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Abstract

In this article we provide global subelliptic estimates for the linearized inhomogeneous Boltzmann equation without angular cutoff, and show that some global gain in the spatial direction is available although the corresponding operator is not elliptic in this direction. The proof is based on a multiplier method and the so-called Wick quantization, together with a careful analysis of the symbolic properties of the Weyl symbol of the Boltzmann collision operator.

Keywords: global hypoellipticity, subellipticity, Boltzmann equation without cut-off, anisotropic diffusion, Wick quantization

2010 MSC: 35S05, 35H10, 35H20, 35B65, 82C40.

Contents

[*]Arts et Métiers ParisTech, Paris 75013 France

[†]Department of Mathematics and MOE-LSEC, Shanghai Jiao Tong University, Shanghai 200240, China

[‡]Laboratoire de Mathématiques Jean Leray, Université de Nantes, 44322 Nantes, France

[§]School of Mathematics and Statistics, and Computational Science Hubei Key Laboratory, Wuhan University, 430072 Wuhan, China

[¶]e-mail: frederic.herau@univ-nantes.fr, wei-xi.li@whu.edu.cn, radjesvarane.alexandre@paristech.fr

[‖]WXL was supported in part by NSFC(11422106) and Fok Ying Tung Education Foundation (151001), WXL and FH were supported by ANR NOSEVOL (2011 BS01019 01)

1 Introduction

In this paper we are interested in giving sharp subellipitic estimates for the non-homogeneous linearized Boltzmann operator

$$\mathcal{P} = v \cdot \partial_x - \mathcal{L}$$

considered as an unbounded operator in $L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)$, where $\mathcal{L}$ is the linearized Boltzmann without cutoff collision kernel whose precise expression is given in (5) in the next subsection. Here x in $\mathbb{R}^3_x$ and v in $\mathbb{R}^3_v$ are respectively the space and velocity variable and ∂_x denotes the gradient in the space variable. The main result of this paper is the sharp estimate given in Theorem 1.1. In this introduction we first present the model, then the main results including Theorem 1.1 and bibliographic comments and we conclude by giving some general comments about the interest of this work and the methodology we followed for the proofs.

1.1 Model and notations

Let us first recall some facts about the non-cutoff inhomogeneous Boltzmann equation. It reads

$$\partial_t F + v \cdot \partial_x F = Q(F, F), \tag{1}$$

with F standing for a probability density function, and a given Cauchy data at $t = 0$, while the position x and velocity v are in $\mathbb{R}^3$, see [14, 42] and references therein for more details on Boltzmann equation. In (1), the collision kernel Q is defined for sufficiently smooth functions F and G by

$$Q(G, F)(t, x, v) = \int_{\mathbb{R}^3} \int_{S^2} B(v - v_*, \sigma) \left(F' G'_* - F G_*\right) dv_* d\sigma$$

where $F' = F(v')$, $F = F(v)$, $G'_* = G(v'_*)$ and $G_* = G(v_*)$ for short. For given velocities after (or before) collision v and v_*, v' and v'_* are the velocities before (or after) collision, with the following energy and momentum conservation rules, expressing the fact that we consider elastic collisions

$$v' + v'_* = v + v_*, \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2. \tag{2}$$

where $|v|$ denotes the canonical euclidian norm in $\mathbb{R}^3$. We will choose the so-called σ representation, for σ on the sphere S^2,

$$\begin{cases} v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma \\ v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \end{cases}$$

and define the deviation angle θ in a standard way by

$$\cos\theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma,$$

where $\cdot$ denotes the usual scalar product in $\mathbb{R}^3$. In the case of inverse power laws, see for example [14], the collisional cross section B looks approximatively as follows

$$B(v - v_*, \sigma) = |v - v_*|^\gamma \mathrm{b}(\cos\theta), \tag{3}$$

for some real parameter γ and some function b.

Without loss of generality, we assume $B(v-v_*, \sigma)$ is supported on the set $(v-v_*)\cdot\sigma \geq 0$ which corresponds to $\theta \in [0, \pi/2]$, since as usual, see [11], B can be eventually replaced by its symetrized version

$$\overline{B}(v - v_*, \sigma) = B(v - v_*, \sigma) + B(v - v_*, -\sigma).$$

Moreover, we assume that we deal with inverse power interaction laws between particles, and thus according to [14], we assume that b has the following singular behavior when $\theta \in]0, \pi/2[$: there exist a constant $c_b > 0$ such that

$$c_b^{-1}\theta^{-1-2s} \leq \sin\theta \mathrm{b}(\cos\theta) \leq c_b\theta^{-1-2s}, \text{ as } \theta \longrightarrow 0^+.$$

In the preceding formulas, we will impose the following range of parameters, coming from the physical derivation,

$$s \in (0, 1), \qquad \gamma \in (-3, \infty).$$

Note that the last condition on $\gamma + 2s$ is weaker than in [7, 21] since we will deal only with the linearized part of Boltzmann collisional operator.

The behavior of this singular kernel is strongly related to the following non-integrability condition

$$\int_0^{\pi/2} \sin\theta b(\cos\theta)d\theta = \infty,$$

which implies some diffusion properties of the (linearized) Boltzmann operator that we will explain more in depth in a moment.

In some expressions involving the integral kernels, it may therefore happen that some non-integrability arise, and in this case these integrals have to be understood as principal values (see the appendix or [11]). Anyway we shall do most of the computations as if B were integrable and use the principal value trick whenever needed.

In this work, we are interested in the linearized Boltzmann operator, around a normalized Maxwellian distribution, which is described as follows. Let this normalized Maxwellian be

$$\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}.$$

Setting $F = \mu + \sqrt{\mu} f$, the perturbation f satisfies the equation

$$\partial_t f + v \cdot \partial_x f - \mu^{-1/2} Q(\mu, \sqrt{\mu} f) - \mu^{-1/2} Q(\sqrt{\mu} f, \mu) = \mu^{-1/2} Q(\sqrt{\mu} f, \sqrt{\mu} f),$$

since $\partial_t F + v \cdot \partial_x F - Q(F, F) = 0$ and $Q(\mu, \mu) = 0$. Using the notation

$$\tilde{\Gamma}(g, f) = \mu^{-1/2} Q(\sqrt{\mu} g, \sqrt{\mu} f),$$

we may rewrite the above equation as

$$\partial_t f + \mathcal{P} f = \tilde{\Gamma}(f,\ f),$$

where the linearized Boltzmann operator $\mathcal{P}$ takes the form

$$\mathcal{P} = v \cdot \partial_x - \mathcal{L} \tag{4}$$

with

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2, \qquad \mathcal{L}_1 f = \tilde{\Gamma}(\sqrt{\mu},\ f), \quad \mathcal{L}_2 f = \tilde{\Gamma}(f,\ \sqrt{\mu}). \tag{5}$$

The operator $\mathcal{P}$ acts only in variables (x, v), is non selfadjoint, and consists of a transport part which is skew-adjoint, and a diffusion part acting only in the v variable.

The elliptic properties of this operator which is the autonomous linear part of the Boltzmann equation are the main subject of this work and we present them below.

Notations

Throughout the paper we shall adopt the following notations : We work in dimension $d = 3$ and denote by $(x, v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v$ the space-velocity variables. For $v \in \mathbb{R}^3$ we denote $\langle v \rangle = (1 + |v|^2)^{1/2}$, where we recall that $|v|$ is the canonical euclidian norm of v in $\mathbb{R}^3$.

The gradient in velocity (resp. space) will be denoted by ∂_v (resp. ∂_x). We shall also denote $D_v = \frac{1}{i}\partial_v$, $D_x = \frac{1}{i}\partial_x$, and denote ξ the dual variable of x and η the dual variable of v.

We shall extensively use the pseudodifferential theory, for which we refer to the appendix here and the reference therein. In particular operators $\langle D_v \rangle$ and $\langle v \wedge D_v \rangle^{2s}$ denotes respectively the pseudo-differential operator with classical symbol $\langle \eta \rangle$ and $\langle v \wedge \eta \rangle^{2s}$.

We will work througout the paper in $L^2(\mathbb{R}^3_v)$ or $L^2\left(\mathbb{R}^3_x \times \mathbb{R}^3_v\right)$ for which we denote (without ambiguity depending on the sections) the scalar product by $(\cdot, \cdot)$ and the norm by $\|\cdot\|$. We shall mainly work with functions in the Schwartz spaces $\mathcal{S}(\mathbb{R}^3_v)$ or $\mathcal{S}(\mathbb{R}^3_x \times \mathbb{R}^3_v)$.

In all the article, the notation $a \approx b$ (resp. $a \lesssim b$) for a and b positive real means that there is some positive constant C not depending on possible free parameters such that $C^{-1}a \leq b \leq Ca$ (resp. $b \leq Ca$).

1.2 Main results and bibliographic comments

The main theorem of this paper deals with operator $\mathcal{P}$, viewed as an unbounded operator in $L^2\left(\mathbb{R}^3_x \times \mathbb{R}^3_v\right)$. We adopt the conventions of notation given at the end of subsection 1.1

Theorem 1.1. *For all $l \in \mathbb{R}$, there exists a constant C_l such that for all $f \in \mathcal{S}(\mathbb{R}^3_x \times \mathbb{R}^3_v)$, we have*

$$\begin{aligned}
&\|\langle v \rangle^\gamma \langle D_v \rangle^{2s} f\| + \|\langle v \rangle^\gamma \langle v \wedge D_v \rangle^{2s} f\| + \|\langle v \rangle^{\gamma+2s} f\| \\
&\quad + \|\langle v \rangle^{\gamma/(2s+1)} \langle D_x \rangle^{2s/(2s+1)} f\| + \|\langle v \rangle^{\gamma/(2s+1)} \langle v \wedge D_x \rangle^{2s/(2s+1)} f\| \\
&\qquad \leq C_l \left(\|\mathcal{P} f\| + \|\langle v \rangle^l f\|\right),
\end{aligned}$$

Note carefully that we do not need to take into account the finite dimensional kernel associated with the linearized Boltzmann operator [7, 21] which is hidden again in the term $\|f\|$.

As an intermediate result, we are also able to give an explicit form of the so-called triple norm introduced in [7]. Previous estimates from below were also given in [39] and [40], but the following coercivity estimate measures now explicitly the global weights and regularity gains of the diffusion kernel $\mathcal{L}$. Note that we again forget in the following result the fact that there is finite dimensional operator kernel.

Theorem 1.2. *For all $l \in \mathbb{R}$, there exists a constant C_l such that for all for all $f \in \mathcal{S}(\mathbb{R}^3_x \times \mathbb{R}^3_v)$, we have*

$$
\begin{aligned}
C_l^{-1} \Big(\| \langle v \rangle^{\gamma/2} \langle D_v \rangle^s f \|^2 + \| \langle v \rangle^{\gamma/2} \langle v \wedge D_v \rangle^s f \|^2 + \| \langle v \rangle^{\gamma/2+s} f \|^2 \Big) \\
\leq -(\mathcal{L}f, f) + \| \langle v \rangle^l f \|^2 \\
\leq C_l \Big(\| \langle v \rangle^{\gamma/2} \langle D_v \rangle^s f \|^2 + \| \langle v \rangle^{\gamma/2} \langle v \wedge D_v \rangle^s f \|^2 + \| \langle v \rangle^{\gamma/2+s} f \|^2 \Big).
\end{aligned}
$$

Theorem 1.1 can be extended to a time dependent version as follows, by considering the time dependent operator

$$\widetilde{P} = \partial_t + v \cdot \partial_x - \mathcal{L},$$

the functional spaces being now $L^2(\mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v)$ with norm denoted by $\|.\|_{L^2_T}$. With this setting, one can show that

Theorem 1.3. *For all $l \in \mathbb{R}$, there exists a constant C_l such that for all $f \in \mathcal{S}(\mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v)$, we have*

$$
\begin{aligned}
\| \langle v \rangle^{\frac{\gamma-2s}{1+2s}} \langle D_t \rangle^{\frac{2s}{1+2s}} f \|_{L^2_T} + \| \langle v \rangle^{\gamma} \langle D_v \rangle^{2s} f \|^2_{L^2_T} + \| \langle v \rangle^{\gamma} \langle v \wedge D_v \rangle^{2s} f \|^2_{L^2_T} + \| \langle v \rangle^{\gamma+2s} f \|^2_{L^2_T} \\
+ \| \langle v \rangle^{\gamma/(2s+1)} \langle D_x \rangle^{2s/(2s+1)} f \|^2_{L^2_T} + \| \langle v \rangle^{\gamma/(2s+1)} \langle v \wedge D_x \rangle^{2s/(2s+1)} f \|^2_{L^2_T} \\
\leq C_l \Big(\| \widetilde{P} f \|^2_{L^2_T} + \| \langle v \rangle^l f \|^2_{L^2_T} \Big)
\end{aligned}
$$

The preceding results are consequences of fundamental pseudodifferential properties of the linearized Boltzmann operator. Indeed, as we shall see in Section 3, the operator $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ can be splitted as

$$\mathcal{L}_1 = -a^w - \mathcal{K}_1, \qquad \mathcal{L}_2 = -\mathcal{K}_2$$

where $a \geq 0$ is real, its Weyl quantization a^w being a pseudodifferential operator of order $2s$, and $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$ is controlled by a^w (see Proposition 1.4 below and the review about Weyl-Hörmander calculus in the appendix, and we refer to [27, Chapter 18] and [29] for more detail on Weyl-Hörmander calculus). Precise expressions of a and $\mathcal{K}_i$ will be given in Section 3. The most significant part of $\mathcal{L}$ is therefore of a pseudo differential type and by the next result, we have fundamental symbolic estimates for a, implying in particular that operator a^w is elliptic in its own calculus (although of infinite order). This very strong property allows to avoid the systematic use of Gårding type inequalities which are not available here.

In the following, we denote $\Gamma = |dv|^2 + |d\eta|^2$ is the flat metric in $\mathbb{R}^6_{v,\eta}$ (recall that η denotes the dual variable of v). Standard notions concerning symbolic estimates and the pseudodifferential calculus are explained at the beginning of section 4.

Proposition 1.4. *Define*

$$\tilde{a}(v,\eta) \stackrel{\text{def}}{=} \langle v \rangle^{\gamma} (1 + |\eta|^2 + |\eta \wedge v|^2 + |v|^2)^s, \text{ for all } (v,\eta) \in \mathbb{R}^6_{v,\eta}.$$

Then we can write $\mathcal{L} = -a^w - \mathcal{K}$, where *i) the symbols a, $\tilde{a}$ are temperate w.r.t. Γ, $a, \tilde{a} \in S(\tilde{a}, \Gamma)$, and there exists a positive constant C such that $C^{-1}\tilde{a}(v,\eta) \leq a(v,\eta) \leq C\tilde{a}(v,\eta)$;*

ii) for all $\varepsilon > 0$ there exists C_ε such that

$$\|\mathcal{K}f\| \leq \varepsilon\|a^w f\| + C_\varepsilon\| \langle v\rangle^{\gamma+2s} f\|;$$

iii) for a sufficiently large constant K depending only on the dimension, $a_K \stackrel{\text{def}}{=} a + K\langle v\rangle^{\gamma+2s}$ belongs to $S(\tilde{a}, \Gamma)$, is invertible as an operator in L^2 and its inverse $(a_K^w)^{-1}$ has the form

$$(a_K^w)^{-1} = H_1 \left(a_K^{-1}\right)^w = \left(a_K^{-1}\right)^w H_2,$$

with H_1, H_2 belonging to $\mathcal{B}(L^2)$, the space of bounded operators on L^2.

Recall that in Hörmander's terminology, $a \in S(\tilde{a}, \Gamma)$ means that for all multi-indices α and β, there exists a constant $C_{\alpha,\beta}$ such that

$$|\partial_v^\alpha \partial_\eta^\beta a(v,\eta)| \leq C_{\alpha,\beta}\tilde{a}(v,\eta).$$

The temperance then implies a correct definition for the associated operators. We postpone to section 3 and the appendix a review of these standard notions of pseudodifferential calculus.

The exponents of derivative terms and weight terms in Theorem 1.1 and Theorem 1.3 seem to be optimal, since the symbolic estimates provided by Proposition 1.4 implies that the operator $\mathcal{P}$ should behave locally like a generalized Kolmogorov type operator

$$\partial_t + v\cdot\partial_x + |D_v|^{2s},$$

for which the exponent $2s/(2s+1)$ for the regularity in the time and space variables is indeed sharp by using a simple scaling argument (see also [32]). In the particular case $s = 1$ we recover formally the Landau equation and our exponents (both in regularity and weight) match perfectly with the exponents in [24].

The main ideas of our proofs of the above theorems rely on some formal computations of symbols in [1], on the method by multiplier used in [24, 35] and some microlocal techniques developed by Lerner while using Wick quantization [30]. We refer to Section 1.3 for some considerations about the methodology we used, and which comes form these previous works. Let us note that functional estimates from a series of work of Alexandre et al. [9, 8, 7, 6] and Gressman et al. [21] are also helpful for a clear understanding of the structure of the collision operator, but a nice feature of our method is that we will be able to completely avoid the use of these previous estimates. Note that there are some other methods to study the regularity of the transport equation; for instance the average arguments used by Bouchut [13] and a version of the uncertainty principle used by Alexandre et al. [5] to prove the regularity in the time and space variables t, x. However, these results do not provide any optimal hypoelliptic estimate for the spatially inhomogeneous Boltzmann equation without angular cutoff.

We give now some bibliographical references about the hypoelliptic properties of the non cutoff Boltzmann equation and related kinetic models. Note that the angular cross-section b is not integrable on the sphere due to the singularity θ^{-2-2s}, which leads to the formal statement that the nonlinear collision operator should behave like a fractional Laplacian; that is,

$$Q(g,f) \approx -C_g(-\triangle_v)^s f + \text{lower order terms},$$

with $C_g > 0$ a constant depending only on the physical properties of g. Initiated by Desvillettes [17, 18], there have been extensive works around this result and regarding the smoothness of solutions for the homogeneous Boltzmann equation without angular cutoff, c.f. [4, 10, 15, 19, 20, 28, 36, 38]. For the inhomogeneous case the study becomes more complicated. We remark that there have been some related works concerned with the linear model of spatially inhomogeneous Boltzmann equation, which takes the following form

$$\partial_t + v \cdot \partial_x + e(t,x,v)(-\triangle_v)^s, \quad \inf_{t,x,v} e(t,x,v) > 0.$$

This model equation was firstly studied by Morimoto and Xu [37], where a global but non optimal hypoelliptic estimate was established. This study was then improved by Chen et al. [16], and also by Lerner et al. in [32] for an optimal local result. We also mention [3] where a simple proof of the subelliptic estimate for the above model operator is given. For general inhomogeneous Boltzmann equation we refer to [9, 8, 7, 6] for recent progress on its qualitative properties. Finally, let us also mention a recent global result by Lerner et al. [33] in the radially symmetric case and the Maxwellian case (which corresponds to $\gamma = 0$ in our notations), and closely related works [21, 22, 34] where the sharp estimates for the Boltzmann collision operator were explored.

1.3 Further comments and methodolodgy

In this subsection, we give some additional comments on this work and explain the general strategy of the proofs.

On the linear approach. First mention that we focus in this article on a linearized Boltzmann operator. We note that a deep knowledge of the linear behavior is of great interest in the study of the non-linear case, at least in a perturbative context (see for example [7, 8, 9, 21] and the references therein for this without cutoff case). These previous works are mainly concerned with the global existence of solutions close to equilibrium for the the full non linear Boltzmann equation, and important parts of the proofs are connected with functional properties of the linearized part of Boltzmann collisional operator. Our main goal here is to understand the functional properties of the linearized part of the full inhomogeneous equation.

On the kernel of the collision operator. We emphasize the fact that we are absolutely not interested in the (finite-dimensionnal) kernel $\mathcal{N}$ of the linearized Boltzmann collision operator. This is an a priori independent question to establish so called hypocoercive estimates on the orthogonal of $\mathcal{N}$ and related exponential return to the equilibrium of the solutions of the Boltzmann equation. We only deal here with regularity or hypoelliptic issues.

On the interest of regularization estimate. In this article we essentially focus on global hypoelliptic estimates concerning the linearized Boltzmann operator $\mathcal{P}$ defined in (4). The main result in Theorem 1.1 just concerns the independent of time problem and implies the following type of result. If one consider an equality $\mathcal{P}f = g$ with given $f,\ g \in L^2$, then in fact f has a better regularity and space/velocity decay given by the inequality in Theorem 1.1 : it has some weighted H^{2s} regularity in velocity and $H^{2s/(2s+1)}$ regularity in space. Note that this kind of conclusion is *not* available if one only use triple norm estimates (see the version given in remark 4.7 here) for which space regularity is not given.

Mention that estimates like in Theorem 1.1 and the careful study of the pseudodifferential and hypoelliptic structure of diffusive inhomogeneous kinetic equations have concrete applications; for example many ideas and tools developed here lead in [25] and [26] to the existence and uniqueness of solutions of the full non-linear inhomogeneous Boltzmann equation without cutoff with close to equilibrium initial data in large spaces (in the spirit of the theory developed recently in [23]).

A multiplier method. In this work we make use of multiplier method to explore the intrinsic hypoelliptic structure of operator $\mathcal{P} = v.\partial_x - \mathcal{L}$ defined in (4). By multiplier method we mean finding a bounded selfadjoint operator $\mathcal{M}$, such that on one side the commutator between the transport part and $\mathcal{M}$

$$\frac{1}{2}\left([\mathcal{M}, v\cdot\partial_x]u,\ u\right)_{L^2} = \mathrm{Re}\left(v\cdot\partial_x u,\ \mathcal{M}u\right)_{L^2}$$

gives some "elliptic" properties in spatial variables, and on the other side we can control the upper bound for the term

$$\left|(\mathcal{L}u,\ \mathcal{M}u)_{L^2}\right|.$$

For the treatment of the latter we need to the representation of $\mathcal{L}$ in term of pseudo-differential operators (see Proposition 1.4) which will be useful to estimate the commutators between $\mathcal{L}$ and $\mathcal{M}$. The choice of the multiplier here is inspired by the Poisson bracket analysis for the transport part and the collision part already done for other diffusive models (see e.g. Fokker-Planck or Landau in [24] or [35]).

The multiplier method explained on a toy model. To clarify the choice of the multiplier $\mathcal{M}$ above we consider the case when $\mathcal{P}$ is replaced by a Kolmogorov type operator $\mathcal{P}_{kol}$

$$\mathcal{P}_{kol} = v\cdot\partial_x - \partial_v^2.$$

(This corresponds to $\gamma = 0$ and $s = 1$ in a simplified case). Then a direct computation gives

$$\left[v\cdot\partial_x, -\partial_v^2\right] = 2\partial_x\cdot\partial_v \qquad \left[v\cdot\partial_x, \left[v\cdot\partial_x, -\partial_v^2\right]\right] = -2\Delta_x,$$

and we observe that this second-commutator analysis exhibit some Laplacian in x. This suggests that the multiplier should be similar to the first-order commutator $2\partial_x\cdot\partial_v$. Since it is not a bounded operator on L^2 we have to modify the multiplier to guarantee its boundedness. It is then easier to see all the computation on the Fourier side : let us ξ be the dual of v and η be the dual of v. then operator $2\partial_x\cdot\partial_v$ is represented by a multiplication by $-2\xi\cdot\eta$ and we note that the laplacian in velocity is a multiplication by $-|\eta|^2$ on the Fourier side. Then a good multiplier $\mathcal{M}$ is given by the quantization of the following *bounded* function

$$m(\xi,\eta) = \frac{\xi\cdot\eta}{\langle\xi\rangle^{4/3}}\chi\left(\frac{\langle\eta\rangle}{\langle\xi\rangle^{1/3}}\right),$$

where $\chi \in C_0^\infty(\mathbb{R};\ [0,1])$ such that $\chi = 1$ in $[-1,1]$ and supp $\chi \subset [-2,2]$. This function is clearly bounded thanks to the localization induced by χ on small η frequencies, and it has to be considered as a (truncated and weighted) modification of the fundamental stone $\xi \cdot \eta$. Computation of all involved commutators on the fourier side give then

$$-\Delta_v + [v \cdot \partial_x, \mathcal{M}] \simeq -\Delta_v + (-\Delta_x)^{1/3} + errors$$

leading after some work to subelliptic estimates of the form

$$\left\| \langle D_v \rangle^2 f \right\| + \left\| \langle D_x \rangle^{2/3} f \right\| \leq C \left(\|\mathcal{P}_{kol} f\| + \|f\| \right),$$

for (compactly) supported smooth functions. We refer to [24] for more developed arguments about this method, and complete computations in some simple cases. Theorem 1.1 is of the same form but global, with weights involving velocity and with regularity $2s$ or $2s/(2s+1)$ instead of 2 or 2/3 because of the structure of the Boltzmann collision operator without cut-off. The proof is also much more complicated than for the previous toy model.

On the use of the Wick quantization. In the example just before, $\mathcal{M}$ was just a standard Fourier multiplier. In the case of the Boltzmann collision operator, the corresponding operator has a more tricky structure and has to be selected into the general family of pseudodifferential operators. Its construction follows anyway exactly the same ideas as before (see Subsection 4.3 for its expression). Now in all these strategies the positivity of the symbols, multipliers and their commutators is an important point, and it appears that one cannot apply standard positivity result of operators having non-negative symbols (as the famous Garding inequality) since they are in bad classes in the sense of Hörmander t(see e.g. [27] chapter 18 or [29]).

Anyway by choosing the Wick quantization of symbols, we can bypass this difficulty : recall indeed that for any symbol $q \geq 0$ we directly have $q^{\text{Wick}} \geq 0$ in the sense of operators. We will use the Wick quantization here instead of the classical or the Weyl ones, and this will simplify our arguments substantially : the computations and inequalities can be directly stated on symbols.

The paper is organized as follows. In Section 2, we provide precise estimates on the nice terms appearing in the splitting of the collision operator $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, involving compact parts and relatively bounded terms w.r.t. the operator of multiplication by $\langle v \rangle^{\gamma+2s}$. In Section 3 we deal with the main terms, which appear to be of pseudodifferential type, and give precise symbolic estimates in the sense of the Weyl-Hörmander calculus. Section 4 is devoted to the proof of the main theorems. An appendix is devoted to a short review of some tools used in this work (Wick quantization, cancellation Lemma and Carleman representation).

2 First estimates on the linearized collision operator

In this section we study the linearized collision part $\mathcal{L}$ defined in (5). We cut it in many pieces and study each of them except the two principal ones, which study is postponed in section 3 (they are indeed of pseudodifferential type). We look here at the properties of the non pseudodifferential parts, and write many estimates in weighted L^2 spaces.

The splitting of the linearized Boltzmann operator $\mathcal{L}$ is as follows. We write of $f \in \mathcal{S}$,

$$
\begin{aligned}
\mathcal{L}f &= \mu^{-1/2}Q(\mu, \mu^{1/2}f) + \mu^{-1/2}Q(\mu^{1/2}f, \mu) \\
&= \mu^{-1/2}\iint dv_* d\sigma B \left(\mu'_*(\mu')^{1/2}f' - \mu_*\mu^{1/2}f + \mu'(\mu'_*)^{1/2}f'_* - \mu(\mu_*)^{1/2}f_*\right) \\
&= \iint dv_* d\sigma B(\mu_*)^{1/2}\left((\mu'_*)^{1/2}f' - (\mu_*)^{1/2}f + (\mu')^{1/2}f'_* - (\mu)^{1/2}f_*\right) \\
&= \iint dv_* d\sigma B(\mu_*)^{1/2}\left((\mu'_*)^{1/2}f' - (\mu_*)^{1/2}f\right) \\
&\qquad + \iint dv_* d\sigma B(\mu_*)^{1/2}\left((\mu')^{1/2}f'_* - (\mu)^{1/2}f_*\right) \\
&= \mathcal{L}_1 f + \mathcal{L}_2 f.
\end{aligned}
\tag{6}
$$

We shall study more precisely each part of $\mathcal{L}$. Let us immediately point out that they have completely different behaviors. The non local term $\mathcal{L}_2$ behaves essentially like a convolution term, with nice estimates, and is relatively compact w.r.t. the main part of $\mathcal{L}_1$ which will appear to be of pseudodifferential type.

2.1 Study of $\mathcal{L}_2$

Starting from the expression of $\mathcal{L}_2$ given by

$$\mathcal{L}_2 f = \iint dv_* d\sigma B(\mu_*)^{1/2}\left((\mu')^{1/2}f'_* - (\mu)^{1/2}f_*\right),$$

we split it into four terms which make sense even for strong singularities of B, i.e. in particular for $s \geq 1/2$. This point will be clear from the proof of Lemma 2.1 below.

$$
\begin{aligned}
\mathcal{L}_2 f &= \iint dv_* d\sigma B(\mu_*)^{1/2}\left((\mu')^{1/2}f'_* - (\mu)^{1/2}f_*\right) \\
&= \iint dv_* d\sigma B\left((\mu^{1/2}f)'_*(\mu')^{1/2} - (\mu^{1/2}f)_*\mu^{1/2}\right) + \iint dv_* d\sigma B(\mu')^{1/2}\left((\mu_*)^{1/2} - (\mu'_*)^{1/2}\right) f'_* \\
&= \iint dv_* d\sigma B(\mu^{1/2}f)'_*\left((\mu')^{1/2} - \mu^{1/2}\right) \\
&\quad + \mu^{1/2}\iint dv_* d\sigma B\left((\mu^{1/2}f)'_* - (\mu^{1/2}f)_*\right) \\
&\qquad + \mu^{1/2}\iint dv_* d\sigma B\left((\mu_*)^{1/2} - (\mu'_*)^{1/2}\right) f'_* \\
&\qquad\quad + \iint dv_* d\sigma B\left((\mu')^{1/2} - (\mu)^{1/2}\right)\left((\mu_*)^{1/2} - (\mu'_*)^{1/2}\right) f'_* \\
&= \mathcal{L}_{2,r}f + \mathcal{L}_{2,ca}f + \mathcal{L}_{2,c}f + \mathcal{L}_{2,d}f.
\end{aligned}
$$

$\mathcal{L}_{2,ca}$ involves essentially a convolution term and can be treated using the cancellation lemma (see [11] and the appendix herein), and the three other ones can be estimated by hands. Let us note that the analysis of $\mathcal{L}_2$ was already given by [7], Lemma 2.15, but we provide a somewhat direct and shorter proof.

Lemma 2.1. *For all $f \in \mathcal{S}(\mathbb{R}^3_v)$ and for all α, $\beta \in \mathbb{R}$ there exists a constant $C_{\alpha,\beta}$ independent of f such that*

$$\|\langle v\rangle^{\alpha}\mathcal{L}_2\langle v\rangle^{\beta} f\| \leq C_{\alpha,\beta}\|f\|.$$

Proof. We start with $\mathcal{L}_{2,ca}f$:

$$\mathcal{L}_{2,ca}f = \mu^{1/2} \iint dv_* d\sigma B \left((\mu^{1/2} f)'_* - (\mu^{1/2} f)_* \right).$$

Applying the Cancellation Lemma (see [11] or the appendix), we get, for some constant c depending only on b:

$$\mathcal{L}_{2,ca}f = c\mu^{1/2} \int dv_* |v - v_*|^\gamma (\mu^{1/2} f)_*.$$

This is an integral operator with the kernel $K(v, v_*) = c\mu^{1/2}(\mu_*)^{1/2}|v - v_*|^\gamma$ for which we can apply Schur's Lemma to get

$$\|\mathcal{L}_{2,ca}f\| \lesssim \|f\|.$$

Note that the assumption $\gamma > -3$ is needed at this point.

More generally, replacing $\mathcal{L}_{2,ca}f$ by $\langle v\rangle^\alpha \mathcal{L}_{2,ca} \langle v\rangle^\beta f$ leads to a kernel

$$K_{\alpha,\beta}(v, v_*) = c\mu^{1/2} \langle v\rangle^\alpha (\mu_*)^{1/2} \langle v_*\rangle^\beta |v - v_*|^\gamma$$

for which we can use the same argument to get

$$\big\| \langle v\rangle^\alpha \mathcal{L}_{2,ca} \langle v\rangle^\beta f\big\| \le C_{\alpha,\beta}\big\|f\big\|.$$

Next, dealing with $\mathcal{L}_{2,c}f$

$$\mathcal{L}_{2,c}f = \mu^{1/2} \iint dv_* d\sigma B \left((\mu_*)^{1/2} - (\mu'_*)^{1/2} \right) f'_*,$$

we split this term into a singular and a non-singular parts. First consider the non singular part defined as

$$\mathcal{L}_{2,c,nonsing}f \stackrel{\text{def}}{=} \mu^{1/2} \iint dv_* d\sigma B \mathbb{1}_{|v'-v|\ge 1} \left((\mu_*)^{1/2} - (\mu'_*)^{1/2} \right) f'_*.$$

As noticed in [7], one has $\mu'_*\mu' = \mu_*\mu \le (\mu'_*\mu)^{1/5}$ due to the kinetic and momentum relations in (2). Therefore

$$Af \stackrel{\text{def}}{=} |\mathcal{L}_{2,c,nonsing}f| \lesssim \mu^{1/10} \iint dv_* d\sigma |B| \mathbb{1}_{|v'-v|\ge 1} \left|(\mu^{1/10} f)'_*\right|$$

which writes in Carleman representation (see the appendix)

$$Af \lesssim \mu^{1/10} \int_{\mathbb{R}^3_h} dh \int_{E_{0,h}} d\alpha \mathbb{1}_{|h|\ge 1} \mathbb{1}_{|\alpha|\ge|h|} \frac{|\alpha + h|^{1+\gamma+2s}}{|h|^{3+2s}} |(\mu^{1/10} f)(\alpha + v)|,$$

where $E_{0,h}$ denotes the hyperplane orthogonal to h and containing 0. By duality, we get, for all $g \in \mathcal{S}$,

$$\begin{aligned} |(Af, g)| &\lesssim \int_{\mathbb{R}^3_v} dv \int_{\mathbb{R}^3_h} dh \int_{E_{0,h}} d\alpha \mathbb{1}_{|h|\ge 1} \mathbb{1}_{|\alpha|\ge|h|} \frac{|\alpha + h|^{1+\gamma+2s}}{|h|^{3+2s}} |(\mu^{1/10} f)(\alpha + v)| \cdot |\mu^{1/10} g(v)| \\ &\lesssim \int_{\mathbb{R}^3_v} dv \int_{\mathbb{R}^3_h} dh \int_{E_{0,h}} d\alpha \mathbb{1}_{|h|\ge 1} \mathbb{1}_{|\alpha|\ge|h|} \frac{|\alpha|^{1+\gamma+2s}}{|h|^{3+2s}} |(\mu^{1/10} f)(\alpha + v)||(\mu^{1/10} g)(v)| \end{aligned}$$

which upon using (82) yields

$$
\begin{aligned}
|(Af,g)| &\lesssim \int_{\mathbb{R}^3_v} dv \int_{\mathbb{R}^3_\alpha} d\alpha \int_{E_{0,\alpha}} dh \mathbb{1}_{|h|\geq 1} \mathbb{1}_{|\alpha|\geq|h|} \frac{|\alpha|^{\gamma+2s}}{|h|^{2+2s}} |\mu^{1/10} f(\alpha+v)||\mu^{1/10} g(v)| \\
&\lesssim \int_{\mathbb{R}^3_v} dv \int_{\mathbb{R}^3_\alpha} d\alpha |\alpha|^{(\gamma+2s)^+} |\mu^{1/10} f(\alpha+v)||\mu^{1/10} g(v)|.
\end{aligned}
$$

Therefore

$$
|(Af,g)| \lesssim \|\mu^{1/20} f\| \|\mu^{1/20} g\|
$$

from which follows that

$$
\| \langle v \rangle^\alpha \mathcal{L}_{2,c,nonsing} \langle v \rangle^\beta f \| \leq C_{\alpha,\beta} \|f\| \tag{7}
$$

for all real α and β.

For the singular part $\mathcal{L}_{2,c,sing}$, again using Carleman's representation (83) gives

$$
\begin{aligned}
\mathcal{L}_{2,c,sing} f = \mu^{1/2} \int_{\mathbb{R}^3_h} dh \int_{E_{0,h}} & d\alpha \tilde{b}(\alpha,h) \mathbb{1}_{|\alpha|\geq|h|} \mathbb{1}_{|h|\leq 1} \\
& \left(\mu^{1/2}(\alpha+v-h) - \mu^{1/2}(\alpha+v)\right) \frac{|\alpha+h|^{1+\gamma+2s}}{|h|^{3+2s}} f(\alpha+v).
\end{aligned}
$$

Changing $h \to -h$ and adding the resulting two formulas (so we see that formally we cancel higher singularities, using also that $\tilde{b}(\alpha,h) = \tilde{b}(\pm\alpha,\pm h)$) yields

$$
\begin{aligned}
\mathcal{L}_{2,c,sing} f = \frac{1}{2}\mu^{1/2} \int_h dh \int_{E_{0,h}} & d\alpha \tilde{b} \mathbb{1}_{|\alpha|\geq|h|} \mathbb{1}_{|h|\leq 1} \times \\
& \left(\mu^{1/2}(\alpha+v-h) + \mu^{1/2}(\alpha+v+h) - 2\mu^{1/2}(\alpha+v)\right) \frac{|\alpha+h|^{1+\gamma+2s}}{|h|^{3+2s}} f(\alpha+v).
\end{aligned}
$$

Factorizing by $\mu^{1/2}(\alpha+v)$ we get

$$
\begin{aligned}
& \mathcal{L}_{2,c,sing} f \\
= \ & \frac{1}{2}\mu^{1/2} \int_h dh \int_{E_{0,h}} d\alpha \tilde{b} \mathbb{1}_{|\alpha|\geq|h|} \mathbb{1}_{|h|\leq 1} \left(e^{-\left(|h|^2 - 2(\alpha+v)\cdot h\right)/4} + e^{-\left(|h|^2 + 2(\alpha+v)\cdot h\right)/4} - 2\right) \\
& \times \frac{|\alpha+h|^{1+\gamma+2s}}{|h|^{3+2s}} \mu^{1/2}(\alpha+v) f(\alpha+v).
\end{aligned}
$$

The term in parentheses is bounded by $|h|^2 \mu^{-1/4}(\alpha+v)$ thanks to the condition on the support for h, and since $|h| \leq |\alpha|$, one has

$$
|\mathcal{L}_{2,c,sing} f| \quad \lesssim \quad \mu^{1/2} \int_h dh \int_{E_{0,h}} d\alpha \mathbb{1}_{|\alpha|\geq|h|} \mathbb{1}_{|h|\leq 1} \frac{|\alpha|^{1+\gamma+2s}}{|h|^{1+2s}} \mu^{1/4}(\alpha + v) |f(\alpha + v)|.
$$

Using again (82) and the duality argument as in the non-singular case (now the singularity in h is integrable), we easily get

$$
\| \langle v \rangle^\alpha \mathcal{L}_{2,c,sing} \langle v \rangle^\beta f \| \leq C_{\alpha,\beta} \|f\| \tag{8}
$$

for all real α and β.

As for $\mathcal{L}_{2,r}f$, recalling that

$$\mathcal{L}_{2,r}f = \iint dv_* d\sigma B (\mu^{1/2}f)'_* \left((\mu')^{1/2} - \mu^{1/2}\right)$$

we see immediately that, using the classical pre-post velocities change of variables that

$$(\mathcal{L}_{2,r}f, g) = (f, \mathcal{L}_{2,c}g)$$

and thus we are done for this term.

It remains to study $\mathcal{L}_{2,d}f$ which is exactly

$$\mathcal{L}_{2,d}f = \iint dv_* d\sigma B \left((\mu')^{1/2} - (\mu)^{1/2}\right)\left((\mu_*)^{1/2} - (\mu'_*)^{1/2}\right) f'_*.$$

Using the equality $a^2 - b^2 = (a-b)(a+b)$ for the Gaussian functions in the above factors, we see again that we can put some power of a gaussian together with f, by using the argument of [7]: that means that for some $c > 0, d > 0$, one has

$$|\mathcal{L}_{2,d}f| \lesssim \mu^d \iint dv_* d\sigma B \left|(\mu')^{1/4} - (\mu)^{1/4}\right| \left|(\mu_*)^{1/4} - (\mu'_*)^{1/4}\right| (\mu^c)'_* |f'_*|$$

and then the remaining analysis is exactly similar to the computations done for $\mathcal{L}_{2,c,sing}f$. □

2.2 Splitting of $\mathcal{L}_1$

The operator $\mathcal{L}_1$ will also be cut into several pieces, which will require two different types of arguments. For some of the nice parts, tools similar to the ones in the previous section will be sufficient. The remaining pseudodifferential parts will be treated in the next Section.

Recall first that

$$\mathcal{L}_1 f = \iint dv_* d\sigma B (\mu_*)^{1/2} \left((\mu'_*)^{1/2} f' - (\mu_*)^{1/2} f\right).$$

Let $0 < \delta \leq 1$ be a fixed parameter (either small or not, this will have no consequence in the study). We first split the above integral according to whether or not $|v' - v| \gtrsim \delta$. To this end, let φ be a positive radial function supported on the unit ball and say 1 in the 1/4 ball. Consider $\varphi_\delta(v) = \varphi(|v|^2/\delta^2)$, which is therefore 0 for $|v| \geq \delta$ and 1 for $|v| \leq \delta/2$. By abuse of notations we shall also denote $\varphi_\delta(r) = \varphi_\delta(v)$ when $|v| = r$. Set $\tilde{\varphi}_\delta(v) = 1 - \varphi_\delta(v)$, which is therefore 0 for small values and 1 for large values.

Then $\mathcal{L}_1 f$ can decomposed as the sum of the following two terms

$$\overline{\mathcal{L}}_{1,\delta} f = \iint dv_* d\sigma B \tilde{\varphi}_\delta(v' - v)(\mu_*)^{1/2} \left((\mu'_*)^{1/2} f' - (\mu_*)^{1/2} f\right)$$

and

$$\mathcal{L}_{1,\delta} f = \iint dv_* d\sigma B \varphi_\delta(v' - v)(\mu_*)^{1/2} \left((\mu'_*)^{1/2} f' - (\mu_*)^{1/2} f\right).$$

Note that $\overline{\mathcal{L}}_{1,\delta}$ is a cutoff type Boltzmann operator. We split it into two terms since there is no singularity any more

$$\begin{aligned}\overline{\mathcal{L}}_{1,\delta} f &= \iint dv_* d\sigma B \tilde{\varphi}_\delta(v'-v)(\mu_*)^{1/2}(\mu'_*)^{1/2} f' \\ &\quad - \left(\iint dv_* d\sigma B \tilde{\varphi}_\delta(v'-v)(\mu_*)^{1/2}(\mu_*)^{1/2}\right) f \\ &= \overline{\mathcal{L}}_{1,\delta,a} f + \overline{\mathcal{L}}_{1,\delta,b} f.\end{aligned} \tag{9}$$

As for $\mathcal{L}_{1,\delta}$, again we split it into four terms:

$$\begin{aligned}\mathcal{L}_{1,\delta} f &= \iint dv_* d\sigma B \varphi_\delta(v'-v)(\mu_*)^{1/2}\left((\mu'_*)^{1/2} f' - (\mu_*)^{1/2} f\right) \\ &= \iint dv_* d\sigma B \varphi_\delta(v'-v)(\mu'_*)^{1/2}(f'-f)\left((\mu_*)^{1/2} - (\mu'_*)^{1/2}\right) \\ &\quad + \left(\iint dv_* d\sigma B \varphi_\delta(v'-v)(\mu'_*)^{1/2}\left((\mu_*)^{1/2} - (\mu'_*)^{1/2}\right)\right) f \\ &\qquad + \iint dv_* d\sigma B \varphi_\delta(v'-v)\mu'_* \left(f'-f\right) \\ &\qquad\quad + \left(\iint dv_* d\sigma B \varphi_\delta(v'-v)\left(\mu'_* - \mu_*\right)\right) f \\ &= \mathcal{L}_{1,1,\delta} f + \mathcal{L}_{1,4,\delta} f + \mathcal{L}_{1,2,\delta} f + \mathcal{L}_{1,3,\delta} f.\end{aligned} \tag{10}$$

Let us immediately notice that this splitting takes into account all values of s. However, for small singularities $0 < s < 1/2$, a simpler decomposition is available and avoids some of the issues dealt with below. We note that $\mathcal{L}_{1,4,\delta} f$ and $\mathcal{L}_{1,3,\delta} f$ are of multiplicative type, and together with $\overline{\mathcal{L}}_{1,\delta,a} f$, they will be studied in the next subsection. They will appear later as relatively bounded terms with respect to $\mathcal{L}_{1,1,\delta} + \mathcal{L}_{1,2,\delta} f$. These last two parts will appear to be of pseudodifferential type, and we shall estimate them very precisely in section 3.

Remark 2.2. In the coming computations, we shall follow the dependence on parameter δ. We point out that it could be fixed at value $\delta = 1$. Anyway, as we shall see in the coming sections, the explicit dependence on δ of the various estimates enlightens the fact that we are the non cutoff case. As already mentioned, the cutoff case corresponds to the case when $\mathcal{L}_{1,\delta} = 0$. It can also be seen as the limiting case $\delta \to 0$ when looking e.g. at $\mathcal{L}_{1,2,\delta}$, for which we give in Proposition 3.1 a lower bound which would be not relevant anymore for $\delta = 0$.

2.3 Relatively bounded terms in $\mathcal{L}_1$

Study of $\mathcal{L}_{1,3,\delta}$

Using some arguments from the proof of the cancellation lemma, see for example [11], we get the following

Lemma 2.3. *For all $f \in \mathcal{S}(\mathbb{R}^3_v)$, we have, for all $s < 1$*

$$\|\mathcal{L}_{1,3,\delta} f\|^2 \lesssim \delta^{2-2s} \| < v >^{\gamma+2s-2} f\|$$

and $\mathcal{L}_{1,3,\delta}$ commutes with the multiplication by $\langle v \rangle^\alpha$ for all $\alpha \in \mathbb{R}$.

Proof. The last assertion is trivial since $\mathcal{L}_{1,3,\delta}$ is a multiplication operator. In order to prove the above inequality, recall first that

$$\mathcal{L}_{1,3,\delta} f(v) = \left(\iint dv_* d\sigma B \varphi_\delta(v' - v) \left(\mu'_* - \mu_*\right) \right) f.$$

Going back to the proof of the cancellation Lemma, it follows that

$$\left(\iint dv_* d\sigma B \varphi_\delta(v' - v) \left(\mu'_* - \mu_*\right) \right) = S *_{v_*} \mu(v)$$

where, writing by abuse of notation $\varphi_\delta(|z|) \stackrel{\text{def}}{=} \varphi_\delta(z)$ for all $z \in \mathbb{R}^3$, S has the following expression

$$\begin{aligned}
S(z) =& |z|^\gamma \int_0^{\pi/2} \sin\theta b(\cos\theta) \left(\varphi_\delta(\frac{|z|}{\cos\frac{\theta}{2}} \sin\frac{\theta}{2}) \cos^{-3-\gamma}\frac{\theta}{2} - \varphi_\delta(|z| \sin\frac{\theta}{2}) \right) d\theta \\
=& |z|^\gamma \int_0^{\pi/2} \sin\theta b(\cos\theta) \varphi_\delta(\frac{|z|}{\cos\frac{\theta}{2}} \sin\frac{\theta}{2}) \left(\cos^{-3-\gamma}\frac{\theta}{2} - 1 \right) d\theta \\
&+ |z|^\gamma \int_0^{\pi/2} \sin\theta b(\cos\theta) \left(\varphi_\delta(\frac{|z|}{\cos\frac{\theta}{2}} \sin\frac{\theta}{2}) - \varphi_\delta(|z| \sin\frac{\theta}{2}) \right) d\theta \\
=& S_1(z) + S_2(z).
\end{aligned}$$

For the first part $S_1(z)$, note that the integrand is now integrable in the θ variable, and we have

$$|S_1(z)| \lesssim |z|^\gamma . \tag{11}$$

The second part $S_2(z)$ is zero if $|z| \leq \delta/2$, and we can suppose therefore that $|z| \geq \delta/2$. Note also that for z bounded, say for $|z| \leq C$ where C is sufficiently large to be fixed later, $S_2(z)$ is also bounded. Since

$$\frac{|z|}{\cos\frac{\theta}{2}} \sin\frac{\theta}{2} \geq |z| \sin\frac{\theta}{2},$$

we get that if $|z| \sin\frac{\theta}{2} \geq \delta$, the integrand is 0, and similarly for small values of θ. In conclusion when $|z| \geq C$, the second integral can be estimated as follows :

$$S_2(z) = |z|^\gamma \int_{c'\delta|z|^{-1}}^{c\delta|z|^{-1}} \sin\theta b(\cos\theta) \left(\varphi_\delta(\frac{|z|}{\cos\frac{\theta}{2}} \sin\frac{\theta}{2}) - \varphi_\delta(|z| \sin\frac{\theta}{2}) \right) d\theta,$$

where C is a posteriori chosen so that $C^{-1} c\delta \leq \pi/2$. Using Taylor formulae, we get

$$\begin{aligned}
|S_2(z)| &\lesssim \delta^{-1} |z|^{\gamma+1} \int_{c'\delta|z|^{-1}}^{c\delta|z|^{-1}} \theta^2 b(\cos\theta) [\cos^{-1}\theta/2 - 1] d\theta \lesssim \delta^{-1} |z|^{\gamma+1} \int_{c'\delta|z|^{-1}}^{c\delta|z|^{-1}} \theta^4 b(\cos\theta) d\theta \\
&\lesssim \delta^{-1} |z|^{\gamma+1} \int_{c'\delta|z|^{-1}}^{c\delta|z|^{-1}} \theta^{2-2s} d\theta \sim \delta^{-1} |z|^{\gamma+1} \delta^{3-2s} |z|^{-3+2s} \\
&\lesssim \delta^{2-2s} |z|^{\gamma+2s-2}.
\end{aligned}$$

This estimate together with (11) yield the proof of the Lemma. □

Study of $\overline{\mathcal{L}}_{1,\delta,a}$

We deal now with the non singular part $\overline{\mathcal{L}}_{1,\delta,a}$ of $\mathcal{L}_1$ for which we have the following result

Lemma 2.4. *(i) For all $f \in \mathcal{S}(\mathbb{R}^3_v)$ and for all $\alpha,\ \beta \in \mathbb{R}$ such that $\alpha+\beta+\gamma+2s \le 0$, we have*

$$\big\| \langle v\rangle^{\alpha}\, \overline{\mathcal{L}}_{1,\delta,a}\, \langle v\rangle^{\beta} f\big\| \le \delta^{-1-2s} C_{\alpha,\beta} \big\|f\big\|.$$

(ii) For all $f \in \mathcal{S}(\mathbb{R}^3_v)$ and for all $\tilde\alpha,\ \tilde\beta \in \mathbb{R}$ such that $\tilde\alpha+\tilde\beta+\gamma+s \le 0$, we have

$$\big\| \langle v\rangle^{\tilde\alpha} \big[\overline{\mathcal{L}}_{1,\delta,a},\ \ \langle v\rangle^{\tilde\beta}\big] f\big\| \le \delta^{-2s} C_{\tilde\alpha,\tilde\beta} \big\|f\big\|,$$

where $\big[\cdot,\ \cdot\big]$ stands for the commutator.

Proof. Recalling that

$$\overline{\mathcal{L}}_{1,\delta,a} f = \iint dv_* d\sigma B \tilde\varphi_\delta(v'-v)(\mu_*)^{1/2}(\mu'_*)^{1/2} f',$$

it follows that

$$\langle v\rangle^{\alpha}\, \overline{\mathcal{L}}_{1,\delta,a}\, \langle v\rangle^{\beta} f = \langle v\rangle^{\alpha} \iint dv_* d\sigma B \tilde\varphi_\delta(v'-v)(\mu_*)^{1/2}(\mu'_*)^{1/2} (\langle v\rangle^{\beta} f)',$$

and

$$\begin{aligned}\langle v\rangle^{\tilde\alpha} \big[\overline{\mathcal{L}}_{1,\delta,a},\ \ \langle v\rangle^{\tilde\beta}\big] f &= \langle v\rangle^{\tilde\alpha}\, \overline{\mathcal{L}}_{1,\delta,a}\, \langle v\rangle^{\tilde\beta} f - \langle v\rangle^{\tilde\alpha} \langle v\rangle^{\tilde\beta}\, \overline{\mathcal{L}}_{1,\delta,a} f \\ &= \langle v\rangle^{\tilde\alpha} \iint dv_* d\sigma B \tilde\varphi_\delta(v'-v)(\mu_*)^{1/2}(\mu'_*)^{1/2} \Big(\langle v'\rangle^{\tilde\beta} - \langle v\rangle^{\tilde\beta}\Big) f'.\end{aligned}$$

(i) We first estimate $\langle v\rangle^{\alpha}\, \overline{\mathcal{L}}_{1,\delta,a}\, \langle v\rangle^{\beta} f$. An application of Carleman's representation (see the appendix for instance) shows that

$$\begin{aligned}|\langle v\rangle^{\alpha}\, \overline{\mathcal{L}}_{1,\delta,a}\, \langle v\rangle^{\beta} f| &\lesssim \langle v\rangle^{\alpha} \int_h dh \int_{E_{0,h}} d\alpha \mathbb{1}_{|\alpha|\ge|h|} \mathbb{1}_{|h|\ge\delta/2} \mu^{1/2}(\alpha+v)\mu^{1/2}(\alpha+v-h) \\ &\qquad \frac{|h+\alpha|^{1+\gamma+2s}}{|h|^{3+2s}} \langle v-h\rangle^{\beta} |f(v-h)| \\ &\lesssim \langle v\rangle^{\alpha} \int_h dh \int_{E_{0,h}} d\alpha \mathbb{1}_{|h|\ge\delta/2} \mu^{1/2}(\alpha+v)\mu^{1/2}(\alpha+v-h) \\ &\qquad \frac{|\alpha|^{1+\gamma+2s}}{|h|^{3+2s}} \langle v-h\rangle^{\beta} |f(v-h)|,\end{aligned} \tag{12}$$

where we used the fact that $|\alpha| \ge |h|$ for the second inequality, and recalling that $E_{0,h}$ denotes the vector plane containing 0 and orthogonal to h. Letting $S(h)$ for the orthogonal projection onto $E_{0,h}$, we can write

$$e^{-|\alpha+v|^2} = e^{-|\alpha+S(h)v|^2} e^{|S(h)v|^2-|v|^2}$$

and similarly

$$e^{-|\alpha+v-h|^2} = e^{-|\alpha+S(h)(v)|^2} e^{|S(h)(v-h)|^2-|v-h|^2},$$

and therefore

$$\mu^{1/2}(\alpha+v)\mu^{1/2}(\alpha+v-h) = (2\pi)^{-3/2}\left(e^{-|\alpha+S(h)v|^2}\left(e^{2(|S(h)v|^2-|v|^2)+|v|^2-|v-h|^2}\right)^{1/2}\right)^{1/2}$$
$$= (2\pi)^{-3/2}\left(e^{-|\alpha+S(h)v|^2}\left(e^{2(|S(h)v|^2-|v|^2)+2v\cdot h-|h|^2}\right)^{1/2}\right)^{1/2}.$$

Going back to (12), we obtain

$$|\langle v\rangle^\alpha \overline{\mathcal{L}}_{1,\delta,a}\langle v\rangle^\beta f| \lesssim \langle v\rangle^\alpha \int_{\mathbb{R}^3_h} dh \int_{E_{0,h}} d\alpha \mathbb{1}_{|\alpha|\geq|h|}\mathbb{1}_{|h|\geq\delta/2}\frac{|\alpha|^{1+\gamma+2s}}{|h|^{3+2s}}\langle v-h\rangle^\beta |f(v-h)|$$
$$\left(e^{-|\alpha+S(h)v|^2}\left(e^{2(|S(h)v|^2-|v|^2)+2v\cdot h-|h|^2}\right)^{1/2}\right)^{1/2}.$$

Performing the integration with respect to α, it follows that

$$\begin{aligned}
|\langle v\rangle^\alpha \overline{\mathcal{L}}_{1,\delta,a}\langle v\rangle^\beta f| &\lesssim \langle v\rangle^\alpha \int_{\mathbb{R}^3_h} dh \mathbb{1}_{|h|\geq\delta/2}\langle S(h)v\rangle^{1+\gamma+2s}\frac{1}{|h|^{3+2s}}\langle v-h\rangle^\beta |f(v-h)| \\
&\qquad \left(e^{2(|S(h)v|^2-|v|^2)+2v\cdot h-|h|^2}\right)^{1/4} \\
&\lesssim \int_{\mathbb{R}^3_z} dz \mathbb{1}_{|v-z|\geq\delta/2}\langle v\rangle^\alpha \langle S(v-z)v\rangle^{1+\gamma+2s}\frac{1}{|v-z|^{3+2s}}\langle z\rangle^\beta |f|(z) \\
&\qquad \left(e^{2(|S(v-z)v|^2-|v|^2)+2v.(v-z)-|v-z|^2}\right)^{1/4} \\
&\overset{\text{def}}{=} \int_{\mathbb{R}^3_z} K_{\alpha,\beta}(v,z)|f|(z)dz
\end{aligned}$$

with

$$K_{\alpha,\beta}(v,z) = \mathbb{1}_{|v-z|\geq\delta/2}\langle v\rangle^\alpha\langle z\rangle^\beta\langle S(v-z)v\rangle^{1+\gamma+2s}\frac{1}{|v-z|^{3+2s}}$$
$$\left(e^{2(|S(v-z)v|^2-|v|^2)+2v.(v-z)-|v-z|^2}\right)^{1/4}.$$

We want to apply Schur's Lemma. To this end, let's first integrate w.r.t. to z, to get

$$\begin{aligned}
\int_{\mathbb{R}^3_z} dz K_{\alpha,\beta}(v,z) &= \int_{\mathbb{R}^3_z} dz \mathbb{1}_{|v-z|\geq\delta/2}\langle v\rangle^\alpha\langle z\rangle^\beta\langle S(v-z)v\rangle^{1+\gamma+2s}\frac{1}{|v-z|^{3+2s}} \\
&\qquad \left(e^{2(|S(v-z)v|^2-|v|^2)+2v.(v-z)-|v-z|^2}\right)^{1/4} \\
&= \int_{\mathbb{R}^3_h} dh \mathbb{1}_{|h|\geq\delta/2}\langle v\rangle^\alpha\langle v-h\rangle^\beta\langle S(h)v\rangle^{1+\gamma+2s}\frac{1}{|h|^{3+2s}} \\
&\qquad \left(e^{2(|S(h)v|^2-|v|^2)+2v\cdot h-|h|^2}\right)^{1/4},
\end{aligned}$$

so that

$$
\begin{aligned}
&\int_{\mathbb{R}^3_z} dz K_{\alpha,\beta}(v,z)\\
&= \int_{\mathbb{R}^3_h} dh \mathbb{1}_{|h|\geq \delta/2} \langle v\rangle^{\alpha} \left(1+|v|^2-|v\cdot \frac{h}{|h|}|^2\right)^{(1+\gamma+2s)/2} \frac{1}{|h|^{3+2s}} \langle v-h\rangle^{\beta}\\
&\qquad \left(e^{-2|v\cdot \frac{h}{|h|}|^2+2v\cdot h-|h|^2}\right)^{1/4}\\
&= \int_{\mathbb{R}^3_h} dh \mathbb{1}_{|h|\geq \delta/2} \langle v\rangle^{\alpha} \left(1+|v|^2-\frac{|v|^2}{|h|^2}|\frac{v}{|v|}\cdot h|^2\right)^{(1+\gamma+2s)/2} \frac{1}{|h|^{3+2s}}\\
&\qquad \left(1+|v|^2-2|v|\frac{v}{|v|}\cdot h+|h|^2\right)^{\beta/2} \left(e^{-2\frac{|v|^2}{|h|^2}|\frac{v}{|v|}\cdot h|^2+2|v|\frac{v}{|v|}\cdot h-|h|^2}\right)^{1/4}.
\end{aligned}
$$

Shifting to polar coordinates, with an axis along direction $v/|v|$, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^3_z} dz K_{\alpha,\beta}(v,z) &\lesssim \int_0^{\pi}\int_{\delta}^{\infty} dr d\varphi \langle v\rangle^{\alpha} \sin\varphi \left(1+|v|^2-|v|^2\cos^2\varphi\right)^{(1+\gamma+2s)/2} \frac{1}{r^{1+2s}}\\
&\qquad (1+|v|^2-2|v|r\cos\varphi+r^2)^{\beta/2} \left(e^{-2|v|^2\cos^2\varphi+2|v|r\cos\varphi-r^2}\right)^{1/4}.
\end{aligned}
$$

Note here that if $|v|\leq 1$, then we directly get that $\int_{\mathbb{R}^3_z} dz K_{\alpha,\beta}(v,z)\lesssim 1$. Therefore we may as well assume that $|v|\geq 1$. Setting $t=\cos\varphi$, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^3_z} dz K_{\alpha,\beta}(v,z)\\
&\lesssim \int_{-1}^{1}\int_{\delta}^{\infty} dr dt \langle v\rangle^{\alpha} (1+|v|^2-|v|^2t^2)^{(1+\gamma+2s)/2} e^{(-2|v|^2t^2+2|v|rt-r^2)/4}\frac{1}{r^{1+2s}}\\
&\qquad (1+|v|^2-2|v|rt+r^2)^{\beta/2}\\
&\approx \langle v\rangle^{\alpha} |v|^{-1} \int_{-|v|}^{|v|}\int_{\delta}^{\infty} dr dt (1+|v|^2-t^2)^{(1+\gamma+2s)/2} e^{-(r-t)^2/4}\frac{1}{r^{1+2s}}(1+|v|^2-2rt+r^2)^{\beta/2}\\
&\approx \langle v\rangle^{\alpha} |v|^{-1} \int_{-|v|}^{|v|}\int_{\delta}^{\infty} dr dt (1+|v|^2-t^2)^{(1+\gamma+2s)/2} e^{-(r-t)^2/4}\frac{1}{r^{1+2s}}(1+|v|^2-t^2+(r-t)^2)^{\beta/2}.
\end{aligned}
$$

In the inner term, note that $|v|^2-t^2\geq 0$. We now use Peetre's inequality

$$
\langle u\rangle^{\beta} \langle u+w\rangle^{-|\beta|} \lesssim \langle w\rangle^{\beta} \lesssim \langle u\rangle^{\beta} \langle u+w\rangle^{|\beta|},
$$

to get here

$$
(1+|v|^2-t^2+(r-t)^2)^{\beta/2} \lesssim (1+|v|^2-t^2)^{\beta/2} \langle r-t\rangle^{|\beta|}.
$$

In addition, since $0<\delta<1$, then $r\geq\delta$ implies that $r\geq C\delta\langle r\rangle$ for some C independent of δ. Thus

$$
\begin{aligned}
&\int_{\mathbb{R}^3_z} dz K_{\alpha,\beta}(v,z) \\
&\lesssim \delta^{-1-2s} \langle v\rangle^{\alpha} |v|^{-1} \int_{-|v|}^{|v|} \int_{-\infty}^{\infty} dr dt (1+|v|^2-t^2)^{(1+\gamma+2s+\beta)/2} \left(\langle r-t\rangle^{|\beta|} e^{-(r-t)^2/4} \right) \frac{1}{\langle r\rangle^{1+2s}} \\
&\lesssim \delta^{-1-2s} \langle v\rangle^{\alpha} |v|^{-1} \int_{-|v|}^{|v|} dt (1+|v|^2-t^2)^{(1+\gamma+2s+\beta)/2} \frac{1}{\langle t\rangle^{1+2s}} \\
&\lesssim \delta^{-1-2s} \langle v\rangle^{\alpha-1} \int_{0}^{|v|} dt (1+|v|^2-t^2)^{(1+\gamma+2s+\beta)/2} \frac{1}{\langle t\rangle^{1+2s}}.
\end{aligned}
\tag{13}
$$

Now for evaluating this quantity, we split the integral into two parts. First note that

$$
\begin{aligned}
\int_0^{|v|/2} dt (1+|v|^2-t^2)^{(1+\gamma+2s+\beta)/2} \frac{1}{\langle t\rangle^{1+2s}} &\lesssim \langle v\rangle^{1+\gamma+2s+\beta} \int_0^{|v|/2} dt \frac{1}{\langle t\rangle^{1+2s}} \\
&\lesssim \langle v\rangle^{1+\gamma+2s+\beta}.
\end{aligned}
\tag{14}
$$

For the remaining part, we write

$$
\begin{aligned}
&\int_{|v|/2}^{|v|} dt (1+|v|^2-t^2)^{(1+\gamma+2s+\beta)/2} \frac{1}{\langle t\rangle^{1+2s}} \\
&\qquad \lesssim \langle v\rangle^{-1-2s} \int_{|v|/2}^{|v|} dt (1+|v|^2-t^2)^{(1+\gamma+2s+\beta)/2} \\
&\qquad \lesssim \langle v\rangle^{-1-2s} \int_{|v|/2}^{|v|} dt (1+(|v|-t)(|v|+t))^{(1+\gamma+2s+\beta)/2} \\
&\qquad \lesssim \langle v\rangle^{-1-2s} \int_{|v|/2}^{|v|} dt (1+|v|(|v|-t))^{(1+\gamma+2s+\beta)/2}
\end{aligned}
$$

Posing $s = |v|(|v|-t)$, $ds = -|v|dt$, we get

$$
\begin{aligned}
&\int_{|v|/2}^{|v|} dt (1+|v|^2-t^2)^{(1+\gamma+2s+\beta)/2} \frac{1}{\langle t\rangle^{1+2s}} \\
&\qquad \lesssim \langle v\rangle^{-1-2s} |v|^{-1} \int_0^{|v|^2/2} ds (1+s)^{(1+\gamma+2s+\beta)/2} \\
&\qquad \lesssim \langle v\rangle^{-1-2s} |v|^{-1} \langle v\rangle^{(1+\gamma+2s+\beta)+2} \\
&\qquad \lesssim \langle v\rangle^{1+\gamma+\beta}.
\end{aligned}
\tag{15}
$$

Putting estimates (14) and (15) in (13) we get

$$
\begin{aligned}
\int_{\mathbb{R}^3_z} dz K_{\alpha,\beta}(v,z) &\lesssim \delta^{-1-2s} \langle v\rangle^{\alpha-1} \langle v\rangle^{1+\gamma+\beta+2s} \\
&\lesssim \delta^{-1-2s} \langle v\rangle^{\alpha+\gamma+\beta+2s} \\
&\lesssim \delta^{-1-2s} \qquad \text{if } \alpha+\beta+\gamma+2s \le 0.
\end{aligned}
$$

In conclusion, we have obtained that if $\alpha+\beta+\gamma+2s \le 0$, then

$$\int_{\mathbb{R}^3_z} dz K_{\alpha,\beta}(v,z) \lesssim \delta^{-1-2s}. \tag{16}$$

Now we look for the integration w.r.t. variable v of $K_{\alpha,\beta}$. We have

$$\int_{\mathbb{R}^3_v} dv K_{\alpha,\beta}(v,z) = \int_{\mathbb{R}^3_v} dv \mathbb{1}_{|v-z|\ge\delta/2} \langle v\rangle^{\alpha} \langle z\rangle^{\beta} \langle S(v-z)v\rangle^{1+\gamma+2s} \left(e^{|S(v-z)v|^2-|v|^2+|S(v-z)(z)|^2-|z|^2}\right)^{1/4} \frac{1}{|v-z|^{3+2s}},$$

since by direct computation

$$2(|S(v-z)v|^2-|v|^2)+2v.(v-z)-|v-z|^2 = |S(v-z)v|^2-|v|^2+|S(v-z)(z)|^2-|z|^2.$$

Taking $h=v-z$, $dh=dv$, we get

$$\begin{aligned}\int_{\mathbb{R}^3_v} dv K_{\alpha,\beta}(v,z) &= \int_{\mathbb{R}^3_h} dh \mathbb{1}_{|h|\ge\delta/2} \langle z+h\rangle^{\alpha} \langle z\rangle^{\beta} \langle S(h)(z+h)\rangle^{1+\gamma+2s} \\ &\qquad \left(e^{|S(h)(z+h)|^2-|z+h|^2+|S(h)(z)|^2-|z|^2}\right)^{1/4} \frac{1}{|h|^{3+2s}} \\ &= \int_{\mathbb{R}^3_h} dh \mathbb{1}_{|h|\ge\delta/2} \langle z+h\rangle^{\alpha} \langle z\rangle^{\beta} \langle S(h)z\rangle^{1+\gamma+2s} \\ &\qquad \left(e^{|S(h)z|^2-|z+h|^2+|S(h)z|^2-|z|^2}\right)^{1/4} \frac{1}{|h|^{3+2s}},\end{aligned}$$

so that expanding again the brackets, we get

$$\begin{aligned}&\int_{\mathbb{R}^3_v} dv K_{\alpha,\beta}(v,z) \\ &= \int_{\mathbb{R}^3_h} dh \mathbb{1}_{|h|\ge\delta/2} \left(1+|z|^2+2z.h+|h|^2\right)^{\alpha/2} \langle z\rangle^{\beta} \left(1+|z|^2-|z.\frac{h}{|h|}|^2\right)^{(1+\gamma+2s)/2} \\ &\qquad \left(e^{-|z.\frac{h}{|h|}|^2-2z.h-|h|^2} e^{-|z.\frac{h}{|h|}|^2}\right)^{1/4} \frac{1}{|h|^{3+2s}}.\end{aligned}$$

We shift to spherical coordinates (along axis w.r.t z) ($h=r\omega$) to get

$$\begin{aligned}&\int_{\mathbb{R}^3_v} dv K_{\alpha,\beta}(v,z) \\ &= \int_0^{\pi}\int_{\delta}^{\infty} d\varphi \sin\varphi dr \langle z\rangle^{\beta} (1+|z|^2+2|z|r\cos\varphi+r^2)^{\alpha/2}(1+|z|^2-|z|^2\cos^2\varphi)^{(1+\gamma+2s)/2} \\ &\qquad \left(e^{-|z|^2\cos^2\varphi-2|z|r\cos\varphi-r^2} e^{-|z|^2\cos^2\varphi}\right)^{1/4} \frac{1}{r^{1+2s}}.\end{aligned}$$

Set $t=\cos\varphi$ to get

$$\begin{aligned}&\int_{\mathbb{R}^3_v} dv K_{\alpha,\beta}(v,z) \\ &= \int_{-1}^{1} dt \int_{\delta}^{\infty} dr \langle z\rangle^{\beta} (1+|z|^2+2|z|rt+r^2)^{\alpha/2}(1+|z|^2-|z|^2t^2)^{(1+\gamma+2s)/2} \\ &\qquad \left(e^{-|z|^2t^2-2|z|rt-r^2} e^{-|z|^2t^2}\right)^{1/4} \frac{1}{r^{1+2s}}.\end{aligned}$$

We note again that if $|z| \leq 1$, this integral is bounded uniformly. We therefore assume in the following that $|z| \geq 1$ and change variable $t' = |z|t$ to deduce that

$$
\begin{aligned}
&\int_{\mathbb{R}^3_v} dv K_{\alpha,\beta}(v,z) \\
&= |z|^{-1} \int_{-|z|}^{|z|} dt \int_{\delta}^{\infty} dr \, \langle z\rangle^{\beta} \, (1+|z|^2+2rt+r^2)^{\alpha/2}(1+|z|^2-t^2)^{(1+\gamma+2s)/2} \\
&\qquad\qquad e^{-(t+r)^2/4} e^{-t^2/4} \frac{1}{r^{1+2s}} \\
&\lesssim |z|^{-1} \int_{-|z|}^{|z|} dt \int_{\delta}^{\infty} dr \, \langle z\rangle^{\beta} \, (1+|z|^2-t^2)^{(1+\gamma+2s+\alpha)/2} \, \langle r+t\rangle^{|\alpha|} \, e^{-(t+r)^2/4} e^{-t^2/4} \frac{1}{r^{1+2s}},
\end{aligned}
$$

where the last inequality is a consequence of Peetre's inequality. With exactly the same argument as before for the integration w.r.t. r, for small δ, we obtain

$$
\int_{\mathbb{R}^3_v} dv K_{\alpha,\beta}(v,z) \lesssim \delta^{-1-2s} \, \langle z\rangle^{\alpha+\beta+\gamma+2s}
$$

and thus

$$
\int_{\mathbb{R}^3_v} dv K_{\alpha,\beta}(v,z) \lesssim \delta^{-1-2s} \tag{17}
$$

when $\alpha+\beta+\gamma+2s \leq 0$. From (16) and (17), we use Schur's Lemma to obtain conclusion (i) in Lemma 2.4.

(ii) Now we prove the second estimate about the commutator in Lemma 2.4. Using the σ representation between v, v_* and v', v'_*, (see Figure 2 in Subsection 5.1 of Appendix), we have, for $\theta \in]0,\pi[$,

$$
|v-v_*| = \frac{|v'-v_*|}{\cos\frac{\theta}{2}} \leq \sqrt{2}\,|v'-v_*| \leq \sqrt{2}\,|v_\lambda - v_*|\,,
$$

where

$$
v_\lambda = v + \lambda(v'-v), \quad \lambda \in [0,1].
$$

As a result,

$$
\langle v\rangle \leq \langle v-v_*\rangle + \langle v_*\rangle \leq \sqrt{2}\,\langle v_\lambda - v_*\rangle + \langle v_*\rangle \leq (1+\sqrt{2})\,\langle v_\lambda\rangle\,\langle v_*\rangle\,,
$$

which along with the estimate

$$
\langle v_\lambda\rangle \leq (1+\sqrt{2})\,\langle v\rangle\,\langle v_*\rangle
$$

due to the fact that $|v'-v| = |v-v_*|\sin\frac{\theta}{2} \leq \frac{\sqrt{2}}{2}\,|v-v_*|$, implies

$$
\forall\, \kappa \in \mathbb{R}, \quad \langle v_\lambda\rangle^{\kappa} \lesssim \langle v\rangle^{\kappa}\,\langle v_*\rangle^{|\kappa|}\,.
$$

Therefore, we have

$$
\begin{aligned}
\left|\langle v'\rangle^{\tilde\beta} - \langle v\rangle^{\tilde\beta}\right| &\lesssim \int_0^1 \langle v_\lambda\rangle^{\tilde\beta-1}\,d\lambda\,|v-v'| \\
&\lesssim \langle v\rangle^{\tilde\beta-1}\,\langle v_*\rangle^{|\tilde\beta-1|}\,|v-v'|\,.
\end{aligned}
$$

Then

$$\left|\langle v\rangle^{\tilde\alpha}\left[\overline{\mathcal{L}}_{1,\delta,a},\ \langle v\rangle^{\tilde\beta}\right]f\right| = \left|\langle v\rangle^{\tilde\alpha}\iint dv_*d\sigma B\tilde\varphi_\delta(v'-v)(\mu_*)^{1/2}(\mu'_*)^{1/2}\left(\langle v'\rangle^{\tilde\beta}-\langle v\rangle^{\tilde\beta}\right)f'\right|$$
$$\lesssim \langle v\rangle^{\tilde\alpha+\tilde\beta-1}\iint dv_*d\sigma B\tilde\varphi_\delta(v'-v)(\mu_*)^{1/2}(\mu'_*)^{1/2}\langle v_*\rangle^{|\tilde\beta-1|}\left|v-v'\right|\left|f'\right|.$$

Using Carleman's representation (see the appendix for instance) shows that

$$\begin{aligned}&\left|\langle v\rangle^{\tilde\alpha}\left[\overline{\mathcal{L}}_{1,\delta,a},\ \langle v\rangle^{\tilde\beta}\right]f\right|\\ \lesssim\ & \langle v\rangle^{\tilde\alpha+\tilde\beta-1}\int_h dh\int_{E_{0,h}}d\alpha \mathbb{1}_{|\alpha|\ge|h|}\mathbb{1}_{|h|\ge\frac{\delta}{2}}\mu^{\frac12}(\alpha+v)\mu^{\frac12}(\alpha+v-h)\\ &\qquad\times\langle\alpha+v\rangle^{|\tilde\beta-1|}\frac{|\alpha+h|^{1+\gamma+2s}}{|h|^{2+2s}}|f(v-h)|\\ \lesssim\ & \langle v\rangle^{\tilde\alpha+\tilde\beta-1}\int_h dh\int_{E_{0,h}}d\alpha \mathbb{1}_{|h|\ge\frac{\delta}{2}}\mu^{\frac14}(\alpha+v)\mu^{\frac14}(\alpha+v-h)\frac{|\alpha|^{1+\gamma+2s}}{|h|^{2+2s}}|f(v-h)|.\end{aligned}$$

The last term is quite similar as the one on the right hand side of (12), with α and β there replaced respectively by $\tilde\alpha+\tilde\beta-1$ and 0, and $\mu^{1/2}$, $|h|^{-(3+2s)}$ there replaced respectively by $\mu^{1/4}$, $|h|^{-(2+2s)}$. Then repeating the arguments after (12), we conclude

$$\left|\langle v\rangle^{\tilde\alpha}\left[\overline{\mathcal{L}}_{1,\delta,a},\ \langle v\rangle^{\tilde\beta}\right]f\right|\lesssim\int\tilde K_{\tilde\alpha,\tilde\beta}(v,z)\left|f\right|(z)dz$$

with

$$\tilde K_{\tilde\alpha,\tilde\beta}(v,z)=\mathbb{1}_{|v-z|\ge\delta/2}\langle v\rangle^{\tilde\alpha+\tilde\beta-1}\langle S(v-z)v\rangle^{1+\gamma+2s}\frac{1}{|v-z|^{2+2s}}$$
$$\left(e^{2(|S(v-z)v|^2-|v|^2)+2v.(v-z)-|v-z|^2}\right)^{1/4}.$$

Arguing as for the analysis of $K_{\alpha,\beta}$ in (i), with $\alpha=\tilde\alpha+\tilde\beta-1$ and $\beta=0$, we obtain a similar estimate as (13), that is,

$$\int_{\mathbb{R}^3_z}dz\tilde K_{\tilde\alpha,\tilde\beta}(v,z)\lesssim\delta^{-2s}\langle v\rangle^{(\tilde\alpha+\tilde\beta-1)-1}\int_0^{|v|}dt(1+|v|^2-t^2)^{(1+\gamma+2s)/2}\frac{1}{\langle t\rangle^{2s}}.$$

It's clear that

$$\int_{\mathbb{R}^3_z}dz\tilde K_{\tilde\alpha,\tilde\beta}(v,z)\lesssim\delta^{-2s}$$

for all v such that $|v|\le1$.

We can therefore assume $|v|\ge1$ in the following. We split the integration into three parts as follows. First

$$\int_0^{1/2}dt(1+|v|^2-t^2)^{(1+\gamma+2s)/2}\frac{1}{\langle t\rangle^{2s}}\lesssim\langle v\rangle^{1+\gamma+2s}.$$

Next, for any $\varepsilon_0>0$,

$$\begin{aligned}\int_{1/2}^{|v|/2}dt(1+|v|^2-t^2)^{(1+\gamma+2s)/2}\frac{1}{\langle t\rangle^{2s}}\ &\lesssim\ \langle v\rangle^{1+\gamma+2s}\int_{1/2}^{|v|/2}\frac{dt}{t^{2s}}\\ &\lesssim\ \begin{cases}\langle v\rangle^{1+\gamma+2s}\left(|v|^{-2s+1}+1\right), & s\ne1/2\\ \langle v\rangle^{1+\gamma+2s}(\ln|v|+1)\lesssim\langle v\rangle^{2+\gamma+\varepsilon_0}, & s=1/2\end{cases}\\ &\lesssim\ \langle v\rangle^{2+\gamma+\varepsilon_0}+\langle v\rangle^{1+\gamma+2s}\end{aligned}$$

Finally, repeating the arguments used to get the estimate (15), we have

$$\begin{aligned}\int_{|v|/2}^{|v|} dt(1+|v|^2-t^2)^{(1+\gamma+2s)/2}\frac{1}{\langle t\rangle^{2s}} &\lesssim \langle v\rangle^{-2s}|v|^{-1}\int_0^{|v|^2/2}(1+\lambda)^{(1+\gamma+2s)/2}\,d\lambda\\ &\lesssim \langle v\rangle^{2+\gamma}.\end{aligned}$$

Combining these inequalities gives, for $|v|\geq 1$,

$$\begin{aligned}\int_{\mathbb{R}^3_z} dz\tilde{K}_{\tilde{\alpha},\tilde{\beta}}(v,z) &\lesssim \delta^{-2s}\langle v\rangle^{(\tilde{\alpha}+\tilde{\beta}-1)-1}\left(\langle v\rangle^{2+\gamma+\varepsilon_0}+\langle v\rangle^{1+\gamma+2s}\right)\\ &\lesssim \delta^{-2s}\langle v\rangle^{\tilde{\alpha}+\tilde{\beta}+\gamma+\varepsilon_0}+\delta^{-2s}\langle v\rangle^{\tilde{\alpha}+\tilde{\beta}+\gamma+2s-1}.\end{aligned}$$

Then choosing $\varepsilon_0 = s$ and using the assumption that $\tilde{\alpha}+\tilde{\beta}+\gamma+s\leq 0$, we conclude

$$\int_{\mathbb{R}^3_z} dz\tilde{K}_{\tilde{\alpha},\tilde{\beta}}(v,z)\lesssim \delta^{-2s}.$$

Similarly as in (i), we can show that

$$\int_{\mathbb{R}^3_z} dv\tilde{K}_{\tilde{\alpha},\tilde{\beta}}(v,z)\lesssim \delta^{-2s}.$$

Then Schur's Lemma applies and this completes the proof of conclusion (ii) in Lemma 2.4. □

Study of $\mathcal{L}_{1,4,\delta}$

Lemma 2.5. *For all $f\in\mathcal{S}(\mathbb{R}^3_v)$, we have*

$$\|\mathcal{L}_{1,4,\delta}f\|^2\lesssim \delta^{2-2s}\|<v>^{\gamma+2s}f\|,$$

and $\mathcal{L}_{1,4,\delta}$ commutes with the multiplication by $\langle v\rangle^{\alpha}$ for all $\alpha\in\mathbb{R}$.

Proof. The last assertion is again trivial since $\mathcal{L}_{1,4,\delta}$ is a multiplication operator. Using the formula $2a(b-a)=b^2-a^2-(b-a)^2$, we get

$$\begin{aligned}\mathcal{L}_{1,4,\delta}f &= \frac{1}{2}f\iint dv_*d\sigma B\varphi_\delta(v'-v)\left((\mu_*)-(\mu'_*)\right)\\ &\quad -\frac{1}{2}f\iint dv_*d\sigma B\varphi_\delta(v'-v)\left((\mu_*)^{1/2}-(\mu'_*)^{1/2}\right)^2\\ &= -\frac{1}{2}\mathcal{L}_{1,3,\delta}f+D(v)f.\end{aligned}$$

It suffices to estimate $D(v)$ in view of Lemma 2.3. To do so we essentially follow the same process, except that we don't need to use a symmetrizing argument to kill higher singularities. We write

$$\begin{aligned}|D(v)| &= \frac{1}{2}\int_{\mathbb{R}^3_h} dh\int_{E_{0,h}} d\alpha \mathbb{1}_{|\alpha|\geq|h|}\varphi_\delta(h)\left(\mu^{1/2}(\alpha+v-h)-\mu^{1/2}(\alpha+v)\right)^2\frac{|\alpha+h|^{1+\gamma+2s}}{|h|^{3+2s}}\\ &\lesssim \int_{\mathbb{R}^3_h} dh\int_{E_{0,h}} d\alpha \mathbb{1}_{|\alpha|\geq|h|}\varphi_\delta(h)\left(e^{(\alpha+v)\cdot h-h^2/2}-1\right)^2\mu(\alpha+v)\frac{|\alpha+h|^{1+\gamma+2s}}{|h|^{3+2s}}\\ &\lesssim \int_{\mathbb{R}^3_h} dh\int_{E_{0,h}} d\alpha \mathbb{1}_{|\alpha|\geq|h|}\varphi_\delta(h)\mu^{1/2}(\alpha+v)\frac{|\alpha|^{1+\gamma+2s}}{|h|^{1+2s}}\\ &\lesssim \delta^{2-2s}\langle v\rangle^{\gamma+2s},\end{aligned}$$

following the same arguments as before. From the estimates on $\mathcal{L}_{1,3,\delta}$ and $D(v)$, the proof is complete. $\square$

3 Pseudodifferential parts

In this section we deal with the remaining parts of $\mathcal{L}_1$, namely:

- a multiplicative operator $\overline{\mathcal{L}}_{1,\delta,b}$;

- the principal term $\mathcal{L}_{1,2,\delta}$ which will appear to be of pseudodifferential type;

- and the term $\mathcal{L}_{1,1,\delta}$ which is also of pseudodifferential type but with lower order (and we therefore call it subprincipal).

Our goal in this section is to prove Proposition 1.4 about the behavior of these pseudodifferential parts of $\mathcal{L}$.

In the following, we keep the notation for φ_δ, the positive compactly supported function equal to 1 in a δ-neighborhood of 0 as introduced previously in the definitions of the operators, and let $E_{0,\omega} = \omega^\perp$ for the hyperplane containing 0 and orthogonal to ω. We study each operator separately. Proposition 1.4 will be obtained as a direct consequence of Proposition 3.5 and Proposition 3.1 below and Definition 3.6.

3.1 Study of the principal term $\mathcal{L}_{1,2,\delta}$

Recall that

$$\mathcal{L}_{1,2,\delta}f = \iint dv_* d\sigma B \varphi_\delta(v'-v)\mu'_* \left(f'-f\right)$$

where

$$B(v,\sigma) = |v-v_*|^\gamma b\left(\left\langle \frac{v-v_*}{|v-v_*|}, \sigma\right\rangle\right).$$

This will appear to be a genuine pseudo differential operator of order $2s$ for which we can control the weigths. Namely one has

Proposition 3.1. *We can write*

$$\mathcal{L}_{1,2,\delta}f = -a_p(v, D_v)f,$$

where a_p is a real symbol in (v,η) (see (19) below for the definition of a_p) satisfying:

i) there exists $C>0$ such that for all $0<\kappa<1$,

$$\begin{aligned} C^{-1}\delta^{2-2s}\left(-\kappa\langle v\rangle^{\gamma+2s} + \kappa\langle v\rangle^\gamma (|\eta|^{2s} + |\eta\wedge v|^{2s})\right) \\ \le a_p(v,\eta) \le C\langle v\rangle^\gamma (1+|\eta|^{2s} + |\eta\wedge v|^{2s}); \end{aligned} \tag{18}$$

ii) $a_p \in S\left(\langle v\rangle^\gamma (1+|v|^{2s}+|\eta|^{2s}+|\eta\wedge v|^{2s}), \Gamma\right)$. Recall $\Gamma = |dv|^2 + |d\eta|^2$ is the flat metric.

Remark 3.2. The first estimates in (18) explain why we don't have regularity estimate for the Boltzmann equation with angular cutoff, since it corresponds to the case $\delta \to 0$ and thus we lose the regularity operator $\langle v\rangle^\gamma \langle D_v\rangle^{2s} + \langle v\rangle^\gamma \langle D_v \wedge v\rangle^{2s}$. Observe we exclude the case $s = 1$, and this corresponds the Landau equation, which is the grazing limit of Boltzmann equation without angular cutoff and still admits the diffusion structure.

Proof. From the expression of $\mathcal{L}_{1,2,\delta}$, using Carleman's transformation as in previous arguments and as in [1] (see also the Appendix), we get

$$\mathcal{L}_{1,2,\delta} f = \int_{\mathbb{R}^3_h} dh \int_{E_{0,h}} d\alpha \tilde{b}(\alpha, h) \mathbb{1}_{|\alpha| \geq |h|} \varphi_\delta(h) \mu(\alpha + v) |\alpha + h|^{1+\gamma+2s} \left(f(v-h) - f(v) \right) \frac{1}{|h|^{3+2s}},$$

where $\tilde{b}(\alpha, h)$ is a function of α and h which is bounded from below and above by positive constants, and satisfies that $\tilde{b}(\alpha, h) = \tilde{b}(\pm\alpha, \pm h)$.

This integral is typically undefined for large values of s, and we have to use its symmetrized version in order to give a meaning in the principal value sense: for this purpose, we change h to $-h$ and add the two expressions to obtain

$$\begin{aligned}\mathcal{L}_{1,2,\delta} f &= \frac{1}{2} \int_h dh \int_{E_{0,h}} d\alpha \tilde{b} \mathbb{1}_{|\alpha| \geq |h|} \varphi_\delta(h) \mu(\alpha + v) |\alpha + h|^{1+\gamma+2s} \\ &\qquad \left(f(v-h) + f(v+h) - 2f(v) \right) \frac{1}{|h|^{3+2s}} \\ &\stackrel{\text{def}}{=} -a_p(v, D_v) f(v) \stackrel{\text{def}}{=} - \int_{\mathbb{R}^3_\eta} a_p(v, \eta) \hat{f}(\eta) e^{i\eta \cdot v} d\eta\end{aligned}$$

with

$$\begin{aligned}a_p(v, \eta) &\stackrel{\text{def}}{=} -\frac{1}{2} \int_{\mathbb{R}^3_h} dh \int_{E_{0,h}} d\alpha \tilde{b} \mathbb{1}_{|\alpha| \geq |h|} \varphi_\delta(h) \mu(\alpha + v) |\alpha + h|^{1+\gamma+2s} \\ &\qquad \left(e^{-i\eta \cdot h} + e^{i\eta \cdot h} - 2 \right) \frac{1}{|h|^{3+2s}} \\ &= \int_{\mathbb{R}^3_h} dh \int_{E_{0,h}} d\alpha \tilde{b} \mathbb{1}_{|\alpha| \geq |h|} \varphi_\delta(h) \mu(\alpha + v) |\alpha + h|^{1+\gamma+2s} \\ &\qquad \left(1 - \cos(\eta \cdot h) \right) \frac{1}{|h|^{3+2s}}.\end{aligned} \tag{19}$$

The non-negativity of $a_p(v, \eta)$ is clear and we shall now work on some properties of this symbol. First recall that on the support of the integrand, we have $|h| \leq \delta \leq 1$ and that $\alpha \perp h$, so that

$$0 \leq a_p(v, \eta) \lesssim \int_{\mathbb{R}^3 h} dh \int_{E_{0,h}} \mathbb{1}_{|\alpha| \geq |h|} \mathbb{1}_{|h| \leq \delta} d\alpha \mu(\alpha + v) \langle \alpha \rangle^{1+\gamma+2s} \left(1 - \cos(\eta \cdot h) \right) \frac{1}{|h|^{3+2s}}.$$

Now we can shift to spherical coordinates $h = r\omega$, and (forgetting the truncation in α) we get

$$a_p(v, \eta) \lesssim \int_0^\delta \int_{S^2_\omega} dr d\omega \int_{E_{0,\omega}} d\alpha \mu(\alpha + v) \langle \alpha \rangle^{1+\gamma+2s} \left(1 - \cos(r\eta.\omega) \right) \frac{1}{r^{1+2s}}.$$

It is possible to integrate directly w.r.t. r, and use the fact that

$$\int_0^\delta \left(1 - \cos(r\eta.\omega) \right) \frac{1}{r^{1+2s}} dr \leq C_s |\omega \cdot \eta|^{2s}.$$

In fact, note that

$$\int_0^\delta (1-\cos(r\eta.\omega))\,\frac{1}{r^{1+2s}}dr = |\omega\cdot\eta|^{2s}\int_0^{\delta|\omega\cdot\eta|}(1-\cos(r))\,\frac{1}{r^{1+2s}}dr$$

Next, we choose a small constant c such that $1-\cos r \gtrsim r^2$ if $r\leq c$.

If $|\omega\cdot\eta|\geq c$, then we get

$$\int_0^\delta (1-\cos(r\eta.\omega))\,\frac{1}{r^{1+2s}}dr \gtrsim |\omega\cdot\eta|^{2s}\int_0^{c\delta}(1-\cos(r))\,\frac{1}{r^{1+2s}}dr \gtrsim \delta^{2-2s}|\omega\cdot\eta|^{2s},$$

while if $|\omega\cdot\eta|\leq c$, then we get

$$\int_0^\delta (1-\cos(r\eta.\omega))\,\frac{1}{r^{1+2s}}dr \gtrsim |\omega\cdot\eta|^{2}\int_0^{\delta} r^2\frac{1}{r^{1+2s}}dr \gtrsim \delta^{2-2s}|\omega\cdot\eta|^{2}.$$

On the whole, we get

$$\int_0^\delta (1-\cos(r\eta.\omega))\,\frac{1}{r^{1+2s}}dr \gtrsim \delta^{2-2s}\min\{|\omega\cdot\eta|^2,|\omega\cdot\eta|^{2s}\}. \tag{20}$$

In fact the same type of arguments show that we get a similar upper bound, and eventually

$$\delta^{2-2s}\min\{|\omega\cdot\eta|^2,|\omega\cdot\eta|^{2s}\} \lesssim \int_0^\delta (1-\cos(r\eta.\omega))\,\frac{1}{r^{1+2s}}dr \lesssim \delta^{2-2s}|\omega\cdot\eta|^{2s}. \tag{21}$$

Next, we deal with the upper bound on a_p. A crude estimate is enough and we get

$$a_p(v,\eta) \lesssim \int_{S^2_\omega} d\omega \int_{E_{0,\omega}} d\alpha\mu(\alpha+v)|\omega\cdot\eta|^{2s}\,\langle\alpha\rangle^{1+\gamma+2s}. \tag{22}$$

Splitting $v=S(\omega)v+(\omega\cdot v)\omega$, we have

$$|\alpha+v|^2 = |\alpha+S(\omega)v+(\omega\cdot v)\omega|^2 = |\alpha+S(\omega)v|^2+|(\omega\cdot v)|^2 \tag{23}$$

since α and ω are orthogonal. We can therefore write

$$\mu(\alpha+v) = (2\pi)^{-3/2}\left(e^{-|\alpha+S(\omega)v|^2}e^{-|(\omega\cdot v)|^2}\right)^{1/2}$$

to get

$$a_p(v,\eta) \lesssim \int_{S^2_\omega} d\omega \int_{E_{0,\omega}} d\alpha\left(e^{-|\alpha+S(\omega)v|^2}e^{-|(\omega\cdot v)|^2}\right)^{1/2}|\omega\cdot\eta|^{2s}\,\langle\alpha\rangle^{1+\gamma+2s}. \tag{24}$$

Next, note that

$$\beta(v,\omega) = \int_{E_{0,\omega}} d\alpha\left(e^{-|\alpha+S(\omega)v|^2}\right)^{1/2}\langle\alpha\rangle^{1+\gamma+2s} \sim < S(\omega)v >^{1+\gamma+2s}$$

and thus

$$a_p(v,\eta) \lesssim \int_{S^2_\omega} d\omega e^{-|(\omega\cdot v)|^2/2} < S(\omega)v >^{1+\gamma+2s}|\omega\cdot\eta|^{2s}. \tag{25}$$

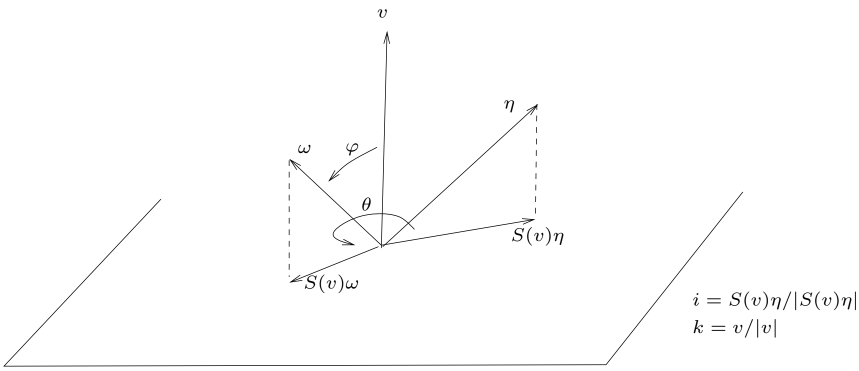

Figure 1: spherical coordinates

We introduce polar coordinates in a coordinate system where $\mathbf{i} = S(v)\eta/|S(v)\cdot\eta|$, $\mathbf{k} = v/|v|$.
In this system, we note that $(\omega\cdot\mathbf{k}) = \cos(\varphi)$. Besides we have $\eta = (\eta.\mathbf{k})\mathbf{k} + S(v)\eta$ so that

$$\begin{aligned}\omega\cdot\eta &= (\eta\cdot\mathbf{k})(\mathbf{k}\cdot\omega) + (S(v)\eta)\cdot\omega\\ &= (\eta\cdot\mathbf{k})(\mathbf{k}\cdot\omega) + (S(v)\eta)\cdot(S(v)\omega)\\ &= (\eta\cdot\mathbf{k})(\mathbf{k}\cdot\omega) + (\mathbf{i}\cdot(S(v)\omega))\,|S(v)\eta|\\ &= \eta\cdot\mathbf{k}\cos(\varphi) + |S(v)\eta|\sin(\varphi)\cos(\theta).\end{aligned}$$

and in a similar way

$$|S(\omega)v|^2 = |v|^2 - |(v.\omega)|^2 = |v|^2(1-\cos^2(\varphi)) = |v|^2\sin^2(\varphi).$$

We therefore get

$$a_p(v,\eta) \lesssim \int_0^{\pi} d\varphi \int_0^{2\pi} d\theta \sin(\varphi) e^{-|v|^2\cos^2(\varphi)}\left(1+|v|^2\sin^2(\varphi)\right)^{(1+\gamma+2s)/2}$$
$$|\eta\cdot\mathbf{k}\cos(\varphi) + |S(v)\eta|\sin(\varphi)\cos(\theta)|^{2s}.$$

Setting $\cos\varphi = t$ in the preceding formula, we get

$$a_p(v,\eta) \lesssim \int_0^{2\pi} d\theta \int_0^{1} dt e^{-|v|^2t^2}\left(1+|v|^2(1-t^2)\right)^{(1+\gamma+2s)/2}$$
$$|\eta\cdot\mathbf{k}t + |S(v)\eta|\sqrt{1-t^2}\cos(\theta)|^{2s}. \quad (26)$$

If we bound roughly $1-t^2$ and $\cos(\varphi)$ by 1 and use the estimates that

$$e^{-|v|^2t^2}\left(1+|v|^2(1-t^2)\right)^{(1+\gamma+2s)/2} \lesssim e^{-|v|^2t^2}\left(1+|v|^2\right)^{(1+\gamma+2s)/2}$$

for $1+\gamma+2s \geq 0$ or $0 \leq t \leq 1/2$, and that

$$e^{-|v|^2t^2}\left(1+|v|^2(1-t^2)\right)^{(1+\gamma+2s)/2} \lesssim e^{-|v|^2t^2} \lesssim e^{-|v|^2t^2/2}\left(1+v^2\right)^{(1+\gamma+2s)/2}$$

for $1+\gamma+2s < 0$ and uniformly w.r.t. $1/2 \leq t \leq 1$, then we get

$$a_p(v,\eta) \lesssim \int_0^{2\pi} d\theta \int_0^1 dt \;\; e^{-|v|^2t^2/2} \left(1+|v|^2\right)^{(1+\gamma+2s)/2} \left(|\eta\cdot \mathbf{k}t|^{2s} + |S(v)\eta|^{2s}\right).$$

If we set $y=|v|t$, we get

$$\begin{aligned} a_p(v,\eta) &\lesssim \frac{1}{|v|}\langle v\rangle^{1+\gamma+2s} \int_0^{2\pi} d\theta \int_0^{|v|} dy \;\; e^{-y^2/2} \left(|\eta\cdot \mathbf{k}|^{2s}\frac{y^{2s}}{|v|^{2s}} + |S(v)\eta|^{2s}\right) \\ &\lesssim \frac{1}{|v|}\langle v\rangle^{1+\gamma+2s} \left(|\eta\cdot \mathbf{k}|^{2s}\frac{1}{|v|^{2s}} + |S(v)\eta|^{2s}\right) \qquad (27) \\ &\lesssim \frac{\langle v\rangle^{1+\gamma+2s}}{|v|^{1+2s}}|\eta|^{2s} + \frac{\langle v\rangle^{1+\gamma+2s}}{|v|}|S(v)\eta|^{2s}. \end{aligned}$$

For $|v|\geq 1$, we therefore get

$$a_p(v,\eta) \lesssim \langle v\rangle^{\gamma}|\eta|^{2s} + \langle v\rangle^{\gamma+2s}|S(v)\eta|^{2s},$$

and thus

$$a_p(v,\eta) \lesssim \langle v\rangle^{\gamma}\left(|\eta|^{2s} + |v\wedge\eta|^{2s}\right),$$

since $|v\wedge \eta| = |v||S(v)\eta|$. For $|v|\leq 1$, a rough estimate gives directly $|a(v,\eta)|\leq \langle \eta\rangle^{2s}$ so that the preceding estimate is also true. The proof of the upper bound is complete.

Now we deal with the lower bound. To this end, we shall use the formula (19)

$$a_p(v,\eta) = \int_{\mathbb{R}^3_h} dh \int_{E_{0,h}} d\alpha \tilde{b}\varphi_\delta(h)\mathbb{1}_{|\alpha|\geq|h|}\mu(\alpha+v)|\alpha+h|^{1+\gamma+2s}\left(1-\cos(\eta\cdot h)\right)\frac{1}{|h|^{3+2s}}.$$

As we want a lower bound we can restrict the integration range to $\{|\alpha|\geq 10\}$ since the integrand is non negative. We use also the facts that $\tilde{b}$ is bounded from below by a positive constant and that $|\alpha+h|\sim|\alpha|$ since $\alpha\perp h$ and $|h|\leq|\alpha|$ in the preceding integral. Therefore, we have

$$a_p(v,\eta) \gtrsim \int_{\mathbb{R}^3_h} dh \int_{E_{0,h}} \varphi_\delta(h)\mathbb{1}_{|\alpha|\geq 10} d\alpha \mu(\alpha+v)\langle\alpha\rangle^{1+\gamma+2s}\left(1-\cos(\eta\cdot h)\right)\frac{1}{|h|^{3+2s}}.$$

We can use some of the previous computations, and from (20)-(21) we get as in (24),

$$a_p(v,\eta) \gtrsim \delta^{2-2s}\int_{S^2_\omega} d\omega \int_{E_{0,\omega}} d\alpha \mathbb{1}_{|\alpha|\geq 10} e^{-|\alpha+S(\omega)v|^2/2}e^{-|(\omega\cdot v)|^2/2}\min\{|\omega\cdot\eta|^2,|\omega\cdot\eta|^{2s}\}\langle\alpha\rangle^{1+\gamma+2s}.$$

Note that

$$\beta_{10}(v,\omega) \stackrel{\text{def}}{=} \int_{E_{0,\omega}} d\alpha 1_{|\alpha|\geq 10} e^{-|\alpha+S(\omega)v|^2/2}\langle\alpha\rangle^{1+\gamma+2s} \sim < S(\omega)v >^{1+\gamma+2s}.$$

Therefore

$$a_p(v,\eta) \gtrsim \delta^{2-2s}\int_{S^2_\omega} d\omega e^{-|(\omega\cdot v)|^2/2} < S(\omega)v >^{1+\gamma+2s}\min\{|\omega\cdot\eta|^2,|\omega\cdot\eta|^{2s}\}. \qquad (28)$$

We now consider an arbitrary real $0 < \kappa < 1$. Using the fact that

$$\min\{|\omega \cdot \eta|^2, |\omega \cdot \eta|^{2s}\} \geq |\omega \cdot \eta|^{2s} - 1,$$

and that the right member in (28) is non-negative, we get that

$$\begin{aligned}
a_p(v,\eta) &\gtrsim \kappa\delta^{2-2s} \int_{S^2_\omega} d\omega e^{-|(\omega\cdot v)|^2/2} < S(\omega)v >^{1+\gamma+2s} \min\{|\omega\cdot\eta|^2, |\omega\cdot\eta|^{2s}\} \\
&\gtrsim \kappa\delta^{2-2s} \int_{S^2_\omega} d\omega e^{-|(\omega\cdot v)|^2/2} < S(\omega)v >^{1+\gamma+2s} \left(|\omega\cdot\eta|^{2s} - 1\right) \\
&\gtrsim \kappa\delta^{2-2s} \int_{S^2_\omega} d\omega e^{-|(\omega\cdot v)|^2/2} < S(\omega)v >^{1+\gamma+2s} |\omega\cdot\eta|^{2s} \qquad (29) \\
&\qquad\qquad - \kappa\delta^{2-2s} \int_{S^2_\omega} d\omega e^{-|(\omega\cdot v)|^2/2} < S(\omega)v >^{1+\gamma+2s} \\
&\stackrel{\text{def}}{=} \kappa\delta^{2-2s} a_{pp} - \kappa\delta^{2-2s} a_{pr}.
\end{aligned}$$

We split the study of the two terms a_{pp} and a_{pr}. For a_{pp}, we can use previous computations yielding to (25). More precisely, we have

$$a_{pp}(v,\eta) = \int_0^{2\pi} d\theta \int_0^1 dt e^{-|v|^2t^2} \left(1 + |v|^2(1-t^2)\right)^{(1+\gamma+2s)/2} \left|\eta\cdot\mathbf{k}t + |S(v)\eta|\sqrt{1-t^2}\cos(\theta)\right|^{2s}. \quad (30)$$

Now an easy remark is that the symbol a_{pp} has the following parity properties:

$$a_{pp}(\pm v, \pm\eta) = a_{pp}(v,\eta).$$

We can therefore assume that $\eta\cdot\mathbf{k} \geq 0$ in all the computations. Moreover we can restrict the above integration to the following subsets

$$t \in [0, \sqrt{3}/2], \qquad \theta \in [0, \pi/3], \qquad (31)$$

which implies that all terms inside the absolute value

$$|\eta\cdot\mathbf{k}t + |S(v)\eta|\sqrt{1-t^2}\cos(\theta)|$$

are non-negative. We therefore get, when (31) is fulfilled, that

$$\left(1 + |v|^2(1-t^2)\right)^{(1+\gamma+2s)/2} \geq \left(1 + \frac{|v|^2}{4}\right)^{(1+\gamma+2s)/2} \geq c_{s,\gamma} \langle v\rangle^{1+\gamma+2s}$$

and

$$\begin{aligned}
|\eta\cdot\mathbf{k}t + |S(v)\eta|\sqrt{1-t^2}\cos(\theta)|^{2s} &\geq 4^{-2s}|\eta\cdot\mathbf{k}t + |S(v)\eta||^{2s} \\
&\geq c_s\left(|\eta\cdot\mathbf{k}t|^{2s} + |S(v)\eta|^{2s}\right).
\end{aligned}$$

Therefore putting the above estimate into (30) gives

$$a_{pp}(v,\eta) \gtrsim \int_0^{\pi/3} d\theta \int_0^{\sqrt{3}/2} dt \;\; e^{-|v|^2t^2} \langle v\rangle^{1+\gamma+2s} \left(|\eta\cdot\mathbf{k}t|^{2s} + |S(v)\eta|^{2s}\right).$$

As in the case of the upper bound, we set $y = |v|t$, and get for $|v| \geq 1$ that

$$\begin{aligned} a_{pp}(v,\eta) &\gtrsim \frac{1}{|v|} \langle v \rangle^{1+\gamma+2s} \int_0^{\pi/3} d\theta \int_0^{\sqrt{3}|v|/2} dy \;\; e^{-y^2} \left(|\eta\cdot\mathbf{k}|^{2s} \frac{y^{2s}}{|v|^{2s}} + |S(v)\eta|^{2s} \right) \\ &\gtrsim \frac{1}{|v|} \langle v \rangle^{1+\gamma+2s} \int_0^{\pi/3} d\theta \int_0^{\sqrt{3}/2} dy \;\; e^{-y^2} \left(|\eta\cdot\mathbf{k}|^{2s} \frac{y^{2s}}{|v|^{2s}} + |S(v)\eta|^{2s} \right) \\ &\gtrsim \frac{1}{|v|} \langle v \rangle^{1+\gamma+2s} \left(|\eta\cdot\mathbf{k}|^{2s} \frac{1}{|v|^{2s}} + |S(v)\eta|^{2s} \right) \\ &\gtrsim \left(\langle v \rangle^{\gamma} |\eta|^{2s} + \langle v \rangle^{\gamma+2s} |S(v)\eta|^{2s} \right), \end{aligned}$$

where in the last inequality we use that $\eta\cdot\mathbf{k} \geq 0$ and the fact that if $\eta\cdot\mathbf{k} \leq |\eta|/2$ then

$$|S(v)\eta| \geq \sqrt{3}|\eta|/2.$$

Since $|v\wedge\eta| = |v||S(v)\eta|$ we get for $|v| \geq 1$ the desired result

$$a_{pp}(v,\eta) \gtrsim \langle v \rangle^{\gamma} \left(|\eta|^{2s} + |v\wedge\eta|^{2s} \right). \tag{32}$$

For $|v| \leq 1$, a direct check, without the change of variables $|v|\, t \to y$, gives

$$\begin{aligned} a_{pp}(v,\eta) \gtrsim \int_0^{\pi/3} d\theta \int_0^{\sqrt{3}/2} dte^{-t^2} \left(|\eta\cdot\mathbf{k}t|^{2s} + |S(v)\eta|^{2s} \right) \;\; &\gtrsim \;\; |\eta\cdot\mathbf{k}|^{2s} + |S(v)\eta|^{2s} \\ &\gtrsim \;\; |\eta|^{2s} + |v\wedge\eta|^{2s}. \end{aligned}$$

So the preceding estimate (32) is also true for $|v| \leq 1$.

For the remainder term in (29), we can use similar computations as the ones done for the upper bound for a_p, and we easily get

$$a_{pr} \lesssim \langle v \rangle^{\gamma+2s}.$$

Putting this estimate and (32) together into (29) completes the proof of the lower bound in (18).

Now we deal with estimates on the derivatives in η and v of a_p. Recall that

$$a_p(v,\eta) = \int_{\mathbb{R}^3_h} dh \int_{E_{0,h}} d\alpha \tilde{b} \mathbb{1}_{|\alpha|\geq|h|} \varphi_\delta(h) \mu(\alpha+v) |\alpha+h|^{1+\gamma+2s} \left(1-\cos(\eta\cdot h)\right) \frac{1}{|h|^{3+2s}}$$

which is clearly smooth with respect to v and η. Let us consider for $\nu_1,\nu_2 \in \mathbb{N}^3$ the derivative

$$\begin{aligned} \partial_v^{\nu_1}\partial_\eta^{\nu_2} a_p(v,\eta) = \int_h dh \int_{E_{0,h}} d\alpha \tilde{b} \mathbb{1}_{|\alpha|\geq|h|} \varphi_\delta(h) \left(\partial_v^{\nu_1}\mu(\alpha+v)\right) |\alpha+h|^{1+\gamma+2s} \\ \left(\partial_\eta^{\nu_2}\left(1-\cos(\eta\cdot h)\right)\right) \frac{1}{|h|^{3+2s}}. \end{aligned}$$

Setting again $h = r\omega$, and (forgetting the truncation in α) we get

$$\left|\partial_v^{\nu_1}\partial_\eta^{\nu_2} a_p(v,\eta)\right| \lesssim \int_0^\delta \int_{S^2_\omega} dr d\omega \int_{E_{0,\omega}} d\alpha |\partial_v^{\nu_1}\mu(\alpha+v)| \langle \alpha \rangle^{1+\gamma+2s} \left|\partial_\eta^{\nu_2}\left(1-\cos(r\eta.\omega)\right)\right| \frac{1}{r^{1+2s}}. \tag{33}$$

Since $r \in [0,\delta]$ we claim that we have the following rough estimate

Lemma 3.3. *Let $0 < s < 1$. Then $\forall\ \nu_2 \in \mathbb{N}^3$, $\int_0^\delta dr \left|\partial_\eta^{\nu_2}(1-\cos(r\omega\cdot\eta))\right| \frac{1}{r^{1+2s}} \leq C_{\delta,s}\langle\omega\cdot\eta\rangle^{2s}$.*

Proof of the Lemma. This is clear for $\nu_2 = 0$ from the previous upper bound computation.

For $|\nu_2| = 1$ we have to estimate

$$I(\nu_2) = \int_0^\delta dr \left|\left(\partial_\eta^{\nu_2}(1-\cos(r\omega\cdot\eta))\right)\right| \frac{1}{r^{1+2s}} \leq \int_0^\delta dr \left|\sin(r\omega\cdot\eta)\right| \frac{1}{r^{2s}}.$$

Firstly, when $0 < s < 1/2$, we directly get

$$I(\nu_2) \leq \int_0^\delta dr \frac{1}{r^{2s}} \leq C_{s\delta} \leq C_{s\delta}\langle\omega\cdot\eta\rangle^{2s}.$$

When $s = 1/2$ then

$$\begin{aligned} I(\nu_2) &\leq \int_0^{\delta|\omega\cdot\eta|} \frac{|\sin t|}{t} dt \leq \int_0^{\langle\delta\omega\cdot\eta\rangle} \frac{|\sin t|}{t} dt \leq \int_0^1 \frac{|\sin t|}{t} dt + \int_1^{\langle\delta\omega\cdot\eta\rangle} 1 dt \\ &\leq 1 + C_\delta\langle\omega\cdot\eta\rangle = C_s\langle\omega\cdot\eta\rangle^{2s} \end{aligned}$$

When $1/2 < s < 1$ we have

$$I(\nu_2) \leq |\omega\cdot\eta|^{2s-1} \int_0^\infty \frac{|\sin t|}{t^{2s}} dt \leq C_s|\omega\cdot\eta|^{2s-1} \leq C_s\langle\omega\cdot\eta\rangle^{2s}.$$

Thus we obtain the estimate for $|\nu_2| = 1$.

It remains to consider the case when $|\nu_2| \geq 2$. Observe $0 < s < 1$, and thus

$$I(\nu_2) = \int_0^\delta dr \left|\left(\partial_\eta^{\nu_2}(1-\cos(r\omega\cdot\eta))\right)\right| \frac{1}{r^{1+2s}} \leq \int_0^\delta \frac{dr}{r^{2s-1}} \leq C_{s\delta} \leq C_{s\delta}\langle\omega\cdot\eta\rangle^{2s}.$$

The proof of the lemma is complete. □

End of the proof of Proposition 3.1 Now we go back to (33). We have also to estimate the term $(\partial_v^{\nu_1}\mu(\alpha+v))$ in this integral. For this purpose, we directly use the fact that for all ν_1,

$$|\partial_v^{\nu_1}\mu(\alpha+v)| \leq C_{\nu_1}\mu^{1/2}(\alpha+v). \tag{34}$$

Thanks to Lemma 3.3 and the preceding estimate, we get from (33) that

$$\left|\partial_v^{\nu_1}\partial_\eta^{\nu_2}a_p(v,\eta)\right| \lesssim \int_\omega d\omega \int_{E_{0,\omega}} d\alpha \mu^{1/2}(\alpha+v)\langle\alpha\rangle^{1+\gamma+2s}\langle\omega\cdot\eta\rangle^{2s}.$$

For the final estimates, we can repeat exactly the proof of the case $\nu_1 = \nu_2 = 0$, to get the desired result. The proof of Proposition 3.1 is complete. □

For further use, we shall also need the following estimate

Proposition 3.4. *The symbol a_p also satisfies the following estimate: for any $0 < \varepsilon < 1$,*

$$\partial_\eta a_p \in S\left(\varepsilon \langle v\rangle^\gamma (1 + |\eta|^{2s} + |\eta \wedge v|^{2s}) + \varepsilon^{-1} \langle v\rangle^{\gamma+2s}, \Gamma\right),$$

with semi-norms (see Subsection A.2 for the definition of semi-norms) independent of ε.

Proof. We can again rely on the preceding arguments. We begin with (33) and we can write for $|\nu_2| \geq 1$,

$$\left|\partial_v^{\nu_1}\partial_\eta^{\nu_2} a_p(v,\eta)\right| \leq C \int_0^\delta dr \int_{S_\omega^2} d\omega \int_{E_{0,\omega}} d\alpha \, (\partial_v^{\nu_1}\mu(\alpha+v))\, | \\ \langle\alpha\rangle^{1+\gamma+2s}\left(\partial_\eta^{\nu_2}\left(1-\cos(r\eta.\omega)\right)\right)\frac{1}{r^{1+2s}}.$$

Suppose that $|\nu_2| \geq 2$. We can verify directly that, observing $0 < s < 1$ and $|\omega| = 1$,

$$\int_0^\delta dr \left|\left(\partial_\eta^{\nu_2}\left(1-\cos(r\omega\cdot\eta)\right)\right)\right| \frac{1}{r^{1+2s}} \leq \int_0^\delta \frac{dr}{r^{2s-1}} \leq C_{\delta,s}.$$

Therefore, using also (34),

$$|\partial_v^{\nu_1}\partial^{\nu_2} a_p(v,\eta)| \lesssim \int_{S_\omega^2} d\omega \int_{E_{0,\omega}} d\alpha \mu^{1/2}(\alpha+v)\langle\alpha\rangle^{1+\gamma+2s} \lesssim \langle v\rangle^{\gamma+2s},$$

the last inequality following the same computation as that after (22) with $|\omega\cdot\eta|^{2s}$ there replaced here by 1.

Consider the case when $|\nu_2| = 1$. Then we have

$$\int_0^\delta dr \left|\left(\partial_\eta^{\nu_2}\left(1-\cos(r\omega\cdot\eta)\right)\right)\right| \frac{1}{r^{1+2s}} \leq \int_0^\delta \frac{|\sin(r\omega\cdot\eta)|}{r^{2s}} dr.$$

Furthermore if $0 < s < 1/2$ then

$$\int_0^\delta \frac{|\sin(r\omega\cdot\eta)|}{r^{2s}} dr \leq C_{\delta,s},$$

and if $1/2 < s < 1$ then

$$\begin{aligned}\int_0^\delta \frac{|\sin(r\omega\cdot\eta)|}{r^{2s}} dr &\leq |\omega\cdot\eta|^{2s-1}\int_0^{\delta|\omega\cdot\eta|} \frac{|\sin\theta|}{\theta^{2s}} d\theta \leq C_{\delta,s}(1+\langle\omega\cdot\eta\rangle^{2s-1}) \\ &\lesssim \varepsilon\langle\omega\cdot\eta\rangle^{2s} + \varepsilon^{-(2s-1)} \lesssim \varepsilon\langle\omega\cdot\eta\rangle^{2s} + \varepsilon^{-1}\end{aligned}$$

for any $0 < \varepsilon < 1$, and finally if $s = 1/2$ then

$$\begin{aligned}\int_0^\delta \frac{|\sin(r\omega\cdot\eta)|}{r^{2s}} dr &\leq \int_0^{\delta|\omega\cdot\eta|} \frac{|\sin\theta|}{\theta} d\theta \leq C_{\delta,s}(1+\ln\langle\omega\cdot\eta\rangle) \leq C_{\delta,s}(1+\langle\omega\cdot\eta\rangle^{s}) \\ &\lesssim \varepsilon\langle\omega\cdot\eta\rangle^{2s} + \varepsilon^{-1}\end{aligned}$$

for any $\varepsilon > 0$. Thus combining the above estimates we conclude, for $0 < s < 1$ and for any $0 < \varepsilon < 1$,

$$\int_0^\delta dr \left|\left(\partial_\eta^{\nu_2}\left(1-\cos(r\omega\cdot\eta)\right)\right)\right| \frac{1}{r^{1+2s}} \lesssim \varepsilon\langle\omega\cdot\eta\rangle^{2s} + \varepsilon^{-1}.$$

Therefore we get that, using again (34) the arguments after (22),

$$
\begin{aligned}
|\partial_v^{\nu_1}\partial^{\nu_2}a_p(v,\eta)| &\lesssim \int_{S^2_\omega} d\omega \int_{E_{0,\omega}} d\alpha \mu^{1/2}(\alpha+v)\left\langle \alpha\right\rangle^{1+\gamma+2s}\left(\varepsilon\left\langle \omega\cdot\eta\right\rangle^{2s}+\varepsilon^{-1}\right) \\
&\lesssim \varepsilon\left\langle v\right\rangle^{\gamma}\left(1+|\eta|^{2s}+|v\wedge\eta|^{2s}\right)+\varepsilon^{-1}\left\langle v\right\rangle^{\gamma+2s},
\end{aligned}
$$

with $|\nu_2|=1$.

Combining the estimates for $|\nu_2|\geq 2$ and for $|\nu_2|=1$ we obtain the statement in Proposition 3.4, completing the proof. □

3.2 Study of the multiplicative term $\overline{\mathcal{L}}_{1,\delta,b}$

Recall that the multiplicative part $\overline{\mathcal{L}}_{1,\delta,b}$ has the following form

$$
\overline{\mathcal{L}}_{1,\delta,b}f=-\left(\iint dv_* d\sigma B\tilde{\varphi}_\delta(v'-v)\mu_*\right)f.
$$

A nice feature of the multiplicative function defining $\overline{\mathcal{L}}_{1,\delta,b}$ is its good symbolic properties.

Proposition 3.5. *We can write*

$$
\overline{\mathcal{L}}_{1,\delta,b}f=-a_m(v)f,
$$

where a_m is a function in v satisfying the following symbolic estimates:

i) there exists $C>0$ such that $C^{-1}\left\langle v\right\rangle^{\gamma+2s}\leq a_m(v,\eta)\leq C\left\langle v\right\rangle^{\gamma+2s}$;

ii) $a_m\in S(\left\langle v\right\rangle^{\gamma+2s},\Gamma)$.

Proof. Let us again use Carleman's representation. We get

$$
a_m(v)=\int_{\mathbb{R}^3_h} dh\int_{E_{0,h}} d\alpha\tilde{b}\mathbb{1}_{|\alpha|\geq|h|}\tilde{\varphi}_\delta(h)\mu(v+\alpha-h)|\alpha+h|^{1+\gamma+2s}\frac{1}{|h|^{3+2s}}. \tag{35}
$$

In this integral $h\perp\alpha$ and $|\alpha|\geq|h|$ so that there exists C_s such that

$$
C_s^{-1}|\alpha|^{1+\gamma+2s}\leq|\alpha+h|^{1+\gamma+2s}\leq C_s|\alpha|^{1+\gamma+2s}. \tag{36}
$$

Therefore, shifting to spherical coordinates, and recalling that we write $\varphi_\delta(h)=\varphi_\delta(r)$ for $r=|h|$ by abuse of notation, we have

$$
\begin{aligned}
a_m(v) &\lesssim \iint d\omega dr\int_{E_{0,\omega}} d\alpha 1_{|\alpha|\geq r}\tilde{\varphi}_\delta(r)\mu(v+\alpha-r\omega)|\alpha|^{1+\gamma+2s}\frac{1}{r^{1+2s}} \\
&\lesssim \iint d\omega dr\int_{E_{0,\omega}} d\alpha 1_{|\alpha|\geq r}\tilde{\varphi}_\delta(r)\mu(v+\alpha-r\omega)|\alpha|^{1+\gamma+2s}\frac{1}{r^{1+2s}}.
\end{aligned}
$$

Note that

$$
|v+\alpha-r\omega|^2=|\alpha+S(\omega)v|^2+|(\omega\cdot v)-r|^2
$$

exactly as in (23) so that

$$
e^{-|v+\alpha-r\omega|^2}=e^{-|\alpha+S(\omega)v|^2}e^{-|(\omega\cdot v)-r|^2}.
$$

Moreover, we have

$$\int_{E_{0,\omega}} d\alpha|\alpha|^{1+\gamma+2s}\mu(\alpha+S(\omega)v) \sim \langle S(\omega)v\rangle^{1+\gamma+2s},$$

and we get (forgetting the truncation function in α)

$$a_m(v) \lesssim \iint d\omega dr \tilde{\varphi}_\delta(r)\,\langle S(\omega)v\rangle^{1+\gamma+2s}\, e^{-|(\omega\cdot v)-r|^2}\frac{1}{r^{1+2s}}.$$

We can now integrate w.r.t. r and compute by virtue of Peetre's inequality (forgetting now the dependence on δ for the constants)

$$\begin{aligned}\int dr\tilde{\varphi}_\delta(r)e^{-|(\omega\cdot v)-r|^2}\frac{1}{r^{1+2s}} &\lesssim \int dr\tilde{\varphi}_\delta(r)e^{-|(\omega\cdot v)-r|^2}\,\langle r-\omega\cdot v\rangle^{1+2s}\,\langle \omega\cdot v\rangle^{-(1+2s)}\\ &\lesssim \langle\omega\cdot v\rangle^{-(1+2s)},\end{aligned}$$

and thus

$$a_m(v) \lesssim \int_{S^2_\omega} d\omega\,\langle\omega\cdot v\rangle^{-(1+2s)}\,\langle S(\omega)v\rangle^{1+\gamma+2s}.$$

We therefore have a similar integral as in (25) and using exactly the same change of polar coordinates and computations as therein with $e^{-|(\omega\cdot v)|^2}$ replaced by $\langle\omega\cdot v\rangle^{-(1+2s)}$ (see Figure 1), we get, just repeating the arguments between (25) and (26),

$$\begin{aligned}a_m(v) &\lesssim \int_0^\pi d\varphi\int_0^{2\pi}d\theta\,\langle|v|\cos\varphi\rangle^{-(1+2s)}\sin\varphi\left(1+|v|^2\sin^2\varphi\right)^{(1+\gamma+2s)/2}\\ &\lesssim \int_0^1 dt\int_0^{2\pi}d\theta\,\langle t\,|v|\rangle^{-(1+2s)}\left(1+|v|^2(1-t^2)\right)^{(1+\gamma+2s)/2}\\ &\lesssim \int_0^{1/2}dt\,\langle t\,|v|\rangle^{-(1+2s)}\left(1+|v|^2(1-t^2)\right)^{(1+\gamma+2s)/2}\\ &\quad+\int_{1/2}^1 dt\,\langle t\,|v|\rangle^{-(1+2s)}\left(1+|v|^2(1-t^2)\right)^{(1+\gamma+2s)/2}\\ &\stackrel{\text{def}}{=} a_{m,1}+a_{m,2}.\end{aligned}$$

One has

$$a_{m,1} \lesssim \int_0^{1/2}dt\,\langle t\,|v|\rangle^{-(1+2s)}\left(1+|v|^2\right)^{(1+\gamma+2s)/2} \lesssim \langle v\rangle^{\gamma+2s},$$

and for the term $a_{m,2}$ we have, by changes of variables and using the fact that $\gamma>-3$,

$$\begin{aligned}a_{m,2} &\lesssim \int_{1/2}^1 dt\,\langle v\rangle^{-(1+2s)}\left(1+|v|^2(1-t)\right)^{(1+\gamma+2s)/2}\\ &\lesssim \langle v\rangle^{-(1+2s)}\,|v|^{-2}\int_0^{|v|^2/2}d\tilde{t}\left(1+\tilde{t}\right)^{(1+\gamma+2s)/2}\\ &\lesssim \langle v\rangle^{-(1+2s)}\,|v|^{-2}\int_0^{|v|^2/2}d\tilde{t}\left(1+\tilde{t}\right)^{-(1+s)}\left(1+\tilde{t}\right)^{(3+\gamma+4s)/2}\\ &\lesssim \langle v\rangle^{-(1+2s)}\,\langle v\rangle^{-2}\,\langle v\rangle^{3+\gamma+4s}\int_0^{|v|^2/2}d\tilde{t}\left(1+\tilde{t}\right)^{-(1+s)}\\ &\lesssim \langle v\rangle^{\gamma+2s}.\end{aligned}$$

Combining these inequalities we conclude

$$a_m \lesssim \langle v\rangle^{\gamma+2s}\,.$$

For the lower bound we can do essentially the same computations : because of the non-negative sign of a_m we can restrict the computations to the following subdomains in (α, h)

$$\{|\alpha| \geq 10\} \quad \text{and} \quad \{|h| \leq 10\}\,,$$

and following (35) and using (36) we get

$$\begin{aligned} a_m(v) \;&\gtrsim\; \iint d\omega dr \int_{E_{0,\omega}} d\alpha \mathbb{1}_{|\alpha|\geq 10} \mathbb{1}_{1\leq r\leq 10} \mu(v+\alpha - r\omega)|\alpha|^{1+\gamma+2s}\frac{1}{r^{1+2s}} \\ &\gtrsim\; \iint d\omega dr \int_{E_{0,\omega}} d\alpha \mathbb{1}_{|\alpha|\geq 10} \mathbb{1}_{1\leq r\leq 10} \mu(\alpha + S(\omega)v) e^{-|\omega\cdot v|^2/2}|\alpha|^{1+\gamma+2s}\frac{1}{r^{1+2s}} \end{aligned}$$

since $\overline{\varphi}_\delta = 1$ in the set $\{1 \leq r \leq 10\}$ (recall $0 < \delta < 1$), and

$$|v+\alpha - r\omega|^2 = |S(\omega)v + \alpha|^2 + |\omega\cdot v - r|^2 \leq |S(\omega)v+\alpha|^2 + |\omega\cdot v|^2 + 100$$

for $r \leq 10$. Then as before we can use the fact that

$$\int d\alpha \mathbb{1}_{|\alpha|\geq 10}|\alpha|^{1+\gamma+2s}\mu(\alpha + S(\omega)v) \sim \langle S(\omega)v\rangle^{1+\gamma+2s}$$

and

$$\int dr \mathbb{1}_{1\leq r\leq 10}\frac{1}{r^{1+2s}} \sim C$$

and we get for a new constant C that

$$a_m(v) \geq C^{-1}\int d\omega\, \langle S(\omega)v\rangle^{1+\gamma+2s}\, e^{-|(\omega\cdot v)|^2/2},$$

and again we can follow the computations as in (28) and thereafter to get

$$a_m(v) \geq C^{-1}\,\langle v\rangle^{\gamma+2s}\,.$$

The proof of i) is thus complete.

As for the proof of ii), we use (35) to get

$$\partial_v^\alpha a_m(v) = \int_{\mathbb{R}^3_h} dh \int_{E_{0,h}} d\alpha \tilde{b}\mathbb{1}_{|\alpha|\geq|h|}\tilde{\varphi}_\delta(h)\left(\partial_v^\alpha \mu(v+\alpha-h)\right)|\alpha+h|^{1+\gamma+2s}\frac{1}{|h|^{3+2s}},$$

which gives

$$|\partial_v^\alpha a_m(v)| \lesssim \int_{\mathbb{R}^3_h} dh \int_{E_{0,h}} d\alpha \tilde{b}\mathbb{1}_{|\alpha|\geq|h|}\tilde{\varphi}_\delta(h)\mu\left(\frac{v+\alpha-h}{2}\right)|\alpha+h|^{1+\gamma+2s}\frac{1}{|h|^{3+2s}}.$$

Then repeating the arguments as in i), we conclude that

$$|\partial_v^\alpha a_m(v)| \lesssim \langle v\rangle^{\gamma+2s}\,.$$

This completes the proof of ii). $\square$

3.3 Proof of Proposition 1.4 i)

In this subsection we prove part i) of Proposition 1.4 concerning the so-called symbol a. We first give its definition, then prove the Proposition, and we shall end this section by giving additional properties of a which will be needed in the sequel.

Definition 3.6. *We define a to be the following real symbol:*

$$a = a_p + a_m,$$

where a_p is defined in Proposition 3.1 and a_m is defined in Proposition 3.5.

We now give the proof of Proposition 1.4 i). From Proposition 3.1 and Proposition 3.5 we know respectively that

$$C^{-1} \langle v \rangle^{\gamma+2s} \leq a_m(v, \eta) \leq C \langle v \rangle^{\gamma+2s}$$

and for all $0 < \kappa \leq 1$,

$$C^{-1} \left(-\kappa \langle v \rangle^{\gamma+2s} + \kappa \langle v \rangle^{\gamma} (1 + |\eta|^{2s} + |\eta \wedge v|^{2s}) \right) \leq a_p(v, \eta) \leq C \langle v \rangle^{\gamma} (1 + |\eta|^{2s} + |\eta \wedge v|^{2s}),$$

where in both cases C denotes a constant independent of κ (but depending on δ, s). Choosing κ sufficiently small and fixed from now on, and adding the two inequalities gives

$$C^{-1} \left(\langle v \rangle^{\gamma+2s} + \langle v \rangle^{\gamma} (|\eta|^{2s} + |\eta \wedge v|^{2s}) \right) \leq a(v, \eta) \leq C \langle v \rangle^{\gamma+2s} + C \langle v \rangle^{\gamma} (1 + |\eta|^{2s} + |\eta \wedge v|^{2s}).$$

so that

$$C^{-1} \langle v \rangle^{\gamma} \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^s \leq a(v, \eta) \leq C \langle v \rangle^{\gamma} \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^s$$

for a new constant C. This proves the lower and upper bounds for a. Using the definition of $\tilde{a}$

$$\tilde{a}(v, \eta) = \langle v \rangle^{\gamma} \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^s \tag{37}$$

we get

$$C^{-1} \tilde{a} \leq a \leq C \tilde{a}. \tag{38}$$

From Proposition 3.1 and Proposition 3.5, we also directly get by addition that

$$a \in S(\tilde{a}, \Gamma).$$

Moreover, we claim that

$$\tilde{a} \in S(\tilde{a}, \Gamma). \tag{39}$$

To see this we use induction on $|\alpha + \beta|$ to prove that for any $\kappa \in \mathbb{R}$ and any $|\alpha + \beta| \geq 0$,

$$\left| \partial_v^{\alpha} \partial_\eta^{\beta} \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^{\kappa} \right| \lesssim \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^{\kappa}, \tag{40}$$

which obviously holds for $|\alpha + \beta| = 0$. Now suppose $|\alpha + \beta| \geq 1$ then we have either $|\alpha| \geq 1$ or $|\beta| \geq 1$, and suppose $|\beta| \geq 1$ without loss of generality. So we can write $\partial_\eta^{\beta} = \partial_\eta^{\tilde{\beta}} \partial_{\eta_j}$ with $|\tilde{\beta}| = |\beta| - 1$ and thus

$$\begin{aligned} &\partial_v^{\alpha} \partial_\eta^{\beta} \Big[\left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^{\kappa} \Big] \\ &\qquad = \partial_v^{\alpha} \partial_\eta^{\tilde{\beta}} \Big[\kappa \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^{\kappa-1} \left(2\eta_j + 2 \left(\eta \wedge v \right) \partial_{\eta_j} \left(\eta \wedge v \right) \right) \Big], \end{aligned}$$

which along with Leibniz's formula and the induction assumption yields

$$\begin{aligned}
&\left|\partial_v^\alpha \partial_\eta^\beta \Big[\left(1+|v|^2+|\eta|^2+|\eta\wedge v|^2\right)^\kappa \Big]\right| \\
\lesssim\ & \left(1+|v|^2+|\eta|^2+|\eta\wedge v|^2\right)^{\kappa-1} \left(1+|\eta|+|\eta\wedge v|\,|v|+|\eta\wedge v|+|\eta|\,|v|+|v|^2\right) \\
\lesssim\ & \left(1+|v|^2+|\eta|^2+|\eta\wedge v|^2\right)^{\kappa}.
\end{aligned}$$

We have proven (40). Now using (40) and Leibniz's formula we conclude

$$\left|\partial_v^\alpha \partial_\eta^\beta \Big[\langle v\rangle^\gamma \left(1+|v|^2+|\eta|^2+|\eta\wedge v|^2\right)^s \Big]\right| \le C_{\alpha,\beta} \langle v\rangle^\gamma \left(1+|v|^2+|\eta|^2+|\eta\wedge v|^2\right)^s.$$

This gives the statement in (39).

It only remains to check the temperance of a and $\tilde a$. From (38) it is sufficient to verify that there exist two constants N and C, both depending only on s and γ, such that for all $Y=(y,\eta)$, $Y'=(y',\eta')$ we have

$$\tilde a(Y) \le C \tilde a(Y')(1+\Gamma(Y-Y'))^N.$$

This is a direct consequence of Peetre's inequality since we have powers of polynomial type quantities. Indeed, we have

$$\frac{\tilde a(Y)}{\tilde a(Y')} \le \frac{\langle y\rangle^\gamma}{\langle y'\rangle^\gamma} \left(\frac{\langle y\rangle^2+\langle \eta\rangle^2+|y\wedge\eta|^2}{1+|y'|^2+|\eta'|^2+|y'\wedge\eta'|^2}\right)^s.$$

On the other hand,

$$\frac{\langle y\rangle^\gamma}{\langle y'\rangle^\gamma} \le 2^{|\gamma|}\left\langle y-y'\right\rangle^{|\gamma|}$$

due to Peetre's inequality. Similarly,

$$\frac{\langle y\rangle^2+\langle \eta\rangle^2}{1+|y'|^2+|\eta'|^2+|y'\wedge\eta'|^2} \le 4\left\langle y-y'\right\rangle^2 + 4\left\langle \eta-\eta'\right\rangle^2.$$

Moreover using the relation

$$y\wedge\eta = \left(y-y'\right)\wedge\left(\eta-\eta'\right) + \left(y-y'\right)\wedge\eta' + y'\wedge\left(\eta-\eta'\right) + y'\wedge\eta',$$

we compute

$$\begin{aligned}
&\frac{|y\wedge\eta|^2}{1+|y'|^2+|\eta'|^2+|y'\wedge\eta'|^2} \\
\le\ & \frac{4\,|y-y'|^2\,|\eta-\eta'|^2+4\,|y-y'|^2\,|\eta'|^2+4\,|y'|^2\,|\eta-\eta'|^2+4\,|y'\wedge\eta'|^2}{1+|y'|^2+|\eta'|^2+|y'\wedge\eta'|^2} \\
\le\ & 4\left|y-y'\right|^2\left|\eta-\eta'\right|^2+4\left|y-y'\right|^2+4\left|\eta-\eta'\right|^2+4 \\
\le\ & 10\left(\left\langle y-y'\right\rangle+\left\langle \eta-\eta'\right\rangle\right)^4.
\end{aligned}$$

Thus,

$$\frac{\langle y\rangle^2+\langle \eta\rangle^2+|y\wedge\eta|^2}{1+|y'|^2+|\eta'|^2+|y'\wedge\eta'|^2} \quad\le\quad 18\left(\left\langle y-y'\right\rangle+\left\langle \eta-\eta'\right\rangle\right)^4.$$

Combining the above inequalities, we get

$$\frac{\tilde{a}(Y)}{\tilde{a}(Y')} \leq C_{s,\gamma} \left(\langle y - y' \rangle + \langle \eta - \eta' \rangle \right)^{4s+|\gamma|} \leq \tilde{C}_{s,\gamma} \left(1 + \Gamma(Y - Y') \right)^{4s+|\gamma|}$$

with $C_{s,\gamma}$ and $\tilde{C}_{s,\gamma}$ two constants depending only on s and γ. The temperance of $\tilde{a}$ follows. The proof is complete. □

For further use we also give here two propositions concerning a and $\tilde{a}$, which will be of great interest in the next section.

Proposition 3.7. *Recall* $\tilde{a}(v, \eta) = \langle v \rangle^{\gamma} \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^s$. *We have*

i) for any $|\alpha| \geq 0$ *and any* $|\beta| \geq 1$ *, there exist two constants* $C_{\alpha,\beta} > 0$ *and* C_β *such that*

$$\left| \partial_v^\alpha \partial_\eta^\beta a \right| \leq C_{\alpha,\beta} \left(\varepsilon \tilde{a} + \varepsilon^{-1} \langle v \rangle^{2s+\gamma} \right)$$

and

$$\left| \partial_\eta^\beta \tilde{a} \right| \leq C_\beta \langle v \rangle^{\gamma+1} \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^{s-1/2};$$

ii) the following estimate is true for any $0 < \varepsilon \leq 1$*, with semi-norms (see Subsection A.2 for the definition of semi-norms) independent of* ε*:*

$$\partial_\eta \tilde{a}, \ \partial_\eta a \in S(\varepsilon a + \varepsilon^{-1} \langle v \rangle^{2s+\gamma}, \ \Gamma); \tag{41}$$

iii) we have

$$|\xi \cdot \partial_\eta \tilde{a}| \lesssim \langle v \rangle^{\gamma} \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^{s-\frac{1}{2}} \left(|\xi|^2 + |v \wedge \xi|^2 \right)^{1/2}. \tag{42}$$

Proof. The point i) for a is just an immediate consequence of Proposition 3.4. Now we check for $\tilde{a}$. Recall

$$\tilde{a}(v, \eta) = \langle v \rangle^{\gamma} \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^s.$$

We claim, for any $\kappa \in \mathbb{R}$ and any $|\beta| \geq 1$,

$$\left| \partial_\eta^\beta \Big[\langle v \rangle^{\gamma} \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^{\kappa} \Big] \right| \quad \lesssim \quad \langle v \rangle^{\gamma+1} \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^{\kappa - \frac{1}{2}},$$

which can be deduced by induction on $|\beta|$. Indeed, by direct computation we see the above estimate holds for $|\beta| = 1$, since

$$\begin{aligned}
& \left| \partial_{\eta_j} \Big[\langle v \rangle^{\gamma} \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^{\kappa} \Big] \right| \\
= & \left| \kappa \langle v \rangle^{\gamma} \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^{\kappa-1} \left(2\eta_j + 2 \left(\eta \wedge v \right) \partial_{\eta_j} \left(\eta \wedge v \right) \right) \right| \\
\lesssim & \ \langle v \rangle^{\gamma} \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^{\kappa-1} \left(|\eta| + |\eta \wedge v| \, |v| \right) \\
\lesssim & \ \langle v \rangle^{\gamma+1} \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2 \right)^{\kappa-\frac{1}{2}}.
\end{aligned}$$

Moreover for any $|\beta| \geq 2$, we may write $\partial_\eta^\beta = \partial_\eta^{\tilde{\beta}} \partial_{\eta_j}$ with $|\tilde{\beta}| = |\beta| - 1$ and thus

$$\begin{aligned}
&\partial_\eta^\beta \Big[\langle v \rangle^\gamma \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2\right)^\kappa \Big] \\
&\qquad = \partial_\eta^{\tilde{\beta}} \Big[\kappa \langle v \rangle^\gamma \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2\right)^{\kappa - 1} \left(2\eta_j + 2\left(\eta \wedge v\right) \partial_{\eta_j} \left(\eta \wedge v\right)\right) \Big].
\end{aligned}$$

As a result, by Leibniz's formula and the induction assumption on $|\beta|$, we obtain

$$\begin{aligned}
& \left| \partial_\eta^\beta \Big[\langle v \rangle^\gamma \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2\right)^\kappa \Big] \right| \\
\lesssim \ & \Big[\langle v \rangle^{\gamma+1} \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2\right)^{\kappa - 1 - \frac{1}{2}} \Big] \left(1 + |\eta| + |\eta \wedge v| \cdot |v| + |v|^2\right) \\
\lesssim \ & \langle v \rangle^{\gamma+1} \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2\right)^{\kappa - 1 - \frac{1}{2}} \left(1 + |\eta|^2 + |\eta \wedge v|^2 + |v|^2\right) \\
\lesssim \ & \langle v \rangle^{\gamma+1} \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2\right)^{\kappa - 1/2}.
\end{aligned}$$

Applying the above inequalities for $\kappa = s$, we obtain the desired estimate for $\tilde{a}$.

Next we prove Point ii). The conclusion for $\partial_\eta a$ follows from the estimates in i). And we have to check $\partial_\eta \tilde{a}$, and we have shown in i) that

$$\begin{aligned}
|\partial_\eta \tilde{a}| \ &\lesssim \ \langle v \rangle^{\gamma+1} \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2\right)^{s - 1/2} \\
&\lesssim \ \langle v \rangle^{\gamma/2 + s} \, \tilde{a}^{1/2}.
\end{aligned}$$

Then arguing as above we can use induction on $|\alpha| + |\beta|$ to obtain, for $|\alpha| + |\beta| \geq 0$,

$$\left| \partial_v^\alpha \partial_\eta^\beta \partial_\eta \tilde{a} \right| \lesssim \langle v \rangle^{\gamma/2+s} \, \tilde{a}^{1/2}.$$

This gives the conclusion for $\partial_\eta \tilde{a}$.

Point iii) in Proposition 3.7 is a direct consequence of the computation on $\tilde{a}$, since

$$\xi \cdot \partial_\eta \tilde{a} = s \, \langle v \rangle^\gamma \left(1 + |v|^2 + |\eta|^2 + |\eta \wedge v|^2\right)^{s-1} \left(2\xi \cdot \eta + 2(v \wedge \xi) \cdot (v \wedge \eta)\right).$$

The proof is complete. □

3.4 Study of the subprincipal term $\mathcal{L}_{1,1,\delta}$

Proposition 3.8. *We can write*

$$\mathcal{L}_{1,1,\delta} f = -a_s(v, D_v) f,$$

where a_s, defined by (44) below, is a (complex valued) classical symbol in (v, η) satisfying that for all $0 < s < 1$ and any $0 < \varepsilon < 1$, we have, with semi-norms independent of ε,

$$a_s(v, \eta) \in S\left(\varepsilon a + \varepsilon^{-1} \langle v \rangle^{\gamma + 2s}, \Gamma\right). \tag{43}$$

Proof. We recall that

$$\mathcal{L}_{1,1,\delta} f = \iint dv_* d\sigma B \varphi_\delta(|v'-v|)(\mu'_*)^{1/2}[f'-f]\left((\mu_*)^{1/2} - (\mu'_*)^{1/2}\right).$$

We shift to Carleman's representation and get

$$\begin{aligned}\mathcal{L}_{1,1,\delta} f &= \int_{\mathbb{R}^3_h} dh \int_{E_{0,h}} d\alpha \tilde{b} 1_{|\alpha|\geq|h|} |\alpha+h|^{1+\gamma+2s} \varphi_\delta(|h|)\mu^{\frac12}(\alpha+v)[f(v-h)-f(v)] \\ &\qquad \left(\mu^{\frac12}(\alpha+v-h) - \mu^{\frac12}(\alpha+v)\right)\frac{1}{|h|^{3+2s}} \\ &= -\int_{\mathbb{R}^3_\eta} \hat{f}(\eta) e^{iv.\eta} a_s(v,\eta) d\eta\end{aligned}$$

with

$$\begin{aligned}a_s(v,\eta) = -\int_{\mathbb{R}^3_h} dh \int_{E_{0,h}} d\alpha \tilde{b} 1_{|\alpha|\geq|h|} |\alpha+h|^{1+\gamma+2s} \varphi_\delta(|h|)\mu^{\frac12}(\alpha+v)[e^{-ih\cdot\eta}-1] \\ \left(\mu^{\frac12}(\alpha+v-h) - \mu^{\frac12}(\alpha+v)\right)\frac{1}{|h|^{3+2s}}. \quad (44)\end{aligned}$$

For the study of this symbol, we shall essentially follow the same computations as in the $\mathcal{L}_{1,2,\delta}$ case. We first note that we have the following bound for all $h \neq 0$

$$\left|\left(\mu^{\frac12}(\alpha+v-h) - \mu^{\frac12}(\alpha+v)\right)\frac{1}{|h|}\right| \leq C.$$

So that using also that $|\alpha| \leq |\alpha+h| \leq 2|\alpha|$ due to the fact that $\alpha \perp h$, we get

$$|a_s(v,\eta)| \lesssim \int_{\mathbb{R}^3_h} dh \int_{E_{0,h}} d\alpha \, |\alpha|^{1+\gamma+2s} \mu^{\frac12}(\alpha+v)\varphi_\delta(|h|)\frac{|e^{-ih\cdot\eta}-1|}{|h|^{2+2s}}.$$

Now we shift to spherical coordinates taking $h = r\omega$ and we get

$$|a_s(v,\eta)| \lesssim \int_0^{+\infty}\int_{S^2_\omega} d\omega dr \int_{E_{0,\omega}} d\alpha \, |\alpha|^{1+\gamma+2s} \mu^{\frac12}(\alpha+v)\varphi_\delta(r)\frac{|e^{-ir\omega\cdot\eta}-1|}{r^{2s}}. \quad (45)$$

We can directly integrate w.r.t. r and this gives

$$\int_0^\infty \varphi_\delta(r)\frac{|e^{-ir\omega\cdot\eta}-1|}{r^{2s}} dr \lesssim \int_0^\delta \frac{|\cos(r\omega\cdot\eta)-1|}{r^{2s}} dr + \int_0^\delta \frac{|\sin(r\omega\cdot\eta)|}{r^{2s}} dr.$$

We have proven in the proof of Proposition 3.4 (see the treatment of the case $|\nu_2| = 1$ threein) that

$$\int_0^\delta \frac{|\sin(r\omega\cdot\eta)|}{r^{2s}} dr \lesssim \varepsilon\,|\omega\cdot\eta|^{2s} + \varepsilon^{-1}$$

for any $0 < \varepsilon < 1$. Furthermore if $0 < s < 1/2$ then

$$\int_0^\delta \frac{|\cos(r\omega\cdot\eta)-1|}{r^{2s}} dr \lesssim \int_0^\delta \frac{1}{r^{2s}} dr \leq C_{\delta,s},$$

and if $1/2 < s < 1$ then

$$\int_0^\delta \frac{|\cos(r\omega\cdot\eta)-1|}{r^{2s}}dr \lesssim |\omega\cdot\eta|^{2s-1}\int_0^{\delta|\omega\cdot\eta|}\frac{|\cos\theta-1|}{\theta^{2s}}d\theta$$

$$\begin{aligned}
&\lesssim |\omega\cdot\eta|^{2s-1}\int_0^{\min\{1,\delta|\omega\cdot\eta|\}}\frac{|\cos\theta-1|}{\theta^{2s}}d\theta + |\omega\cdot\eta|^{2s-1}\int_{\min\{1,\delta|\omega\cdot\eta|\}}^{\delta|\omega\cdot\eta|}\frac{|\cos\theta-1|}{\theta^{2s}}d\theta\\
&\lesssim |\omega\cdot\eta|^{2s-1}\int_0^1\frac{1}{\theta^{2s-1}}d\theta + |\omega\cdot\eta|^{2s-1}\int_1^{+\infty}\frac{1}{\theta^{2s}}d\theta\\
&\lesssim |\omega\cdot\eta|^{2s-1} \lesssim \varepsilon|\omega\cdot\eta|^{2s}+\varepsilon^{-(2s-1)} \lesssim \varepsilon|\omega\cdot\eta|^{2s}+\varepsilon^{-1},
\end{aligned}$$

and finally if $s = 1/2$ then

$$\begin{aligned}
\int_0^\delta \frac{|\cos(r\omega\cdot\eta)-1|}{r^{2s}}dr &\le \int_0^{\min\{\varepsilon,\delta\}}\frac{|\cos(r\omega\cdot\eta)-1|}{r}dr + \int_{\min\{\varepsilon,\delta\}}^{\delta}\frac{|\cos(r\omega\cdot\eta)-1|}{r}dr\\
&\lesssim |\omega\cdot\eta|\int_0^{\min\{\varepsilon,\delta\}}dr+\varepsilon^{-1} \lesssim \varepsilon|\omega\cdot\eta|+\varepsilon^{-1} = \varepsilon|\omega\cdot\eta|^{2s}+\varepsilon^{-1}.
\end{aligned}$$

Combining the above estimate we have

$$\int_0^\infty \varphi_\delta(r)\frac{|e^{-ir\omega\cdot\eta}-1|}{r^{2s}}dr \lesssim \varepsilon|\omega\cdot\eta|^{2s}+\varepsilon^{-1},$$

and thus, in view of (45),

$$\begin{aligned}
|a_s(v,\eta)| \lesssim\ & \varepsilon\int_{S^2_\omega}d\omega\int_{E_{0,\omega}}d\alpha\,|\alpha|^{1+\gamma+2s}\mu^{\frac12}(\alpha+v)|\omega\cdot\eta|^{2s}\\
&+\varepsilon^{-1}\int_{S^2_\omega}d\omega\int_{E_{0,\omega}}d\alpha\,|\alpha|^{1+\gamma+2s}\mu^{\frac12}(\alpha+v).
\end{aligned}$$

This enables us to do exactly the same computations as in the $\mathcal{L}_{1,2,\delta}$ case, with the factors $\mu(\alpha+v)$ in formula (22) replaced respectively by $\mu^{1/2}(\alpha+v)$ here and the factor $|\omega\cdot\eta|^{2s}$ by 1. We directly get, following the computations after (22) , that

$$|a_s(v,\eta)| \lesssim \varepsilon\langle v\rangle^\gamma(1+|\eta|^{2s}+|v\wedge\eta|^{2s})+\varepsilon^{-1}\langle v\rangle^{\gamma+2s} \lesssim \varepsilon a+\varepsilon^{-1}\langle v\rangle^{\gamma+2s},$$

the last inequality using (38).

Again the proof of the estimates for higher order derivatives of a_s is similar to the one of order 0, and we skip this part of the proof for brevity. This completes the proof of Proposition 3.8. □

4 Proof of the main results

This section is devoted to the proof of the main results mentionned in the introduction, including in particular Theorems 1.1 and 1.3. We shall use extensively properties of the classical Weyl and Wick quantizations, for which we postpone a brief review in the Appendix. In Subsection 4.1 we make the reduction to the hypoelliptic problems for a simplified operator, by virtue of Proposition 1.4 whose proof is also presented in this subsection. In Subsection 4.2, we give some coercivity estimates, and recover a result of coercivity of [7] implying the so-called triple norm. The proof of the main results is then achieved in the last subsection 4.3.

4.1 Proof of Proposition 1.4 ii) and iii) and related results

In the previous sections, we splitted operator $\mathcal{L}$ into several pieces in the following way, with $a = a_p + a_m$ defined in Proposition 3.1 and Proposition 3.5, and a_s defined in Proposition 3.8,

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_1 + \mathcal{L}_2 = -a(v, D_v) + \mathcal{L}_2 + \overline{\mathcal{L}}_{1,\delta,a} + \mathcal{L}_{1,3,\delta} + \mathcal{L}_{1,4,\delta} - a_s(v, D_v) \\ &= -a^w - \underbrace{\left(-\mathcal{L}_2 - \overline{\mathcal{L}}_{1,\delta,a} - \mathcal{L}_{1,3,\delta} - \mathcal{L}_{1,4,\delta} + a_s(v, D_v) + (a(v, D_v) - a^w)\right)}_{\mathcal{K}},\end{aligned}$$

recalling that $\mathcal{L}_1, \mathcal{L}_2$ are defined by (6), $a(v, D_v) = -\mathcal{L}_{1,2,\delta} - \overline{\mathcal{L}}_{1,\delta,b}$ and $a_s(v, D_v) = -\mathcal{L}_{1,1,\delta}$, and $\overline{\mathcal{L}}_{1,\delta,a}, \overline{\mathcal{L}}_{1,\delta,b}$ and $\mathcal{L}_{1,j,\delta}, 1 \le j \le 4$, are given by (9)-(10). Thus we can write

$$P = v \cdot \partial_x + a^w + \mathcal{K}.$$

Notice that the diffusion term $a^w + \mathcal{K}$ above is only an operator with respect to the velocity variable v. So we only work on the resulting operator after performing partial Fourier transform in the x variables, considering the dual variables ξ of x as parameter. More precisely we will study the operator

$$\hat{P}_K = i\,(v \cdot \xi) + a_K^w,$$

where

$$a_K = a + K \langle v \rangle^{2s+\gamma}.$$

with K a fixed number, constructed in Lemma 4.2 and Lemma 4.8 below, depending only on the integer N in (85). Accordingly we also introduce the weight function

$$\tilde{a}_K = \tilde{a} + K \langle v \rangle^{2s+\gamma},$$

where $\tilde{a}$ is the weight function given in Proposition 1.4. We claim that $\tilde{a}_K$ is temperate uniformly with respect to K. Indeed, by Proposition 1.4 i), whose proof is given in Subsection 3.3, we see $\tilde{a}$ is temperate weight with respect to Γ, i.e., there exist two constants C and N, both depending only on γ, s, such that

$$\forall\ (v,\eta),(w,\zeta) \in \mathbb{R}^6, \quad \frac{\tilde{a}(v,\eta)}{\tilde{a}(w,\zeta)} \le C\left(\langle v-w\rangle + \langle \eta-\zeta\rangle\right)^N.$$

Thus for any $(v,\eta),(w,\zeta) \in \mathbb{R}^6$,

$$\begin{aligned}\frac{\tilde{a}_K(v,\eta)}{\tilde{a}_K(w,\zeta)} &= \frac{\tilde{a}(v,\eta)}{\tilde{a}(w,\zeta) + K\langle w\rangle^{2s+\gamma}} + \frac{K\langle v\rangle^{2s+\gamma}}{\tilde{a}(w,\zeta) + K\langle w\rangle^{2s+\gamma}} \\ &\le \frac{\tilde{a}(v,\eta)}{\tilde{a}(w,\zeta)} + \frac{K\langle v\rangle^{2s+\gamma}}{K\langle w\rangle^{2s+\gamma}} \\ &\le C\left(\langle v-w\rangle + \langle \eta-\zeta\rangle\right)^N + 2^{|2s+\gamma|}\langle v-w\rangle^{|2s+\gamma|} \\ &\le \left(C + 2^{|2s+\gamma|}\right)\left(\langle v-w\rangle + \langle \eta-\zeta\rangle\right)^{N+|2s+\gamma|},\end{aligned}$$

the second inequality using peetre's inequality. This gives $\tilde{a}_K$ is temperate uniformly with respect to K, since the constant C above is independent of K.

We note that $a_K \in S(\tilde{a}_K, \Gamma)$ uniformly in K, since for any multi-index $\alpha, \beta \in \mathbb{Z}_+^3$, we have

$$\begin{aligned} \left|\partial_v^\alpha \partial_\eta^\beta a_K(v,\eta)\right| &\leq \left|\partial_v^\alpha \partial_\eta^\beta a(v,\eta)\right| + \left|\partial_v^\alpha \partial_\eta^\beta \left(K \langle v\rangle^{2s+\gamma}\right)\right| \\ &\leq C_{\alpha,\beta}\, a(v,\eta) + K C_{\alpha,\beta} \langle v\rangle^{2s+\gamma} \\ &\leq 2C_{\alpha,\beta}\, a_K(v,\eta) \leq C_{\alpha,\beta,\gamma,s}\, \tilde{a}_K(v,\eta), \end{aligned}$$

with $C_{\alpha,\beta}$ a constant depending only on α, β, and $C_{\alpha,\beta,\gamma,s}$ a constant depending only on α, β, γ and s. Thus $a_K \in S(\tilde{a}_K, \Gamma)$ uniformly in K. More generally we can show, for $r \in [-1, 1]$,

$$\forall\ \alpha \in \mathbb{Z}_+^6, \quad |\partial^\alpha a_K^r| \leq C_\alpha a_K^r \leq \tilde{C}_\alpha \tilde{a}_K^r$$

by induction on $|\alpha|$, which gives $a_K^r \in S(\tilde{a}_K^r, \Gamma)$ uniformly w.r.t. K for all $r \in [-1, 1]$. Working with a_K^w instead of a^w will enable us to construct the inverse of the former, see Lemma 4.2 below. This is of big importance in the following analysis of hypo-elliptic estimates.

Notations. In the following, let K be fixed, satisfying the assumptions in Lemma 4.2 and Lemma 4.8 below, and let $\ell \in \mathbb{R}$ be an arbitrary number, fixed and as small as we want. To simplify the notation, by $A \lesssim B$ we mean there exists a positive constant C, which may depend on K and ℓ but is *independent of the parameters* ξ, such that $A \leq CB$, and similarly for $A \gtrsim B$. While the notation $A \approx B$ means both $A \lesssim B$ and $B \lesssim A$ hold. Given a symbol q and a weight function M, by $q \in S(M, \Gamma)$ we alway mean, in the following discussion, q lies in $S(M, \Gamma)$ *uniformly w.r.t.* K *and* ξ.

Now we state the main result of this subsection, which shows that it is sufficient to study the operator $\hat{P}_K$ instead of the original one.

Proposition 4.1. *The conclusion in Theorem 1.1 holds true if the estimate*

$$\left\|\tilde{a}(v,\xi)^{\frac{1}{1+2s}} f\right\| + \left\|a_K^w f\right\| \lesssim \left\|\hat{P}_K f\right\|_{L^2} + \left\|f\right\|_{L^2_\ell} \tag{46}$$

holds uniformly with respect to ξ.

We proceed to prove the above proposition through several lemmas. Firstly we begin with the construction of the inverses of operators.

Lemma 4.2. *There exists a K_0 sufficiently large, depending only on a fixed finite number of semi-norms of a, such that for all $K \geq K_0$ we have*

(i) a_K^w is invertible and its inverse $(a_K^w)^{-1}$ has the form

$$(a_K^w)^{-1} = H_1 \left(a_K^{-1}\right)^w = \left(a_K^{-1}\right)^w H_2,$$

with H_1, H_2 belonging to $\mathcal{B}(L^2)$, the space of bounded operators on L^2, and $\left\|H_j\right\|_{\mathcal{B}(L^2)}$ bounded from above by some constant independent of K for $j = 1, 2$;

(ii) $\left(a_K^{1/2}\right)^w$ is invertible and its inverse $\left[\left(a_K^{1/2}\right)^w\right]^{-1}$ has the form

$$\left[\left(a_K^{1/2}\right)^w\right]^{-1} = G_1 \left(a_K^{-1/2}\right)^w = \left(a_K^{-1/2}\right)^w G_2$$

with $G_1, G_2 \in \mathcal{B}(L^2)$ and $\left\|G_j\right\|_{\mathcal{B}(L^2)}$ bounded from above by some constant independent of K for $j = 1, 2$;

(iii) $\left(\tilde{a}_K^{1/2} a_K^{1/2}\right)^w$ *is invertible and its inverse* $\left[\left(\tilde{a}_K^{1/2} a_K^{1/2}\right)^w\right]^{-1}$ *has the form*

$$\left[\left(\tilde{a}_K^{1/2} a_K^{1/2}\right)^w\right]^{-1} = Q_1 \left(\tilde{a}_K^{-1/2} a_K^{-1/2}\right)^w = \left(\tilde{a}_K^{-1/2} a_K^{-1/2}\right)^w Q_2$$

with $Q_1, Q_2 \in \mathcal{B}(L^2)$ *and* $\|Q_j\|_{\mathcal{B}(L^2)}$ *bounded from above by some constant independent of* K *for* $j = 1, 2$.

Proof. Note first that in all what follows, we shall crucially use the fact that only a finite number N (depending only on the dimension $n = 3$ here) of seminorms of a symbol is needed to control the norm of the corresponding pseudodifferential operator (see (85) here and e.g. [29, Lemma 2.5.2]).

Let us now prove the conclusion in (i). Using (86) and (87), we may write

$$a_K^w (a_K^{-1})^w = \mathrm{Id} - R_K^w, \tag{47}$$

where

$$R_K = -\int_0^1 (\partial_\eta a_K) \sharp_\theta \left(\partial_v (a_K^{-1})\right) d\theta + \int_0^1 (\partial_v a_K) \sharp_\theta \left(\partial_\eta (a_K^{-1})\right) d\theta$$

with $g\sharp_\theta h$ defined by

$$g\sharp_\theta h(Y) = \iint e^{-2i\sigma(Y-Y_1, Y-Y_2)/\theta} \frac{1}{2i} g(Y_1) h(Y_2) dY_1 dY_2 / (\pi\theta)^6. \tag{48}$$

Let now N be the integer which is given in (85) (and therefore depending only on the dimension $n = 3$ here). By [12, Proposition 1.1] we can find a constant C_N and a positive integer ℓ_N, both depending only on N but independent of K and θ, such that

$$\left\| (\partial_\eta a_K) \sharp_\theta \left(\partial_v (a_K^{-1})\right) \right\|_{N; S(1,\Gamma)} \le C_N \|\partial_\eta a_K\|_{\ell_N; S(\tilde{a}_K, \Gamma)} \left\| \left(\partial_v (a_K^{-1})\right) \right\|_{\ell_N; S(\tilde{a}_K^{-1}, \Gamma)},$$

where the semi-norm $\|\cdot\|_{k; S(M,\Gamma)}$ is defined by (84). Moreover, using (41) for $\varepsilon = K^{-1/2}$ yields

$$\|\partial_\eta a_K\|_{\ell_N; S(\tilde{a}_K, \Gamma)} \le \tilde{C}_N K^{-\frac{1}{2}}$$

and from the fact $a_K \in S(\tilde{a}_K, \Gamma)$ it follows that $a_K^{-1} \in S(\tilde{a}_K^{-1}, \Gamma)$, and thus

$$\|\partial_v (a_K^{-1})\|_{\ell_N; S(\tilde{a}_K^{-1}, \Gamma)} \le \tilde{C}_N$$

with $\tilde{C}_N$ a constant depending only on N but independent of K. As a result,

$$\left\| (\partial_\eta a_K) \sharp_\theta \left(\partial_v (a_K^{-1})\right) \right\|_{N; S(1,\Gamma)} \le C_N \tilde{C}_N^2 K^{-\frac{1}{2}}.$$

Similarly,

$$\left\| (\partial_v a_K) \sharp_\theta \left(\partial_\eta (a_K^{-1})\right) \right\|_{N; S(1,\Gamma)} \le C_N \tilde{C}_N^2 K^{-\frac{1}{2}}.$$

Then

$$\|R_K\|_{N; S(1,\Gamma)} \le 2 C_N \tilde{C}_N^2 K^{-\frac{1}{2}},$$

and thus by (85)

$$\left\|R_K^w\right\|_{\mathcal{B}(L^2)} \leq 2CC_N\tilde{C}_N^2 K^{-\frac{1}{2}}$$

with C a constant depending only on the dimension. This implies $\mathrm{Id} - R_K^w$ is invertible in the space $\mathcal{B}(L^2)$ of bounded operators on L^2 if we choose K in such a way that $K \geq \left(4CC_N\tilde{C}_N^2\right)^2$. Moreover

$$(\mathrm{Id} - R_K^w)^{-1} = \sum_{j=0}^{\infty} (R_K^w)^j \in \mathcal{B}(L^2).$$

Taking into account (47), we conclude

$$a_K^w \left((a_K^{-1})^w \, (\mathrm{Id} - R_K^w)^{-1}\right) = \mathrm{Id}.$$

Similarly we can find a $\tilde{R}_K \in S(1,\Gamma)$ such that

$$\left(\left(\mathrm{Id} - \tilde{R}_K^w\right)^{-1} (a_K^{-1})^w\right) a_K^w = \mathrm{Id}.$$

These facts imply a_K^w is invertible and its inverse $(a_K^w)^{-1}$ has the form

$$(a_K^w)^{-1} = (a_K^{-1})^w \, (\mathrm{Id} - R_K^w)^{-1} = \left(\mathrm{Id} - \tilde{R}_K^w\right)^{-1} (a_K^{-1})^w.$$

We have proved the conclusion in (i) in Lemma 4.2. The remaining proofs in (ii) and (iii) can be deduced quite similarly and are therefore omitted. The proof of Lemma 4.2 is thus complete. □

In the following, we always let K be fixed satisfying the condition in the above lemma 4.2.

Corollary 4.3. *Let ε be an arbitrarily small number and let $g \in S\left(\varepsilon a_K + \varepsilon^{-1} \langle v\rangle^{2s+\gamma},\ \Gamma\right)$ uniformly with respect to ε. Then*

$$\left\|g(v,D_v)f\right\|_{L^2} + \left\|g^w f\right\|_{L^2} \lesssim \varepsilon\left\|a_K^w f\right\| + \varepsilon^{-1}\left\| \langle v\rangle^{2s+\gamma} f\right\|_{L^2}.$$

Proof. We first show that $\varepsilon a_K + \varepsilon^{-1} \langle v\rangle^{2s+\gamma}$ is a temperate weight uniformly with respect to ε. Recall $a_K(v,\eta) = a(v,\eta) + K\langle v\rangle^{\gamma}$. By Proposition 1.4 i), whose proof is given in Subsection 3.3, we see a is temperate weight with respect to Γ, i.e., there exist two constants N and C, both depending only on γ, s, such that

$$\forall\ (v,\eta),(\tilde{v},\tilde{\eta}) \in \mathbb{R}^6, \quad \frac{a(v,\eta)}{a(\tilde{v},\tilde{\eta})} \leq C\left(\langle v-\tilde{v}\rangle + \langle \eta-\tilde{\eta}\rangle\right)^N.$$

As a result,

$$\forall\ (v,\eta),(\tilde{v},\tilde{\eta}) \in \mathbb{R}^6, \quad \frac{\varepsilon a(v,\eta)}{\varepsilon a_K(\tilde{v},\tilde{\eta}) + \varepsilon^{-1}\langle \tilde{v}\rangle^{2s+\gamma}} \leq \frac{\varepsilon a(v,\eta)}{\varepsilon a(\tilde{v},\tilde{\eta})} \leq C\left(\langle v-\tilde{v}\rangle + \langle \eta-\tilde{\eta}\rangle\right)^N.$$

Moreover, for any $(v,\eta),(\tilde v,\tilde\eta)\in\mathbb{R}^6$,

$$\frac{\varepsilon K\,\langle v\rangle^{2s+\gamma}}{\varepsilon a_K(\tilde v,\tilde\eta)+\varepsilon^{-1}\langle\tilde v\rangle^{2s+\gamma}}\le\frac{\varepsilon K\,\langle v\rangle^{2s+\gamma}}{\varepsilon K\,\langle\tilde v\rangle^{2s+\gamma}}\le 2^{|2s+\gamma|}\langle v-\tilde v\rangle^{|2s+\gamma|},$$

the last inequality following from Peetre's inequality. Similarly,

$$\frac{\varepsilon^{-1}\langle v\rangle^{2s+\gamma}}{\varepsilon a_K(\tilde v,\tilde\eta)+\varepsilon^{-1}\langle\tilde v\rangle^{2s+\gamma}}\le\frac{\varepsilon^{-1}\langle v\rangle^{2s+\gamma}}{\varepsilon^{-1}\langle\tilde v\rangle^{2s+\gamma}}\le 2^{|2s+\gamma|}\langle v-\tilde v\rangle^{|2s+\gamma|}.$$

The above inequalities yield, for any $(v,\eta),(\tilde v,\tilde\eta)\in\mathbb{R}^6$,

$$\frac{\varepsilon a_K(v,\eta)+\varepsilon^{-1}\langle v\rangle^{2s+\gamma}}{\varepsilon a_K(\tilde v,\tilde\eta)+\varepsilon^{-1}\langle\tilde v\rangle^{2s+\gamma}}\le\left(C+2^{1+|2s+\gamma|}\right)\left(\langle v-\tilde v\rangle+\langle\eta-\tilde\eta\rangle\right)^{N+|2s+\gamma|}.$$

Observe the constant C above is independent of ε, and thus $\varepsilon a_K+\varepsilon^{-1}\langle v\rangle^{2s+\gamma}$ is a temperate weight uniformly with respect to ε.

Now we will prove the conclusion in the corollary. This is just a consequence of the conclusion (i) in Lemma 4.2. In fact note that $K\ge K_0$ with K_0 the constant given in Lemma 4.2, and thus $K+\varepsilon\ge K_0$. Then the assumption in Lemma 4.2 is fulfilled and we may apply the conclusion (i) in Lemma 4.2 to conclude that $a^w_{K+\varepsilon^{-2}}$ is invertible and its inverse has the form

$$\left(a^w_{K+\varepsilon^{-2}}\right)^{-1}=\left(a^w_K+\varepsilon^{-2}\langle v\rangle^{2s+\gamma}\right)^{-1}=\left(a^{-1}_{K+\varepsilon^{-2}}\right)^w H$$

with H a bounded operator in L^2. The assumption on g shows

$$\varepsilon^{-1}g\in S\left(a_K+\varepsilon^{-2}\langle v\rangle^{2s+\gamma},\ \Gamma\right),$$

and thus we can write

$$g^w=\underbrace{\left(\varepsilon^{-1}g\right)^w\left(a^{-1}_{K+\varepsilon^{-2}}\right)^w H}_{\in\,\mathcal{B}(L^2)}\,\varepsilon\left(a^w_K+\varepsilon^{-2}\langle v\rangle^{2s+\gamma}\right),$$

which yields the desired estimate for g^w. The estimate for $g(v,D_v)$ is similar, since $g(v,D_v)=\left(J^{-1/2}g\right)^w$ with $J^{-1/2}g$ belonging to the same symbol class as g. We have obtained the desired estimate in Corollary 4.3. The proof is complete. □

We will apply the preceding lemma to specific pseudodifferential operators:

Lemma 4.4. *The symbols of $a_s(v,D_v)$ and $a^w-a(v,D_v)$ lie in $S\left(\varepsilon a+\varepsilon^{-1}\langle v\rangle^{2s+\gamma},\ \Gamma\right)$ for all $\varepsilon>0$ with seminorms independent of ε.*

Proof. For the first operator $a_s(v,D_v)$, this is point ii) of Proposition 3.8. For the second one $a^w-a(v,D_v)$, we need more facts from the theory of Weyl and classical quantizations. In order to get the result, we use the expansion of $J^{1/2}a$, which reads (c.f. [29, Lemma 4.1.5] and the appendix)

$$a^w-a(v,D_v)=\left(J^{1/2}a\right)(v,D_v)-a(v,D_v)=R(v,D_v)$$

with

$$R(v,\eta)=\frac{1}{2}\int\Big(J^{\theta/2}\left(D_\eta\cdot\partial_v a\right)\Big)(v,\eta)d\theta.$$

Proposition 3.7 implies that $D_\eta\cdot\partial_v a\in S\left(M_\varepsilon,\ \Gamma\right)$ uniformly with respect to ε, where

$$M_\varepsilon=\varepsilon\tilde a+\varepsilon^{-1}\left\langle v\right\rangle^{2s+\gamma}.$$

Then proceeding as in the proof of [29, Lemma 4.1.2], we conclude that $J^{\theta/2}\left(D_\eta\cdot\partial_v a\right)$ belongs to the same symbol class $S\left(M_\varepsilon,\ \Gamma\right)$ as $D_\eta\cdot\partial_v a$, due to the fact that

$$M_\varepsilon(v+z,\eta+\zeta)\leq CM_\varepsilon(v,\eta)H(\left\langle z\right\rangle,\left\langle \zeta\right\rangle)$$

with $H(\left\langle z\right\rangle,\left\langle \zeta\right\rangle)$ being some polynomial of $\left\langle z\right\rangle,\left\langle \zeta\right\rangle$ and C a constant independent of ε. Observe $\tilde a\lesssim a_K$ due to Proposition 1.4 i). Then we have proven that the classical symbol of the difference $a(v,D_v)-a^w$ lies in $S\left(\varepsilon a+\varepsilon^{-1}\left\langle v\right\rangle^{2s+\gamma},\ \Gamma\right)$. The Weyl symbol therefore also belongs to this class by direct transformation. The proof is complete.

Proposition 4.5. *Let ξ be the dual variable of x and let ℓ be an arbitrarily real number. Then for any ε, there exists a constant C_ε such that*

$$\forall\ f\in\mathcal{S}(\mathbb{R}^3_v),\quad \left\|\left\langle v\right\rangle^{2s+\gamma}f\right\|_{L^2}\leq\varepsilon\left\|a_K^wf\right\|_{L^2}+C_\varepsilon\left(\left\|(iv\cdot\xi-\mathcal{L})f\right\|_{L^2}+\left\|f\right\|_{L^2_\ell}\right).\tag{49}$$

Proof. Let us first recall the coercivity estimate (see for instance Theorem 1.1 and Proposition 2.2 in [7] and [39, 40]) : for $0<s<1$ and $\gamma>-3$,

$$\forall\ f\in\mathcal{S}(\mathbb{R}^6),\quad \left\|\left\langle v\right\rangle^{s+\frac{\gamma}{2}}(\mathbf{Id}-\mathbf{P})f\right\|_{L^2}^2\lesssim(-\mathcal{L}f,\ f)_{L^2},$$

where $\mathbf{Id}$ stands for the identity operator and $\mathbf{P}$ is the L^2-orthogonal projection onto the null space

$$\mathrm{Span}\left\{\mu^{1/2},v_1\mu^{1/2},v_2\mu^{1/2},v_3\mu^{1/2},\left|v\right|^2\mu^{1/2}\right\}.$$

Consequently we have, for any $\ell\in\mathbb{R}$,

$$\forall\ f\in\mathcal{S}(\mathbb{R}^6),\quad \left\|\left\langle v\right\rangle^{s+\frac{\gamma}{2}}f\right\|_{L^2}^2\lesssim\mathrm{Re}\left((iv\cdot\xi-\mathcal{L})f,\ f\right)_{L^2}+\left\|\left\langle v\right\rangle^{\ell-s-\gamma/2}f\right\|_{L^2}^2.\tag{50}$$

Now applying estimate (50) to the function $\left\langle v\right\rangle^{s+\frac{\gamma}{2}}f$ yields

$$\begin{aligned}\left\|\left\langle v\right\rangle^{2s+\gamma}f\right\|_{L^2}^2&\lesssim\mathrm{Re}\left((iv\cdot\xi-\mathcal{L})\left\langle v\right\rangle^{s+\frac{\gamma}{2}}f,\ \left\langle v\right\rangle^{s+\frac{\gamma}{2}}f\right)_{L^2}+\left\|f\right\|_{L^2_\ell}^2\\&\lesssim\left|\left((iv\cdot\xi-\mathcal{L})f,\ \left\langle v\right\rangle^{2s+\gamma}f\right)_{L^2}\right|+\left|\left(\left\langle v\right\rangle^{s+\frac{\gamma}{2}}\left[\mathcal{L},\ \left\langle v\right\rangle^{s+\frac{\gamma}{2}}\right]f,\ f\right)_{L^2}\right|+\left\|f\right\|_{L^2_\ell}^2,\end{aligned}$$

and therefore

$$\left\|\left\langle v\right\rangle^{2s+\gamma}f\right\|_{L^2}^2\lesssim\left\|(iv\cdot\xi-\mathcal{L})f\right\|_{L^2}^2+\left\|f\right\|_{L^2_\ell}^2+\left|\left(\left\langle v\right\rangle^{s+\frac{\gamma}{2}}\left[\mathcal{L},\ \left\langle v\right\rangle^{s+\frac{\gamma}{2}}\right]f,\ f\right)_{L^2}\right|.\tag{51}$$

We have to treat the last term in the above estimate, which is bounded from above by

$$\left|\left(\left\langle v\right\rangle^{s+\frac{\gamma}{2}}\left[a^w,\ \left\langle v\right\rangle^{s+\frac{\gamma}{2}}\right]f,\ f\right)_{L^2}\right|+\left|\left(\left\langle v\right\rangle^{s+\frac{\gamma}{2}}\left[\mathcal{K},\ \left\langle v\right\rangle^{s+\frac{\gamma}{2}}\right]f,\ f\right)_{L^2}\right|.\tag{52}$$

We apply (41) and [29, Theorem 2.3.8] to conclude that for any $\varepsilon \in]0,1[$ the symbol of the operator

$$\langle v\rangle^{-(2s+\gamma-1)} \langle v\rangle^{s+\frac{\gamma}{2}} \left[a^w,\ \langle v\rangle^{s+\frac{\gamma}{2}}\right]$$

belongs to

$$S\left(\varepsilon a_K + \varepsilon^{-1} \langle v\rangle^{2s+\gamma},\ \Gamma\right)$$

uniformly with respect to ε. Then Corollary 4.3 gives, with $\tilde{\varepsilon}$ arbitrarily small,

$$\begin{aligned}\left|\left(\langle v\rangle^{s+\frac{\gamma}{2}} \left[a^w,\ \langle v\rangle^{s+\frac{\gamma}{2}}\right] f,\ f\right)_{L^2}\right| &\lesssim \left(\varepsilon\|a_K^w f\|_{L^2} + \varepsilon^{-1}\| \langle v\rangle^{2s+\gamma} f\|_{L^2}\right) \| \langle v\rangle^{2s+\gamma-1} f\|_{L^2} \\ &\lesssim \varepsilon\|a_K^w f\|_{L^2}^2 + \tilde{\varepsilon}\| \langle v\rangle^{2s+\gamma} f\|_{L^2}^2 + C_{\varepsilon,\tilde{\varepsilon}}\|f\|_{L^2_\ell}^2,\end{aligned}$$

where in the last inequality we used the interpolation inequality:

$$\| \langle v\rangle^{2s+\gamma-1} f\|_{L^2} \le \tilde{\varepsilon}\| \langle v\rangle^{2s+\gamma} f\|_{L^2} + C_{\tilde{\varepsilon}}\|f\|_{L^2_\ell}.$$

Now we have to deal with the operator

$$\langle v\rangle^{s+\frac{\gamma}{2}} \left[\mathcal{K},\ \langle v\rangle^{s+\frac{\gamma}{2}}\right]$$

in (52). For this purpose, we split $\mathcal{K}$ into three parts :

$$\mathcal{K} \quad = \quad \underbrace{-\mathcal{L}_2 - \overline{\mathcal{L}}_{1,\delta,a}}_{\mathcal{K}_{small}} \underbrace{-\mathcal{L}_{1,3,\delta} - \mathcal{L}_{1,4,\delta}}_{\mathcal{K}_{mult}} + \underbrace{a_s(v,D_v) + (a(v,D_v) - a^w)}_{\mathcal{K}_{pseudo}}. \tag{53}$$

For the second part $\mathcal{K}_{mult}$, the estimate is easy since, as recalled in lemma 2.3 and 2.5, operators $\mathcal{L}_{1,3,\delta}$ and $\mathcal{L}_{1,4,\delta}$ commute with the multiplication with a function of v. We therefore have

$$\left|\left(\langle v\rangle^{s+\frac{\gamma}{2}} \left[\mathcal{K}_{mult},\ \langle v\rangle^{s+\frac{\gamma}{2}}\right] f,\ f\right)_{L^2}\right| \quad = \quad 0.$$

For the first part $\mathcal{K}_{small}$ of $\mathcal{K}$ in (53), we expand the commutators and use Cauchy-Schwarz inequality to get

$$\begin{aligned}&\left|\left(\langle v\rangle^{s+\frac{\gamma}{2}} \left[\mathcal{K}_{small},\ \langle v\rangle^{s+\frac{\gamma}{2}}\right] f,\ f\right)_{L^2}\right| \\ &\lesssim C_\varepsilon\| \langle v\rangle^{-s-\frac{\gamma}{2}} \left[\mathcal{K}_{small},\ \langle v\rangle^{s+\frac{\gamma}{2}}\right] \langle v\rangle^{-(s+\gamma)} \langle v\rangle^{s+\gamma} f\|_{L^2}^2 + \varepsilon\| \langle v\rangle^{2s+\gamma} f\|^2 \\ &\lesssim C_\varepsilon\| \langle v\rangle^{-s-\frac{\gamma}{2}} \left[\mathcal{K}_{small},\ \langle v\rangle^{-\frac{\gamma}{2}}\right] \langle v\rangle^{s+\gamma} f\|_{L^2}^2 + C_\varepsilon\| \left[\mathcal{K}_{small},\ \langle v\rangle^{-s-\gamma}\right] \langle v\rangle^{s+\gamma} f\|_{L^2}^2 \\ &\quad + \varepsilon\| \langle v\rangle^{2s+\gamma} f\|^2 \\ &\lesssim C_\varepsilon \left(\| \langle v\rangle^{-s-\frac{\gamma}{2}} \mathcal{L}_2 \langle v\rangle^{-\frac{\gamma}{2}} \langle v\rangle^{s+\gamma} f\|^2 + \| \langle v\rangle^{-s-\gamma} \mathcal{L}_2 \langle v\rangle^{s+\gamma} f\|^2 + \|\mathcal{L}_2 \langle v\rangle^{-s-\gamma} \langle v\rangle^{s+\gamma} f\|^2\right) \\ &\quad + C_\varepsilon\| \langle v\rangle^{-s-\frac{\gamma}{2}} \left[\overline{\mathcal{L}}_{1,\delta,a},\ \langle v\rangle^{-\frac{\gamma}{2}}\right] \langle v\rangle^{s+\gamma} f\|_{L^2}^2 + C_\varepsilon\| \left[\overline{\mathcal{L}}_{1,\delta,a},\ \langle v\rangle^{-s-\gamma}\right] \langle v\rangle^{s+\gamma} f\|_{L^2}^2 \\ &\quad + \varepsilon\| \langle v\rangle^{2s+\gamma} f\|^2.\end{aligned}$$

Then, we use Lemma 2.1 and conclusion (ii) in Lemma 2.4 with either $\tilde{\alpha} = -s - \gamma/2$, $\tilde{\beta} = -\gamma/2$ or $\tilde{\alpha} = 0$, $\tilde{\beta} = -s - \gamma$ (for which we have in both cases $\tilde{\alpha} + \tilde{\beta} + \gamma + s \le 0$) and we get

$$\begin{aligned}\left|\left(\langle v\rangle^{s+\frac{\gamma}{2}} \left[\mathcal{K}_{small},\ \langle v\rangle^{s+\frac{\gamma}{2}}\right] f,\ f\right)_{L^2}\right| &\lesssim \tilde{C}_\varepsilon\| \langle v\rangle^{s+\gamma} f\|^2 + \varepsilon\| \langle v\rangle^{2s+\gamma} f\|^2 \\ &\lesssim \widetilde{C}_\varepsilon\|f\|_{L^2_\ell}^2 + 2\varepsilon\| \langle v\rangle^{2s+\gamma} f\|^2\end{aligned}$$

since $s > 0$.

Next we deal with the last part $\mathcal{K}_{pseudo}$ of $\mathcal{K}$ in (53). From Lemma 4.4, we already know that $\mathcal{K}_{pseudo}$ belongs to

$$S\left(\varepsilon a + \varepsilon^{-1} \langle v \rangle^{2s+\gamma},\ \Gamma\right)$$

with uniform semi-norms with respect to ε. We follow the same strategy as in the lines just after (52) for commutators involving a^w. Using that $\partial_v \langle v \rangle^{\mu} = \mathcal{O}(\langle v \rangle^{\mu-1})$ for all $\mu \in \mathbb{R}$, and applying [29, Theorem 2.3.8] (see also appendix), we get that for any $\varepsilon \in]0, 1[$ the symbol of the operator

$$\langle v \rangle^{-(2s+\gamma-1)} \langle v \rangle^{s+\frac{\gamma}{2}} \left[\mathcal{K}_{pseudo},\ \langle v \rangle^{s+\frac{\gamma}{2}}\right]$$

belongs to

$$S\left(\varepsilon a_K + \varepsilon^{-1} \langle v \rangle^{2s+\gamma},\ \Gamma\right)$$

uniformly with respect to ε. Then Corollary 4.3 gives, with $\tilde{\varepsilon}$ arbitrarily small,

$$\begin{aligned}
\left|\left(\langle v \rangle^{s+\frac{\gamma}{2}} \left[\mathcal{K}_{pseudo},\ \langle v \rangle^{s+\frac{\gamma}{2}}\right] f,\ f\right)_{L^2}\right| &\lesssim \left(\varepsilon \left\|a_K^w f\right\|_{L^2} + \varepsilon^{-1} \left\| \langle v \rangle^{2s+\gamma} f\right\|_{L^2}\right) \left\| \langle v \rangle^{2s+\gamma-1} f\right\|_{L^2} \\
&\lesssim \varepsilon \left\|a_K^w f\right\|_{L^2}^2 + \tilde{\varepsilon} \left\| \langle v \rangle^{2s+\gamma} f\right\|_{L^2}^2 + C_{\varepsilon,\tilde{\varepsilon}} \left\|f\right\|_{L^2_\ell}^2.
\end{aligned}$$

Combining these estimates we obtain

$$\left|\left(\langle v \rangle^{s+\frac{\gamma}{2}} \left[\mathcal{K},\ \langle v \rangle^{s+\frac{\gamma}{2}}\right] f,\ f\right)_{L^2}\right| \lesssim \varepsilon \left\|a_K^w f\right\|_{L^2}^2 + \tilde{\varepsilon} \left\| \langle v \rangle^{2s+\gamma} f\right\|_{L^2}^2 + C_{\varepsilon,\tilde{\varepsilon}} \left\|f\right\|_{L^2_\ell}^2.$$

Now taking into account (51), the desired estimate (49) follows if we choose $\tilde{\varepsilon}$ small enough. The proof is thus complete. □

In order to prove the main result, Proposition 4.1, we will need the conclusion in Proposition 1.4. So let us firstly present the proof of this Proposition.

Proof of Proposition 1.4 ii) and iii). We have shown Proposition 1.4 iii) in Lemma 4.2. For the conclusion ii), let us rewrite the linearized Boltzmann operator $\mathcal{L}$ as

$$\mathcal{L} = -a^w + \underbrace{\mathcal{L}_2 + \overline{\mathcal{L}}_{1,\delta,a} + \mathcal{L}_{1,3,\delta} + \mathcal{L}_{1,4,\delta} - a_s(v, D_v) - (a(v, D_v) - a^w)}_{-\mathcal{K}}.$$

As a direct consequence of Lemma 2.1, conclusion (i) in Lemma 2.4, Lemma 2.3, Lemma 2.5 we have

$$\left\| \left(\mathcal{L}_2 + \overline{\mathcal{L}}_{1,\delta,a} + \mathcal{L}_{1,3,\delta} + \mathcal{L}_{1,4,\delta}\right) f\right\|_{L^2} \lesssim \left\| \langle v \rangle^{2s+\gamma} f\right\|_{L^2}.$$

Moreover from Lemma 4.4 we know that for any $\varepsilon > 0$,

$$-\mathcal{K}_{pseudo} = -a_s(v, D_v) - (a^w - a(v, D_v)) \in Op_{weyl}\left(\varepsilon a + \varepsilon^{-1} \langle v \rangle^{2s+\gamma},\ \Gamma\right)$$

uniformly with respect to ε, and thus

$$\left\|\mathcal{K}_{pseudo} f\right\|_{L^2} \lesssim \varepsilon \left\|a_K^w f\right\| + \varepsilon^{-1} \left\| \langle v \rangle^{2s+\gamma} f\right\|_{L^2}$$

due to Corollary 4.3.

The proof of point ii) of Proposition 1.4 is complete. $\square$

The rest of this subsection is devoted to the

Proof of Proposition 4.1. Now assuming that (46) holds, we have

$$\|\tilde{a}(v,\xi)^{\frac{1}{1+2s}} f\| + \|a_K^w f\| \lesssim \|\left(iv\cdot\xi - \mathcal{L}\right) f\|_{L^2} + \|f\|_{L^2_\ell} + \|\left(iv\cdot\xi - \mathcal{L} - \hat{P}_K\right) f\|_{L^2}.$$

On the other hand, note that

$$iv\cdot\xi - \mathcal{L} - \hat{P}_K = a^w + \mathcal{K} - (a + K\langle v\rangle^{2s+\gamma})^w = \mathcal{K} - K\langle v\rangle^{2s+\gamma},$$

and thus Proposition 1.4 yields, with ε arbitrarily small,

$$\begin{aligned}\|\left(iv\cdot\xi - \mathcal{L} - \hat{P}_K\right) f\|_{L^2} &\lesssim \varepsilon\|a_K^w f\|_{L^2} + C_\varepsilon\|\langle v\rangle^{2s+\gamma} f\|_{L^2}\\ &\lesssim \varepsilon\|a_K^w f\| + C_\varepsilon\left(\|(iv\cdot\xi - \mathcal{L})f\|_{L^2} + \|f\|_{L^2_\ell}\right),\end{aligned}$$

the last inequality following from (49) . Combining these inequalities and letting the above ε be sufficiently small, we get

$$\|\tilde{a}(v,\xi)^{\frac{1}{1+2s}} f\| + \|a_K^w f\| \lesssim \|(iv\cdot\xi - \mathcal{L})f\|_{L^2} + \|f\|_{L^2_\ell}.$$

Taking into account the facts that

$$\langle v\rangle^{\gamma/(2s+1)}\langle\xi\rangle^{2s/(2s+1)} + \langle v\rangle^{\gamma/(2s+1)}\langle v\wedge\xi\rangle^{2s/(2s+1)} \lesssim \tilde{a}(v,\xi)^{1/(2s+1)}$$

and that

$$\|\langle v\rangle^{\gamma}\langle D_v\rangle^{2s} f\|_{L^2} + \|\langle v\rangle^{\gamma}\langle v\wedge D_v\rangle^{2s} f\|_{L^2} + \|\langle v\rangle^{2s+\gamma} f\|_{L^2} \lesssim \|a_K^w f\|_{L^2}$$

due to the conclusion (i) in Lemma 4.2, we obtain the desired estimate in Theorem 1.1. The proof of Proposition 4.1 is complete. $\square$

4.2 Proof of Theorem 1.2 and boundedness estimates

In this section we prove first Theorem 1.2 about coercivity. As mentioned in the introduction it can be understood as an exact estimate for the so called triple norm introduced in [7] and recalled in Remark 4.7 below. It involves the pseudodifferential part a^w, for which we have elliptic properties stated in Proposition 1.4. Theorem 1.2 is a direct consequence of the following Lemma:

Lemma 4.6. *We have for a sufficiently large constant C and for all $l\in\mathbb{R}$ with $l\le\gamma/2+s$,*

$$\begin{aligned}&\|\langle v\rangle^{\gamma/2}\langle D_v\rangle^{s} f\|^2 + \|\langle v\rangle^{\gamma/2}\langle v\wedge D_v\rangle^{s} f\|^2 + \|\langle v\rangle^{\gamma/2+s} f\|^2\\ &\qquad\approx (a^w f, f) + C\|\langle v\rangle^{\gamma/2+s} f\|^2 \approx -(\mathcal{L}f,f) + \|\langle v\rangle^{l} f\|^2,\end{aligned}$$

where in the last equivalence the constant depends on l.

Proof. We first show the second equivalence. To do so rewrite the linearized Boltzmann operator $\mathcal{L}$ as

$$\mathcal{L} \;=\; -a^w + \underbrace{\mathcal{L}_2 + \overline{\mathcal{L}}_{1,\delta,a} + \mathcal{L}_{1,3,\delta} + \mathcal{L}_{1,4,\delta} - a_s(v,D_v) - (a(v,D_v) - a^w)}_{-\mathcal{K}}.$$

As a direct consequence of Lemma 2.1, conclusion (i) in Lemma 2.4, Lemma 2.3, Lemma 2.5 we have

$$\left|\left(\left(\mathcal{L}_2 + \overline{\mathcal{L}}_{1,\delta,a} + \mathcal{L}_{1,3,\delta} + \mathcal{L}_{1,4,\delta}\right) f,\ f\right)_{L^2}\right| \lesssim \left\| \langle v\rangle^{\gamma/2+s} f\right\|_{L^2}^2.$$

Moreover from (43) and Lemma 4.4 we know that

$$-\mathcal{K}_{pseudo} = -a_s(v,D_v) - (a^w - a(v,D_v)) \in \mathrm{Op}\left(\varepsilon a_K + \varepsilon^{-1}\langle v\rangle^{\gamma+2s},\ \Gamma\right),$$

and thus for any $\varepsilon > 0$,

$$\left|(\mathcal{K}_{pseudo} f,\ f)_{L^2}\right| \lesssim \varepsilon \left\| \left(a_K^{1/2}\right)^w f\right\|^2 + \varepsilon^{-1}\left\| \langle v\rangle^{\gamma/2+s} f\right\|_{L^2}^2$$

due to (ii) Lemma 4.2. Combining these estimates we conclude

$$-\left(\mathcal{L}f,\ f\right)_{L^2} = (a^w f,\ f)_{L^2} + (\mathcal{K}f,\ f)_{L^2}$$

with

$$\left|(\mathcal{K}f,\ f)_{L^2}\right| \lesssim \varepsilon \left\| \left(a_K^{1/2}\right)^w f\right\|^2 + \varepsilon^{-1}\left\| \langle v\rangle^{\gamma/2+s} f\right\|_{L^2}^2.$$

Moreover by (54) we have

$$(a^w f,\ f)_{L^2} + \left(K\langle v\rangle^{\gamma+2s} f, f\right)_{L^2} = (a_K^w f,\ f)_{L^2} = \left\| \left(a_K^{1/2}\right)^w f\right\|^2$$

and thus choosing ε small enough, we get

$$-\left(\mathcal{L}f,\ f\right)_{L^2} = (a^w f,\ f)_{L^2} + \left(\tilde{\mathcal{K}}f,\ f\right)_{L^2}$$

with

$$\left|\left(\tilde{\mathcal{K}}f,\ f\right)_{L^2}\right| \lesssim \left\| \langle v\rangle^{\gamma/2+s} f\right\|_{L^2}^2.$$

As a result, combining the estimate (see (50) for instance)

$$\left\| \langle v\rangle^{\gamma/2+s} f\right\|_{L^2}^2 \lesssim -\left(\mathcal{L}f,\ f\right)_{L^2} + \left\| \langle v\rangle^{\ell} f\right\|_{L^2}^2$$

we obtain the second equivalence.

Next we show the first equivalence. Using the estimate (54) below, we see that

$$(a^w f,\ f)_{L^2} + C\left\|K\langle v\rangle^{\gamma+2s} f, f\right\|_{L^2} = (a_K^w f,\ f)_{L^2} \approx \left\| \left(a_K^{1/2}\right)^w f\right\|^2.$$

On the other hand by conclusion (ii) in Lemma 4.2 we can deduce that

$$\left\| \left(a_K^{1/2}\right)^w f\right\|_{L^2}^2 \approx \left\| \langle v\rangle^{\gamma/2}\langle D_v\rangle^s f\right\|^2 + \left\| \langle v\rangle^{\gamma/2}\langle v\wedge D_v\rangle^s f\right\|^2 + \left\| \langle v\rangle^{\gamma/2+s} f\right\|^2.$$

Then the first equivalence follows, and the proof is complete. $\square$

Remark 4.7. In [7], the authors introduced the following non-isotropic norm

$$|||f|||^2 \stackrel{\text{def}}{=} \iiint \Phi(|v-v_*|)b(\cos\theta)\mu_*\left(f-f'\right)^2 + \iiint \Phi(|v-v_*|)b(\cos\theta)f_*^2\left(\sqrt{\mu'}-\sqrt{\mu}\right)^2,$$

where the integration is over $\mathbb{R}^3_v \times \mathbb{R}^3_{v_*} \times \mathbb{S}^2_\sigma$. For such a norm, Theorem 1.1 of [7]) says, with $l \in \mathbb{R}$ arbitrary (and equivalence norm depending on l),

$$\left\| \langle v\rangle^{\gamma/2} \langle D_v\rangle^s f\right\|^2 + \left\| \langle v\rangle^{\gamma/2+s} f\right\| \lesssim |||f|||^2 \lesssim -\left(\mathcal{L}f,\ f\right) + C_2\left\| \langle v\rangle^l f\right\|^2,$$

provided the Boltzmann cross-section B satisfies (3) with $0 < s < 1$ and $\gamma > -3$. In Lemma 4.6 above, we are therefore able to exhibit the complete form of this triple norm $|||f|||$.

Now we focus on the more difficult subelliptic estimate stated in 1.1. We begin with another coercivity estimate for the Weyl quantization a_K^w.

Lemma 4.8. *Let $\hat{P}_K$ be the operator defined at the beginning of Subsection 4.1. Then there exists a positive number $k_0 > 0$ such that for all $K \geq k_0$ and any $f \in \mathcal{S}(\mathbb{R}^3)$, we have*

$$\left\| \left(a_K^{1/2}\right)^w f\right\|^2 \approx (a_K^w f,\ f)_{L^2} = \mathrm{Re}\left(\hat{P}_K f,\ f\right)_{L^2} \tag{54}$$

and

$$\left\| \left(\tilde{a}_K^{1/2} a_K^{1/2}\right)^w f\right\|^2 \approx ((\tilde{a}_K a_K)^w f,\ f)_{L^2}. \tag{55}$$

Proof. The arguments are similar to the ones used in the proof of Lemma 4.2. Together with (86) and (87), we may write

$$(a_K^{1/2})^w (a_K^{1/2})^w = a_K^w - R^w, \tag{56}$$

where

$$R = -\int_0^1 \left(\partial_\eta (a_K^{1/2})\right) \sharp_\theta \left(\partial_v (a_K^{1/2})\right) d\theta + \int_0^1 \left(\partial_v (a_K^{1/2})\right) \sharp_\theta \left(\partial_\eta (a_K^{1/2})\right) d\theta$$

with $g\sharp_\theta h$ defined in (48). Using (41) for $\varepsilon = K^{-1/4}$, we conclude that

$$\partial_\eta (a_K^{1/2}) \in S(K^{-1/4} a_K^{1/2},\ \Gamma)$$

uniformly with respect to K. On the other hand, it is clear that $\partial_v (a_K^{1/2}) \in S(a_K^{1/2},\ \Gamma)$. As a result, [12, Proposition 1.1] yields

$$\left(\partial_\eta (a_K^{1/2})\right) \sharp_\theta \left(\partial_v (a_K^{1/2})\right),\ \left(\partial_v (a_K^{1/2})\right) \sharp_\theta \left(\partial_\eta (a_K^{1/2})\right) \in S(K^{-1/4} a_K,\ \Gamma)$$

uniformly w.r.t. K. Thus $R \in S(K^{-1/4} a_K,\ \Gamma)$ uniformly w.r.t. K. Then the conclusion (ii) in Lemma 4.2 allows us to rewrite R^w as

$$R^w = K^{-1/4} (a_K^{1/2})^w \underbrace{K^{1/2}\left[(a_K^{1/2})^w\right]^{-1} R^w \left[(a_K^{1/2})^w\right]^{-1}}_{\in \mathcal{B}(L^2) \text{ uniformly w.r.t. } K} (a_K^{1/2})^w,$$

which gives

$$|(R^w f,\ f)_{L^2}| \le C_0 K^{-1/4} \big\|(a_K^{1/2})^w f\big\|^2$$

with C_0 some constant independent of K. Taking into account the relation (56) we obtain

$$(a_K^w f,\ f)_{L^2} \le \Big((a_K^{1/2})^w (a_K^{1/2})^w f,\ f\Big)_{L^2} + C_0 K^{-1/4} \big\|(a_K^{1/2})^w f\big\|^2 \le (C_0+1) \big\|(a_K^{1/2})^w f\big\|^2$$

and

$$\Big((a_K^{1/2})^w (a_K^{1/2})^w f,\ f\Big)_{L^2} \le (a_K^w f,\ f)_{L^2} + C_0 K^{-1/4} \big\|(a_K^{1/2})^w f\big\|^2.$$

The desired estimate (54) follows if we take K sufficiently large such that $K \ge k_0 \stackrel{\text{def}}{=} 16 C_0^4$. Since the second estimate (55) can be deduced similarly by virtue of (iii) in Lemma 4.2, we omit it here. The proof is thus complete. □

Corollary 4.9. *Let ℓ be an arbitrary real number. The following estimate*

$$\forall\ f \in \mathcal{S}(\mathbb{R}_v^3), \quad \big\| \langle v \rangle^{2s+\gamma} f \big\|_{L^2} + \big\|(a_K^{1/2})^w \langle v \rangle^{s+\gamma/2} f \big\|_{L^2} \lesssim \big\|\hat{P}_K f\big\|_{L^2} + \big\|f\big\|_{L^2_\ell} \tag{57}$$

holds uniformly with respect to ξ.

Proof. We have obtained in the proof of Lemma 4.6 the estimate

$$\big\| \langle v \rangle^{2s+\gamma} f \big\|_{L^2} \lesssim \big\|(a_K^{1/2})^w \langle v \rangle^{s+\gamma/2} f \big\|_{L^2}.$$

Moreover using the coercivity estimate (54) applied to the function $\langle v \rangle^{s+\gamma/2} f$, we have

$$\begin{aligned}
&\big\| \langle v \rangle^{2s+\gamma} f \big\|_{L^2}^2 + \big\|(a_K^{1/2})^w \langle v \rangle^{s+\gamma/2} f \big\|_{L^2}^2 \\
&\lesssim \Big| \Big(\hat{P}_K \langle v \rangle^{s+\gamma/2} f,\ \langle v \rangle^{s+\gamma/2} f \Big)_{L^2} \Big| \\
&\lesssim \Big| \Big(\big[\hat{P}_K,\ \langle v \rangle^{s+\gamma/2}\big] f,\ \langle v \rangle^{s+\gamma/2} f \Big)_{L^2} \Big| + \Big| \Big(\hat{P}_K f,\ \langle v \rangle^{2s+\gamma} f \Big)_{L^2} \Big| \\
&\lesssim \Big| \Big(\big[a^w,\ \langle v \rangle^{s+\gamma/2}\big] f,\ \langle v \rangle^{s+\gamma/2} f \Big)_{L^2} \Big| + \varepsilon^{-1} \big\|\hat{P}_K f\big\|_{L^2}^2 + \varepsilon \big\| \langle v \rangle^{2s+\gamma} f \big\|_{L^2}^2.
\end{aligned}$$

We apply (41) and [29, Theorem 2.3.8] to conclude that the symbol of the operator

$$\big[a^w,\ \langle v \rangle^{s+\frac{\gamma}{2}}\big]$$

belongs to

$$S\Big(a_K^{1/2} \langle v \rangle^{2s+\gamma-1},\ \Gamma\Big).$$

This fact, together with Lemma 4.2 (ii), allows us to write

$$\begin{aligned}
&\big[a^w,\ \langle v \rangle^{s+\frac{\gamma}{2}}\big] \\
&= \epsilon^{-1} \langle v \rangle^{s-1+\frac{\gamma}{2}} \underbrace{\langle v \rangle^{-(s-1+\frac{\gamma}{2})} \big[a^w,\ \langle v \rangle^{s+\frac{\gamma}{2}}\big] \langle v \rangle^{-(s+\frac{\gamma}{2})} \Big[\Big(a_K^{1/2}\Big)^w\Big]^{-1}}_{\in\ \mathcal{B}(L^2)} \epsilon \Big(a_K^{1/2}\Big)^w \langle v \rangle^{s+\frac{\gamma}{2}}.
\end{aligned}$$

Then

$$\begin{aligned}\left|\Big(\big[a^w,\ \langle v\rangle^{s+\gamma/2}\big]f,\ \langle v\rangle^{s+\gamma/2} f\Big)_{L^2}\right| &\lesssim \varepsilon\big\|\left(a_K^{1/2}\right)^w \langle v\rangle^{s+\frac{\gamma}{2}} f\big\|_{L^2}^2+\varepsilon^{-1}\big\|\langle v\rangle^{2s+\gamma-1} f\big\|_{L^2}^2\\ &\lesssim \varepsilon\big\|\left(a_K^{1/2}\right)^w \langle v\rangle^{s+\frac{\gamma}{2}} f\big\|_{L^2}^2+\varepsilon\big\|\langle v\rangle^{2s+\gamma} f\big\|_{L^2}^2\\ &\quad +C_\varepsilon\big\|\langle v\rangle^{\ell} f\big\|_{L^2}^2.\end{aligned}$$

Letting ε be small sufficiently gives the conclusions. □

Corollary 4.10.

$$\Big(\left(\langle v\rangle^{2s+\gamma}\right)^{\mathrm{Wick}} f,\ f\Big)_{L^2}\lesssim\Big(\left(\tilde a(v,\eta)\right)^{\mathrm{Wick}} f,\ f\Big)_{L^2}\lesssim\left|\Big(\hat P_K f,\ f\Big)_{L^2}\right|. \tag{58}$$

Proof. The first inequality is due to the positivity of Wick quantization. The second one is just an immediate consequence of (54) and Lemma 4.2, since we may write

$$\left(\tilde a(v,\eta)\right)^{\mathrm{Wick}}=\left(a_K^{1/2}\right)^w\underbrace{\left[\left(a_K^{1/2}\right)^w\right]^{-1}\left(\tilde a(v,\eta)\right)^{\mathrm{Wick}}\left[\left(a_K^{1/2}\right)^w\right]^{-1}}_{\in\mathcal B(L^2)}\left(a_K^{1/2}\right)^w,$$

where we use the fact (see the appendix) that $\tilde a^{\mathrm{Wick}}=b^w$ with b belonging to the same symbol class as $\tilde a$. The proof is complete.

□

4.3 Hypoelliptic estimates and proof of Theorems 1.1 and 1.3

This last subsection is devoted to the proofs of the main results, Theorem 1.1 and Theorem 1.3. As explained in Proposition 4.1, we only work on $\hat P_K$ instead of P. Therefore, in this subsection, ξ and τ are considered as parameters. Recall that $\tilde a$ is defined in (37), whose explicit form, as to be seen below, will be of convenient use. The main result to be shown here can be stated as follows

Proposition 4.11. *Under the conditions of Theorem 1, we have, for any $\ell\in\mathbb R$,*

$$\big\|\tilde a(v,\xi)^{\frac{1}{1+2s}}f\big\|+\big\|a_K^w f\big\|\lesssim\big\|\hat P_K f\big\|_{L^2}+\big\|\langle v\rangle^{\ell} f\big\|_{L^2}.$$

The above proposition will be proved in several steps, following the multiplier strategy introduced in [24]. To this end, throughout this section, we let $\chi\in C_0^\infty(\mathbb R;\ [0,1])$ such that $\chi=1$ in $[-1,1]$ and supp $\chi\subset[-2,2]$, and let g be a symbol given by

$$g(v,\eta)=g_\xi(v,\eta)=\frac{a_3(v,\eta)}{\tilde a(v,\xi)^{\frac{2s}{1+2s}}}\psi(v,\eta), \tag{59}$$

where

$$\psi(v,\eta)=\chi\left(\frac{\tilde a(v,\eta)}{\tilde a(v,\xi)^{\frac{1}{1+2s}}}\right) \tag{60}$$

and

$$a_3(v,\eta)=\langle v\rangle^{\gamma}\Big(1+|v|^2+|\xi|^2+|v\wedge\xi|^2\Big)^{s-1}\Big(\xi\cdot\eta+(v\wedge\xi)\cdot(v\wedge\eta)\Big). \tag{61}$$

The main property linking a_3 and $\tilde{a}$ is that

$$\{a_3(v,\eta), v\cdot\xi\} = \tilde{a}(v,\xi) - \langle v\rangle^{\gamma+2}\left(1+|v|^2+|\xi|^2+|v\wedge\xi|^2\right)^{s-1}. \tag{62}$$

where $\{\cdot,\ \cdot\}$ is the Poisson bracket defined in (92). Thanks to the explicit symbolic estimates for $\tilde{a}$, g and ψ also have good behavior as symbols, that is,

$$g,\ \psi \in S(1, |dv|^2+|d\eta|^2)$$

uniformly with respect to ξ, where we use the estimate

$$|a_3(v,\eta)| \lesssim \tilde{a}(v,\xi)^{\frac{2s-1}{2s}}\tilde{a}(v,\eta)^{\frac{1}{2s}}.$$

Moreover direct computation shows that

$$|\xi\cdot\partial_\eta\psi| \lesssim \tilde{a}(v,\eta). \tag{63}$$

Lemma 4.12. *Under the conditions in Theorem 1, we have*

$$\forall\ f\in\mathcal{S}(\mathbb{R}^3),\quad \left\|\tilde{a}(v,\xi)^{\frac{1}{1+2s}}f\right\|^2 \lesssim \left\|\hat{P}_K f\right\|_{L^2}^2 + \left\|f\right\|_{L^2_\ell}.$$

Proof. The proof is divided into three steps.

Step 1) Let g^{Wick} be the Wick quantization of the symbol g given in (59). We claim

$$\left|\left(a_K^w f,\ g^{\mathrm{Wick}}f\right)_{L^2}\right| \lesssim \left|\left(\hat{P}_K f,\ f\right)_{L^2}\right|. \tag{64}$$

Indeed, let us write, denoting by H the inverse of $\left(a_K^{1/2}\right)^w$,

$$\left(a_K^w f,\ g^{\mathrm{Wick}}f\right)_{L^2} = \left(Ha_K^wH\left(a_K^{1\,2}\right)^w f,\ \left(a_K^{1/2}\right)^w g^{\mathrm{Wick}}H\left(a_K^{1/2}\right)^w f\right)_{L^2}.$$

Note that Ha_K^wH and $\left(a_K^{1/2}\right)^w g^{\mathrm{Wick}}H$ are bounded operators on L^2 due to Lemma 4.2 and the fact that $g^{\mathrm{Wick}} = \tilde{g}^w$ with $\tilde{g}\in S(1,\Gamma)$ (see the appendix). Then one has

$$\left|\left(a_K^w f,\ g^{\mathrm{Wick}}f\right)_{L^2}\right| \lesssim \|\left(a_K^{1/2}\right)^w f\|_{L^2}^2 \lesssim \left|\left(\hat{P}_K f,\ f\right)_{L^2}\right|,$$

the last inequality following from (54).

Step 2) We now prove

$$\left\|\tilde{a}(v,\xi)^{\frac{1}{2+4s}}f\right\|_{L^2} \lesssim \left\|\tilde{a}(v,\xi)^{-\frac{1}{2+4s}}\hat{P}_K f\right\|_{L^2}. \tag{65}$$

Note that $g\in S(1,\Gamma)$ and $\tilde{a}(v,\xi)^r \in S\left(\tilde{a}(v,\xi)^r,\Gamma\right)$ for any $r\in\mathbb{R}$. Then the above estimate will follow if we can show that

$$\left\|\tilde{a}(v,\xi)^{\frac{1}{2+4s}}f\right\|_{L^2}^2 \lesssim \left|\left(\hat{P}_K f,\ f\right)_{L^2}\right| + \left|\left(\hat{P}_K f,\ g^{\mathrm{Wick}}f\right)_{L^2}\right|. \tag{66}$$

To prove the above inequality we make use of the relation

$$\mathrm{Re}\left(i\left(v\cdot\xi\right)f,\ g^{\mathrm{Wick}}f\right)_{L^2} = \mathrm{Re}\left(\hat{P}_K f,\ g^{\mathrm{Wick}}f\right)_{L^2} - \mathrm{Re}\left(a_K^w f,\ g^{\mathrm{Wick}}f\right)_{L^2}$$

and (64), to conclude that

$$\mathrm{Re}\left(i\,(v\cdot\xi)\,f,\ g^{\mathrm{Wick}}f\right)_{L^2}\lesssim\left|\left(\hat{P}_K f,\ f\right)_{L^2}\right|+\left|\left(\hat{P}_K f,\ g^{\mathrm{Wick}}f\right)_{L^2}\right|. \tag{67}$$

Next we will give a lower bound of the term on the left hand side. Observe that by (90),

$$v\cdot\xi=(v\cdot\xi)^{\mathrm{Wick}}.$$

Then we have, by (91),

$$\mathrm{Re}\left(i\,(v\cdot\xi)\,f,\ g^{\mathrm{Wick}}f\right)_{L^2}=\frac{1}{4\pi}\left(\{g,\ v\cdot\xi\}^{\mathrm{Wick}}f,\ f\right)_{L^2}. \tag{68}$$

Using (62) we compute

$$\begin{aligned}
&\{g,\ v\cdot\xi\}\\
=\ &\tilde{a}(v,\xi)^{\frac{1}{1+2s}}\psi-\frac{\langle v\rangle^{\gamma+2}\left(1+|v|^2+|\xi|^2+|v\wedge\xi|^2\right)^{s-1}}{\tilde{a}(v,\xi)^{\frac{2s}{1+2s}}}\psi+\frac{a_3(v,\eta)}{\tilde{a}(v,\xi)^{\frac{2s}{1+2s}}}\xi\cdot\partial_\eta\psi\\
=\ &\tilde{a}(v,\xi)^{\frac{1}{1+2s}}-\tilde{a}(v,\xi)^{\frac{1}{1+2s}}\left(1-\psi\right)-\frac{\langle v\rangle^{\gamma+2}\left(1+|v|^2+|\xi|^2+|v\wedge\xi|^2\right)^{s-1}}{\tilde{a}(v,\xi)^{\frac{2s}{1+2s}}}\psi\\
&+\frac{a_3(v,\eta)}{\tilde{a}(v,\xi)^{\frac{2s}{1+2s}}}\xi\cdot\partial_\eta\psi.
\end{aligned}$$

This along with (67) and (68) yields

$$\left(\left(\tilde{a}(v,\xi)^{\frac{1}{1+2s}}\right)^{\mathrm{Wick}}f,\ f\right)_{L^2}\lesssim\sum_{j=1}^{3}T_j+\left|\left(\hat{P}_K f,\ f\right)_{L^2}\right|+\left|\left(\hat{P}_K f,\ g^{\mathrm{Wick}}f\right)_{L^2}\right|, \tag{69}$$

with

$$\begin{aligned}
T_1&=\left(\left(\tilde{a}(v,\xi)^{\frac{1}{1+2s}}\left(1-\psi\right)\right)^{\mathrm{Wick}}f,\ \ f\right)_{L^2},\\
T_2&=\left(\left(\langle v\rangle^{\gamma+2}\left(1+|v|^2+|\xi|^2+|v\wedge\xi|^2\right)^{s-1}\tilde{a}(v,\xi)^{-\frac{2s}{1+2s}}\psi\right)^{\mathrm{Wick}}f,\ \ f\right)_{L^2},\\
T_3&=\left(\left(-\frac{a_3(v,\eta)}{\tilde{a}(v,\xi)^{\frac{2s}{1+2s}}}\xi\cdot\partial_\eta\psi\right)^{\mathrm{Wick}}f,\ \ f\right)_{L^2}.
\end{aligned}$$

Note that $\tilde{a}(v,\xi)^{\frac{1}{1+2s}}\leq\tilde{a}(v,\eta)$ on the support of $1-\psi$, and thus

$$\tilde{a}(v,\xi)^{\frac{1}{1+2s}}(1-\psi)\leq\tilde{a}(v,\eta).$$

Then the positivity of Wick quantization gives

$$T_1\lesssim\left((\tilde{a}(v,\eta))^{\mathrm{Wick}}f,\ f\right)_{L^2}\lesssim\left|\left(\hat{P}_K f,\ f\right)_{L^2}\right|, \tag{70}$$

where the last inequality follows from (58). Similarly, observing that

$$\langle v\rangle^{\gamma+2}\left(1+|v|^2+|\xi|^2+|v\wedge\xi|^2\right)^{s-1}\tilde{a}(v,\xi)^{-\frac{2s}{1+2s}}\psi\leq\langle v\rangle^{2s+\gamma},$$

we have

$$T_2 \lesssim \left(\left(\langle v\rangle^{2s+\gamma}\right)^{\mathrm{Wick}} f,\ f\right)_{L^2} \lesssim \left|\left(\hat{P}_K f,\ f\right)_{L^2}\right|. \tag{71}$$

As for T_3, it follows from (63) that

$$-\frac{a_3(v,\eta)}{\tilde{a}(v,\xi)^{\frac{2s}{1+2s}}}\xi\cdot\partial_\eta\psi \lesssim \tilde{a}(v,\eta).$$

Thus

$$T_3 \lesssim \left((\tilde{a}(v,\eta))^{\mathrm{Wick}} f,\ f\right)_{L^2} \lesssim \left|\left(\hat{P}_K f,\ f\right)_{L^2}\right|.$$

This, together with (69), (70) and (71), gives

$$\left(\left(\tilde{a}(v,\xi)^{\frac{1}{1+2s}}\right)^{\mathrm{Wick}} f,\ f\right)_{L^2} \lesssim \left|\left(\hat{P}_K f,\ f\right)_{L^2}\right| + \left|\left(\hat{P}_K f,\ g^{\mathrm{Wick}} f\right)_{L^2}\right|.$$

As a result, the desired estimate (66) follows, since by (90),

$$\left(\tilde{a}(v,\xi)^{\frac{1}{1+2s}}\right)^{\mathrm{Wick}} = \int \tilde{a}(v-\tilde{v},\xi)^{\frac{1}{1+2s}} e^{-2\pi\tilde{v}^2} 2^3 d\tilde{v},$$

which is bounded from below by $\tilde{a}(v,\xi)^{1/(1+2s)}$ by a direct check (see for instance the arguments used in the proof of [24, Lemma 3.14]).

Step 3) Now applying inequality (66) to the function $\tilde{a}(v,\xi)^{\frac{1}{2+4s}} f$, we get

$$\begin{aligned}\left\|\tilde{a}(v,\xi)^{\frac{1}{1+2s}} f\right\|_{L^2} &\lesssim \left\|\tilde{a}(v,\xi)^{-\frac{1}{2+4s}} \hat{P}_K \tilde{a}(v,\xi)^{\frac{1}{2+4s}} f\right\|_{L^2} \\ &\lesssim \left\|\hat{P}_K f\right\|_{L^2} + \left\|\tilde{a}(v,\xi)^{-\frac{1}{2+4s}}\left[a_K^w,\ \tilde{a}(v,\xi)^{\frac{1}{2+4s}}\right] f\right\|_{L^2}.\end{aligned}$$

In view of (87), the symbol of $\tilde{a}(v,\xi)^{-1/(2+4s)}\left[a_K^w,\ \tilde{a}(v,\xi)^{1/(2+4s)}\right]$ has the form

$$\tilde{a}(v,\xi)^{-\frac{1}{2+4s}} \int_0^1 (\partial_\eta a_K)\,\sharp_\theta \left(\partial_v(\tilde{a}^{1/(2+4s)})\right) d\theta,$$

which, arguing as in the proof of Lemma 4.2, belongs to

$$S\left(a^{1/2}\langle v\rangle^{s+\gamma/2},\ \Gamma\right).$$

As a result, we can use (ii) in Lemma 4.2 to write

$$\begin{aligned}&\tilde{a}(v,\xi)^{-\frac{1}{2+4s}}\left[a_K^w,\ \tilde{a}(v,\xi)^{\frac{1}{2+4s}}\right] \\ =\ &\underbrace{\tilde{a}(v,\xi)^{-\frac{1}{2+4s}}\left[a_K^w,\ \tilde{a}(v,\xi)^{\frac{1}{2+4s}}\right]\langle v\rangle^{-(s+\gamma/2)}\left(\left(a_K^{1/2}\right)^w\right)^{-1}}_{\in\ \mathcal{B}(L^2)}(a_K^{1/2})^w\langle v\rangle^{s+\gamma/2}.\end{aligned}$$

This gives

$$\begin{aligned}\left\|\tilde{a}(v,\xi)^{-\frac{1}{2+4s}}\left[a_K^w,\ \tilde{a}(v,\xi)^{\frac{1}{2+4s}}\right] f\right\|_{L^2} &\lesssim \left\|(a_K^{1/2})^w\langle v\rangle^{s+\gamma/2} f\right\|_{L^2} \\ &\lesssim \left\|\hat{P}_K f\right\|_{L^2} + \left\|f\right\|_{L^2_\ell},\end{aligned}$$

where the last inequality follows from (57). Combining these inequalities, we get the desired estimate

$$\left\|\tilde{a}(v,\xi)^{\frac{1}{1+2s}} f\right\|_{L^2} \lesssim \left\|\hat{P}_K f\right\|_{L^2} + \left\|f\right\|_{L^2_\ell}.$$

The proof of Lemma 4.12 is thus complete. □

Lemma 4.13. *Under the conditions in Theorem 1, we have, for any $\ell \in \mathbb{R}$,*

$$\left\|a_K^w f\right\|_{L^2} \lesssim \left\|\hat{P}_K f\right\|_{L^2} + \left\| \langle v\rangle^\ell f\right\|_{L^2}.$$

Proof. The proof is divided into four steps. In the following, let $\varepsilon > 0$ be an arbitrarily small number, to be fixed later on, and denote by C_ε different suitable constants depending only on ε and appearing in the the estimations below.

Step 1) We define ρ_ε by

$$\rho_\varepsilon(v,\eta) = \chi\left(\frac{\tilde{a}(v,\xi)^{\frac{1}{1+2s}}}{\varepsilon\tilde{a}(v,\eta)}\right),$$

where $\chi \in C_0^\infty(\mathbb{R};\ [0,1])$ such that $\chi = 1$ in $[-1,1]$ and $\operatorname{supp}\chi \subset [-2,2]$. Let $\lambda_{1,\varepsilon}$ and $\lambda_{2,\varepsilon}$ be two symbols defined by

$$\lambda_{1,\varepsilon}(v,\eta) = \rho_\varepsilon(v,\eta)\tilde{a}(v,\eta) \tag{72}$$

and

$$\lambda_{2,\varepsilon}(v,\eta) = \big(1-\rho_\varepsilon(v,\eta)\big)\tilde{a}(v,\eta). \tag{73}$$

Then $\rho_\varepsilon(v,\eta) \in S\,(1,\ \Gamma)$,

$$\lambda_{1,\varepsilon},\ \lambda_{2,\varepsilon} \in S\,(\tilde{a}(v,\eta),\ \Gamma) \quad \text{and} \quad \lambda_{2,\varepsilon} \in S\left(\varepsilon^{-1}\tilde{a}(v,\xi)^{\frac{1}{1+2s}},\ \Gamma\right), \tag{74}$$

uniformly with respect to ξ and ε, due to the conclusion (i) in Proposition 1.4 and the fact that $\tilde{a}(v,\eta) \le \varepsilon^{-1}\tilde{a}(v,\xi)^{\frac{1}{1+2s}}$ on the support of $\lambda_{2,\varepsilon}$.

Step 2) Let $\lambda_{1,\varepsilon}(v,\eta)$ be given in (72). In this step we show that

$$\left|\Big(\big[v\cdot\xi,\quad \lambda_{1,\varepsilon}^w\big]f,\ f\Big)_{L^2}\right| \le \varepsilon\left\|a_K^w f\right\|_{L^2}^2. \tag{75}$$

In fact, the symbol of the above commutator $\big[v\cdot\xi,\quad \lambda_{1,\varepsilon}^w\big]$ is

$$-\frac{1}{2i\pi}\xi\cdot\partial_\eta\lambda_{1,\varepsilon}(v,\eta),$$

which belongs to $S\left(\varepsilon^{(1+2s)/2s}\tilde{a}(v,\eta)^2,\ \Gamma\right)$ uniformly with respect to ξ and ε, due to (42) and the fact that

$$|\xi| + |v\wedge\xi| \lesssim \tilde{a}(v,\xi)^{\frac{1}{2s}}\langle v\rangle^{-\frac{\gamma}{2s}} \le \varepsilon^{\frac{1+2s}{2s}}\tilde{a}(v,\eta)^{\frac{1+2s}{2s}}\langle v\rangle^{-\frac{\gamma}{2s}}$$

on the support of $\lambda_{1,\varepsilon}$. Thus writing

$$\left[v\cdot\xi,\quad \lambda_{1,\varepsilon}^w\right]=\varepsilon a_K^w\underbrace{(a_K^w)^{-1}\left[v\cdot\xi,\quad \lambda_{1,\varepsilon}^w\right](a_K^w)^{-1}}_{\in\mathcal{B}(L^2)}a_K^w,$$

we obtain

$$\left|\left(\left[v\cdot\xi,\quad \lambda_{1,\varepsilon}^w\right]f,\ f\right)_{L^2}\right|\lesssim\varepsilon\left\|a_K^wf\right\|_{L^2}^2.$$

This gives the desired upper bound and therefore the proof of (75).

Step 3) Let $\lambda_{2,\varepsilon}(v,\eta)$ be given in (73). We claim that

$$\left|\left(\left[v\cdot\xi,\quad \lambda_{2,\varepsilon}^w\right]f,\ f\right)_{L^2}\right|\lesssim\varepsilon\left\|(v\cdot\xi)f\right\|_{L^2}^2+C_\varepsilon\left(\left\|\hat{P}_Kf\right\|_{L^2}^2+\left\|f\right\|_{L^2_\ell}^2\right).\tag{76}$$

Indeed, we write $\left[v\cdot\xi,\quad \lambda_{2,\varepsilon}^w\right]=v\cdot\xi\lambda_{2,\varepsilon}^w-\lambda_{2,\varepsilon}^wv\cdot\xi$ to get

$$\left|\left(\left[v\cdot\xi,\quad \lambda_{2,\varepsilon}^w\right]f,\ f\right)_{L^2}\right|\leq 2\left\|\,(v\cdot\xi)\,f\right\|_{L^2}\left\|\lambda_{2,\varepsilon}^wf\right\|_{L^2}.$$

Moreover it follows from (74) that

$$\left\|\lambda_{2,\varepsilon}^wf\right\|_{L^2}\lesssim\varepsilon^{-1}\left\|\tilde{a}(v,\xi)^{1/(1+2s)}f\right\|_{L^2}\lesssim\varepsilon^{-1}\left(\left\|\tilde{\mathcal{P}}f\right\|_{L^2}+\left\|f\right\|_{L^2_\ell}\right),$$

the last inequality using Lemma 4.12. Combining these inequalities, we obtain the desired estimate (76).

Step 4) Now we are ready to prove that

$$\left\|a_K^wf\right\|_{L^2}^2\lesssim\left\|\hat{P}_Kf\right\|_{L^2}^2+\left\|f\right\|_{L^2_\ell}^2.\tag{77}$$

This inequality will be obtained if we can show that

$$\left|\mathrm{Re}\,(i(v\cdot\xi)f,\ \tilde{a}_K^wf)_{L^2}\right|\lesssim\varepsilon\left\|a_K^wf\right\|_{L^2}^2+C_\varepsilon\left(\left\|\hat{P}_Kf\right\|_{L^2}^2+\left\|f\right\|_{L^2_\ell}^2\right)\tag{78}$$

and

$$\left\|a_K^wf\right\|^2\leq\mathrm{Re}\ \left(a_K^wf,\ \tilde{a}_K^wf\right)_{L^2}+\varepsilon\left\|a_K^wf\right\|^2+C_\varepsilon\left(\left\|\hat{P}_Kf\right\|^2+\left\|f\right\|^2\right),\tag{79}$$

due to the relation

$$\mathrm{Re}\left(\hat{P}_Kf,\ \tilde{a}_K^wf\right)_{L^2}=\mathrm{Re}\,(i(v\cdot\xi)f,\ \tilde{a}_K^wf)_{L^2}+\mathrm{Re}\,(a_K^wf,\ \tilde{a}_K^wf)_{L^2}\,.$$

To prove (78), we compute

$$\begin{aligned}\left|\mathrm{Re}\,(i(v\cdot\xi)f,\ \tilde{a}_K^wf)_{L^2}\right|&=\left|\frac{i}{2}\left(\left[v\cdot\xi,\ \tilde{a}_K^w\right]f,\ f\right)_{L^2}\right|=\left|\frac{i}{2}\left(\left[v\cdot\xi,\ \tilde{a}^w\right]f,\ f\right)_{L^2}\right|\\ &\lesssim\ \left|\left(\left[v\cdot\xi,\ \lambda_{1,\varepsilon}^w\right]f,\ f\right)_{L^2}\right|+\left|\left(\left[v\cdot\xi,\ \lambda_{2,\varepsilon}^w\right]f,\ f\right)_{L^2}\right|\end{aligned}$$

with $\lambda_{1,\varepsilon}$, $\lambda_{2,\varepsilon}$ defined in (72) and (73). Combining the above inequalities and the conclusion in the previous two steps, we have

$$\left|\mathrm{Re}\,(i(v\cdot\xi)f,\ \tilde{a}_K^wf)_{L^2}\right|\lesssim\varepsilon\left\|a_K^wf\right\|_{L^2}^2+\varepsilon\left\|(v\cdot\xi)f\right\|_{L^2}^2+C_\varepsilon\left(\left\|\hat{P}_Kf\right\|_{L^2}^2+\left\|f\right\|_{L^2_\ell}^2\right).$$

This inequality along with the relation

$$\|(v\cdot\xi)f\|_{L^2}^2 \lesssim \|\hat{P}_K f\|_{L^2}^2 + \|a_K^w f\|_{L^2}^2$$

implies the desired estimate (78).

We now prove (79). In view of (87) we may write

$$(\tilde{a}_K \sharp a_K)^w = (\tilde{a}_K a_K)^w + r^w, \tag{80}$$

where

$$r(Y) = \int_0^1 \iint e^{-2i\sigma(Y-Y_1,Y-Y_2)/\theta}\frac{1}{2i}\sigma(\partial_{Y_1},\partial_{Y_2})\tilde{a}(Y_1)a_K(Y_2)dY_1dY_2d\theta/(\pi\theta)^6.$$

Note that (41) also holds true, with a replaced by $\tilde{a}_K$ or a_K. Then in view of [12, Proposition 1.1], we can check that

$$r \in S\left(a_K^{3/2}\langle v\rangle^{s+\gamma/2},\ \Gamma\right),$$

and thus we may use Lemma 4.2 to rewrite r^w as

$$r^w = \varepsilon^{1/2}a_K^w \underbrace{(a_K^w)^{-1}r^w\langle v\rangle^{-(s+\gamma/2)}\left[\left(a_K^{1/2}\right)^w\right]^{-1}}_{\in\mathcal{B}(L^2)}\varepsilon^{-1/2}\left(a_K^{1/2}\right)^w\langle v\rangle^{s+\gamma/2}.$$

This gives

$$\begin{aligned}|(r^wf,\ f)_{L^2}| &\lesssim \varepsilon\|a_K^wf\|_{L^2}^2 + \varepsilon^{-1}\|\left(a_K^{1/2}\right)^w\langle v\rangle^{s+\gamma/2}f\|_{L^2}^2\\ &\lesssim \varepsilon\|a_K^wf\|_{L^2}^2 + \varepsilon^{-1}\left(\|\hat{P}_Kf\|_{L^2}^2 + \|f\|_{L^2_\ell}^2\right),\end{aligned}$$

the last inequality following from (57). Taking into account (80), one has

$$\mathrm{Re}\left((\tilde{a}_Ka_K)^wf,\ f\right)_{L^2} \lesssim \mathrm{Re}\left(a_K^wf,\ \tilde{a}_Kf\right)_{L^2} + \varepsilon\|a_K^wf\|_{L^2}^2 + \varepsilon^{-2}\left(\|\hat{P}_Kf\|_{L^2}^2 + \|f\|_{L^2_\ell}^2\right),$$

which along with (55) yields

$$\|\left(\tilde{a}_K^{1/2}a_K^{1/2}\right)^wf\|_{L^2}^2 \lesssim \mathrm{Re}\left(a_K^wf,\ \tilde{a}_Kf\right)_{L^2} + \varepsilon\|a_K^wf\|_{L^2}^2 + \varepsilon^{-2}\left(\|\hat{P}_Kf\|_{L^2}^2 + \|f\|_{L^2_\ell}^2\right).$$

Moreover note that

$$\|a_K^wf\|_{L^2}^2 \lesssim \|\left(\tilde{a}_K^{1/2}a_K^{1/2}\right)^wf\|_{L^2}^2$$

due to the conclusion (iii) in Lemma 4.2. Then the desired estimate (79) follows from the above inequalities, completing the proof of Lemma 4.13. □

Combining the conclusions in Lemma 4.12 and Lemma 4.13, we obtain Proposition 4.11. Thus Theorem 1.1 follows due to Proposition 4.1. Now it remains to do the

Proof of Theorem 1.3. Let τ be the dual variable of t and let $\hat{P}_\tau$ be the operator defines as follows

$$\hat{P}_\tau = i\tau + iv\cdot\xi - \mathcal{L} = i\left(\tau + v\cdot\xi\right) + a^w + \mathcal{K}.$$

Just proceeding as in the proof of Lemma 4.12 and Lemma 4.13, we have the maximal hypoelliptic estimate

$$\left\| \langle v\rangle^{2s+\gamma} f\right\|_{L^2} + \left\| (\tau + v\cdot\xi) f\right\|_{L^2} + \left\|a^w f\right\|_{L^2} + \left\| \langle v\rangle^{\frac{\gamma}{1+2s}} |\xi|^{\frac{2s}{1+2s}} f\right\|_{L^2} \lesssim \left\|\hat{P}_\tau f\right\|_{L^2} + \left\|f\right\|_{L^2_\ell}. \tag{81}$$

Now it remains to prove

$$\left\| \langle v\rangle^{\frac{\gamma-2s}{1+2s}} \langle \tau\rangle^{\frac{2s}{1+2s}} f\right\|_{L^2} \lesssim \left\|\hat{P}_\tau f\right\|_{L^2} + \left\|f\right\|_{L^2_\ell}.$$

To do so, we compute

$$\begin{aligned}
\langle v\rangle^{\frac{\gamma-2s}{1+2s}} |\tau|^{\frac{2s}{1+2s}} &\lesssim \langle v\rangle^{\frac{\gamma-2s}{1+2s}} |\tau+v\cdot\xi|^{\frac{2s}{1+2s}} + \langle v\rangle^{\frac{\gamma-2s}{1+2s}} |v\cdot\xi|^{\frac{2s}{1+2s}}\\
&\lesssim \langle v\rangle^{\frac{\gamma-2s}{1+2s}} |\tau+v\cdot\xi|^{\frac{2s}{1+2s}} + \langle v\rangle^{\frac{\gamma}{1+2s}} |\xi|^{\frac{2s}{1+2s}}\\
&\lesssim \langle v\rangle^{\gamma-2s} + |\tau+v\cdot\xi| + \langle v\rangle^{\frac{\gamma}{1+2s}} |\xi|^{\frac{2s}{1+2s}},
\end{aligned}$$

where the last inequality follows from the Young's inequality:

$$\langle v\rangle^{\frac{\gamma-2s}{1+2s}} |\tau+v\cdot\xi|^{\frac{2s}{1+2s}} \le \frac{\left(\langle v\rangle^{\frac{\gamma-2s}{1+2s}}\right)^{1+2s}}{1+2s} + \frac{2s}{1+2s}\left(|\tau+v\cdot\xi|^{\frac{2s}{1+2s}}\right)^{(1+2s)/(2s)}.$$

As a result we have,

$$\begin{aligned}
\left\| \langle v\rangle^{\frac{\gamma-2s}{1+2s}} |\tau|^{\frac{2s}{1+2s}} f\right\|_{L^2} &\lesssim \left\| (\tau+v\cdot\xi) f\right\|_{L^2} + \left\| \langle v\rangle^{\gamma-2s} f\right\|_{L^2} + \left\| \langle v\rangle^{\frac{\gamma}{1+2s}} |\xi|^{\frac{2s}{1+2s}} f\right\|_{L^2}\\
&\lesssim \left\|\hat{P}_\tau f\right\|_{L^2} + \left\|f\right\|_{L^2_\ell},
\end{aligned}$$

where the last inequality follows from (81). The proof of Theorem 1.3 is complete. □

A Appendix

In this section we briefly review some tools used through the proofs. The first section is devoted to the links between some integrals concerning the Boltzmann kernel. In the second one, we recall some basic facts about the Weyl-Hörmander quantization, and the last section will recall some ideas and results about the Wick quantization.

A.1 Integral representations

Principal values

Let $q(\theta)$ be a given measurable function such that

$$\int_{\mathbb{R}} |q(\theta)|\, d\theta d\theta = \infty, \quad \int_{\mathbb{R}} \theta^2 |q(\theta)|\, d\theta < \infty.$$

Then for any $\psi(\theta) \in C^2(\mathbb{R})$, the function

$$\theta \longrightarrow q(\theta)\left(\psi(\theta)+\psi(-\theta)-2\psi(0)\right)$$

belongs to L^1 locally. In particular, when $q(\theta)$ is moreover an even and compactly supported function, we use the notation

$$\int_{\mathbb{R}} q(\theta)\psi(\theta)d\theta \overset{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}} q(\theta)\left(\psi(\theta)+\psi(-\theta)-2\psi(0)\right)d\theta.$$

In our paper, we use it for the function $q(\theta) = |\theta|^{-1-2s}\mathbf{1}_{|\theta|\le\pi/2}$.

A basic formula

The first tool we use is the following Fubini-type formula, derived by rather explicit computation.

Consider a measurable function $0 \leq F(\alpha, h)$ of variables h and $\alpha \in \mathbb{R}^3$. For any $h \in \mathbb{R}^3$, we denote by $E_{0,h}$ the (hyper-)vector plane orthogonal to h. Then

$$\int_{\mathbb{R}^3_h} dh \int_{E_{0,h}} d\alpha F(\alpha, h) = \int_{\mathbb{R}^3_\alpha} d\alpha \int_{E_{0,\alpha}} dh \frac{|h|}{|\alpha|} F(\alpha, h). \tag{82}$$

Carleman representation

The second formula is the so-called ω-representation. It says that we have the following (almost everywhere) equalities when all sides are well-defined :

$$\begin{aligned}
&\iint dv_* d\sigma b(\cos\theta)|v - v_*|^\gamma F(v, v_*, v', v'_*) \\
&\quad = 4 \int_{\mathbb{R}^3_h} dh \int_{E_{0,h}} d\alpha \frac{1}{|\alpha + h|\ |h|} b(\cos\theta)|\alpha - h|^\gamma F(v, v + \alpha - h, v - h, v + \alpha) \\
&\quad \approx \int_{\mathbb{R}^3_h} dh \int_{E_{0,h}} d\alpha \mathbb{1}_{|\alpha| \geq |h|} \frac{1}{|\alpha + h|\ |h|} b\left(\frac{|\alpha|^2 - |h|^2}{|\alpha + h|^2}\right) |\alpha + h|^\gamma \\
&\qquad\qquad F(v, v + \alpha - h, v - h, v + \alpha).
\end{aligned}$$

These formulae are consequences of the following properties (see picture 2):

1. We make the change of variables $(v_*, \sigma) \longmapsto (\alpha, h)$ with $v' = v - h$, $v_* = v + \alpha - h$, $v'_* = v + \alpha$.

2. Since we restricted by symmetrization to the case $\sigma \cdot (v - v_*) \geq 0$ (which is equivalent to $\cos\theta \geq 0$), this implies $|\alpha| \geq |h|$. Note also that $h \perp \alpha$ and therefore $|\alpha + h|^2 = |\alpha - h|^2 = |\alpha|^2 + |h|^2$.

3. By immediate trigonometric properties we have $\cos\theta = \frac{|\alpha|^2 - |h|^2}{|\alpha + h|^2}$ and $\sin\theta = \frac{2|\alpha|\ |h|}{|\alpha + h|^2}$.

From the singular behavior of the singular kernel we deduce

$$0 \leq b(\cos\theta) \approx K\theta^{-2-2s} \approx \tilde{K}(\sin\theta)^{-2-2s} \approx \tilde{K} \frac{|\alpha + h|^{4+4s}}{|\alpha|^{2+2s}|h|^{2+2s}} \approx \frac{|\alpha + h|^{2+2s}}{|h|^{2+2s}},$$

since $|\alpha|^2 \leq |\alpha + h|^2 \leq 2|\alpha|^2$. At the end we get

$$\begin{aligned}
&\iint dv_* d\sigma b(\cos\theta)|v - v_*|^\gamma F(v, v_*, v', v'_*) \\
&\quad = \int_{\mathbb{R}^3_h} dh \int_{E_{0,h}} d\alpha \tilde{b}(\alpha, h) \mathbb{1}_{|\alpha| \geq |h|} \frac{|\alpha + h|^{\gamma+1+2s}}{|h|^{3+2s}} F(v, v + \alpha - h, v - h, v + \alpha).
\end{aligned} \tag{83}$$

where $\tilde{b}(\alpha, h)$ is bounded from below and above by positive constants, and $\tilde{b}(\alpha, h) = \tilde{b}(\pm\alpha, \pm h)$. Figure 2 shows the preceding relations between all vectors and angles.

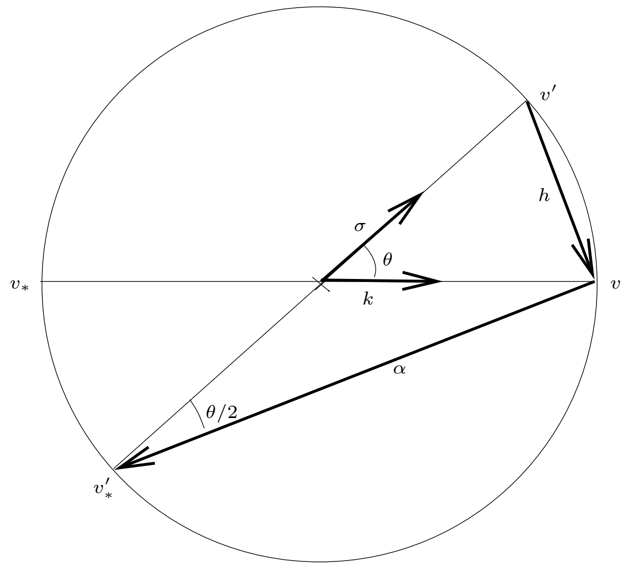

Figure 2: σ and Carleman representations

The cancellation lemma

We give here an other formula, in a slightly different version than the original one presented in [11]. We consider a function $G(|v-v_*|,|v-v'|)$. Then for smooth f, we have

$$\left(\iint dv_* d\sigma G(|v-v_*|,|v-v'|)b(\cos\theta)\left(f'_*-f_*\right)\right)=S*_{v_*}f(v),$$

where for all $z\in\mathbb{R}^3$, S has the following expression

$$S(z)=2\pi\int_0^{\pi/2}d\theta\sin\theta b(\cos\theta)\left(G\Big(\frac{|z|}{\cos\frac{\theta}{2}},\frac{|z|}{\cos\frac{\theta}{2}}\sin\frac{\theta}{2}\Big)\cos^{-3}\frac{\theta}{2}-G(|z|,|z|\sin\frac{\theta}{2})\right)$$

This applies in particular to functions of type

$$G(|v-v_*|,|v-v'|,\cos\theta)=b(\cos\theta)|v-v_*|^\gamma\varphi(v-v').$$

A.2 Weyl-Hörmander calculus

We recall here some notations and basic facts of symbolic calculus, and refer to [27, Chapter 18] for detailed discussions on the pseudodifferential calculus.complete

From now on, we set $\Gamma=|dv|^2+|d\eta|^2$, and let M be an admissible weight function w.r.t. Γ, that is the weight function M satisfies the following conditions:

(a) (slowly varying condition) there exists a constant δ such that

$$\forall X,Y\ \ |X-Y|\le\delta,\quad M(X)\approx M(Y);$$

(b) (temperance) there exist two constants C and N such that

$$\forall\ X,Y\in\mathbb{R}^6,\quad M(X)/M(Y)\le C\left\langle X-Y\right\rangle^N.$$

Considering symbols $q(\xi, v, \eta)$ as a function of (v, η) with parameters ξ, we say that $q \in S(M, \Gamma)$ uniformly with respect to ξ, if

$$\forall\ \alpha, \beta \in \mathbb{Z}_+^3, \quad \forall\ v, \eta \in \mathbb{R}^3, \quad \left|\partial_v^\alpha \partial_\eta^\beta q(\xi, v, \eta)\right| \leq C_{\alpha,\beta} M,$$

with $C_{\alpha,\beta}$ a constant depending only on α and β, but independent of ξ. For simplicity of notations, in the following discussion, we omit the parameters dependence in the symbols, and by $q \in S(M, \Gamma)$ we always mean that q satisfies the above inequality, uniformly with respect to ξ. The space $S(M, \Gamma)$ endowed with the semi-norms

$$\|q\|_{k;S(M,\Gamma)} = \max_{0\leq|\alpha|+|\beta|\leq k} \sup_{(v,\eta)\in\mathbb{R}^6} \left|M(v, \eta)^{-1} \partial_v^\alpha \partial_\eta^\beta q(v, \eta)\right|, \tag{84}$$

becomes a Fréchet space. Let $q \in \mathcal{S}'(\mathbb{R}_v^3 \times \mathbb{R}_\eta^3)$ be a tempered distribution and let $t \in \mathbb{R}$. We define the operator $\mathrm{op}_t q : \mathcal{S}(\mathbb{R}_v^3) \to \mathcal{S}^*(\mathbb{R}_v^3)$, with $\mathcal{S}^*(\mathbb{R}_v^3)$ the antidual of $\mathcal{S}(\mathbb{R}_v^3)$, by the formula

$$\langle (\mathrm{op}_t q) f, h\rangle_{\mathcal{S}^*,\ \mathcal{S}} = \langle q,\ \Omega_{f,h}\rangle_{\mathcal{S}',\ \mathcal{S}},$$

where

$$\Omega_{f,h}(t)(v, \eta) = \int e^{-2i\pi z\cdot\eta} f(v + (1 - t)z)\overline{h}(v - tz) dz.$$

In particular we denote $q(v, D_v) = \mathrm{op}_0 q$ and $q^w = \mathrm{op}_{1/2} q$. Here q^w is called the Weyl quantization of symbol q.

An elementary property to be used frequently is the L^2 continuity theorem in the class $S(1,\ g)$, see [29, Theorem 2.5.1] for instance, which says that there exists a constant C and a positive integer N depending only the dimension, such that

$$\forall\ u \in L^2, \quad \|q^w u\|_{L^2} \leq C\|q\|_{N;S(1,\Gamma)}\|u\|_{L^2}. \tag{85}$$

Let us also recall here the composition formula of Weyl quantization. Given $p_i \in S(M_i, \Gamma)$ we have

$$p_1^w p_2^w = (p_1 \sharp p_2)^w \tag{86}$$

with $p_1 \sharp p_2 \in S(M_1 M_2,\ \Gamma)$ admitting the expansion

$$p_1 \sharp p_2 = p_1 p_2 + \int_0^1 \iint e^{-2i\sigma(Y-Y_1,Y-Y_2)/\theta} \frac{1}{2i} \sigma(\partial_{Y_1}, \partial_{Y_2}) p_1(Y_1) p_2(Y_2) dY_1 dY_2 d\theta/(\pi\theta)^6, \tag{87}$$

where σ is the symplectic form in $\mathbb{R}^6$ given by

$$\sigma\left((z, \zeta), (\tilde{z}, \tilde{\zeta})\right) = \zeta \cdot \tilde{z} - \tilde{\zeta} \cdot z.$$

For the relation between the classical pseudodifferential operator $q(v, D_v)$ and Weyl quantization q^w, we have the formula (see Proposition 1.1.10 and Lemma 4.1.2 of [29]):

$$q^w = \left(J^{1/2} q\right)(v, D_v), \tag{88}$$

where $J^{1/2} : \mathcal{S}' \to \mathcal{S}'$ is defined by

$$(J^{1/2} q)(v.\eta) = 2^3 \iint e^{-4i\pi z\cdot\zeta} q(v + z, \eta + \zeta) dz d\zeta. \tag{89}$$

A.3 Wick quantization

Finally let us recall some basic properties of the Wick quantization, and refer the reader to the works of Lerner [31, 30, 29] for extensive presentations of this quantization and some of its applications. Let $Y = (v, \eta)$ be a point in $\mathbb{R}^6$. The wave-packets transform of a function $u \in \mathcal{S}\left(\mathbb{R}^3_v\right)$ is defined by

$$Wu(Y) = (u, \varphi_Y)_{L^2(\mathbb{R}^3_v)} = 2^{3/4} \int_{\mathbb{R}^3} u(z) e^{-\pi|z-v|^2} e^{2i\pi(z-v/2)\cdot\eta}\, dz,$$

with

$$\varphi_Y(z) = 2^{3/4} e^{-\pi|z-v|^2} e^{2i\pi(z-v/2)\cdot\eta}, \quad z \in \mathbb{R}^3.$$

Then W is an isometric mapping from $L^2(\mathbb{R}^3_v)$ to $L^2(\mathbb{R}^6)$ with adjoint W^*. We define the Wick quantization of any L^∞ symbol q as

$$p^{\mathrm{Wick}} = W^* p W.$$

The main property of the Wick quantization is its positivity, i.e.,

$$q(v, \eta) \geq 0 \quad \text{for all } (v, \eta) \in \mathbb{R}^6 \text{ implies } \quad q^{\mathrm{Wick}} \geq 0.$$

According to Proposition 2.4.3 in [29], the Wick and Weyl quantizations of a symbol q are linked by the following identities

$$q^{\mathrm{Wick}} = \left(q * 2^3 e^{-2\pi|\cdot|^2}\right)^w = q^w + r^w \tag{90}$$

with

$$r(Y) = \int_0^1 \int_{\mathbb{R}^6} (1-\theta) q''(Y + \theta Z) Z^2 e^{-2\pi|Z|^2} 2^3\, dZ d\theta$$

if we use here the normalization chosen in [19] for the Weyl quantization

$$(r^w u)(x) = \int e^{2i\pi(x-y).\xi} r\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi$$

which differs (without any influence) from the one chosen in the rest of this paper. As a result, q^{Wick} is a bounded operator in L^2 if $q \in S(1, g)$ due to (85).

We also recall the following composition formula obtained in the proof of Proposition 3.4 in [31]

$$q_1^{\mathrm{Wick}} q_2^{\mathrm{Wick}} = \left[q_1 q_2 - \frac{1}{4\pi} q_1' \cdot q_2' + \frac{1}{4i\pi}\{q_1,\ q_2\}\right]^{\mathrm{Wick}} + T, \tag{91}$$

with T a bounded operator in $L^2(\mathbb{R}^{2n})$, when $q_1 \in L^\infty(\mathbb{R}^{2n})$ and q_2 is a smooth symbol whose derivatives of order ≥ 2 are bounded on $\mathbb{R}^6$. The notation $\{q_1, q_2\}$ denotes the Poisson bracket defined by

$$\{q_1,\ q_2\} = \frac{\partial q_1}{\partial \eta} \cdot \frac{\partial q_2}{\partial v} - \frac{\partial q_1}{\partial v} \cdot \frac{\partial q_2}{\partial \eta}. \tag{92}$$

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