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UNIPOTENT GROUP ACTIONS ON DEL PEZZO CONES

TAKASHI KISHIMOTO, YURI PROKHOROV, AND MIKHAIL ZAIDENBERG

Abstract. In our previous paper [KPZ11b] we showed that for any del Pezzo surface $Y$ of degree $d \geq 4$ and for any $r \geq 1$, the affine cone $X = \text{cone}_{r(-K_Y)}(Y)$ admits an effective $\mathbb{G}_a$-action. In particular, the group $\text{Aut}(X)$ is infinite dimensional. In this note we prove that for a del Pezzo surface $Y$ of degree $\leq 2$ the generalized cones $X$ as above do not admit any non-trivial action of a unipotent algebraic group.

1. Introduction

We are working over an algebraically closed field $k$ of characteristic 0. Let $Y$ be a smooth projective variety with a polarization $H$, where $H$ is an ample Cartier divisor. A generalized affine cone over $(Y, H)$ is the normal affine variety

$$\text{cone}_H(Y) = \text{Spec} \bigoplus_{\nu \geq 0} H^0(Y, \nu H).$$

This variety $\text{cone}_H(Y)$ is the usual affine cone over $Y$ embedded in a projective space $\mathbb{P}^n$ by the linear system $[H]$ provided that $H$ is very ample and the image of $Y$ in $\mathbb{P}^n$ is projectively normal.

In this paper we deal with a del Pezzo surface $Y$ and a pluri-anticanonical divisor $H = -rK_Y$ on $Y$, where $r \geq 1$; we call then $\text{cone}_H(Y)$ a del Pezzo cone. This is a usual cone if $r \geq 4 - d$ (see e.g. [Dol12, Theorem 8.3.4]) and a generalized cone otherwise.

It is known [KPZ11b, 3.1.13] that for any smooth rational surface there is an ample polarization such that the associated affine cone admits an effective $\mathbb{G}_a$-action. Furthermore, for any del Pezzo surface of degree $\geq 4$ the corresponding del Pezzo cones $\text{cone}_{-rK_Y}(Y) \ (r \geq 1)$ admit such an action (loc.cit.). The latter holds also for some smooth rational Fano threefolds with Picard number 1 [KPZ11b, KPZ11a]. However, for del Pezzo surfaces of small degrees the consideration turns out to be more complicated. It is unknown so far whether the affine cone over a smooth cubic surface in $\mathbb{P}^3$ admits a $\mathbb{G}_a$-action (cf. [KPZ11b, §4]). In this paper we investigate the cases $d = 1$ and $d = 2$. Our main result can be stated as follows.

Theorem 1.1. Let $Y$ be a del Pezzo surface of degree $d = K_Y^2 \leq 2$. Then for any $r \geq 1$ there is no non-trivial action of a unipotent group on the generalized affine cone

$$X_r = \text{cone}_{-rK_Y}(Y) = \text{Spec} \ A, \quad \text{where} \quad A = \bigoplus_{\nu \geq 0} H^0(Y, -\nu rK_Y).$$

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Corollary 1.2. In the notation as before assume that \( d \leq 2 \) and \( r \geq 4 - d \) so that \( X_r = \text{cone}_{-rK_Y}(Y) \) is a usual del Pezzo cone. Then any algebraic subgroup \( G \subseteq \text{Aut}(X_r) \) is isomorphic to a subgroup of \( \mathbb{G}_m \times \text{Aut}(Y) \), where \( \text{Aut}(Y) \) is finite.

Proof. As follows from Theorem 1.1 \( G \) is a reductive group. Thus by Lemma 2.3.1 and Proposition 2.2.6 in [KPZ11b] there are an injection and an isomorphism

\[
G \hookrightarrow \text{Lin}(X_r) \simeq \mathbb{G}_m \times \text{Lin}(Y) \subseteq \mathbb{G}_m \times \text{Aut}(Y),
\]

where the group \( \text{Aut}(Y) \) is finite, see [Dol12]. \( \square \)

We suggest the following

1.3. Conjecture. If \( d \leq 2 \) then for any \( r \geq d - 4 \) the full automorphism group \( \text{Aut}(X_r) \) is a finite extension of the multiplicative group \( \mathbb{G}_m \).

Likewise in [KPZ11a, KPZ11b] we use a geometric criterion of existence of an effective \( \mathbb{G}_a \)-action on the affine cone \( \text{cone}_H(Y) \) (see [KPZ12] and Theorem 2.1 below).

Sections 2, 3, and 4 contain necessary preliminaries. Theorem 1.1 is proven in section 5. The proof proceeds as follows. Assumining to the contrary that there exists a non-trivial unipotent group action on \( X_r = \text{cone}_{(-rK_Y)}(Y) \), there also exists an effective \( \mathbb{G}_a \)-action on \( X_r \). By Theorem 2.1 there is an effective \( \mathbb{Q} \)-divisor \( D \) on \( Y \) such that \( D \sim_q -K_Y \) and \( U = Y \setminus D \cong Z \times \mathbb{A}^1 \), where \( Z \) is a smooth rational affine curve. Such a principal open subset \( U \) is called in [KPZ11b] a \((-K_Y)\)-polar cylinder. One of the key points consists in an estimate for the singularities of the pair \((Y, D)\). More precisely, we consider the linear pencil \( \mathcal{L} \) on \( Y \) generated by the closures of the fibers of the projection \( U \cong Z \times \mathbb{A}^1 \rightarrow Z \). Letting \( S \) be the last exceptional divisor appearing in the process of the minimal resolution of the base locus of \( \mathcal{L} \) we compute the discrepancy \( a(S; D) \). Using this and some subtle geometrical properties of the pair \((Y, D)\) we finally come to a contradiction.

2. Criterion

Let \( Y \) be a projective variety and \( H \) be an ample divisor on \( Y \). Recall [KPZ11b] that an \( H \)-polar cylinder in \( Y \) is an open subset \( U = Y \setminus \text{supp}(D) \) isomorphic to \( Z \times \mathbb{A}^1 \) for some affine variety \( Z \), where \( D = \sum_i \delta_i \Delta_i \) with \( \delta_i > 0 \ \forall i \) is an effective \( \mathbb{Q} \)-divisor on \( Y \) such that \( qD \) is integral and \( qD \sim H \) for some \( q \in \mathbb{N} \). Corollary 2.12 in [KPZ12] provides the following useful criterion of existence of an effective \( \mathbb{G}_a \)-action on the affine cone.

Theorem 2.1. Let \( Y \) be a normal projective algebraic variety with an ample polarization \( H \in \text{Div}(Y) \), and let \( X = \text{cone}_H(Y) \) be the corresponding generalized affine cone. If \( X \) is normal then \( X \) admits an effective \( \mathbb{G}_a \)-action if and only if \( Y \) contains an \( H \)-polar cylinder.

We apply this criterion to a del Pezzo surface \( Y \) of degree \( d \leq 2 \) and a generalized cone

\[
X_r = \text{Spec} \bigoplus_{\nu \geq 0} H^0(Y, -\nu rK_Y)
\]

associated with \( H = -rK_Y \), where \( r \geq 1 \). It follows, in particular, that if the cone \( X_r \) admits an effective \( \mathbb{G}_a \)-action then \( Y \) contains a cylinder \( Y \setminus \text{supp} D \) with \( qD \sim -rK_Y \).

\(^1\text{Cf. also [KPZ11b, 3.1.9].}\)
Hence $\frac{2}{r}D \sim_Q -K_Y$. Replacing $D$ by $\frac{2}{r}D$ we assume in the sequel that $D \sim_Q -K_Y$. This assumption leads finally to a contradiction, which proves Theorem 1.1.

3. Preliminaries on weak del Pezzo surfaces

A smooth projective surface $Y$ is called a del Pezzo surface if the anticanonical divisor $-K_Y$ is ample, and a weak del Pezzo surface if $-K_Y$ is big and nef. The degree of such a surface is $\deg Y = K_Y^2 \in \{1, \ldots, 9\}$.

Lemma 3.1 (see e.g. [Dol12, Proposition 8.1.23]). Blowing up a point on a del Pezzo surface of degree $d \geq 2$ yields a weak del Pezzo surface of degree $d - 1$.

Theorem 3.2 (see e.g. [Dol12, Thm. 8.3.2]). Let $Y$ be a del Pezzo surface of degree $d$. Then the following hold.

(i) If $d \geq 3$ then $| -K_Y|$ defines an embedding $Y \hookrightarrow \mathbb{P}^d$.

(ii) If $d = 2$ then $| -K_Y|$ defines a double cover $\Phi : Y \rightarrow \mathbb{P}^2$ branched along a smooth curve $B \subset \mathbb{P}^2$ of degree 4.

(iii) If $d = 1$ then $| -K_Y|$ is a pencil with a single base point, say $O$. The linear system $| -2K_Y|$ defines a double cover $\Phi : Y \rightarrow Q' \subset \mathbb{P}^3$, where $Q'$ is a quadric cone with vertex at $\Phi(O)$. Furthermore $\Phi$ is branched along a smooth curve $B \subset Q'$ cut out on $Q'$ by a cubic surface.

The Galois involution $\tau : Y \rightarrow Y$ associated to the double cover $\Phi$ is a regular morphism. It is called Geiser involution in the case $d = 2$ and Bertini involution in the case $d = 1$.

Remark 3.3. Recall the following facts (see e.g. [Dol12]). For an irreducible curve $C$ on $Y$ we have $C^2 \geq -1$ if $Y$ is a del Pezzo surface and $C^2 \geq -2$ if $Y$ is a weak del Pezzo surface. In both cases $C^2 = -1$ if and only if $C$ is a $(1, -1)$-curve, and $C^2 = -2$ if and only if $C$ is a $(2, -2)$-curve, and if and only if $-K_Y \cdot C = 0$.

A weak del Pezzo surface is del Pezzo if and only if it has no $(2, -2)$-curve.

If $d \geq 2$ then any curve $C$ on $Y$ such that $-K_Y \cdot C = 1$ is an irreducible smooth rational curve by (i) and (ii). By the adjunction formula such $C$ must be a $(1, -1)$-curve.

Lemma 3.4. Let $Y$ be a del Pezzo surface of degree $d \leq 2$. Then any member $R \in | -K_Y|$ is reduced and $p_a(R) = 1$. Moreover, $R$ is reducible except in the case where

- $d = 2$, $R = R_1 + R_2$, $R_i^2 = -1$, $i = 1, 2$, $R_1 \cdot R_2 = 2$, and $R_2 = \tau(R_1)$.

Furthermore, $\text{Sing}(R) \subset \Phi^{-1}(B)$ and for any $P \in \Phi^{-1}(B)$ there is a unique member $R \in | -K_Y|$ singular at $P$.

Proof. We have $p_a(R) = 1$ by adjunction. Let $R_1 \subsetneq R$ be a reduced irreducible component. Then $(-K_Y) \cdot R_1 < (-K_Y) \cdot R = d$ and so $d = 2$ and $R_1$ is a $(1, -1)$-curve by Remark 3.3. Since $R_i^2 = d = 2$, $R \neq 2R_i$. Therefore $R = R_1 + R_2$, where the $R_i$ ($i = 1, 2$) are $(1, -1)$-curves and $R_1 \cdot R_2 = \frac{1}{2}(R^2 - R_1^2 - R_2^2) = 2$. Finally, in both cases we have $R = \Phi^{-1}(L)$, where $L$ is a line in $\mathbb{P}^2$. Thus $R$ is singular at $P$ if and only if $\Phi(P) \subset B$ and $L$ is tangent to $B$ at $\Phi(P)$. \[\square\]

Remark 3.5. Let $R_1$ and $R_2$ be $(1, -1)$-curves on a del Pezzo surface $Y$ of degree 2 such that $R_1 \cdot R_2 \geq 2$. Then $R_2 = \tau(R_1)$, $R_1 \cdot R_2 = 2$, and $R_1 + R_2 \in | -K_Y|$. Indeed, $R_1 + \tau(R_1) \sim -K_Y$. Hence $\tau(R_1) \cdot R_2 = -1$ and so $\tau(R_1) = R_2$. 


4. \((-K\))-polar cylinders on del Pezzo surfaces

We adjust here some lemmas in [KPZ11b, §4] to our setting.

**Notation 4.1.** Let \( Y \) be a del Pezzo surface of degree \( d \). Suppose that \( Y \) admits a \((-K_Y\))-polar cylinder

\[
U = Y \setminus \text{supp}(D) \cong Z \times \mathbb{A}^1,
\]

where

\[
D = \sum_{i=1}^{n} \delta_i \Delta_i \sim -K_Y \quad (\delta_i > 0)
\]

and \( Z \) is a smooth rational affine curve. We let \( \mathcal{L} \) be the linear pencil on \( Y \) defined by the rational map \( \Psi : Y \to \mathbb{P}^1 \) which extends the projection \( \text{pr}_1 : U \cong Z \times \mathbb{A}^1 \to Z \).

Resolving, if necessary, the base locus of the pencil \( \mathcal{L} \) we obtain a diagram

\[
\begin{array}{ccc}
W & \xrightarrow{p} & Y \\
\downarrow p & \nearrow \Psi & \downarrow q \\
\mathbb{P}^1 & \rightarrow & \mathbb{P}^1
\end{array}
\]

where we let \( p : W \to Y \) be the shortest succession of blowups such that the proper transform \( \mathcal{L}_W := p_*^{-1} \mathcal{L} \) is base point free. Let \( S \) be the last exceptional curve of the modification \( p \) unless \( p \) is the identity map, i.e., \( \text{Bs} \mathcal{L} = \emptyset \). Notice that \( S \) is a unique \((-1)\)-curve in the exceptional locus \( p^{-1}(P) \) and a section of \( q \). The restriction \( \Phi_{\mathcal{L}_W|U} \) is an \( \mathbb{A}^1 \)-fibration and its fibers are reduced, irreducible affine curves with one place at infinity, situated on \( S \).

**Lemma 4.4.** One of the following holds.

(i) \( \text{Bs} \mathcal{L} \) consists of a single point, say \( P \);

(ii) \( \text{Bs} \mathcal{L} = \emptyset \) and \( 5 \leq d \leq 8 \).

**Proof.** Since the general members of \( \mathcal{L} \) are disjoint in \( U \) and each one meets the cylinder \( U \) along an \( \mathbb{A}^1 \)-curve, \( \text{Bs} \mathcal{L} \) consists of at most one point, which we denote by \( P \). Suppose that \( \text{Bs} \mathcal{L} = \emptyset \). Then the pencil \( \mathcal{L} \) yields a conic bundle \( \Psi : Y \to \mathbb{P}^1 \) with a section, which is a component of \( D \), say \( \Delta_0 \). In particular \( d \leq 8 \). For a general fiber \( L \) of \( \Psi \) we have

\[
L^2 = 0, \quad -K_Y \cdot L = 2 = D \cdot L = \delta_0.
\]

Note that \( \Psi \) has exactly \( 8 - d \) degenerate fibers \( L_1, \ldots, L_{8-d} \). Each of these fibers is reduced and consists of two \((-1)\)-curves meeting transversally at a point. Let \( C_i \) be the component of \( L_i \) that meets \( \Delta_0 \). We claim that each \( C_i \) is a component of \( D \). Indeed, otherwise

\[
1 = -K_Y \cdot C_i = D \cdot C_i \geq \delta_0 \Delta_0 \cdot C_i = \delta_0 = 2,
\]

a contradiction. Therefore we may assume that \( C_i = \Delta_i \) and so

\[
1 = D \cdot C_i \geq \delta_0 \Delta_0 \cdot C_i + \delta_i C_i^2 = 2 - \delta_i.
\]

Hence \( \delta_i \geq 1, \ i = 1, \ldots, 8 - d \). We obtain

\[
d = -K_Y \cdot D \geq \sum \delta_i \geq \delta_0 + \sum_{i=1}^{8-d} \delta_i \geq 2 + 8 - d = 10 - d.
\]

Thus \( d \geq 5 \) as stated. \( \square \)
Remark 4.5. If $B_L = \{P\}$ (or $B_L = \emptyset$, respectively) then all components of $D$ (all components of $D$ except for $\Delta_0$, respectively) are contained in the fibers of $\Psi$. Indeed, otherwise not all the fibers of $\Psi|U$ were $\mathbb{A}^1$-curves, contrary to the definition of a cylinder.

Lemma 4.6. The number of irreducible components of the reduced curve $\text{supp}(D)$, say $n$, is greater than or equal to $10 - d$.

Proof. Consider the exact sequence
\[ \bigoplus_{i=1}^{n} \mathbb{Z}[\Delta_i] \to \text{Pic}(Y) \to \text{Pic}(U) \to 0. \]
Since $\text{Pic}(Z) = 0$ and $U \cong Z \times \mathbb{A}^1$ we have $\text{Pic}(U) = 0$. Hence $n \geq \rho(Y) = 10 - d$, as stated.

Lemma 4.7. Assume that $B_L = \{P\}$. Let $L$ be a member of $\mathcal{L}$ and $C$ be an irreducible component of $L$. Then the following hold.

(i) $\text{supp}(L)$ is simply connected and $\text{supp}(L) \setminus \{P\}$ is an SNC divisor;
(ii) $C$ is rational and smooth outside $P$;
(iii) if $P \in C$ then $C \setminus \{P\} \cong \mathbb{A}^1$.

Proof. All the assertions follow from the fact that $q$ in (4.3) is a rational curve fibration and the exceptional locus of $p$ coincides with $p^{-1}(P)$.

In the next lemma we study the singularities of the pair $(Y, D)$. We refer to [Kol97] or to [KM98, Chapter 2] for the standard terminology on singularities of pairs.

Lemma 4.8 (Key Lemma). Assume that $B_L = \{P\}$. Then the pair $(Y, D)$ is not log canonical at $P$. More precisely, in notation as in 4.1 the discrepancy $a(S; D)$ of $S$ with respect to $K_Y + D$ is equal to $-2$.

Proof. We write
\[ K_W + D_W \sim_q p^*(K_Y + D) + a(S; D)S + \sum a(E; D)E, \]
where the summation on the right hand side ranges over the components of the exceptional divisor of $p$ except for $S$, and $D_W$ is the proper transform of $D$ on $W$. Letting $l$ be a general fiber of $q$, by (4.9) we obtain
\[ -2 = (K_W + D_W) \cdot l = a(S; D). \]
Indeed, $K_Y + D \sim_q 0$ and $l$ does not meet the curve $\text{supp}(D_W + p^*(P) - S)$. This proves the assertion.

Corollary 4.10. If $B_L = \{P\}$ then $\text{mult}_P(D) > 1$.

Proof. Indeed, otherwise the pair $(Y, D)$ would be canonical by [Kol97, Ex. 3.14.1], and in particular, log canonical at $P$, which contradicts Lemma 4.8.

Corollary 4.11. If $B_L = \{P\}$ then every $(-1)$-curve $C$ on $Y$ passing through $P$ is contained in $\text{supp}(D)$.

Proof. Assume to the contrary that $C$ is not a component of $D$. Then
\[ \text{mult}_P D \leq C \cdot D = -K_Y \cdot C = 1, \]
which contradicts Corollary 4.10.
**Convention 4.12.** From now on we assume that \( d \leq 3 \). By Lemma 4.4 we have \( \text{Bs} \mathcal{L} = \{ P \} \).

**Lemma 4.13.** We have \([D] = 0\) i.e. \( \delta_i < 1 \) for all \( i = 1, \ldots, n \).

*Proof.* For the case \( d = 3 \) see [KPZ11b, Lemma 4.1.5]. Consider the case \( d = 1 \). By Lemma 4.6 \( n \geq 9 \). For any \( i = 1, \ldots, n \) we have

\[
1 = -K_Y \cdot D = \sum_{j=1}^{n} \delta_j (-K_Y) \cdot \Delta_j > \delta_1 (-K_Y) \cdot \Delta_1.
\]

Since the anticanonical divisor \(-K_Y\) is ample, it follows that \( \delta_1 < 1 \), as required.

Let further \( d = 2 \). Assuming that \( \delta_1 \geq 1 \) we obtain:

\[
(4.14) \quad 2 = -K_Y \cdot D = \sum_{i=1}^{n} \delta_i (-K_Y) \cdot \Delta_i \geq \delta_1 (-K_Y) \cdot \Delta_1,
\]

where \( n \geq 8 \) by Lemma 4.6. It follows that \(-K_Y \cdot \Delta_1 = 1\), i.e. \( \Delta_1 \) is a \((-1)\)-curve. Then \( C := \tau(\Delta_1) \) is also a \((-1)\)-curve, where \( \tau \) is the Geiser involution, and \( \Delta_1 + C \sim -K_Y \).

If \( C \subset \text{supp}(D) \), e.g. \( C = \Delta_2 \), then by (4.14) we obtain that \( \delta_2 < 1 \). Now \( \Delta_1 + \Delta_2 \sim_D 0 \) yields a relation with positive coefficients

\[
(1 - \delta_2) \Delta_2 \sim_D (\delta_1 - 1) \Delta_1 + \sum_{i=3}^{n} \delta_i \Delta_i.
\]

This implies that \( C^2 = \Delta_2^2 \geq 0 \), a contradiction.

Hence \( C \not\subset \text{supp}(D) \). Thus \( C \sim_D D - \Delta_1 \), where the right hand side is effective. This leads to a contradiction as before. \( \square \)

**Lemma 4.15.** \(^2\) For a member \( L \) of \( \mathcal{L} \), any irreducible component of \( L \) passes through the base point \( P \) of \( \mathcal{L} \).

*Proof.* Assume to the contrary that there exists a component \( C \) of \( L \) such that \( P \not\in C \). Then clearly \( C^2 < 0 \) (see the proof of Lemma 4.4). Since also \(-K_Y \cdot C > 0\), \( C \) is a \((-1)\)-curve. Let \( C' \) be a component of \( L \) meeting \( C \). If \( P \not\in C' \), then \( C \) and \( C' \) are both \((-1)\)-curves and so \( L = C + C' \). Thus \( \mathcal{L} = [C + C'] \) is base point free, which contradicts Lemma 4.4. Hence \( C' \) passes through \( P \). Since \( P \) is a unique base point of \( \mathcal{L} \), \( C \) does not meet any member \( L' \in \mathcal{L} \) different from \( L \). By Lemma 4.7 \( L \) is simply connected, so \( C' \) is the only component of \( L \) meeting \( C \). Note that \( \text{supp}(D) \) is connected because \( D \) is ample. Hence \( C' \) must be contained in \( \text{supp}(D) \). In fact, supposing to the contrary that \( C' \) is not contained in \( \text{supp}(D) \), the curve \( C \) must be contained in \( \text{supp}(D) \). Indeed, the affine surface \( U = Y \setminus \text{supp}(D) \) does not contain any complete curve. Since \( \text{supp}(D) \) is connected there is an irreducible component of \( \text{supp}(D) \) intersecting \( C \) and passing through \( P \). This contradicts Lemma 4.7. Thus we may suppose that \( C' = \Delta_1 \).

If \( C \subset \text{supp}(D) \), say, \( C = \Delta_2 \), then

\[
1 = -K_Y \cdot C = \left( \sum_{i=1}^{n} \delta_i \Delta_i \right) \cdot \Delta_2 = \delta_1 - \delta_2.
\]

Hence \( \delta_1 = \delta_2 + 1 > 1 \), which contradicts Lemma 4.13.

\(^2\)Cf. [KPZ11b, Lemma 4.1.6].
Therefore $C \not\subset \text{supp}(D)$ and so
\[ 1 = -K_Y \cdot C = \left( \sum_{i=1}^{n} \delta_i \Delta_i \right) \cdot C = \delta_1, \]
which again gives a contradiction by Lemma 4.13. \(\square\)

5. PROOF OF THEOREM 1.1

According to our geometric criterion 2.1, Theorem 1.1 is a consequence of the following proposition.

**Proposition 5.1.** Let $Y$ be a del Pezzo surface of degree $d \leq 2$. Then $Y$ does not admit any $(-K_Y)$-polar cylinder.

**Convention 5.2.** We let $Y$ be a del Pezzo surface of degree $d \leq 2$. We assume to the contrary that $Y$ possesses a $(-K_Y)$-polar cylinder $U$ as in (4.2). By Lemma 4.4 we have $B_s \mathcal{L} = \{ P \}$.

**Lemma 5.3.** For any $R \in |-K_Y|$ we have $\text{supp}(R) \not\subset \text{supp}(D)$.  

**Proof.** Suppose to the contrary that $\text{supp}(R) \subset \text{supp}(D)$. Let $\lambda \in \mathbb{Q} > 0$ be maximal such that $D - \lambda R$ is effective. We can write $D = \lambda R + D_{\text{res}}$, where $D_{\text{res}}$ is an effective $\mathbb{Q}$-divisor such that $\text{supp}(R) \not\subset \text{supp}(D_{\text{res}})$. For $t \in \mathbb{Q}_{\geq 0}$ we consider the following linear combination
\[ D_t := D - tR + \frac{t}{1 - \lambda} D_{\text{res}} \sim_{\mathbb{Q}} -K_Y. \]

We have $D_0 = D$ and $D_\lambda = \frac{1}{1 - \lambda} D_{\text{res}}$. For $t < \lambda$, the $\mathbb{Q}$-divisor $D_t$ is effective with $\text{supp}(D_t) = \text{supp}(D)$. By Lemma 4.8 applied to $D_t$ instead of $D$, for any $t < \lambda$ the pair $(Y, D_t)$ is not log canonical at $P$, with discrepancy $a(S; D_t) = -2$. Since the function $t \mapsto a(S; D_t)$ is continuous, passing to the limit we obtain $a(S; D_\lambda) = -2$. Hence the pair $(Y, D_\lambda)$ is not log canonical at $P$ either and so $\text{mult}_P(D_\lambda) > 1$.

Assume that $R$ is irreducible. Since $R \subset \text{supp}(D)$, $R$ is a component of a member of $\mathcal{L}$. Hence the curve $R$ is smooth outside $P$ and rational (see Lemma 4.7(ii)). Since $p_a(R) = 1$, $R$ is singular at $P$ and $\text{mult}_P(R) = 2$. Since $R$ is different from the components of $D_\lambda$ and $\text{mult}_P(D_\lambda) > 1$ we obtain
\[ 2 \geq K_Y^2 = D_\lambda \cdot R \geq \text{mult}_P(D_\lambda) \text{mult}_P(R) > 2, \]
a contradiction.

Let further $R$ be reducible. By Lemma 3.4 we have $d = 2$ and $R = R_1 + R_2$, where, say, $R_i = \Delta_i$, $i = 1, 2$, are $(-1)$-curves passing through $P$ (see Lemma 4.15). We may assume that $\delta_1 \leq \delta_2$ and so $\lambda = \delta_1$. Since $\Delta_1$ is not a component of $D_\lambda$ we obtain
\[ 1 = -K_Y \cdot R_1 = D_\lambda \cdot \Delta_1 \geq \text{mult}_P(D_\lambda) > 1, \]
a contradiction. This finishes the proof. \(\square\)
Proof of Proposition 5.1 in the case $d = 1$. Since $\dim | - K_Y | = 1$ there is $C \in | - K_Y |$ passing through $P$. Furthermore, by Lemma 3.4 $C$ is irreducible. By Lemma 5.3 $C$ is not contained in $\text{supp}(D)$. Likewise in (5.4) we get a contradiction. Indeed, by Corollary 4.10 we have

$$1 = C^2 = D \cdot C \geq \text{mult}_P D \cdot \text{mult}_P C > 1.$$  

holds.

Convention 5.5. We assume in the remaining part that $d = 2$.

Lemma 5.6. A member $R \in | - K_Y |$ cannot be singular at $P$.

Proof. Assume that $P \in \text{Sing}(R)$. By Lemma 3.4 we have two possibilities for $R$. Suppose first that $R$ is irreducible. By Lemma 5.3 $R \not\subset \text{supp}(D)$ and we get a contradiction likewise in (5.4). In the second case $R = R_1 + R_2$, where $R_1$ and $R_2$ are $(-1)$-curves passing through $P$. Hence $R_1$, $R_2 \subset \text{supp}(D)$ by Corollary 4.11. The latter contradicts Lemma 5.3.

Notation 5.7. We let $f : Y' \to Y$ be the blowup of $P$ and $E' \subset Y'$ be the exceptional divisor. By Lemma 3.1 $Y'$ is a weak del Pezzo surface of degree 1.

5.8. Applying Proposition 5.1 with $d = 1$, we can conclude that $Y'$ is not del Pezzo because it contains a $-K_Y$-polar cylinder. Indeed, let $D'$ be the crepant pull-back of $D$ on $Y'$, that is,

$$K_{Y'} + D' = f^*(K_Y + D) \quad \text{and} \quad f_*D' = D.$$  

Then

$$D' = \sum_{i=1}^{6} \delta_i \Delta'_i + \delta_0 E', \quad \text{where} \quad \delta_0 = \text{mult}_P(D) - 1 > 0$$  

(see Lemma 4.10) and $\Delta'_i$ is the proper transform of $\Delta_i$ on $Y'$. Thus $D'$ is an effective $\mathbb{Q}$-divisor on $Y'$ such that $D' \sim_{\mathbb{Q}} -K_{Y'}$ and $Y' \setminus \text{supp} D' \simeq U \simeq Z \times \mathbb{A}^1$ is a $-K_Y$-polar cylinder.

Lemma 5.10. We have $\text{mult}_P(D) < 2$ and $|D'| = 0$.

Proof. Suppose first that all components of $D$ are $(-1)$-curves. Then $\Delta_i \cdot \Delta_j = 1$ for $i \neq j$ by Remark 3.5 and Lemma 5.3. Hence $f$ is a log resolution of the pair $(Y, D)$. Therefore $1 - \sum \delta_i = a(Y, E') < -1$ by Lemma 4.8, so $\sum \delta_i > 2$. On the other hand $2 = -K_Y \cdot D = \sum \delta_i$, a contradiction. This shows that there exists a component $\Delta_i$ of $D$ which is not a $(-1)$-curve. By the dimension count there exists an effective divisor $R \in | - K_Y |$ passing through $P$ and a general point $Q \in \Delta_i$. On the other hand, there is no $(-1)$-curve in $Y$ passing through $Q$. So by Lemma 3.4 we may assume that $R$ is reduced and irreducible. By Lemma 5.3 $R$ is different from the components of $D$. Assuming that $\text{mult}_P(D) \geq 2$ we obtain

$$2 = R \cdot D \geq \text{mult}_P(D) + \delta_i > 2,$$  

a contradiction. This proves the first assertion. Now the second follows since $\delta_0 > 0$ in (5.9).

Corollary 5.11. The pair $(Y', D')$ is Kawamata log terminal in codimension one and is not log canonical at some point $P' \in E'$. 

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Proof. This follows from Lemma 5.10 taking into account that $D'$ is the crepant pullback of $D$, see [Kol97, L. 3.10].

Since dim $| - K_Y| = 1$ there exists an element $C' \in | - K_Y|$ passing through the point $P'$ as in Corollary 5.11.

**Lemma 5.12.** The point $P \in Y$ is a smooth point of the image $C = f_! C'$.

**Proof.** This follows by Lemma 5.6 since $C \in | - K_Y|$ passes through $P$. □

**Corollary 5.13.** $E'$ is not a component of $C'$.

**Proof.** We can write $f^* C = C' + kE'$ for some $k \in \mathbb{Z}$. Then $k = -kE'^2 = C' \cdot E' = 1$. By Lemma 5.12 the coefficient of $E'$ in $f^* C$ is equal to 1 as well. Now the assertion follows. □

**Lemma 5.14.** $C$ is reducible.

**Proof.** Indeed, otherwise $C'$ is irreducible by Corollary 5.13. Since mult$_P D' > 1$ by Corollary 5.11 and $D' \cdot C' = K_Y^2$, $1 = C'$ is a component of $D'$. Hence $C$ is a component of $D$. This contradicts Lemma 5.3. □

**Lemma 5.15.** We have $C' = C'_1 + C'_2$, where $C_1$ is a $(-1)$-curve, $C'_2$ is a $(-2)$-curve, and $C'_1 \cdot C'_2 = 2$. Furthermore, $P' \in C'_2 \setminus C'_1$, and $C_2 = f(C'_2)$ is a $(-1)$-curve.

**Proof.** Since $C$ is reducible and $C \in | - K_Y|$, by Lemma 3.4 $C = C_1 + C_2$, where $C_1$, $C_2$ are $(-1)$-curves with $C_1 \cdot C_2 = 2$. By Lemma 5.12 $P \notin C_1 \cap C_2$, where $C_2$ is a component of $D$ by Corollary 4.11, while by Lemma 5.3 $C_1$ is not. So we may assume that $P \in C_2 \setminus C_1$. Now the lemma follows from Corollary 5.11. □

5.16. Letting in the sequel $C_2 = \Delta_1$ we can write $D = \delta_1 C_2 + D_{res}$, where $\delta_1 > 0$, $D_{res}$ is an effective $\mathbb{Q}$-divisor, and $C_2$ is not a component of $D_{res}$. Similarly $D' = \delta_1 C_2' + D_{res}' + \delta_0 E'$, where $D_{res}'$ is the proper transform of $D_{res}$ and $\delta_0 = \text{mult}_P(D) - 1$ (cf. (5.9)).

**Lemma 5.17.** We have $2\delta_1 \leq 1$.

**Proof.** This follows from $0 \leq D_{res} \cdot C_1 = (D - \delta_1 C_2) \cdot C_1 = 1 - 2\delta_1$. □

**Lemma 5.18.** In the notation as before $\delta_0 + D_{res}' \cdot C'_2 > 1$.

**Proof.** Let us show first that $\{P'\} = C'_2 \cap E' = C'_2 \cap \text{supp}(D_{res}')$. Indeed, $P' \in E'$ by construction, $P' \in C'_2$ by Lemma 5.15, and $P' \in \text{supp}(D_{res}')$ because otherwise $P'$ would be a node of $D'$ (indeed, $E'$ meets $C'_2$ transversally at $P'$) and so the pair $(Y', D')$ would be log canonical at $P'$ contrary to Corollary 5.11. On the other hand, the curves $C'_2$ and $D_{res}'$ have only one point in common by Lemma 4.7(i).

Since $\delta_1 < 1$ the pair $(Y', C'_2 + D_{res}' + \delta_0 E')$ is not log canonical at $P'$. Now applying [KM98, Corollary 5.57] we obtain $1 < (D_{res}' + \delta_0 E') \cdot C'_2 = \delta_0 + D_{res}' \cdot C'_2$, as stated. □
Proof of Proposition 5.1 in the case $d = 2$. We use the notation as above. Since $C'_2$ is a $(-2)$-curve, by virtue of Lemmas 5.17 and 5.18 we obtain

$$1 - \delta_0 < D'_{\text{res}} \cdot C'_2 = (D' - \delta_1 C'_2 - \delta_0 E') \cdot C'_2 = 2\delta_1 - \delta_0 \leq 1 - \delta_0,$$

a contradiction. Now the proof of Proposition 5.1 is completed. □

Remark 5.19. Our proof of Proposition 5.1 goes along the lines of that of Lemmas 3.1 and 3.5 in [Chel08]. However, this proposition does not follow immediately from the results in [Chel08]. Indeed, in notation of [Chel08] by Lemma 4.8 we have $\text{lct}(Y, D) < 1$. This is not sufficient to get a contradiction with [Chel08, Theorem 1.7]. The point is that our boundary $D$ is not arbitrary, in contrary, it is rather special (see Lemma 4.7).

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