Bayesian Inference for Partially Observed Branching Processes

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Supplementary material

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A Proof of identifiability

We prove identifiability in the special case corresponding to $n_{obs} = 0$ and $X(\tau_0)$ unobserved which is the least favorable case. Let $\theta = (\nu_k, k = 0, \ldots, K)$ and $\theta' = (\nu'_k, k = 0, \ldots, K)$ and consider any distributions π and π' on $X(\tau_0)$, possibly depending on θ and θ' . For $N(\tau) = 0$, integrating out $X(\tau_0)$, we obtain

$$\begin{aligned} \mathcal{L}(N(\tau), T_1, \ldots, T_{N(\tau)}; \pi, \theta) &= \mathcal{L}(N(\tau), X(\tau_0), T_1, \ldots, T_{N(\tau)}, \pi', \theta') \quad \Leftrightarrow \\ e^{-\mu\tau} E_\pi[e^{-\nu_0 X(\tau_0)\tau}] &= e^{-\mu'\tau} E_{\pi'}[e^{-\nu'_0 X(\tau_0)\tau}] \end{aligned} \tag{1}$$

Similarly considering $N(\tau) = 1$ and integrating out $X(\tau_0)$ and Z_1 we obtain

$$\mathcal{L}(T_1; \theta, \pi) \;=\; \nu_0 e^{-\nu_0 j_0(\tau+\tau_0-T_1)} e^{-\mu\tau} E_\pi[X(\tau_0) e^{-\nu_0 X(\tau_0)\tau}] + e^{-\mu\tau} E_\pi[e^{-\nu_0\tau X(\tau_0)}] \textstyle\sum_{k=1}^K e^{-\nu_0(\tau+\tau_0-T_1) j_k} \nu_k$$

Then $\mathcal{L}(T_1; \theta, \pi) = \mathcal{L}(T_1; \theta', \pi_{\theta'})$ for all $T_1 \in [\tau_0, \tau_0+\tau]$ leads to the identifiability of the exponents $\nu_0 j_l$ for $l = 0, \cdots, K$ since the j_k's are all different (see condition $\mathcal{C}$) so that $\nu_0 = \nu'_0$ and $\nu_k = \nu'_k$ for $k \geq 1$. This proves identifiability of the coefficients ν_k even though the prior on $X(\tau_0)$ may be miss-specified.

Identifiability in the cases where $X(\tau_0)$ and/or $\mathbf{Z}$ are observed is deduced directly from the previous result.

B Asymptotic study of $(X(t))_{t \geq 0}$: proof of Proposition 4 and Theorem 1

In Proposition 4 we give an integral expression of the generating function of the j_0-Yule process with multi-size immigration described in Section 2 of the paper. More precisely, let $X(t)$ by a branching process such that each particle gives birth to j_0 particles at random time distributed with $\mathcal{E}(\nu_0)$ and such that groups of j_k immigrants arrive at random times distributed with $\mathcal{E}(\nu_k)$, for $k = 1 \dots K$. We assume that $X(0) = x_0$.

B.1 Proof of Proposition 4

The proof of Proposition 4 can be divided into two parts. Indeed, from one part, the x_0 particles present at time 0 will give birth to x_0 independent j_0-Yule process (without immigration). Once the processes derived from those particles have been taken into account, we can reduce the study to a j_0-Yule process with multi-size immigration starting from 0 particles.

In lemma 1, we recall the expression of the generating function of a j_0-Yule process starting with one particle and study its asymptotic distribution. The generating function of a j_0-Yule process with multi-size immigration starting with $X(0) = 0$ particle is then detailed. The proof is similar to the one given in (1) but adapted to our particular case. In the particular case where the sizes of the immigration groups are proportional to j_0 we derive a explicit expression of the limit distribution in subsection B.2.

Lemma 1. *Let $Y(t)$ be a branching process starting with $Y(0)$ particle such that each particle gives birth to j_0 particles at interval time distributed with an exponential distribution of parameter ν_0. Then we have $\Psi(s,t) = E[s^{Y(t)}] = \left[1 - (1 - s^{-j_0})\exp(\nu_0 j_0 t)\right]^{-Y(0)/j_0}$. Moreover,*

$$\lim_{t\to\infty} e^{-j_0\nu_0 t} Y(t) = \Gamma\left(\frac{Y(0)}{j_0}, \frac{1}{j_0}\right) \quad (\mathcal{L})$$

Proof of Lemma 1 We first assume that $Y(0) = 1$. By construction of the process $Y(t)$,

$Q_{i,j}(h) = P(Y(t+h) = j|Y(t) = i)$ only depends on h, i, j. When h is small, $Q_{i,j}(h)$ verifies:

$$Q_{i,j}(h) = \begin{cases} \nu_0 ih + o(h) & \text{if} \quad j = i + j_0 \\ 1 - \nu_0 ih + o(h) & \text{if} \quad j = i \\ o(h) & \text{if} \quad j \notin \{i, i + j_0\} \end{cases} \tag{2}$$

We now derive a partial differential equation fulfilled by the probability-generating function. Using the fact that $Y(t)$ take its values in $\{j_0 k + 1, k \in \mathbb{N}\}$ we have:

$$\Psi(s,t) \quad = \quad E[s^{Y(t)}] = \sum_{k \in \mathbb{N}} P(Y(t) = j_0 k + 1 | Y(0) = 1) s^{j_0 k + 1} = \sum_{k \in \mathbb{N}} Q_{1, j_0 k + 1}(t) s^{j_0 k + 1} \tag{3}$$

Using a backward-equation we derive an expression for $Q_{1,j_0 k+1}(t) = P(Y(t) = j_0 k + 1)$. Indeed,

$$Q_{1,j_0 k+1}(t+h) = P(Y(t+h) = j_0 k+1 | Y(0) = 1) = Q_{1,j_0 k+1}(t)(1-\nu_0 h) + Q_{j_0+1, j_0 k+1}(t)\nu_0 h + o(h)$$

using (2). We directly obtain the following ODE:

$$Q'_{1,j_0 k+1}(t) = \lim_{h \to 0} \frac{Q_{1,j_0 k+1}(t+h) - Q_{1,j_0 k+1}(t)}{h} = -\nu_0 Q_{1,j_0 k+1}(t) + \nu_0 Q_{j_0+1, j_0 k+1}(t) \tag{4}$$

We now derive from (4) and (3) a partial differential equation for $\Psi(s,t)$

$$\begin{aligned} \frac{\partial}{\partial t}\Psi(s,t) \quad &= \quad s \sum_{k \in \mathbb{N}} Q'_{1,j_0 k+1}(t)(s^{j_0})^k = \sum_{k=1}^{\infty} s^{j_0 k+1} \left[-\nu_0 Q_{1,j_0 k+1}(t) + \nu_0 Q_{j_0+1, j_0 k+1}(t) \right] \\ &= \quad -\nu_0 \Psi(s,t) + \nu_0 \sum_{k=1}^{\infty} s^{j_0 k+1} Q_{j_0+1, j_0 k+1}(t) \end{aligned} \tag{5}$$

$\sum_{k=1}^{\infty} s^{j_0 k+1} Q_{j_0+1, j_0 k+1}(t)$ is the probability-generating function of a process with the same division mechanism as $Y(t)$ by which would start with $j_0 + 1$ particles. By properties inherent to branching processes, this is equivalent to the sum of $j_0 + 1$ independent processes starting with one particle. As a consequence, let $Y_1, \ldots Y_{j_0+1}$ by independent processes distributed as Z, we have:

$$\sum_{k=1}^{\infty} s^{j_0k+1}Q_{j_0+1,j_0k+1}(t) = E[s^{Y_1(t)+\cdots+Y_{j_0+1}(t)}] = E[s^{Y(t)}]^{j_0+1} = \Psi(s,t)^{j_0+1}$$

As a consequence the partial differential equation (5) becomes : $\frac{\partial}{\partial t}\Psi(s,t) = \nu_0\Psi(s,t)(\Psi^{j_0}(s,t)-1)$ with $\Psi(s,0) = s$. The solution of this equation is :

$$\Psi(s,t) = \left[1 - e^{\nu_0 j_0 t}(1 - s^{-j_0})\right]^{-1/j_0}$$

We now study the asymptotic distribution of $\tilde{Z}(t) = e^{-\nu_0 j_0 t}Y(t)$ through its moments-generating function :

$$\Phi_{\tilde{Z}(t)}(\theta) = E[e^{\theta\tilde{Z}(t)}] = E[e^{\theta e^{-\nu_0 j_0 t}Y(t)}] = E[(e^{\theta e^{-\nu_0 j_0 t}})^{Y(t)}] = E[s_t^{Y(t)}] = \Psi(s_t,t)$$

where $s_t = e^{\theta e^{-\nu_0 j_0 t}} \simeq_{t\to\infty} 1 + \theta e^{-\nu_0 j_0 t}$. We easily obtain the following limit

$$\lim_{t\to\infty}\Phi_{\tilde{Z}(t)}(\theta) = \frac{1}{(1-j_0\theta)^{1/j_0}}$$

and recognize the moment generating function of the $\Gamma\left(\frac{1}{j_0},\frac{1}{j_0}\right)$.

Now, if the process starts with $Y(0) = X(0)$ particles, each of them initiates a j_0-Yule process which is independent of the other ones, leading to :

$$\Psi(s,t) = \left[1 - (1 - s^{-j_0})\exp(\nu_0 j_0 t)\right]^{-X(0)/j_0}$$

and

$$\lim_{t\to\infty} e^{-j_0\nu_0 t}Y(t) = \Gamma\left(\frac{X(0)}{j_0},\frac{1}{j_0}\right) \quad (\mathcal{L})$$

We now use Lemma 1 to study the distribution of $X(t)$, the number of particles issued from the multi-immigration j_0- Yule process described in the paper. We first assume that $X(0) = 0$. Let $\phi(s,t)$ denote the probability-generating function function of $(X(t))_{t\geq 0}$

$$\phi(s,t) = E[s^{X(t)}] = \sum_{n=0}^{\infty} P_n(t) s^n$$

where $P_n(t) = P(X(t) = n)$ is the probability to have n particles at time t. This probability can be decomposed into :

$$P_n(t) = P_{n|0}(t) m_0(t) + \sum_{k=1}^{\infty} P_{n|k}(t) m_k(t) \tag{6}$$

where $m_k(t)$ is the probability that k immigration groups arrived in the time interval $[0,t)$ and $P_{n|k}(t)$ denotes the probability there are n particles at time t given that k immigration groups arrived during $[0,t)$. Moreover $P_{n|0}(t) = \delta_{n0}$ because $X(0) = 0$.

Using the independence of the immigration events, $P_{n|k}(t)$ can also be decomposed as :

$$P_{n|k}(t) = \sum_{i_1+\cdots+i_k=n} U_{i_1}(t) \ldots U_{i_k}(t) \tag{7}$$

where $U_m(t)$ denotes the probability that an immigration group leads (by the branching mechanism) to m particles at time t given that the group immigrates during the interval $[0,t)$. Combining (7) and (6), we can rewrite the probability-generating function $\phi(s,t)$:

$$\begin{aligned} \phi(s,t) &= \sum_{n=0}^{\infty} P_n(t) s^n = m_0(t) + \sum_{n=0}^{\infty} s^n \sum_{k=1}^{\infty} m_k(t) \sum_{i_1+\cdots+i_k=n} U_{i_1}(t) \ldots U_{i_k}(t) \\ &= \sum_{k=0}^{\infty} m_k(t) \left(\sum_{n=0}^{\infty} s^n U_n(t) \right)^k \end{aligned}$$

Denoting $J(s,t) = \sum_{n=0}^{\infty} s^n U_n(t)$, we obtain:

$$\phi(s,t) = \sum_{k=0}^{\infty} m_k(t) J^k(s,t) \tag{8}$$

where $m_k(t)$ is the probability that k immigration groups arrived in the time interval $[0,t)$. Using the Poisson properties of our immigration process we have : $m_k(t) = e^{-\mu t} \frac{(\mu t)^k}{k!}$ with

$\mu = \nu_1 + \cdots + \nu_K$ and so by (8):

$$\phi(s,t) \quad = \quad \sum_{k=0}^{\infty} e^{-\mu t} \frac{(\mu t)^k}{k!} J^k(s,t) = e^{-\mu t} \exp\{\mu t J(s,t)\} \tag{9}$$

We now compute $J(s,t)$ and to that purpose we first study $U_n(t)$. Recall that $U_n(t)$ is the probability that an immigration group leads (by the branching mechanism) to n particles at time t given that the group immigrates during the interval $[0,t)$. As a consequence, using the infinitesimal probabilities, $U_n(t)$ can be decomposed into :

$$U_n(t) = \int_0^t \sum_{k=1}^{K} r_k(u) Q(n,t|j_k,u) d_u N(u|t) \tag{10}$$

where

• $d_u N(u|t)$ is the conditional infinitesimal immigration rate i.e. the probability that there is exactly one immigration group during the infinitesimal interval $[u, u+du) \subset [0,t)$ given there is exactly one immigration group in the interval $[0,t)$. In the case of a Poisson process, $d_u N(u|t) = \frac{du}{t}$

• $r_k(u)$ is the probablity that the immigration group is of size j_k given that it arrived at time u, $k = 1, \ldots, K$. Using the Poisson properties of our immigration process we have :

$$r_k(u) = \frac{\nu_k}{\nu_1 + \cdots + \nu_K} = \frac{\nu_k}{\mu}$$

and so is independent of u.

• $Q(n,t|j_k,u)$ denotes the probability that an immigration occuring at time u and consisting of j_k particles leads to n particles at time t. $Q(n,t|j_k,u)$ only relies on the branching part of the process and can be decomposed as previously:

$$Q(n,t|j_k,u) = \sum_{i_1+\cdots+i_{j_k}=n} Q_{i_1}(t-u) \ldots Q_{i_{j_k}}(t-u) \tag{11}$$

where $Q_i(t)$ is the probability that one particle leads to i particles by division process in a period of length t. Indeed, once arrived, each particle is the initial point of a branching process which evolves independently during the remaining time $t-u$.

As a consequence, we can express $J(s,t)$ as:

$$\begin{aligned}
J(s,t) &= \sum_{n=0}^{\infty} s^n U_n(t) = \sum_{n=0}^{\infty}\sum_{k=1}^{K} \frac{\nu_k}{\mu} s^n \int_0^t Q(n,t|j_k,u)\frac{du}{t} \\
&= \sum_{k=1}^{K} \frac{\nu_k}{\mu} \int_0^t \sum_{n=0}^{\infty} s^n \sum_{i_1+\cdots+i_{j_k}=n} Q_{i_1}(t-u)\dots Q_{i_{j_k}}(t-u)\frac{du}{t} \\
&= \sum_{k=1}^{K} \frac{\nu_k}{\mu} \int_0^t \left[\sum_{n=0}^{\infty} s^n Q_n(t-u)\right]^{j_k} \frac{du}{t}.
\end{aligned}$$

Let $\Psi(s,t) = \sum_{n=0}^{\infty} s^n Q_n(t)$ be the probability-generating function of a j_0-Yule process without immigration, starting with one particle, then

$$J(s,t) = \sum_{k=1}^{K} \frac{\nu_k}{\mu} \int_0^t (\Psi(s,t-u))^{j_k} \frac{du}{t} \tag{12}$$

which combined with Lemma 1 leads to :

$$J(s,t) = \sum_{k=1}^{K} \frac{\nu_k}{\mu} \int_0^t \left[1-(1-s^{-j_0})\exp(\nu_0 j_0(t-u))\right]^{-j_k/j_0} \frac{du}{t}$$

Substituting $v = (1-s^{-j_0})\exp(\nu_0 j_0(t-u))$ into the integral gives :

$$\int_0^t \left[1-(1-s^{-j_0})\exp(\nu_0 j_0(t-u))\right]^{-j_k} \frac{du}{t} = \frac{1}{t\nu_0 j_0} \int_{1-\frac{1}{s^{j_0}}}^{(1-\frac{1}{s^{j_0}})\exp(\nu_0 j_0 t)} \frac{1}{(1-v)^{j_k/j_0} v} dv$$

and proposition 4 is proved.

B.2 Proof of Theorem 1

Recall that assumption $\mathcal{A}$ states that $\forall k \in \{1 \dots K\}, \frac{j_k}{j_0} = r_k \in \mathbb{N}^*$. The usual decomposition of $\frac{1}{(1-v)^{r_k} v}$ into fractions leads to

$$\int_0^t \left[1 - (1 - s^{-j_0}) \exp(\nu_0 j_0 (t-\tau))\right]^{-j_k/j_0} \frac{d\tau}{t} = \frac{1}{t\nu_0 j_0} \left[\log(v) - \log(1-v) + \sum_{l=1}^{r_k-1} \frac{1/l}{(1-v)^l}\right]_{1-\frac{1}{s^{j_0}}}^{(1-\frac{1}{s^{j_0}})\exp(\nu_0 j_0 t)}.$$

Introducing the following notations :

$$J_0(s,t) = \frac{1}{t\nu_0 j_0} \left[\log(v) - \log(1-v)\right]_{1-\frac{1}{s^{j_0}}}^{(1-\frac{1}{s^{j_0}})\exp(\nu_0 j_0 t)} = -\frac{1}{t\nu_0 j_0} \log\left[1 - s^{j_0}(1 - e^{-\nu_0 j_0 t})\right]$$

And for $1 \leq l \leq R_k - 1$

$$J_l(s,t) = \frac{1}{t\nu_0 j_0} \left[\frac{1/l}{(1-v)^l}\right]_{1-\frac{1}{s^{j_0}}}^{(1-\frac{1}{s^{j_0}})\exp(\nu_0 j_0 t)} = \frac{1}{tl\nu_0 j_0} \left[\left(1 - (1 - \frac{1}{s^{j_0}}) e^{\nu_0 j_0 t)}\right)^{-l} - s^{l j_0}\right]$$

we can write : $J(s,t) = \frac{1}{\mu} \sum_{k=1}^K \nu_k \sum_{l=0}^{r_k-1} J_l(s,t)$. A rearrangement is the sums (using the fact that $r_1 < r_2 < \dots < r_K$) leads to :

$$\begin{aligned} J(s,t) &= \tfrac{1}{\mu} \quad (\nu_1 + \dots + \nu_K)(J_0(s,t) + J_1(s,t) + \dots + J_{r_1-1}(s,t)) \\ &+ \tfrac{1}{\mu} \quad (\nu_2 + \dots + \nu_K)(J_{r_1}(s,t) + \dots + J_{r_2-1}(s,t)) \\ &+ \quad \dots \\ &+ \tfrac{1}{\mu} \quad \nu_K(J_{r_{K-1}}(s,t) + \dots + J_{r_K-1}(s,t)) \end{aligned}$$

Setting

$$\alpha_l = \begin{cases} 1 & \text{for } 0 \leq l \leq r_1 - 1 \\ \frac{\nu_k + \dots + \nu_K}{\mu} & \text{for any } r_{k-1} \leq l \leq r_k - 1 \text{ and for any } k = 2 \dots K \end{cases}$$

we have the following expression for the probability-generating function of $X(t)$:

$$\phi(s,t) = e^{-\mu t} \exp\{\mu t J(s,t)\} = \exp\left[-\mu t + \mu t \sum_{l=0}^{r_K-1} \alpha_l J_l(s,t)\right]$$

We can now study the asymptotic comportment of $\tilde{X}(t) = e^{-\nu_0 j_0 t} X(t)$. Let $\Phi_{\tilde{X}(t)}(\theta) = E[e^{-\theta \tilde{X}(t)}]$ be the moment generating function of $\tilde{X}(t)$. We have :

$$\Phi_{\tilde{X}(t)}(\theta) = E[(e^{\theta e^{-\nu_0 j_0 t}})^{X(t)}] = \phi(e^{\theta e^{-\nu_0 j_0 t}}, t) = \exp\left[-\mu t + \mu t \sum_{l=0}^{r_K - 1} \alpha_l J_l(s_t, t)\right] \tag{13}$$

with $s_t = \exp\left[\theta e^{-\nu_0 j_0 t}\right] \simeq_{t\to\infty} 1 + \theta e^{-j_0 \nu_0 t}$. We study the convergence of each term of the product.

$J_0(s_t, t) = -\frac{1}{t\nu_0 j_0} \log\left[1 - s_t^{j_0}(1 - e^{-\nu_0 j_0 t})\right]$ and so $\exp\left[-\mu t + \mu t J_0(s_t, t)\right] = \left[e^{\nu_0 j_0 t} - s_t^{j_0}(e^{\nu_0 j_0 t} - 1)\right]^{\frac{-\mu}{\nu_0 j_0}}$. Using $s_t^{j_0} = e^{\theta j_0 e^{-\nu_0 j_0 t}} \approx_{t\to\infty} 1 + j_0 \theta e^{-\nu_0 j_0 t}$, we obtain :

$$\exp\left[-\mu t + \mu t J_0(s_t, t)\right] \approx_{t\to\infty} (1 - j_0\theta)^{\frac{-\mu}{\nu_0 j_0}}.$$

We know consider the terms of the form $\phi_l(s_t, t) = \exp\left[\mu t \alpha_l J_l(s_t, t)\right]$. We have

$$\begin{aligned}
\phi_l(s_t, t) &= \exp\left[\frac{\mu \alpha_l}{l \nu_0 j_0}\left[\left(1 - (1 - s_t^{-j_0}) e^{\nu_0 j_0 t}\right)^{-l} - s_t^{j_0 l}\right]\right] \\
&\approx_{t\to\infty} \exp\left[\frac{\mu \alpha_l}{l \nu_0 j_0}\left[(1 - j_0\theta)^{-l} - 1\right]\right]
\end{aligned}$$

Finally for all $\theta \leq 0$,

$$\begin{aligned}
\lim_{t\to\infty} E[e^{\theta \tilde{X}(t)}] &= (1 - j_0\theta)^{\frac{-\mu}{\nu_0 j_0}} \prod_{l=1}^{r_K - 1} \exp\left[\frac{\mu \alpha_l}{l \nu_0 j_0}\left[(1 - j_0\theta)^{-l} - 1\right]\right] \\
&= E[e^{\theta Y_0}] \prod_{l=1}^{r_K - 1} E[e^{\theta Y_l}]
\end{aligned}$$

where $Y_0 \sim \Gamma\left(\frac{\mu}{\nu_0 j_0}, \frac{1}{j_0}\right)$ with moment generating function equal to $M(\theta) = (1 - j_0\theta)^{\frac{-\mu}{\nu_0 j_0}}$. Moreover Y_l is such that its moment generating function is $\exp\left[\frac{\mu \alpha_l}{l \nu_0 j_0}\left[(1 - j_0\theta)^{-l} - 1\right]\right]$ which can be rewritten as

$$
\begin{aligned}
\exp\left[\frac{\mu\alpha_l}{l\nu_0 j_0}\left[(1-j_0\theta)^{-l}-1\right]\right] &= \sum_{k=0}^{\infty}\exp\left[-\frac{\mu\alpha_l}{l\nu_0 j_0}\right]\left(\frac{\mu\alpha_l}{l\nu_0 j_0}\right)^k \frac{1}{k!}\frac{1}{(1-j_0\theta)^{kl}} \\
&= \sum_{k=0}^{\infty}\rho_{l,k}\frac{1}{(1-j_0\theta)^{kl}}
\end{aligned}
$$

where $\rho_{kl} = \exp\left[-\frac{\mu\alpha_l}{l\nu_0 j_0}\right]\left(\frac{\mu\alpha_l}{l\nu_0 j_0}\right)^k \frac{1}{k!}$ is the probability that a Poisson random variable of parameter $\frac{\mu\alpha_l}{l\nu_0 j_0}$ is equal to k. So Y_l is distributed with an infinite mixture of $\Gamma\left(kl, \frac{1}{j_0}\right)$ with Poisson weights. Finally, $e^{-\nu_0 j_0 t}X(t)$ converges in distribution towards a sum of r_K independent variables $\sum_{l=0}^{j_K-1} Y_l$ where the $Y_0 \sim \Gamma\left(\frac{\mu}{\nu_0 j_0}, \frac{1}{j_0}\right)$ and $Y_l \sim \sum_{k=0}^{\infty}\rho_{kl}\Gamma\left(kl, \frac{1}{j_0}\right)$ with $\rho_{kl} = \exp\left[-\frac{\mu\alpha_l}{l\nu_0 j_0}\right]\left(\frac{\mu\alpha_l}{l\nu_0 j_0}\right)^k \frac{1}{k!}$.

References

[1] Ron Shonkwiler. On age-dependent branching processes with immigration. *Comput. Math. Appl.*, 6(3):289–296, 1980.