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Only the first term of some series counts

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Abstract

Let $X, X_1, X_2, \dots$ be i.i.d. random variables, and set $S_n = X_1 + \dots + X_n$. We prove that for three important distributions of X , namely normal, exponential and geometric, series of the type $\sum_{n \geq 1} a_n P(|S_n| \geq xb_n)$ or $\sum_{n \geq 1} a_n P(S_n \geq xb_n)$ behave like their first term as $x \rightarrow \infty$.

Keywords: Tail probabilities of sums of i.i.d. random variables, normal distribution, exponential distribution, geometric distribution.

1. Introduction

Let $X, X_1, X_2, \dots$ be i.i.d. random variables with partial sums $S_n = X_1 + \dots + X_n, n \geq 1$. Several authors considered series of the type

$$f(x) = \sum_{n \geq 1} a_n P(|S_n| \geq xb_n), \quad x > 0, \quad (1.1)$$

or, in case $X \geq 0$,

$$f(x) = \sum_{n \geq 1} a_n P(S_n \geq xb_n), \quad x > 0, \quad (1.2)$$

where $a_n, b_n > 0$. If $\sum_{n \geq 1} a_n = \infty$, the primary focus was to obtain necessary and sufficient moment conditions for the convergence of these series, i.e. to find a threshold α such that $f(x) = \infty$ for $x < \alpha$, while $f(x) < \infty$ for $x > \alpha$. Then it was sensible to look for a normalizing

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elementary function $g(x)$, $x > \alpha$, such that the ratio $f(x)/g(x)$ have a nontrivial limit as $x \searrow \alpha$. The first result in this direction was due to Heyde (1975) who proved that

$$\lim_{x \searrow 0} x^2 \sum_{n \geq 1} P(S_n \geq xn) = EX^2$$

whenever $EX^2 < \infty$ and $EX = 0$. (Hsu and Robbins (1947) and Erdős (1949,1950) showed that $\sum_{n \geq 1} P(S_n \geq xn) < \infty$, $x > 0$, if and only if $EX^2 < \infty$ and $EX = 0$.) References to the rich literature on this so-called precise asymptotics problem are given in Spătaru (2010).

The purpose of this paper is to deal with a dual problem motivated also by insurance applications. Namely, if $f(x_0) < \infty$ for some $x_0 > 0$, then $\lim_{x \rightarrow \infty} f(x) = 0$ by dominated convergence. Then it is interesting to determine a normalizing simple function $g(x)$, $x \geq x_0$, such that $f(x)/g(x)$ have a nondegenerate limit as $x \rightarrow \infty$. It turns out that in three important cases presented below this limit is 1, and $g(x)$ equals the first term of the series. This phenomenon might be more general. Actually, if $EX^2 < \infty$ and $EX = 0$, Pruss (1997) proved that there are absolute constants $C_1, C_2 > 0$ such that

$$C_1 x^{-2} E[X^2 I\{|X| \geq x\}] \leq \sum_{n \geq 1} P(|S_n| \geq xn) \leq C_2 x^{-2} E[X^2 I\{|X| \geq x\}], \quad x > 0. \quad (1.3)$$

Since $P(|S_1| \geq x) = P(|X| \geq x) \leq x^{-2} E[X^2 I\{|X| \geq x\}]$, $x > 0$, (1.3) may suggest that $\sum_{n \geq 1} P(|S_n| \geq xn) \sim P(|S_1| \geq x)$ as $x \rightarrow \infty$ whenever the series is positive for all $x > 0$.

The three cases refers to the normal distribution and series (1.1), where the assumption that $X_1, X_2, \dots$ have a common distribution is dropped, and to the exponential and geometric distributions and series (1.2).

2. Normal distribution

For $n \geq 1$, we assume that X_n has a normal distribution with zero mean and σ_n^2 variance, and set $B_n^2 = \sigma_1^2 + \dots + \sigma_n^2$.

Lemma 1. Let Φ denote the standard normal distribution function, and put $\Psi(x) = 1 - \Phi(x) + \Phi(-x)$, $x \geq 0$. For $c > 1$, the following assertions hold:

- (i) the function $\Psi(cx)/\Psi(x)$, $x \geq 0$, is decreasing;
- (ii) $\lim_{x \rightarrow \infty} \Psi(cx)/\Psi(x) = 0$.

Proof. To verify (i), it suffices to show that the function

$$R(x) = \left(\int_{cx}^{\infty} e^{-y^2/2} dy \right) / \int_x^{\infty} e^{-y^2/2} dy, \quad x \geq 0,$$

is decreasing. For $0 \leq x \leq y$, we have

$$\frac{x^2}{2}(c^2 - 1) \leq \frac{y^2}{2}(c^2 - 1) \implies -\frac{x^2}{2} - \frac{c^2 y^2}{2} \leq -\frac{y^2}{2} - \frac{c^2 x^2}{2} \implies e^{-x^2/2} e^{-c^2 y^2/2} \leq e^{-y^2/2} e^{-c^2 x^2/2}.$$

Therefore, for $x \geq 0$,

$$\begin{aligned} ce^{-x^2/2} \int_x^{\infty} e^{-c^2 y^2/2} dy &\leq ce^{-c^2 x^2/2} \int_x^{\infty} e^{-y^2/2} dy \implies e^{-x^2/2} \int_{cx}^{\infty} e^{-y^2/2} dy \leq ce^{-c^2 x^2/2} \int_x^{\infty} e^{-y^2/2} dy \\ &\implies R'(x) \leq 0. \end{aligned}$$

Next, as $\Psi(x) \sim \frac{2}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2}$ as $x \rightarrow \infty$ (see, e.g., Corollary 2, p. 49 of Chow and Teicher (1978)), we obtain

$$\frac{\Psi(cx)}{\Psi(x)} \sim \left(\frac{e^{-c^2 x^2/2}}{cx} \right) \frac{x}{e^{-x^2/2}} = \frac{1}{c} e^{-(c^2-1)x^2/2} \rightarrow 0 \text{ as } x \rightarrow \infty. \quad \square$$

Theorem 1. Suppose that $\sum_{n \geq 1} a_n P(|S_n| \geq x_0 b_n)$ converges for some $x_0 > 0$, where $b_n/b_1 > B_n/B_1$, $n \geq 2$. Then

$$\sum_{n \geq 1} a_n P(|S_n| \geq x b_n) \sim a_1 P(|S_1| \geq x b_1) = a_1 \Psi(x b_1 / \sigma_1) \text{ as } x \rightarrow \infty. \quad (2.1)$$

Proof. Write

$$\sum_{n \geq 1} a_n P(|S_n| \geq x b_n) = a_1 P(|S_1| \geq x b_1) + \sum_{n \geq 2} a_n P(|S_n| \geq x b_n), \quad x > 0. \quad (2.2)$$

For $x \geq x_0$, applying Lemma 1.i with $c = \frac{b_n}{B_n} \cdot \frac{B_1}{b_1}$, $n \geq 2$, we get

$$\begin{aligned} \sum_{n \geq 2} \frac{a_n P(|S_n| \geq x b_n)}{a_1 P(|S_1| \geq x b_1)} &= \sum_{n \geq 2} \frac{a_n \Psi(x \frac{b_n}{B_n})}{a_1 \Psi(x \frac{b_1}{B_1})} = \sum_{n \geq 2} \frac{a_n \Psi(x \frac{b_1}{B_1} \cdot \frac{b_n}{B_n} \cdot \frac{B_1}{b_1})}{a_1 \Psi(x \frac{b_1}{B_1})} \\ &\leq \sum_{n \geq 2} \frac{a_n \Psi(x_0 \frac{b_1}{B_1} \cdot \frac{b_n}{B_n} \cdot \frac{B_1}{b_1})}{a_1 \Psi(x_0 \frac{b_1}{B_1})} = \frac{1}{a_1 \Psi(x_0 \frac{b_1}{B_1})} \sum_{n \geq 2} a_n P(|S_n| \geq x_0 b_n) < \infty. \end{aligned}$$

Consequently, by dominated convergence and Lemma 1.ii, we have

$$\lim_{x \rightarrow \infty} \sum_{n \geq 2} \frac{a_n P(|S_n| \geq x b_n)}{a_1 P(|S_1| \geq x b_1)} = \sum_{n \geq 2} \frac{a_n}{a_1} \lim_{x \rightarrow \infty} \frac{\Psi(x b_1 b_n / b_1 \sqrt{n})}{\Psi(x b_1)} = 0. \quad (2.3)$$

Now (2.1) follows from (2.2) and (2.3). $\square$

3. Exponential distribution

In this section we assume that X has an exponential distribution with rate parameter λ .

We need the following lemma.

Lemma 2. For $n \geq 1$, define

$$h_n(x) = e^{-x b_n} \sum_{k=0}^{n-1} \frac{(x b_n)^k}{k!}, \quad x \geq 0.$$

where $b_n > 0$ and $b_n - b_1 \geq C^{-1}(n-1)$ for some $C > 0$. Then, for $n \geq 2$, we have:

- (i) the function $h_n(x)/h_1(x)$, $x \geq 0$, is decreasing for $x > C$;
- (ii) $\lim_{x \rightarrow \infty} h_n(x)/h_1(x) = 0$.

Proof. For $n \geq 2$ and $1 \leq k \leq n-1$, the derivative of the function $e^{-x(b_n-b_1)}x^k$, $x \geq 0$, is negative whenever $x > k/(b_n - b_1)$. This entails that the derivative of the function $h_n(x)/h_1(x)$, $x \geq 0$, is negative whenever $x > C \geq (n-1)/(b_n - b_1)$. Thus (i) holds, and (ii) is obvious. $\square$

Theorem 2. Assume that b_n , $n \geq 1$, are as in Lemma 2 and $\sum_{n \geq 1} a_n P(S_n \geq x_0 b_n) < \infty$ for some $x_0 > 0$. Then

$$\sum_{n \geq 1} a_n P(S_n \geq x b_n) \sim a_1 P(S_1 \geq x b_1) = a_1 e^{-\lambda b_1 x} \text{ as } x \rightarrow \infty. \quad (3.1)$$

Proof. Without loss of generality, suppose that $\lambda = 1$. Then S_n has an Erlang distribution with shape parameter n and rate parameter 1, and so $P(S_n \geq x) = e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!}$, $n \geq 1$, $x \geq 0$. We have

$$\sum_{n \geq 1} a_n P(S_n \geq xb_n) = a_1 P(S_1 \geq xb_1) + \sum_{n \geq 2} a_n P(S_n \geq xb_n), \quad x > 0. \quad (3.2)$$

For $x \geq x_0 \vee C$, on account of Lemma 2.i, we obtain

$$\begin{aligned} \sum_{n \geq 2} \frac{a_n P(S_n \geq xb_n)}{a_1 P(S_1 \geq xb_1)} &= \sum_{n \geq 2} \frac{a_n h_n(x)}{a_1 h_1(x)} \leq \sum_{n \geq 2} \frac{a_n h_n(x_0 \vee C)}{a_1 h_1(x_0 \vee C)} \\ &= \frac{1}{a_1 h_1(x_0 \vee C)} \sum_{n \geq 2} a_n P(S_n \geq (x_0 \vee C)b_n) < \infty. \end{aligned}$$

Therefore, by dominated convergence and Lemma 2.ii, we get

$$\lim_{x \rightarrow \infty} \sum_{n \geq 2} \frac{a_n P(S_n \geq xb_n)}{a_1 P(S_1 \geq xb_1)} = \sum_{n \geq 2} \frac{a_n}{a_1} \lim_{x \rightarrow \infty} \frac{h_n(x)}{h_1(x)} = 0. \quad (3.3)$$

By (3.2) and (3.3), (3.1) obtains. $\square$

4. Geometric distribution

Now we assume that X is geometrically distributed with parameter p and taking values in the set $\{0, 1, \dots\}$. Then S_n follows a negative binomial distribution with parameters n and p expressed in terms of the regularized incomplete beta function I_p , namely we have

$$P(S_n \geq k) = I_p(k, n) = \sum_{j=k}^{k+n-1} \binom{k+n-1}{j} p^j q^{k+n-1-j}, \quad n, k \geq 1, \quad (4.1)$$

where $q = 1 - p$.

Lemma 3. Let $0 < p < 1$. For $n \geq 2$, the following assertions hold:

- (i) the sequence $\{I_p(mn, n)/I_p(m, 1) : m \geq -1/\log p\}$ is decreasing;
- (ii) $\lim_{m \rightarrow \infty} I_p(mn, n)/I_p(m, 1) = 0$.

Proof. By (10), we get $I_p(m, 1) = P(S_1 \geq m) = p^m$, $m \geq 1$. For $n \geq 2$ and $m \geq 1$, on account of (10), we have

$$\begin{aligned} I_p(mn, n) &= \binom{mn+n-1}{n-1} p^{mn} q^{n-1} + \binom{mn+n-1}{n-2} p^{mn+1} q^{n-2} \\ &\quad + \dots + \binom{mn+n-1}{1} p^{mn+n-2} q + p^{mn+n-1} \\ &= \frac{(mn+n-1)(mn+n-2)\dots(mn+1)}{(n-1)!} p^{mn} q^{n-1} \\ &\quad + \frac{(mn+n-1)(mn+n-2)\dots(mn+2)}{(n-2)!} p^{mn+1} q^{n-2} \\ &\quad + \dots + (mn+n-1) p^{mn+n-2} q + p^{mn+n-1}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \frac{I_p(mn, n)}{I_p(m, 1)} &= \frac{(mn+n-1)(mn+n-2)\dots(mn+1)}{(n-1)!} p^{m(n-1)} q^{n-1} \\ &\quad + \frac{(mn+n-1)(mn+n-2)\dots(mn+2)}{(n-2)!} p^{m(n-1)+1} q^{n-2} \\ &\quad + \dots + (mn+n-1) p^{m(n-1)+n-2} q + p^{m(n-1)+n-1} \\ &= \frac{(mp^m n + (n-1)p^m)(mp^m n + (n-2)p^m) \dots (mp^m n + p^m)}{(n-1)!} q^{n-1} \\ &\quad + \frac{(mp^m n + (n-1)p^m)(mp^m n + (n-2)p^m) \dots (mp^m n + 2p^m)}{(n-2)!} p^{m+1} q^{n-2} \\ &\quad + \dots + (mp^m n + (n-1)p^m) p^{(m+1)(n-2)} q + p^{(m+1)(n-1)}. \end{aligned} \tag{4.2}$$

As the function $x p^x$, $x \geq 1$, is decreasing for $x \geq -1/\log p$, (4.2) shows that (i) holds. Since

$$\lim_{m \rightarrow \infty} m p^m = \lim_{m \rightarrow \infty} p^m = \lim_{m \rightarrow \infty} p^{(m+1)l} = 0, \text{ (ii) follows as well. } \square$$

Theorem 3. Suppose that $\sum_{n \geq 1} a_n P(S_n \geq m_0 n) < \infty$ for some $m_0 \geq 1$. Then

$$\sum_{n \geq 1} a_n P(S_n \geq mn) \sim a_1 P(S_1 \geq m) = a_1 p^m \text{ as } m \rightarrow \infty. \tag{4.3}$$

Proof. Notice that $\lim_{m \rightarrow \infty} \sum_{n \geq 1} a_n P(S_n \geq mn) = 0$ by dominated convergence. Write

$$\sum_{n \geq 1} a_n P(S_n \geq mn) = a_1 P(S_1 \geq m) + \sum_{n \geq 2} a_n P(S_n \geq mn), \quad m \geq 1. \quad (4.4)$$

Let $m_1 \geq -1/\log p$. Then, for $m \geq m_0 \vee m_1$, in view of Lemma 3.i, we have

$$\begin{aligned} \sum_{n \geq 2} \frac{a_n P(S_n \geq mn)}{a_1 P(S_1 \geq m)} &= \sum_{n \geq 2} \frac{a_n I_p(mn, n)}{a_1 I_p(m, 1)} \leq \sum_{n \geq 2} \frac{a_n I_p((m_0 \vee m_1)n, n)}{a_1 I_p(m_0 \vee m_1, 1)} \\ &= \frac{1}{a_1 I_p(m_0 \vee m_1, 1)} \sum_{n \geq 2} a_n P(S_n \geq (m_0 \vee m_1)n) < \infty. \end{aligned}$$

Hence, by dominated convergence and Lemma 3.ii, we get

$$\lim_{m \rightarrow \infty} \sum_{n \geq 2} \frac{a_n P(S_n \geq mn)}{a_1 P(S_1 \geq m)} = \sum_{n \geq 2} \frac{a_n}{a_1} \lim_{m \rightarrow \infty} \frac{I_p(mn, n)}{I_p(m, 1)} = 0. \quad (4.5)$$

Thus (4.3) follows from (4.4) and (4.5). $\square$

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