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Abstract

A proper vertex coloring of a graph is said to be locally identifying if the sets of colors in the closed neighborhood of any two adjacent non-twin vertices are distinct. The lid-chromatic number of a graph is the minimum number of colors used by a locally identifying vertex-coloring. In this paper, we prove that for any graph class of bounded expansion, the lid-chromatic number is bounded. Classes of bounded expansion include minor closed classes of graphs. For these latter classes, we give an alternative proof to show that the lid-chromatic number is bounded. This leads to an explicit upper bound for the lid-chromatic number of planar graphs. This answers in a positive way a question of Esperet et al. [L. Esperet, S. Gravier, M. Montassier, P. Ochem and A. Parreau. Locally identifying coloring of graphs. Electronic Journal of Combinatorics, 19(2), 2012.].

1 Introduction

A vertex-coloring is said to be locally identifying if (i) the vertex-coloring is proper (i.e. no adjacent vertices receive the same color), and (ii) for any adjacent vertices \( u, v \), the set of colors assigned to the closed neighborhood of \( u \) differs from the set of colors assigned to the closed neighborhood of \( v \) whenever these neighborhoods are distinct. The locally identifying chromatic number of the graph \( G \) (or lid-chromatic number, for short), denoted by \( \chi_{lid}(G) \), is the smallest number of colors required in any locally identifying coloring of \( G \).

Locally identifying colorings of graphs have been recently introduced by Esperet et al. [6] and later studied by Foucaud et al. [7]. They are related to identifying codes [8, 9], distinguishing colorings [1, 3, 5] and locating-colorings [4].

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example, upper bounds on lid-chromatic number have been obtained for bipartite graphs, $k$-trees, outerplanar graphs and bounded degree graphs. An open question asked by Esperet et al. [6] was to know whether $\chi_{lid}$ is bounded for the class of planar graphs. In this paper, we answer positively to this question proving more generally that $\chi_{lid}$ is bounded for any class of bounded expansion.

In Section 3, we first give a tight bound of $\chi_{lid}$ in term of the tree-depth. Then we use the fact that any class of bounded expansion admits a low tree-depth coloring (that is a $k$-coloring such that each triplet of colors induces a graph of tree-depth 3, for some constant $k$) to prove that it has bounded lid-chromatic number.

In Section 4, we focus on minor closed classes of graphs which have bounded expansion and give an alternative bound on the lid-chromatic number, which gives an explicit bound for planar graphs.

The next section is devoted to introduce notation and preliminary results.

2 Notation and preliminary results

Let $G = (V,E)$ be a graph. For any vertex $u$, we denote by $N_G(u)$ its neighborhood in $G$ and by $N_G[u]$ its closed neighborhood in $G$ (together with its adjacent vertices). The notion of neighborhood can be extended to sets as follows: for $X \subseteq V$, the notion of neighborhood can be extended to sets as follows: for $X \subseteq V$, we have $N_G[X] = \{w \in V(G) \mid \exists v \in X, w \in N[v]\}$ and $N_G(X) = N_G[X] \setminus X$. When the considered graph is clearly identified, the subscript is dropped.

The degree of vertex $u$ is the size of its neighborhood. The distance between two vertices $u$ and $v$ is the number of edges in a shortest path between $u$ and $v$. For $X \subseteq V$, we denote by $G[X]$ the subgraph of $G$ induced by $X$.

We say that two vertices $u$ and $v$ are twins (i.e. $u \sim v$) if $N[u] = N[v]$ (although they are often called true twins in the literature, we call them twins for convenience). In particular, $u$ and $v$ are adjacent vertices. Note that if $u$ and $v$ are adjacent but not twins, there exists a vertex $w$ which is adjacent to exactly one vertex among $\{u,v\}$, i.e. $w \in N[u] \triangle N[v]$ (where $\triangle$ is the symmetric difference between sets). We say that $w$ distinguishes $u$ and $v$, or simply $w$ distinguishes the edge $uv$. For a subset $X \subseteq V$, we say that a subset $Y \subseteq V$ distinguishes $X$ if for every pair $u,v$ of non-twin vertices of $X$, there exists a vertex $w \in Y$ that distinguishes the edge $uv$.

Let $c : V \rightarrow \mathbb{N}$ be a vertex-coloring of $G$. The coloring $c$ is proper if adjacent vertices have distinct colors. We denote by $\chi(G)$ the chromatic number of $G$, i.e. the minimum number of colors in a proper coloring of $G$. For any $X \subseteq V$, let $c(X)$ be the set of colors that appear on the vertices of $X$. A locally identifying coloring (lid-coloring for short) of $G$ is a proper vertex-coloring $c$ of $G$ such that for any two adjacent vertices $u$ and $v$ that are not twins (i.e. $N[u] \neq N[v]$), we have $c(N[u]) \neq c(N[v])$. A graph $G$ is $k$-lid-colorable if it admits a locally identifying coloring using at most $k$ colors and the minimum number of colors needed for any locally identifying coloring of $G$ is called the locally identifying chromatic number (lid-chromatic number for short) denoted by $\chi_{lid}(G)$. For a vertex $u$, we say that $u$ sees color $a$ if $a \in c(N[u])$. For two adjacent vertices...
u and v, a color that is in the set \(c(N[u]) \Delta c(N[v])\) separates u and v, or simply separates the edge uv. The notion of chromatic number (resp. \(\chi\text{-}\text{lid}
abla\text{-}\text{chromatic number}) can be extended to a class of graphs \(\mathcal{C}\) as follows: \(\chi(\mathcal{C}) = \sup\{\chi(G), G \in \mathcal{C}\}\) (resp. \(\chi_{\text{lid}}(\mathcal{C}) = \sup\{\chi_{\text{lid}}(G), G \in \mathcal{C}\}\)).

The following theorem is due to Bondy [2]:

**Theorem 1** (Bondy’s theorem [2]). Let \(\mathcal{A} = \{A_1, \ldots, A_n\}\) be a collection of \(n\) distinct subsets of a finite set \(X\). There exists a subset \(X'\) of \(X\) of size at most \(n-1\) such that the sets \(A_i \cap X'\) are all distinct.

**Corollary 2.** Let \(C\) be a \(n\)-clique subgraph of \(G\). There exists a vertex subset \(S(C) \subseteq V(G)\) of size at most \(n-1\) that distinguishes all the pair of non-twin vertices of \(C\).

**Proof.** Let \(C\) be a \(n\)-clique subgraph of \(G\) induced by the vertex set \(V(C) = \{v_1, v_2, \ldots, v_n\}\). Let \(\mathcal{A} = \{N[v_i] \mid v_i \in V(C)\}\) be a collection of distinct subsets of the finite set \(X = \bigcup_{1 \leq i \leq n} N[v_i]\). Note that some \(v_i\)'s might be twins in \(G\) (i.e. \(N[v_i] = N[v_j]\) for some \(v_i, v_j \in V(C)\)) and therefore \(|\mathcal{A}|\) could be smaller than \(n\). By Bondy Theorem, there exists \(S(C) \subseteq X\) of size at most \(|\mathcal{A}| - 1 \leq n - 1\) such that for any distinct elements \(A_1, A_2\) of \(\mathcal{A}\), we have \(A_1 \cap S(C) \neq A_2 \cap S(C)\).

Let us prove that \(S(C)\) is a set of vertices that distinguish all the pairs of non-twin vertices of \(C\). For a pair of non-twin vertices \(v_i, v_j\) of \(C\), we have \(N[v_i] \neq N[v_j]\). By definition of \(S(C)\), we have \(N[v_i] \cap S(C) \neq N[v_j] \cap S(C)\), then there exists \(w \in S(C)\) that belongs to \(N[v_i] \Delta N[v_j]\). Therefore, \(w\) distinguishes the edge \(v_iv_j\).

\[\square\]

### 3 Bounded expansion classes of graphs

A rooted tree is a tree with a special vertex, called the root. The **height** of a vertex \(x\) in a rooted tree is the number of vertices on a path from the root to \(x\) (hence, the height of the root is 1). The **height** of a rooted tree \(T\) is the maximum height of the vertices of \(T\). If \(x\) and \(y\) are two vertices of \(T\), \(x\) is an **ancestor** of \(y\) in \(T\) if \(x\) belongs to the path between \(y\) and the root. The **closure** \(\text{clos}(T)\) of a rooted tree \(T\) is the graph with vertex set \(V(T)\) and edge set \(\{xy \mid x\) is an ancestor of \(y\) in \(T, x \neq y\}\). The **tree-depth** \(\text{td}(G)\) of a connected graph \(G\) is the minimum height of a rooted tree \(T\) such that \(G\) is a subgraph of \(\text{clos}(T)\). If \(G\) is not connected, the tree-depth of \(G\) is the maximum tree-depth of its connected components.

Let \(p\) be a fixed integer. A **low tree-depth coloring** of a graph \(G\) (relatively to \(p\)) is a coloring of the vertices of \(G\) such that the union of any \(i \leq p\) color classes induces a graph of tree-depth at most \(i\). Let \(\chi_{\text{td}}^p(G)\) be the minimum number of colors required in such a coloring. Note that as tree-depth one graphs and tree-depth two graphs are respectively the stables and star forests, \(\chi_{\text{td}}^1\) and \(\chi_{\text{td}}^2\) respectively correspond to the usual chromatic number and the star chromatic number.
In the following of this section, we first give a tight bound on the lid-chromatic number in terms of tree-depth.

**Proposition 3.** For any graph $G$, $\chi_{\text{lid}}(G) \leq 2td(G) - 1$ and this is tight.

Using this bound, we then bound the lid-chromatic number in terms of $\chi^d_3$.

**Theorem 4.** For any graph $G$,

$$\chi_{\text{lid}}(G) \leq 6(\chi^d_3(G))$$

Classes of graphs of bounded expansion have been introduced by Nešetřil and Ossona de Mendez [10]. These classes contain minor closed classes of graphs and any class of graphs defined by an excluded topological minor. Actually, these classes of graphs are closely related to low tree-depth colorings:

**Theorem 5** (Theorem 7.1 [10]). A class of graphs $\mathcal{C}$ has bounded expansion if and only if $\chi^d_p(\mathcal{C})$ is bounded for any $p$.

We therefore deduce the following corollary from Theorems 4 and 5:

**Corollary 6.** For any class $\mathcal{C}$ of bounded expansion, $\chi_{\text{lid}}(\mathcal{C})$ is bounded.

It is in particular true for a class of bounded tree-width. A consequence is that $\chi_{\text{lid}}$ is bounded for chordal graphs by a function of the clique number (which is equals to the tree-width plus 1 for a chordal graph). It is conjectured by Esperet et al. [6] that $\chi_{\text{lid}}(G) \leq 2\omega(G)$ if $G$ is chordal.

We now prove Proposition 3.

**Proof of Proposition 3.** Let us first prove that the bound is tight. Consider the graph $H_n$ obtained from a complete graph, with vertex set $\{a_1, \ldots, a_n\}$, by adding a pendant vertex $b_i$ to every $a_i$ but one, say for $1 \leq i < n$. The tree-depth of this graph is at least $n$ as it contains a $n$-clique. Indeed, given a rooted tree $T$, two vertices at the same height are non-adjacent in $\text{clos}(T)$, we thus need at least $n$ levels. Actually the tree-depth of this graph is at most $n$ since the tree $T$ rooted at $a_1$ and such that $a_i$ has two sons $a_{i+1}$ and $b_i$, for $1 \leq i < n$, has height $n$ and is such that $\text{clos}(T)$ contains $H_n$ as a subgraph.

Let us show that in any lid-coloring of $H_n$ all the vertices must have distinct colors, and thus use $2n - 1 = 2td(H_n) - 1$ colors. Indeed, two vertices $a_i$ must have different colors as the coloring is proper. A vertex $b_i$ cannot use the same color as a vertex $a_i$, as otherwise the vertex $a_i$ would only see the $n$ colors used in the clique, just as $a_n$. Similarly if two vertices $b_i$ and $b_j$ would use the same color, the vertices $a_i$ and $a_j$ would see the same set of colors.

Let us now focus on the upper bound. We prove the result for a connected graph and by induction on the tree-depth of $G$, denoted by $k$. The result is clear for $k = 1$ (the graph is a single vertex).

Let $G$ be a graph of tree-depth $k > 1$ and let $T$ be a rooted tree of height $k$ such that $G$ is a subgraph of $\text{clos}(T)$. If $T$ is a path, the result is clear since there are only $k$ vertices. So assume that $T$ is not a path, and let $r$ be the root
of $T$. Let $s$ be the smallest height such that there are at least two vertices of height $s + 1$. We name $r_i$, for $i \in \{1, \ldots, s\}$, the unique vertex of height $i$. Let $R = \{r_1, \ldots, r_s\}$. Note that each of the vertices of $R$ is adjacent to all the vertices of $\text{clos}(T)$. Therefore, we can choose the way we label the $s$ vertices in $R$ (i.e. we can choose the height of each of them in $T$) without changing $\text{clos}(T)$.

Necessarily, $G \setminus R$ has at least two connected components. Let $G_1, \ldots, G_\ell$ be its connected components and thus $\ell \geq 2$. We choose $T$ such that $s$ is minimal. It implies that for each $i \in \{1, \ldots, s\}$, $r_i$ has neighbors in all the components $G_1, \ldots, G_\ell$. Indeed, if it is not the case, by permuting the elements of $R$ (this is possible by the above remark), we can assume without loss of generality that $r_s$ does not have a neighbor in $G_\ell$. Therefore, the set of edges $e(r_s, G_\ell) = \{r_s x : x \in V(G_\ell)\}$ of $\text{clos}(T)$ are not used by $G$. Then let $T'$ be the tree obtained from $T$ by moving the whole component $G_\ell$ one level up in such a way that the root of the subtree corresponding to $G_\ell$ is now the son of $r_{s-1}$ (instead of $r_s$ previously). Note that $\text{clos}(T')$ is isomorphic to $\text{clos}(T) \setminus e(r_s, G_\ell)$ and thus $G$ is a subgraph of $\text{clos}(T')$. This new tree $T'$ has two vertices at height $s$, contradicting the minimality of $s$.

Any connected component $G_j$ has tree-depth at most $k' = k - s < k$. By induction, for each $j \in \{1, \ldots, \ell\}$, there exists a lid-coloring $c_j$ of $G_j$ using colors in $\{1, \ldots, 2k' - 1\}$. For each $c_j$, there is a minimum value $s_j$ such that every vertex $r_i$ sees a color in $\{1, \ldots, s_j\}$ in $G_j$. We choose a $(2k' - 1)$-lid-coloring $c_j$ of $G_j$ such that $s_j$ is minimized. Note that for each color $a \leq s_j$, there exists $r_i \in R$ such that $r_i$ sees color $a$ in $G_j$ but no other color of $\{1, \ldots, s_j\}$. Otherwise, after permuting colors $a$ and $s_j$, every vertex $r_i \in R$ would see a color in $\{1, \ldots, s_j - 1\}$, contradicting the minimality of $s_j$. Assume without loss of generality that $s_1 \geq s_2 \geq \ldots \geq s_\ell$.

We replace in $c_j$ the colors $1, 2, \ldots, s_1$ by $1', 2', \ldots, s_1'$. Note that now each vertex $r_i$ sees a color in $\{1', \ldots, s_1'\}$ (in $G_1$) and a color in $\{1, \ldots, s_2\}$ (in $G_2$). Furthermore, the other vertices of $G'$ (that is the vertices in $G_1, \ldots, G_\ell$) do not have this property since $s_1 \geq s_2$. Thus at this step every edge $x r_i$ with $x$ in some $G_j$ is separated.

Now we color each vertex $r_i$ with color $i^\circ$. Let $c : V(G) \rightarrow \{1^\circ, \ldots, s^*\} \cup \{1', \ldots, s_1'\} \cup \{1, \ldots, 2k' - 1\}$ be the current coloring of $G$.

Note that now every distinguishable edge $xy$ in some $G_j$ is separated. Indeed, either $xy$ was distinguished in $G_j$ and it has been separated by $c_j$, or $xy$ is distinguished by some $r_i$ and it is separated by the color $i^\circ$. Note also that $c$ is a proper coloring.

It remains to deal with the edges $r_i r_j$. For that purpose we will refine some color classes. In the following lemma we show that such refinements do not damage what we have done so far.

Claim. Consider a graph $G$ and a coloring $\varphi : V(G) \rightarrow \{1, \ldots, k\}$. Consider any refinement $\varphi'$ of $\varphi$, obtained from $\varphi$ by recoloring with color $k + 1$ some vertices colored $i$, for some $i$. Any edge $xy$ of $G$ properly colored (resp. separated) by $\varphi$ is properly colored (resp. separated) by $\varphi'$.

Indeed if $\varphi(x) \neq \varphi(y)$ then $\varphi'(x) \neq \varphi'(y)$, and if $i \in \varphi(N[x]) \Delta \varphi(N[y])$ then
Let us define a relation $R$ among vertices in $G$ by $r_i R r_j$ if and only if $c(N[r_i]) = c(N[r_j])$. Let $R_1, \ldots, R_s$ be the equivalence classes of the relation $R$ (note that each $R_i$ forms a clique since every $r_i$ has distinct colors). We have $s \geq s_1$. Indeed, by definition of $s_1$ and the coloring $c_1$, for each color $a \in \{1', \ldots, s_1'\}$, there exists $r_i \in R$ that sees $a$ in $G_1$ but no other color of $\{1', \ldots, s_1'\}$. This vertex $r_i$ belongs to some equivalence class $R_j$ and thus all the vertices of $R_j$ sees color $a$ in $G_1$ but no other color of $\{1', \ldots, s_1'\}$.

By Corollary 2, there is a vertex set $S(R_i)$ of size at most $|R_i| - 1$ which distinguishes all pairs of non-twin vertices in $R_i$. We give to the vertices of $S(R_i)$ new distinct colors. By the previous claim, this last operation does not damage the coloring, and now all the distinguishable edges are separated.

Since for this last operation we need $s - s$ new colors, since we used $2k' - 1$ colors $\{1, \ldots, 2k' - 1\}$, $s_1$ colors $\{1', \ldots, s_1'\}$ and $s$ colors $\{1^*, \ldots, s^*\}$, the total number of colors is $(s - s) + (2k' - 1) + s_1 + s = 2k - 1 + s_1 - s \leq 2k - 1$. This concludes the proof of the theorem.

We are now ready to prove Theorem 4:

**Proof of Theorem 4.** Let $a$ be a low tree-depth coloring of $G$ with parameter $p = 3$ and using $\chi^{td}_3(G)$ colors. Let $A = \{\alpha_1, \alpha_2, \alpha_3\}$ be a triple of three distinct colors and let $H_A$ be the subgraph of $G$ induced by the vertices colored by a color of $A$. Since $H_A$ has tree-depth at most 3, by Proposition 3, $H_A$ admits a lid-coloring $c_A$ with five colors (says colors 1 to 5). We extend $c_A$ to the whole graph by giving color 0 to the vertices in $V(G) \setminus V(H_A)$.

Let $A_1, A_2, \ldots, A_k$ be the $k = (\chi^{td}_3(G))$ distinct triplets of colors. We now construct a coloring $c$ of $G$ giving to each vertex $x$ of $G$ the $k$-uplet

$$(c_{A_1}(x), c_{A_2}(x), \ldots, c_{A_k}(x)).$$

The coloring $c$ is using $6^k$ colors. Clearly it is a proper coloring: each pair of adjacent vertices will be in some common graph $H_A$ and will receive distinct colors in this graph. Let $x$ and $y$ be two adjacent vertices with $N[x] \neq N[y]$. Let $w$ be a vertex adjacent to only one vertex among $x$ and $y$. Let $A = \{\alpha(x), \alpha(y), \alpha(w)\}$. Vertices $x$ and $y$ are not twins in the graph $H_A$. Hence $c_A(N[x]) \neq c_A(N[y])$ and therefore, $c(N[x]) \neq c(N[y])$.

## 4 Minor closed classes of graphs

Let $G$ and $H$ be two graphs. $H$ is a minor of $G$ if $H$ can be obtained from $G$ with successive edge deletions, vertex deletions and edge contractions. A class $\mathcal{C}$ is minor closed if for any graph $G$ of $\mathcal{C}$, for any minor $H$ of $G$, we have $H \in \mathcal{C}$. The class $\mathcal{C}$ is proper if it is not the class of all graphs. Let $H$ be a graph. A $H$-minor free graph is a graph that does not have $H$ as a minor. We denote by $\mathcal{K}_n$ the $K_n$-minor-free class of graphs. It is clear that any proper minor closed class of graphs is included in the class $\mathcal{K}_n$ for some $n$. It is folklore that any proper minor closed class of graphs $\mathcal{C}$ has a bounded chromatic number $\chi(\mathcal{C})$. 


The class of graphs of bounded expansion includes all the proper minor closed classes of graphs. Thus, by Corollary 6, proper minor closed classes have bounded lid-chromatic number. In this section, we focus on these latter classes and give an alternative upper bound on the lid-chromatic number. This gives us an explicit upper bound for the lid-chromatic number of planar graphs.

Consider any proper minor closed class of graphs $C$. Since $C$ is proper, there exists $n$ such that $G$ does not contain $K_n$, that is $G \subseteq \mathcal{K}_n$. Let $\mathcal{E}^N$ be the class of graphs defined by $H \in \mathcal{E}^N$ if and only if there exists $G \in C$ and $v \in G$ such that $H = G[N(v)]$. Note that $\mathcal{E}^N$ is a minor-closed class of graphs. Indeed, given any $H \in \mathcal{E}^N$, let $G \in C$ and $v \in V(G)$ such that $H = G[N(v)]$. Let $H'$ be any minor of $H$. Since $\mathcal{E}$ is minor-closed and $H$ is a subgraph of $G$, there exists a minor $G'$ of $G$ such that $H' = G'[N(v)]$. Therefore, $H'$ belongs to $\mathcal{E}^N$.

We prove the following result on minor-closed classes of graphs:

**Theorem 7.** Let $C$ be a proper minor closed class of graphs and let $n \geq 3$ be such that $C \subseteq \mathcal{K}_n$. Then

$$\chi_{lid}(C) \leq 4 \cdot \chi_{lid}(\mathcal{E}^N) \cdot \chi(C)^{n-3}$$

The class of trees is exactly the class $\mathcal{K}_3$. Esperet et al. [6] proved the following result.

**Proposition 8 ([6]).** $\chi_{lid}(\mathcal{K}_3) \leq 4$.

It is clear that $\mathcal{K}_3^N$ is the class of stable graphs and therefore, $\chi_{lid}(\mathcal{K}_3^N) = 1$. Note that Theorem 7 implies Proposition 8.

Assume that $\chi_{lid}(\mathcal{K}_{n-1})$ is bounded for some $n \geq 4$. It is clear that $\mathcal{K}_n^N = \mathcal{K}_{n-1}$. Then, by Theorem 7, we have $\chi_{lid}(\mathcal{K}_n) \leq 4 \cdot \chi_{lid}(\mathcal{K}_{n-1}) \cdot \chi(\mathcal{K}_n)^{n-3}$. Since $\chi_{lid}(\mathcal{K}_{n-1})$ and $\chi(\mathcal{K}_n)$ are bounded, $\chi_{lid}(\mathcal{K}_n)$ is bounded.

Esperet et al. [6] also proved the following result.

**Proposition 9 ([6]).** If $G$ is an outerplanar graph, $\chi_{lid}(G) \leq 20$.

We can then deduce from Theorem 7 and Proposition 9 the following corollary:

**Corollary 10.** Let $\mathcal{P}$ be the class of planar graphs. Then $\chi_{lid}(\mathcal{P}) \leq 1280$.

**Proof.** Any graph $G \in \mathcal{P}$ is a $\{K_{3,3}, K_5\}$-minor free and thus $\mathcal{P}$ is a proper minor closed class of graphs. Moreover, the neighborhood of any vertex of $G \in \mathcal{P}$ is an outerplanar graph. By Proposition 9, we have $\chi_{lid}(\mathcal{P}^N) \leq 20$. Furthermore, the Four-Color-Theorem gives $\chi(\mathcal{P}) = 4$. By Theorem 7, $\chi_{lid}(\mathcal{P}) \leq 4 \times 20 \times 4^2 = 1280$.

We finally give the proof of Theorem 7.

**Proof of Theorem 7.** Let $G \in C$ and let $u$ be a vertex of minimum degree. For any $i$, define $V_{u,i}$ as the set of vertices of $G$ at distance exactly $i$ from $u$ and let $G_{u,i} = G[V_{u,i}]$. Let $s$ be the largest distance from a vertex of $V$ to $u$. In other words, there are $s + 1$ nonempty sets $V_{u,i}$ (note that $V_{u,0} = \{u\}$).

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For any \( i \), contracting in \( G \) the subgraph \( G[V_{u,0} \cup V_{u,1} \cup \ldots \cup V_{u,i-1}] \) in a single vertex \( x \) gives a graph \( G' \in \mathcal{C} \) such that \( x \) is exactly adjacent to every vertex of \( G_{u,i} \). Therefore, for any \( i \), \( G_{u,i} \in \mathcal{C}^N \). Hence, \( \chi_{\text{id}}(G_{u,i}) \leq \chi_{\text{id}}(\mathcal{C}^N) \) for any \( i \). Moreover, \( \mathcal{C}^N \subseteq \mathcal{K}_{n-1} \). Indeed, suppose that there exists \( H \in \mathcal{C}^N \) that admits \( K_{n-1} \) as a minor. Therefore there exists \( G \in \mathcal{C} \) such that \( H = G[N(v)] \) for some \( v \in G \). Taking \( v \) together with its neighborhood would give \( K_n \) as a minor, that contradicts the fact that \( \mathcal{C} \subseteq \mathcal{K}_{n-1} \). Hence, any \( G_{u,i} \in \mathcal{K}_{n-1} \).

We construct a lid-coloring of \( G \) using \( 4 \cdot \chi_{\text{id}}(\mathcal{C}^N) \cdot \chi(\mathcal{C})^{n-3} \) colors. This coloring is constructed with three different colorings of the vertices of \( G \): \( c_1 \) which uses 4 colors, \( c_2 \) which uses \( \chi_{\text{id}}(\mathcal{C}^N) \) colors and \( c_3 \) which is itself composed of \( n-3 \) colorings with \( \chi(\mathcal{C}) \) colors. The final color \( c(v) \) of a vertex \( v \) will be the triplet \( (c_1(v), c_2(v), c_3(v)) \). Hence the coloring \( c \) uses at most \( 4 \chi_{\text{id}}(\mathcal{C}^N) \chi(\mathcal{C})^{n-3} \) colors. The coloring \( c_3 \) is used to separate the pairs of vertices that lie in distinct sets \( V_{u,i} \). The coloring \( c_2 \) separates the pairs of vertices that lie in the same set \( V_{u,i} \) and are not twins in \( G_{u,i} \). Finally, the coloring \( c_3 \) separates the pairs of vertices that lie in the same set \( V_{u,i} \), that are twins in \( G_{u,i} \) but that are not twins in \( G \).

The coloring \( c_1 \) is simply defined by \( c_1(v) \equiv i \mod 4 \) if \( v \in V_{u,i} \).

To define \( c_2 \), we define for each \( i \), \( 0 \leq i \leq s \), a lid-coloring \( c_2^i \) of \( G_{u,i} \) using colors 1 to \( \chi_{\text{id}}(\mathcal{C}^N) \). Then \( c_2 \) is defined by \( c_2(v) = c_2^i(v) \) if \( v \in V_{u,i} \).

We now define the coloring \( c_3 \). Let \( V_{u,i}^{id} \) be the set of vertices of \( V_{u,i} \) that have a twin in \( G_{u,i} \):

\[
V_{u,i}^{id} = \{ v \in V_{u,i} \mid \exists w \in V_{u,i}, N_{G_{u,i}}[v] = N_{G_{u,i}}[w] \}.
\]

Let \( G_{u,i}^{id} = G_{u,i}[V_{u,i}^{id}] \). Since the relation “be twin” is transitive (i.e. if \( u \) and \( v \) are twins, and \( v \) and \( w \) are twins, then \( u \) and \( w \) are twins), then \( G_{u,i}^{id} \) is clearly a union of cliques. In addition, since \( G_{u,i} \in \mathcal{K}_{n-1} \), the connected components of \( G_{u,i}^{id} \) are cliques of size at most \( n-2 \).

Let \( C \) be a clique of \( G_{u,i}^{id} \). By Corollary 2, there exists a subset \( S(C) \subseteq V(G) \) of at most \( n-3 \) vertices that distinguishes all the pairs of non-twin vertices of \( C \). Note that by definition of \( C \), \( S(C) \cap V_{u,i} = \emptyset \), and thus \( S(C) \subseteq V_{u,i-1} \cup V_{u,i+1} \).

Let \( S = \{(v, C) \mid v \in S(C) \text{ and } C \text{ is a clique in a graph } G_{u,i}^{id}\} \). We partition \( S \) in \( s \times (n-3) \) sets \( S^k \), \( 1 \leq i \leq s \), \( 1 \leq k \leq n-3 \), such that:

- if \((v, C) \in S^k \) for some \( k \), then \( v \in V_{u,i} \);
- if \((v, C) \) and \((w, C') \) are two elements of \( S^k \), then \( C \neq C' \).

This partition can be done because each set \( S(C) \) has size at most \( n-3 \).

For each \( S^k \), \( \{(x_1, C_1), (x_2, C_2), \ldots, (x_1, C_3)\} \), we define a graph \( H^k_i \) as follows. We start from the graph induced by \( V_{u,i} \cup V(C_1) \cup V(C_2) \cup \ldots \cup V(C_3) \). Then, for each \( (x_j, C_j) \) in \( S^k \), we contract \( C_j \) in a single vertex \( y_j \) and finally, we contract the edge \( x_jy_j \) on the vertex \( x_j \). Note that \( V_{u,i} \) is the vertex set of \( H^k_i \). Note also that \( H^k_i \in \mathcal{C} \) since it is obtained from a subgraph of \( G \) by successive edge-contractions. Therefore, \( \chi(H^k_i) \leq \chi(\mathcal{C}) \).
We now define a proper coloring \( c^k_{i,j} \) of \( H^k_i \) with colors 1 to \( \chi(G) \). Let \( c^k_i \) be the coloring of vertices of \( G \) defined by \( c^k_3(v) = c^k_{i,j}(v) \) if \( v \in V_{u,i} \). Finally, \( c_3 \) is defined by \( c_3(v) = (c_3^1(v), \ldots, c_3^{n-3}(v)) \), and the final color of \( v \) is \( c(v) = (c_1(v), c_2(v), c_3(v)) \).

We now prove that \( c \) is a lid-coloring of \( G \). First, \( c \) is a proper coloring. Indeed, two adjacent vertices that are not in the same set \( V_{u,i} \) have different colors in \( c_1 \), and two adjacent vertices in the same set \( V_{u,i} \) have different colors in \( c_2 \) (which induces a proper coloring on \( V_{u,i} \)).

Let now \( x \) and \( y \) be two adjacent vertices with \( N[x] \neq N[y] \). We will prove that \( c(N[x]) \neq c(N[y]) \). We distinguish three cases.

Case 1: \( x \in V_{u,i} \) and \( y \in V_{u,i+1} \).

If \( x = u \), then \( y \) has a neighbor \( v \) in \( V_{u,i+2} = V_{u,2} \). Indeed, \( u \) is taken with minimum degree, so \( y \) has at least as many neighbors as \( u \) and does not have the same neighborhood than \( u \), implying that \( y \) has a neighbor in \( V_{u,2} \). Then \( c_1(v) = 2 \not\in c_1(N[u]) \) and \( c(N[x]) \neq c(N[y]) \).

Otherwise, \( x \) has neighbor \( v \) in \( V_{u,i-1} \) and \( c_1(v) \equiv i - 1 \) (mod 4) \( \in c_1(N[x]) \). On the other hand, all the neighbors of \( y \) belong to \( V_{u,i} \cup V_{u,i+1} \cup V_{u,i+2} \) and therefore \( c_1(N[y]) \subseteq \{i, i + 1, i + 2 \ (\text{mod } 4)\} \). Thus, \( c(N[x]) \neq c(N[y]) \).

Case 2: \( x \) and \( y \) belong to \( V_{u,i} \) and they are not twins in \( V_{u,i} \) (i.e. \( N_{V_{u,i}}[x] \neq N_{V_{u,i}}[y] \)).

By definition of the coloring \( c_2 \), there exists a color \( a \) that separates \( x \) and \( y \), i.e. \( a \in c_2(N_{V_{u,i}}[x]) \not\Delta c_2(N_{V_{u,i}}[y]) \). Then we necessarily have \( c(N[x]) \neq c(N[y]) \).

Case 3: \( x \) and \( y \) belong to \( V_{u,i} \) and they are twins in \( V_{u,i} \) (i.e. \( N_{V_{u,i}}[x] = N_{V_{u,i}}[y] \)).

In this case, vertices \( x \) and \( y \) are in the set \( V_{u,i}^{2j} \). Let \( C \) be the clique of \( G_{u,i} \) containing \( x \) and \( y \). Let \( v \in S(C) \) that distinguishes \( x \) and \( y \); thus, \( v \in V_{u,j} \) for \( j = i - 1 \) or \( j = i + 1 \). Wlog, \( v \in N[x] \) but \( v \not\in N[y] \). Let \( S^k_i \) be the part of \( S \) that contains \( (v, C) \). Suppose that there exists a neighbor \( w \) of \( y \) such that \( c(v) = c(w) \). Then \( w \) lies in \( V_{u,j} \) because of the coloring \( c_1 \). However, in the graph \( H^k_j \), the vertex \( v \) is adjacent to all the neighbors of \( y \) in \( V_{u,j} \), and in particular is adjacent to \( w \); therefore, \( c^k_{i,j}(v) \neq c^k_{i,j}(w) \), a contradiction. Therefore, the vertex \( y \) does not have any neighbor that has the same color as \( v \). Hence, \( c(v) \not\in c(N[y]) \), and \( c(N[x]) \neq c(N[y]) \).

\[ \square \]
References


