\mathrm{W}_{12'1'2})$

Only three terms in the off-block-diagonal part are not obviously zero.

$$\mathrm{W}_{11'11'} \quad = \quad \mathrm{R}_{11'11'}-\frac{1}{2}(-r_{11}+r_{1'1'})-\frac{\mathrm{Scal}}{6}=\mathrm{R}_{11'11'}+r_{11}-\frac{\mathrm{Scal}}{6}$$

$$\mathrm{W}_{22'22'}=\mathrm{R}_{22'22'}-\frac{1}{2}(-r_{22}+r_{2'2'})-\frac{\mathrm{Scal}}{6}=\mathrm{R}_{22'22'}+r_{22}-\frac{\mathrm{Scal}}{6}$$

but $r_{11}=-\mathrm{R}_{11'11'}+\mathrm{R}_{1212}-\mathrm{R}_{12'12'}$ and $r_{22}=\mathrm{R}_{2121}-\mathrm{R}_{21'21'}-\mathrm{R}_{22'22'}=\mathrm{R}_{1212}-\mathrm{R}_{12'12'}-\mathrm{R}_{22'22'}$ so that

$$\mathrm{W}_{11'11'}-\mathrm{W}_{22'22'}=\mathrm{R}_{11'11'}-\mathrm{R}_{22'22'}+r_{11}-r_{22}=0.$$

Similarly

$$\mathrm{W}_{11'12}+\mathrm{W}_{1222'}=\mathrm{R}_{11'12}-\frac{r_{1'2}}{2}+\mathrm{R}_{1222'}+\frac{r_{12'}}{2}=\mathrm{R}_{11'12}+\mathrm{R}_{1222'}-r_{1'2}=0$$

$$\mathrm{W}_{11'12'}+\mathrm{W}_{12'22'}=\mathrm{R}_{11'12'}-\frac{r_{1'2'}}{2}+\mathrm{R}_{12'22'}+\frac{r_{12}}{2}=\mathrm{R}_{11'12'}+\mathrm{R}_{12'22'}+r_{12}=0$$

which proves that W is block-diagonal, i.e. commutes with the Hodge operator.

Let us now look more closely at the W^- term

$$\begin{pmatrix} \mathrm{W}_{11'11'}+\mathrm{W}_{22'22'}+2\mathrm{W}_{11'22'} & 0 & 0 \\ & 2(\mathrm{W}_{1212}+\mathrm{W}_{121'2'}) & 2(\mathrm{W}_{1212'}+\mathrm{W}_{121'2}) \\ & & 2(\mathrm{W}_{12'12'}+\mathrm{W}_{12'1'2}) \end{pmatrix}$$

$$
\begin{aligned}
&\mathrm{W}_{11'11'} + \mathrm{W}_{22'22'} + 2\mathrm{W}_{11'22'} \\
&\qquad = \mathrm{R}_{11'11'} + r_{11} - \frac{\mathrm{Scal}}{6} + \mathrm{R}_{22'22'} + r_{22} - \frac{\mathrm{Scal}}{6} + 2\mathrm{R}_{11'22'} \\
&\qquad = \mathrm{R}_{11'11'} + \mathrm{R}_{22'22'} + 2\mathrm{R}_{11'22'} + \frac{\mathrm{Scal}}{2} - \frac{\mathrm{Scal}}{3} \\
&\qquad = \mathrm{R}_{11'11'} + \mathrm{R}_{22'22'} + 2(-\mathrm{R}_{1212} + \mathrm{R}_{12'12'}) + \frac{\mathrm{Scal}}{6} = -\frac{\mathrm{Scal}}{3}
\end{aligned}
$$

where we have used the first Bianchi identity (and the invariance of Rm)

$$\mathrm{R}_{11'22'} = -\mathrm{R}_{1'212'} - \mathrm{R}_{211'2'} = \mathrm{R}_{12'12'} - \mathrm{R}_{1212}.$$

$$
\begin{aligned}
\mathrm{W}_{1212} + \mathrm{W}_{121'2'} &= \mathrm{R}_{1212} - \frac{r_{22} + r_{11}}{2} + \frac{\mathrm{Scal}}{6} + \mathrm{R}_{121'2'} \\
&= \mathrm{R}_{1212} - \frac{\mathrm{Scal}}{4} + \frac{\mathrm{Scal}}{6} + \mathrm{R}_{121'2'} = -\frac{\mathrm{Scal}}{12}
\end{aligned}
$$

$$
\mathrm{W}_{12'12'} + \mathrm{W}_{12'1'2} = \mathrm{R}_{12'12'} + \frac{\mathrm{Scal}}{4} - \frac{\mathrm{Scal}}{6} + \mathrm{R}_{12'1'2} = \frac{\mathrm{Scal}}{12}
$$

$$
\mathrm{W}_{1212'} + \mathrm{W}_{121'2} = \mathrm{R}_{1212'} - \frac{r_{22'}}{2} + \mathrm{R}_{121'2} - \frac{r_{11'}}{2} = 0.
$$

Finally,

$$
\mathrm{W}^- = \mathrm{Scal} \begin{pmatrix} -1/3 & & \\ & -1/6 & \\ & & 1/6 \end{pmatrix}
$$

vanishes if and only if Scal $= 0$. (The Reader will check that this matrix is indeed traceless w.r.t. the pseudo-metric g.)

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