Locally identifying coloring in bounded expansion classes of graphs
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Abstract
A proper vertex coloring of a graph is said to be locally identifying if (i) the vertex-coloring is proper (i.e. no adjacent vertices receive the same color), and (ii) for any adjacent vertices $u, v$, the set of colors assigned to the closed neighborhood of $u$ differs from the set of colors assigned to the closed neighborhood of $v$ whenever these neighborhoods are distinct. The locally identifying chromatic number of a graph $G$ (or lid-chromatic number, for short), denoted by $\chi_{lid}(G)$, is the smallest number of colors required in any locally identifying coloring of $G$.

1 Introduction
A vertex-coloring is said to be locally identifying if (i) the vertex-coloring is proper (i.e. no adjacent vertices receive the same color), and (ii) for any adjacent vertices $u, v$, the set of colors assigned to the closed neighborhood of $u$ differs from the set of colors assigned to the closed neighborhood of $v$ whenever these neighborhoods are distinct. The locally identifying chromatic number of the graph $G$ (or lid-chromatic number, for short), denoted by $\chi_{lid}(G)$, is the smallest number of colors required in any locally identifying coloring of $G$.

Locally identifying colorings of graphs have been recently introduced by Esperet et al. [6] and later studied by Foucaud et al. [7]. They are related to identifying codes [8, 9], distinguishing colorings [1, 3, 5] and locating-colorings [4]. For
example, upper bounds on lid-chromatic number have been obtained for bipartite graphs, $k$-trees, outerplanar graphs and bounded degree graphs. An open question asked by Esperet et al. [6] was to know whether $\chi_{lid}$ is bounded for the class of planar graphs. In this paper, we answer positively to this question proving more generally that $\chi_{lid}$ is bounded for any class of bounded expansion.

In Section 3, we first give a tight bound of $\chi_{lid}$ in term of the tree-depth.

Then we use the fact that any class of bounded expansion admits a low tree-depth coloring (that is a $k$-coloring such that each triplet of colors induces a graph of tree-depth 3, for some constant $k$) to prove that it has bounded lid-chromatic number.

In Section 4, we focus on minor closed classes of graphs which have bounded expansion and give an alternative bound on the lid-chromatic number, which gives an explicit bound for planar graphs.

The next section is devoted to introduce notation and preliminary results.

2 Notation and preliminary results

Let $G = (V, E)$ be a graph. For any vertex $u$, we denote by $NC(u)$ its neighborhood in $G$ and by $NC[u]$ its closed neighborhood in $G$ ($u$ together with its adjacent vertices). The notion of neighborhood can be extended to sets as follows: for $X \subseteq V$, $NC[X] = \{w \in V(G) \mid \exists v \in X, w \in N[v]\}$ and $NC(X) = NC[X] \setminus X$. When the considered graph is clearly identified, the subscript is dropped.

The degree of vertex $u$ is the size of its neighborhood. The distance between two vertices $u$ and $v$ is the number of edges in a shortest path between $u$ and $v$. For $X \subseteq V$, we denote by $G[X]$ the subgraph of $G$ induced by $X$.

We say that two vertices $u$ and $v$ are twins if $N[u] = N[v]$ (although they are often called true twins in the literature, we call them twins for convenience). In particular, $u$ and $v$ are adjacent vertices. Note that if $u$ and $v$ are adjacent but not twins, there exists a vertex $w$ which is adjacent to exactly one vertex among $\{u, v\}$, i.e. $w \in N[u] \Delta N[v]$ (where $\Delta$ is the symmetric difference between sets). We say that $w$ distinguishes $u$ and $v$, or simply $w$ distinguishes the edge $uv$. For a subset $X \subseteq V$, we say that a subset $Y \subseteq V$ distinguishes $X$ if for every pair $u, v$ of non-twin vertices of $X$, there exists a vertex $w \in Y$ that distinguishes the edge $uv$.

Let $c : V \rightarrow \mathbb{N}$ be a vertex-coloring of $G$. The coloring $c$ is proper if adjacent vertices have distinct colors. We denote by $\chi(G)$ the chromatic number of $G$, i.e. the minimum number of colors in a proper coloring of $G$. For any $X \subseteq V$, let $c(X)$ be the set of colors that appear on the vertices of $X$. A locally identifying coloring (lid-coloring for short) of $G$ is a proper vertex-coloring $c$ of $G$ such that for any two adjacent vertices $u$ and $v$ that are not twins (i.e. $N[u] \neq N[v]$), we have $c(N[u]) \neq c(N[v])$. A graph $G$ is $k$-lid-colorable if it admits a locally identifying coloring using at most $k$ colors and the minimum number of colors needed for any locally identifying coloring of $G$ is the locally identifying chromatic number (lid-chromatic number for short) denoted by $\chi_{lid}(G)$. For a vertex $u$, we say that $u$ sees color $a$ if $a \in c(N[u])$. For two adjacent vertices
of non-twin vertices of $C$

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The edge $v \rightarrow x$ where $x$ is a vertex such that for any distinct elements $N \mid A$ there exists $w \in \mathcal{S}(G)$ of size at most $n − 1$ such that the sets $A_i \cap X'$ are all distinct.

**Theorem 1** (Bondy’s theorem [2]). Let $A = \{A_1, . . . , A_n\}$ be a collection of $n$ distinct subsets of a finite set $X$. There exists a subset $X'$ of $X$ of size at most $n − 1$ such that the sets $A_i \cap X'$ are all distinct.

**Corollary 2.** Let $C$ be a $n$-clique subgraph of $G$. There exists a vertex subset $S(C) \subseteq V(G)$ of size at most $n − 1$ that distinguishes all the pair of non-twin vertices of $C$.

**Proof.** Let $C$ be a $n$-clique subgraph of $G$ induced by the vertex set $V(C) = \{v_1, v_2, . . . , v_n\}$. Let $A = \{N[v_i] \mid v_i \in V(C)\}$ be a collection of distinct subsets of the finite set $X = \bigcup_{1 \leq i \leq n} N[v_i]$. Note that some $v_i$'s might be twins in $G$ (i.e., $N[v_i] = N[v_j]$ for some $v_i, v_j \in V(C)$) and therefore $|A|$ could be smaller than $n$. By Bondy Theorem, there exists $S(C) \subseteq X$ of size at most $|A| − 1 \leq n − 1$ such that for any distinct elements $A_1, A_2$ of $A$, we have $A_1 \cap S(C) \neq A_2 \cap S(C)$.

Let us prove that $S(C)$ is a set of vertices that distinguish all the pairs of non-twin vertices of $C$. For a pair of non-twin vertices $v_i, v_j$ of $C$, we have $N[v_i] \neq N[v_j]$. By definition of $S(C)$, we have $N[v_i] \cap S(C) \neq N[v_j] \cap S(C)$, then there exists $w \in S(C)$ that belongs to $N[v_i] \Delta N[v_j]$. Therefore, $w$ distinguishes the edge $v_i v_j$.

3 Bounded expansion classes of graphs

A rooted tree is a tree with a special vertex, called the root. The height of a vertex $x$ in a rooted tree is the number of vertices on a path from the root to $x$ (hence, the height of the root is 1). The height of a rooted tree $T$ is the maximum height of the vertices of $T$. If $x$ and $y$ are two vertices of $T$, $x$ is an ancestor of $y$ in $T$ if $x$ belongs to the path between $y$ and the root. The closure $\text{clos}(T)$ of a rooted tree $T$ is the graph with vertex set $V(T)$ and edge set $\{xy \mid x$ is an ancestor of $y$ in $T, x \neq y\}$. The tree-depth $\text{td}(G)$ of a connected graph $G$ is the minimum height of a rooted tree $T$ such that $G$ is a subgraph of $\text{clos}(T)$. If $G$ is not connected, the tree-depth of $G$ is the maximum tree-depth of its connected components.

Let $p$ be a fixed integer. A low tree-depth coloring of a graph $G$ (relatively to $p$) is a coloring of the vertices of $G$ such that the union of any $i \leq p$ color classes induces a graph of tree-depth at most $i$. Let $\chi_{td}^i(G)$ be the minimum number of colors required in such a coloring. Note that as tree-depth one graphs and tree-depth two graphs are respectively the stables and star forests, $\chi_{td}^1$ and $\chi_{td}^2$ respectively correspond to the usual chromatic number and the star chromatic number.
In the following of this section, we first give a tight bound on the lid-chromatic number in terms of tree-depth.

**Proposition 3.** For any graph $G$, $\chi_{lid}(G) \leq 2td(G) - 1$ and this is tight.

Using this bound, we then bound the lid-chromatic number in terms of $\chi^{td}_{3}$.

**Theorem 4.** For any graph $G$,
\[\chi_{lid}(G) \leq 6^{\left(\frac{\chi^{td}_{3}}{3(G)}\right)}\]

Classes of graphs of bounded expansion have been introduced by Nešetřil and Ossona de Mendez [10]. These classes contain minor closed classes of graphs and any class of graphs defined by an excluded topological minor. Actually, these classes of graphs are closely related to low tree-depth colorings:

**Theorem 5** (Theorem 7.1 [10]). A class of graphs $\mathcal{C}$ has bounded expansion if and only if $\chi^{td}_{p}(\mathcal{C})$ is bounded for any $p$.

We therefore deduce the following corollary from Theorems 4 and 5:

**Corollary 6.** For any class $\mathcal{C}$ of bounded expansion, $\chi_{lid}(\mathcal{C})$ is bounded.

It is in particular true for a class of bounded tree-width. A consequence is that $\chi_{lid}$ is bounded for chordal graphs by a function of the clique number (which is equals to the tree-width plus 1 for a chordal graph). It is conjectured by Esperet et al. [6] that $\chi_{lid}(G) \leq 2\omega(G)$ if $G$ is chordal.

We now prove Proposition 3.

**Proof of Proposition 3.** Let us first prove that the bound is tight. Consider the graph $H_n$ obtained from a complete graph, with vertex set $\{a_1, \ldots, a_n\}$, by adding a pendant vertex $b_i$ to every $a_i$ but one, say for $1 \leq i < n$. The tree-depth of this graph is at least $n$ as it contains a $n$-clique. Indeed, given a rooted tree $T$, two vertices at the same height are non-adjacent in clos($T$), we thus need at least $n$ levels. Actually the tree-depth of this graph is at most $n$ since the tree $T$ rooted at $a_1$, and such that $a_i$ has two sons $a_{i+1}$ and $b_i$, for $1 \leq i < n$, has height $n$ and is such that clos($T$) contains $H_n$ as a subgraph.

Let us show that in any lid-coloring of $H_n$ all the vertices must have distinct colors, and thus use $2n - 1 = 2td(H_n) - 1$ colors. Indeed, two vertices $a_i$ must have different colors as the coloring is proper. A vertex $b_j$ cannot use the same color as a vertex $a_i$, as otherwise the vertex $a_j$ would only see the $n$ colors used in the clique, just as $a_n$. Similarly if two vertices $b_i$ and $b_j$ would use the same color, the vertices $a_i$ and $a_j$ would see the same set of colors.

Let us now focus on the upper bound. We prove the result for a connected graph and by induction on the tree-depth of $G$, denoted by $k$. The result is clear for $k = 1$ (the graph is a single vertex).

Let $G$ be a graph of tree-depth $k > 1$ and let $T$ be a rooted tree of height $k$ such that $G$ is a subgraph of clos($T$). If $T$ is a path, the result is clear since there are only $k$ vertices. So assume that $T$ is not a path, and let $r$ be the root
of $T$. Let $s$ be the smallest height such that there are at least two vertices of height $s + 1$. We name $r_i$, for $i \in \{1, \ldots, s\}$, the unique vertex of height $i$. Let $R = \{r_1, \ldots, r_s\}$. Note that each of the vertices of $R$ is adjacent to all the vertices of $\text{clos}(T)$, Therefore, we can choose the way we label the $s$ vertices in $R$ (i.e. we can choose the height of each of them in $T$) without changing $\text{clos}(T)$.

Necessarily, $G \setminus R$ has at least two connected components. Let $G_1, \ldots, G_\ell$ be its connected components and thus $\ell \geq 2$. We choose $T$ such that $s$ is minimal. It implies that for each $i \in \{1, \ldots, s\}$, $r_i$ has neighbors in all the components $G_1, \ldots, G_\ell$. Indeed, if it is not the case, by permuting the elements of $R$ (this is possible by the above remark), we can assume without loss of generality that $r_s$ does not have a neighbor in $G_\ell$. Therefore, the set of edges $e(r_s, G_\ell) = \{r_s x : x \in V(G_\ell)\}$ of $\text{clos}(T)$ are not used by $G$. Then let $T'$ be the tree obtained from $T$ by moving the whole component $G_\ell$ one level up in such a way that the root of the subtree corresponding to $G_\ell$ is now the son of $r_{s-1}$ (instead of $r_s$ previously). Note that $\text{clos}(T')$ is isomorphic to $\text{clos}(T) \setminus e(r_s, G_\ell)$ and thus $G$ is a subgraph of $\text{clos}(T')$. This new tree $T'$ has two vertices at height $s$, contradicting the minimality of $s$.

Any connected component $G_j$ has tree-depth at most $k' = k - s < k$. By induction, for each $j \in \{1, \ldots, \ell\}$, there exists a lid-coloring $c_j$ of $G_j$ using colors in $\{1, \ldots, 2k' - 1\}$. For each $c_j$, there is a minimum value $s_j$ such that every vertex $r_i$ sees a color in $\{1, \ldots, s_j\}$ in $G_j$. We choose a $(2k' - 1)$-lid-coloring $c_j$ of $G_j$ such that $s_j$ is minimized. Note that for each color $a \leq s_j$, there exists $r_i \in R$ such that $r_i$ sees color $a$ in $G_j$ but no other color of $\{1, \ldots, s_j\}$. Otherwise, after permuting colors $a$ and $s_j$, every vertex $r_i \in R$ would see a color in $\{1, \ldots, s_j - 1\}$, contradicting the minimality of $s_j$. Assume without loss of generality that $s_1 \geq s_2 \geq \ldots \geq s_\ell$.

We replace in $c_j$ the colors $1, 2, \ldots, s_1$ by $1', 2', \ldots, s'_1$. Note that now each vertex $r_i$ sees a color in $\{1', \ldots, s'_1\}$ (in $G_1$) and a color in $\{1, \ldots, s_2\}$ (in $G_2$). Furthermore, the other vertices of $G$ (that is the vertices in $G_1, \ldots, G_\ell$) do not have this property since $s_1 \geq s_2$. Thus at this step every edge $xr_i$ with $x$ in some $G_j$ is separated.

Now we color each vertex $r_i$ with color $i^*$. Let $c : V(G) \rightarrow \{1^*, \ldots, s^*\} \cup \{1', \ldots, s'_1\} \cup \{1, \ldots, 2k' - 1\}$ be the current coloring of $G$.

Note that now every distinguishable edge $xy$ in some $G_j$ is separated. Indeed, either $xy$ was distinguished in $G_j$ and it has been separated by $c_j$, or $xy$ is distinguished by some $r_i$ and it is separated by the color $i^*$. Note also that $c$ is a proper coloring.

It remains to deal with the edges $r_ir_j$. For that purpose we will refine some color classes. In the following lemma we show that such refinements do not damage what we have done so far.

**Claim.** Consider a graph $G$ and a coloring $\varphi : V(G) \rightarrow \{1, \ldots, k\}$. Consider any refinement $\varphi'$ of $\varphi$, obtained from $\varphi$ by recoloring with color $k + 1$ some vertices colored $i$, for some $i$. Any edge $xy$ of $G$ properly colored (resp. separated) by $\varphi$ is properly colored (resp. separated) by $\varphi'$.

Indeed if $\varphi(x) \neq \varphi(y)$ then $\varphi'(x) \neq \varphi'(y)$, and if $i \in \varphi(N[x]) \Delta \varphi(N[y])$ then
The class $C$ is proper minor closed if and only if $c(N[r_i]) = c(N[r_j])$. Let $R_1, \ldots, R_n$ be the equivalence classes of the relation $R$ (note that each $R_i$ forms a clique since every $r_i$ has distinct colors). We have $s \geq s_1$. Indeed, by definition of $s_1$ and the coloring $c_1$, for each color $a \in \{1', \ldots, s'_1\}$, there exists $r_i \in R$ that sees $a$ in $G_1$ but no other color of $\{1', \ldots, s'_1\}$. This vertex $r_i$ belongs to some equivalence class $R_j$ and thus all the vertices of $R_j$ sees color $a$ in $G_1$ but no other color of $\{1', \ldots, s'_1\}$.

By Corollary 2, there is a vertex set $S(R_i)$ of size at most $|R_i| - 1$ which distinguishes all pairs of non-twin vertices in $R_i$. We give to the vertices of $S(R_i)$ new distinct colors. By the previous claim, this last operation does not damage the coloring, and now all the distinguishable edges are separated.

Since for this last operation we need $s - s'$ new colors, since we used $2k' - 1$ colors $\{1, \ldots, 2k' - 1\}$, $s_1$ colors $\{1', \ldots, s'_1\}$ and $s$ colors $\{1^*, \ldots, s^*\}$, the total number of colors is $(s - s') + (2k' - 1) + s_1 + s = 2k - 1 + s_1 - s \leq 2k - 1$. This concludes the proof of the theorem. □

We are now ready to prove Theorem 4:

**Proof of Theorem 4.** Let $\alpha$ be a low tree-depth coloring of $G$ with parameter $p = 3$ and using $x^\text{td}(G)$ colors. Let $A = \{\alpha_1, \alpha_2, \alpha_3\}$ be a triplet of three distinct colors and let $H_A$ be the subgraph of $G$ induced by the vertices colored by a color of $A$. Since $H_A$ has tree-depth at most 3, by Proposition 3, $H_A$ admits a lid-coloring $c_A$ with five colors (says colors 1 to 5). We extend $c_A$ to the whole graph by giving color 0 to the vertices in $V(G) \setminus V(H_A)$.

Let $A_1, A_2, \ldots, A_k$ be the $k = \{x^\text{td}(G)\}$ distinct triplets of colors. We now construct a coloring $c$ of $G$ giving to each vertex $x$ of $G$ the $k$-uplet

$$(c_{A_1}(x), c_{A_2}(x), \ldots, c_{A_k}(x)).$$

The coloring $c$ is using $6^k$ colors. Clearly it is a proper coloring: each pair of adjacent vertices will be in some common graph $H_A$ and will receive distinct colors in this graph. Let $x$ and $y$ be two adjacent vertices with $N[x] \neq N[y]$. Let $w$ be a vertex adjacent to only one vertex among $x$ and $y$. Let $A = \{\alpha(x), \alpha(y), \alpha(w)\}$. Vertices $x$ and $y$ are not twins in the graph $H_A$. Hence $c_A(N[x]) \neq c_A(N[y])$ and therefore, $c(N[x]) \neq c(N[y])$. □

## 4 Minor closed classes of graphs

Let $G$ and $H$ be two graphs. $H$ is a minor of $G$ if $H$ can be obtained from $G$ with successive edge deletions, vertex deletions and edge contractions. A class $\mathcal{C}$ is minor closed if for any graph $G$ of $\mathcal{C}$, for any minor $H$ of $G$, we have $H \in \mathcal{C}$. The class $\mathcal{C}$ is proper if it is not the class of all graphs. Let $H$ be a graph. A $H$-minor free graph is a graph that does not have $H$ as a minor. We denote by $\mathcal{K}_n$ the $K_n$-minor-free class of graphs. It is clear that any proper minor closed class of graphs is included in the class $\mathcal{K}_n$ for some $n$. It is folklore that any proper minor closed class of graphs $\mathcal{C}$ has a bounded chromatic number $\chi(\mathcal{C})$. 6
The class of graphs of bounded expansion includes all the proper minor closed classes of graphs. Thus, by Corollary 6, proper minor closed classes have bounded lid-chromatic number. In this section, we focus on these latter classes and give an alternative upper bound on the lid-chromatic number. This gives us an explicit upper bound for the lid-chromatic number of planar graphs.

Consider any proper minor closed class of graphs \( \mathcal{C} \). Since \( \mathcal{C} \) is proper, there exists \( n \) such that \( \mathcal{C} \) does not contain \( K_n \), that is \( \mathcal{C} \subseteq \mathcal{K}_n \). Let \( \mathcal{C}^N \) be the class of graphs defined by \( H \in \mathcal{C}^N \) if and only if there exists \( G \in \mathcal{C} \) and \( v \in G \) such that \( H = G[N(v)] \). Note that \( \mathcal{C}^N \) is a minor-closed class of graphs. Indeed, given any \( H \in \mathcal{C}^N \), let \( G \in \mathcal{C} \) and \( v \in V(G) \) such that \( H = G[N(v)] \). Let \( H' \) be any minor of \( H \). Since \( \mathcal{C} \) is minor-closed and \( H \) is a subgraph of \( G \), there exists a minor \( G' \) of \( G \) such that \( H' = G'[N(v)] \). Therefore, \( H' \) belongs to \( \mathcal{C}^N \).

We prove the following result on minor-closed classes of graphs:

**Theorem 7.** Let \( \mathcal{C} \) be a proper minor closed class of graphs and let \( n \geq 3 \) be such that \( \mathcal{C} \subseteq \mathcal{K}_n \). Then

\[
\chi_{\text{lid}}(\mathcal{C}) \leq 4 \cdot \chi_{\text{lid}}(\mathcal{C}^N) \cdot \chi(\mathcal{C})^{n-3}
\]

The class of trees is exactly the class \( \mathcal{K}_3 \). Esperet et al. [6] proved the following result.

**Proposition 8** ([6]). \( \chi_{\text{lid}}(\mathcal{K}_3) \leq 4 \).

It is clear that \( \mathcal{K}_3^N \) is the class of stable graphs and therefore, \( \chi_{\text{lid}}(\mathcal{K}_3^N) = 1 \). Note that Theorem 7 implies Proposition 8.

Assume that \( \chi_{\text{lid}}(\mathcal{K}_{n-1}) \) is bounded for some \( n \geq 4 \). It is clear that \( \mathcal{K}_n^N = \mathcal{K}_{n-1} \). Then, by Theorem 7, we have \( \chi_{\text{lid}}(\mathcal{K}_n) \leq 4 \cdot \chi_{\text{lid}}(\mathcal{K}_{n-1}) \cdot \chi(\mathcal{K}_n)^{n-3} \).

Since \( \chi_{\text{lid}}(\mathcal{K}_{n-1}) \) and \( \chi(\mathcal{K}_n) \) are bounded, \( \chi_{\text{lid}}(\mathcal{K}_n) \) is bounded.

Esperet et al. [6] also proved the following result.

**Proposition 9** ([6]). If \( G \) is an outerplanar graph, \( \chi_{\text{lid}}(G) \leq 20 \).

We can then deduce from Theorem 7 and Proposition 9 the following corollary:

**Corollary 10.** Let \( \mathcal{P} \) be the class of planar graphs. Then \( \chi_{\text{lid}}(\mathcal{P}) \leq 1280 \).

**Proof.** Any graph \( G \in \mathcal{P} \) is \( \{K_{3,3}, K_5\} \)-minor free and thus \( \mathcal{P} \) is a proper minor closed class of graphs. Moreover, the neighborhood of any vertex of \( G \in \mathcal{P} \) is an outerplanar graph. By Proposition 9, we have \( \chi_{\text{lid}}(\mathcal{P}^N) \leq 20 \). Furthermore, the Four-Color-Theorem gives \( \chi(\mathcal{P}) = 4 \). By Theorem 7, \( \chi_{\text{lid}}(\mathcal{P}) \leq 4 \times 20 \times 4^2 = 1280 \). \( \square \)

We finally give the proof of Theorem 7.

**Proof of Theorem 7.** Let \( G \in \mathcal{C} \) and let \( u \) be a vertex of minimum degree. For any \( i \), define \( V_{u,i} \) as the set of vertices of \( G \) at distance exactly \( i \) from \( u \) and let \( G_{u,i} = G[V_{u,i}] \). Let \( s \) be the largest distance from a vertex of \( V \) to \( u \). In other words, there are \( s + 1 \) nonempty sets \( V_{u,i} \) (note that \( V_{u,0} = \{u\} \)).
For any $i$, contracting in $G$ the subgraph $G[V_{1,0} \cup V_{1,1} \cup \ldots \cup V_{n,1-1}]$ in a single vertex $x$ gives a graph $G' \in \mathcal{C}$ such that $x$ is exactly adjacent to every vertex of $G_{u,i}$. Therefore, for any $i$, $G_{u,i} \in \mathcal{C}_N$. Hence, $\chi_{lid}(G_{u,i}) \leq \chi_{lid}(\mathcal{C}_N)$ for any $i$. Moreover, $\mathcal{C}_N \subseteq \mathcal{K}_{n-1}$. Indeed, suppose that there exists $H \in \mathcal{C}_N$ that admits $K_{n-1}$ as a minor. Therefore there exists $G \in \mathcal{C}$ such that $H \cong G[N(v)]$ for some $v \in G$. Taking $v$ together with its neighborhood would give $K_n$ as a minor, that contradicts the fact that $\mathcal{C} \subseteq \mathcal{K}_n$. Hence, any $G_{u,i} \in \mathcal{K}_{n-1}$.

We construct a lid-coloring of $G$ using $4 \cdot \chi_{lid}(\mathcal{C}_N) \cdot \chi(\mathcal{C})^{n-3}$ colors. This coloring is constructed with three different colorings of the vertices of $G$: $c_1$ which uses $4$ colors, $c_2$ which uses $\chi_{lid}(\mathcal{C}_N)$ colors and $c_3$ which is itself composed of $n-3$ colorings with $\chi(\mathcal{C})$ colors. The final color $c(v)$ of a vertex $v$ will be the triplet $(c_1(v), c_2(v), c_3(v))$. Hence the coloring $c$ uses at most $4 \chi_{lid}(\mathcal{C}_N) \chi(\mathcal{C})^{n-3}$ colors. The coloring $c_1$ is used to separate the pairs of vertices that lie in distinct sets $V_{u,i}$. The coloring $c_2$ separates the pairs of vertices that lie in the same set $V_{u,i}$ and are not twins in $G_{u,i}$. Finally, the coloring $c_3$ separates the pairs of vertices that lie in the same set $V_{u,i}$, that are twins in $G_{u,i}$ but that are not twins in $G$.

The coloring $c_1$ is simply defined by $c_1(v) \equiv i \mod 4$ if $v \in V_{u,i}$.

To define $c_2$, we define for each $i$, $0 \leq i \leq s$, a lid-coloring $c_2^i$ of $G_{u,i}$ using colors $1$ to $\chi_{lid}(\mathcal{C}_N)$. Then $c_2$ is defined by $c_2(v) = c_2^i(v)$ if $v \in V_{u,i}$.

We now define the coloring $c_3$. Let $V_{u,i}^{lid}$ be the set of vertices of $V_{u,i}$ that have a twin in $G_{u,i}$:

$$V_{u,i}^{lid} = \{ v \in V_{u,i} \mid \exists w \in V_{u,i}, N_{G_{u,i}}[v] = N_{G_{u,i}}[w] \}.$$

Let $G_{u,i}^{lid} = G_{u,i}[V_{u,i}^{lid}]$. Since the relation “be twin” is transitive (i.e. if $u$ and $v$ are twins, and $v$ and $w$ are twins, then $u$ and $w$ are twins), then $G_{u,i}^{lid}$ is clearly a union of cliques. In addition, since $G_{u,i} \in \mathcal{K}_{n-1}$, the connected components of $G_{u,i}^{lid}$ are cliques of size at most $n-2$.

Let $C$ be a clique of $G_{u,i}^{lid}$. By Corollary 2, there exists a subset $S(C) \subseteq V(G)$ of at most $n-3$ vertices that distinguishes all the pairs of non-twin vertices of $C$. Note that by definition of $C$, $S(C) \cap V_{u,i} = \emptyset$, and thus $S(C) \subseteq V_{u,i} \cup V_{u,i+1}$.

Let $S = \{(v, C) \mid v \in S(C) \text{ and } C \text{ is a clique in a graph } G_{u,i}\}$. We partition $S$ in $s \times (n-3)$ sets $S_i^k$, $1 \leq i \leq s$, $1 \leq k \leq n-3$, such that:

- if $(v, C) \in S_i^k$ for some $k$, then $v \in V_{u,i}$;
- if $(v, C)$ and $(w, C')$ are two elements of $S_i^k$, then $C \neq C'$.

This partition can be done because each set $S(C)$ has size at most $n-3$.

For each $S_i^k = \{(x_1, C_1), (x_2, C_2), \ldots, (x_t, C_t)\}$, we define a graph $H_i^k$ as follows. We start from the graph induced by $V_{u,i} \cup V(C_1) \cup V(C_2) \cup \ldots \cup V(C_t)$. Then, for each $(x_j, C_j)$ in $S^k_i$, we contract $C_j$ in a single vertex $y_j$ and finally, we contract the edge $x_jy_j$ on the vertex $x_j$. Note that $V_{u,i}$ is the vertex set of $H_i^k$. Note also that $H_i^k \in \mathcal{C}$ since it is obtained from a subgraph of $G$ by successive edge-contractions. Therefore, $\chi(H_i^k) \leq \chi(\mathcal{C})$. 

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Indeed, two adjacent vertices that are not in the same set \(V\), have any neighbor that has the same color as \(c\). Thus, \(x\) and \(y\) are in different sets, implying that \(x\) and \(y\) are twins in \(V\), which is a contradiction. Therefore, the vertex \(y\) does not have any neighbor that has the same color as \(v\). Hence, \(c(v) \notin c(N[y])\), and \(c(N[x]) \neq c(N[y])\).

\[\Box\]
References


