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Multi-country real business cycle models: Accuracy tests and test bench

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Abstract

This paper describes the methodology used to compare the results of different solution algorithms for a multi-country real business cycle model. It covers in detail the structure of the model, the choice of values for the parameters, the accuracy tests used in the comparison, and the computer program specifically developed for performing the tests.

Introduction

In order to compare complex algorithms such as the ones necessary to compute the solution of nonlinear dynamic stochastic general equilibrium models, one needs to define clearly the dimensions along which the results will be compared. Furthermore, it is also important to compare the results of the different algorithms using the same implementation of the tests so that one can exclude differences in the administration of the testing procedure. For this reason, separately from the implementation of each algorithm by the participants to the comparison project, we have developed a computer program that takes as input the solution provided by each group and subjects it to an identical set of tests.

Following previous literature (*i.e.* Aruoba et al., 2006; Den Haan and Marcet, 1994; Heer and Maussner, 2008; Judd, 1992), we focus on evaluating errors of approximation: the errors that one obtains by plugging back the

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approximated solution in the original problem. We implement this approach in two different setups: the first test computes errors of approximation at different points in the state space located on a sphere around the deterministic steady state of the problem. This test informs us on the ability of a given solution method to provide accurate approximations of the solution further and further away from a point "easy" to solve for: the deterministic steady state. Of course, ordering points by their distance to the steady state does not indicate how likely these points are in simulated data. This remark motivates the second test: measuring the errors of approximation along a given simulation path. More precisely, each solution is simulated starting from the same initial point, the deterministic steady state, and subjected to the same set of random shocks. In the literature, the test proposed by Den Haan and Marcet (1994) is often used to test whether prediction errors implied by the numerical solutions are orthogonal to the elements of information set. Finally, we didn't retain it in the testing procedure used for this exercise, for reasons that we present below.

The multi-country RBC models used in this comparison exercise are introduced in Den Haan et al. (2010). The four basic specifications try to introduce both a simple model such as the first one that considers fixed labor and a simple Cobb-Douglas function, and more complicated setups with endogenous labor, CES production functions and asymmetries between countries. The number of countries varies from two to ten. Altogether the comparison involves 30 different models.

All approaches provide an approximated solution to the first-order conditions of the social planner optimization problem. In the first section, we provide a formal presentation of the model and derive the first-order conditions of the social planner problem. The detailed specifications and the parameterization of the 30 models are given in section 2. The approximation errors are defined in section 3 along with the specification of the tests retained in the comparison. In section 4, we describe the computer program that serves as the test bench. It is this software that is used in the comparison paper by Kollmann et al. (2010).

1 Description of the model economies

1.1 General setup

We consider a multi-country RBC model with complete asset markets. We denote by N the number of countries. A single homogenous good is produced, traded and consumed across countries.

Production of country $j \in \{1, \ldots, N\}$ at date t is equal to $a_t^j f^j(k_t^j, \ell_t^j)$, where f^j is the production function, k_t^j is the beginning-of-period capital stock, ℓ_t^j is hours worked, and a_t^j is productivity level. The law of motion of capital is:

$$k_{t+1}^j = (1 - \delta)k_t^j + i_t^j \tag{1}$$

where i_t^j is investment and δ is the depreciation rate of capital.

The law of motion of productivity is:

$$\ln a_t^j = \rho \ln a_{t-1}^j + \sigma(e_t^j + e_t) \tag{2}$$

where e_t^j is a country specific shock and e_t is a worldwide shock. Shocks are assumed to be independent and identically distributed Gaussian variables with zero mean and unit variance. Parameters ρ and σ determine the autocorrelation and variance of the logarithm of the productivity level, respectively.

There is an adjustment cost on the capital stock:

$$\Gamma_t^j = \frac{\phi}{2} k_t^j \left(\frac{i_t^j}{k_t^j} - \delta \right)^2 \tag{3}$$

where parameter ϕ governs the intensity of the friction. The value of this parameter has an impact on the dimension of the state space. When $\phi > 0$, all the country-level capital stocks are state variables. In contrast, when $\phi = 0$ (*i.e.* when there is no adjustment cost), only the world capital stock matters, since in that case capital is perfectly mobile. In this project, we are only interested in specifications where the stock of capital of each country matters ($\phi > 0$).

Each country has a representative agent, whose instantaneous utility is $u^j(c_t^j, \ell_t^j)$, where u^j is the utility function and c_t^j is consumption.

The aggregate world resource constraint is:

$$\sum_{j=1}^{N} \left(c_t^j + i_t^j - \delta\, k_t^j \right) \leq \sum_{j=1}^{N} \left(a_t^j\, f^j(k_t^j, \ell_t^j) - \Gamma_t^j \right) \tag{4}$$

It is a well known result that the decentralized market equilibrium with complete asset markets is equivalent to the solution of a corresponding social planner problem, where each country has a weight τ^j (Negishi weight) in the planner's objective, the weights depending on initial endowments.

We therefore formulate the problem as a social planner problem. Since we want to evaluate algorithms used for general equilibrium problems, solution algorithms are required to use Euler equations together with dynamic and static constraints to compute the decision rules; in particular, this rules out algorithms that compute the value function of the social planner.

The problem of the social planner is:

$$\max_{\{c_t^j, i_t^j, k_{t+1}^j, \ell_t^j\}_{t=0,\dots,+\infty}^{j=1,\dots,N}} \mathbb{E}_0 \sum_{j=1}^{N} \tau^j \left(\sum_{t=0}^{+\infty} \beta^t u^j(c_t^j, \ell_t^j) \right) \tag{5}$$

subject to (1), (3) and (4), where β is the subjective time discount factor.

We call λ_t the Lagrange multiplier of the aggregate resource constraint, which is binding in equilibrium. The first-order conditions are:

$$\tau^j u_c^j(c_t^j, \ell_t^j) = \lambda_t \tag{6}$$

$$\tau^j u_\ell^j(c_t^j, \ell_t^j) = -\lambda_t a_t^j f_\ell^j(k_t^j, \ell_t^j) \tag{7}$$

$$\lambda_t \left[1 + \phi\left(\frac{i_t^j}{k_t^j} - \delta \right) \right] = \beta\, \mathbb{E}_t \Bigg\{ \lambda_{t+1}\bigg[1 + a_{t+1}^j f_k^j(k_{t+1}^j, \ell_{t+1}^j) \tag{8}$$

$$+ \phi\left(1 - \delta + \frac{i_{t+1}^j}{k_{t+1}^j} - \frac{1}{2}\left(\frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right) \right)\left(\frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right) \bigg] \Bigg\}$$

$$k_{t+1}^j = (1-\delta)k_t^j + i_t^j \tag{9}$$

$$\sum_{j=1}^{N} \left(c_t^j + i_t^j - \delta k_t^j \right) = \sum_{j=1}^{N} \left[a_t^j f^j(k_t^j, \ell_t^j) - \frac{\phi}{2} k_t^j \left(\frac{i_t^j}{k_t^j} - \delta \right)^2 \right] \tag{10}$$

$$\ln a_t^j = \rho \ln a_{t-1}^j + \sigma\left(e_t + e_t^j \right) \tag{11}$$

where u_c^j (u_ℓ^j) is the derivative of u^j with respect to c (ℓ), and f_k^j (f_ℓ^j) is the

derivative of f^j with respect to k (ℓ).

1.2   Specifications of utility and production functions

Four specifications are used for the utility function:

1. A constant relative risk aversion (CRRA) function, incorporating only consumption (no labor):

$$u^j(c_t^j, \ell_t^j) = \frac{(c_t^j)^{1-\frac{1}{\gamma^j}}}{1 - \frac{1}{\gamma^j}} \tag{12}$$

   where γ^j is the intertemporal elasticity of substitution.

2. A utility function separable in consumption and labor:

$$u^j(c_t^j, \ell_t^j) = \frac{(c_t^j)^{1-\frac{1}{\gamma^j}}}{1 - \frac{1}{\gamma^j}} - b^j \frac{(\ell_t^j)^{1+\frac{1}{\eta^j}}}{1 + \frac{1}{\eta^j}} \tag{13}$$

   where η^j is the Frisch elasticity of labor supply, and b^j governs the relative weight of consumption and leisure.

3. A Cobb-Douglas utility function (hence with unit elasticity of substitution between consumption and leisure):

$$u^j(c_t^j, \ell_t^j) = \frac{\gamma^j}{\gamma^j - 1} \left[(c_t^j)^{\psi^j} \left(L^e - \ell_t^j \right)^{(1-\psi^j)} \right]^{1-\frac{1}{\gamma^j}} \tag{14}$$

   where L^e is the labor endowment of the representative agent in each country, and ψ^j governs the relative weight of consumption and leisure.

4. A constant elasticity of substitution (CES) function embedded inside a CRRA function:

$$u^j(c_t^j, \ell_t^j) = \frac{\gamma^j}{\gamma^j - 1} \left[(c_t^j)^{1-\frac{1}{\chi^j}} + b^j \left(L^e - \ell_t^j \right)^{1-\frac{1}{\chi^j}} \right]^{\frac{1-\frac{1}{\gamma^j}}{1-\frac{1}{\chi^j}}} \tag{15}$$

   where χ^j is the elasticity of substitution between consumption and leisure.

Three specifications are used for the production function:

1. A production function with decreasing returns to scale, using only capital:

$$f^j(k_t^j, \ell_t^j) = A(k_t^j)^\alpha \tag{16}$$

where A is a constant and $\alpha < 1$.

2. A Cobb-Doublas specification with constant returns to scale:

$$f^j(k_t^j, \ell_t^j) = A(k_t^j)^\alpha(\ell_t^j)^{1-\alpha} \tag{17}$$

where α is the share of capital in the production function.

3. A constant elasticity of substitution (CES) function:

$$f^j(k_t^j, \ell_t^j) = A\left[\alpha(k_t^j)^{\mu^j} + (1-\alpha)(\ell_t^j)^{\mu^j}\right]^{\frac{1}{\mu^j}} \tag{18}$$

where $\frac{1}{1-\mu^j}$ is the elasticity of substitution between capital and labor.

1.3 Classification of variables and policy function

The model described above has the following characteristics:[1][2]

- There are $4N + 1$ endogenous (control) variables: consumption (c_t^j), labor (ℓ_t^j), capital stock (k_{t+1}^j), investment (i_t^j), Lagrange multiplier (λ_t).

- Consequently, there are $4N + 1$ model equations given by (6)-(10).

- There are N exogenous variables (technology, a_t^j), governed by equation (11).

- Among the endogenous variables, only previous period capital stock k_t^j is a predetermined state variable.

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[1] In models A1 and A5, there is no labor. There are therefore only $3N + 1$ endogenous variables and as many model equations.

[2] An alternative categorization of variables would include technology level a_t^j in the set of endogenous variables, thus making a total of $5N + 1$ endogenous variables and as many model equations. The set of exogenous variables would then consist of the $N+1$ shocks e_t^j and e_t, and there would be $2N$ predetermined endogenous state variables (k_t^j and a_{t-1}^j). We don't adopt this point of view since it artificially expands the dimension of the state space (it becomes of dimension $3N + 1$).

- The state space is therefore of dimension $2N$.[3]

We denote by $z_t = (c_t, \ell_t, i_t, \lambda_t)$ the vector of the $3N + 1$ endogenous variables (excluding capital stocks). The model can then be written in the following compact form:

$$\mathbb{E}_t \mathcal{H}(z_{t+1}, z_t, k_{t+1}, k_t, a_{t+1}, a_t) = 0$$

The model solution is given by policy functions $k_{t+1} = F(k_t, a_t)$ and $z_t = G(k_t, a_t)$, which satisfy:

$$\mathbb{E}_t \mathcal{H}(G(F(k_t, a_t), a_{t+1}), G(k_t, a_t), F(k_t, a_t), k_t, a_{t+1}, a_t) = 0$$

2 Specifications, values and deterministic steady state

We tried to select examples that are both representative of useful models and challenging from a numerical point of view. Finally, thirty different models were selected. Table 1 gives the values of the parameters which are constant across all specifications. Table 2 lists the various specifications with the functional form of the utility and production functions, and gives the values of the parameters which vary across specifications.[4] Note that, for parameter ϕ (the investment adjustment cost), we choose a low value, thereby increasing the variance of the endogenous variables: in test 2, which considers accuracy along a simulated time path, this leads to considering state points further away from the steady state.

The number of countries, N, considered in each model is different so as to check the ability of each algorithm to deal with both small and larger models.

Some parameters still need to be specified: the Negishi weights τ^j, and the parameters of the utility and production functions: A, b^j (only for A2, A4, A6, A8), ψ^j (only for A3, A7) and L^e (only for A3, A4, A7, A8). These parameters are chosen so that:

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[3]As noted above, this relies on the fact that $\phi > 0$. If there was no adjustment cost, the state space would be of dimension $N + 1$.

[4]In this table, for asymmetric specifications A5-A8, the range notation $[a, b]$ for a parameter (say γ) means that $\gamma^j = a + \frac{(j-1)(b-a)}{N-1}$. In other words, a (b) is the lowest (highest) value of γ across countries, and the N values for γ are evenly spread over $[a, b]$.

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Parameter	Value
β	0.99
α	0.36
δ	0.025
σ	0.01
ρ	0.95
ϕ	0.5

Table 1: Values of parameters common to all specifications

Model	N	u	f	γ	η	μ	χ
A1	2,4,6,8,10	(12)	(16)	1			
A2	2,4,6,8	(13)	(17)	0.25	0.1		
A3	2,4,6	(14)	(17)	0.25			
A4	2,4,6	(15)	(18)	0.25		-0.2	0.83
A5	2,4,6,8,10	(12)	(16)	[0.25,1]			
A6	2,4,6,8	(13)	(17)	[0.25,1]	[0.1,1]		
A7	2,4,6	(14)	(17)	[0.25,1]			
A8	2,4,6	(15)	(18)	[0.2,0.4]		[-0.3,0.3]	[0.75,0.9]

Notes: N is the number of countries; u the equation number for the utility specification; f the equation number for the production function specification; γ, η, μ and χ, the value of the parameter when it appears in a given model specification.

Table 2: Model specifications

- At the steady state, all the countries consume exactly their net domestic production. This implies that in the dynamic model, net foreign assets remain small, which is a rough approximation of reality.

- At the steady state, all the countries have the same size. More precisely we assume that steady capital and labor supply are equal to unity (which implies that production of a country equals A, and therefore $\bar{c^j} = A$ given the previous assumption).

- In all models, the steady state share of capital income in production is equal to α. This will be even true in A4 and A8, because we impose $\bar{k^j} = \bar{\ell^j} = 1$.

- In models with a time endowment (A3, A4, A7, A8), the steady state ratio of labor supply over the time endowment is equal to 40%.

These assumptions lead to the following values for the remaining parameters:

$$
\begin{aligned}
A &= \frac{1-\beta}{\alpha\beta} \\
\tau^j &= \frac{1}{u_c^j(\bar{c^j}, \bar{\ell^j})} = \frac{1}{u_c^j(A, 1)} \\
b^j &= (1-\alpha)A^{1-\frac{1}{\gamma^j}} \quad \text{(for A2, A6)} \\
L^e &= 2.5 \quad \text{(for A3, A4, A7, A8)} \\
b^j &= (1-\alpha)\frac{A^{1-\frac{1}{\chi^j}}}{(L^e - \bar{\ell^j})^{-\frac{1}{\chi^j}}} = (1-\alpha)\frac{A^{1-\frac{1}{\chi^j}}}{(L^e - 1)^{-\frac{1}{\chi^j}}} \quad \text{(for A4, A8)} \\
\psi^j &= \frac{1}{L^e - \alpha(L^e - \bar{\ell^j})} = \frac{1}{L^e - \alpha(L^e - 1)} \quad \text{(for A3, A7)}
\end{aligned}
$$

Given these restrictions on the parameters, the deterministic steady state

for all the model specifications is:

$$\bar{\ell}^j \;=\; 1 \;\;(\text{except A1, A5})$$
$$\bar{k}^j \;=\; 1$$
$$\bar{a}^j \;=\; 1$$
$$\bar{i}^j \;=\; \delta$$
$$\bar{c}^j \;=\; A$$
$$\bar{\lambda} \;=\; 1$$

3 Accuracy tests

3.1 Approximation errors

We now turn to the definition of approximation errors. For given approximated policy functions $(\hat{F}, \hat{G})$, we denote by $\mathcal{R}_{(\hat{F},\hat{G})}(k_t, a_t)$ the vector of unit-free approximation errors at point (k_t, a_t), determined as follows:

$$
\mathcal{R}^j_{(\hat{F},\hat{G})}(k_t, a_t) \;=\; \frac{\tau^j u^j_c(c^j_t, \ell^j_t) - \lambda_t}{\tau^j u^j_c(c^j_t, \ell^j_t)}
$$

$$
\mathcal{R}^{N+j}_{(\hat{F},\hat{G})}(k_t, a_t) \;=\; \frac{\tau^j u^j_\ell(c^j_t, \ell^j_t) + \lambda_t a^j_t f^j_\ell(k^j_t, \ell^j_t)}{\tau^j u^j_\ell(c^j_t, \ell^j_t)}
$$

$$
\mathcal{R}^{2N+j}_{(\hat{F},\hat{G})}(k_t, a_t) \;=\; \left(\lambda_t \left[1 + \phi \left(\frac{i^j_t}{k^j_t} - \delta \right) \right] - \beta\, \mathbb{E}_t \left\{ \lambda_{t+1} \left[1 + a^j_{t+1} f^j_k(k^j_{t+1}, \ell^j_{t+1}) \right. \right. \right.
$$
$$
\left. \left. \left. + \phi \left(1 - \delta + \frac{i^j_{t+1}}{k^j_{t+1}} - \frac{1}{2} \left(\frac{i^j_{t+1}}{k^j_{t+1}} - \delta \right) \right) \left(\frac{i^j_{t+1}}{k^j_{t+1}} - \delta \right) \right] \right\} \right)
$$
$$
\Big/ \left(\lambda_t \left[1 + \phi \left(\frac{i^j_t}{k^j_t} - \delta \right) \right] \right)
$$

$$
\mathcal{R}^{3N+j}_{(\hat{F},\hat{G})}(k_t, a_t) \;=\; \frac{k^j_{t+1} - (1-\delta)k^j_t - i^j_t}{k^j_{t+1}}
$$

$$
\mathcal{R}^{4N+1}_{(\hat{F},\hat{G})}(k_t, a_t) \;=\; \frac{\sum_{j=1}^{N} \left[c^j_t + i^j_t - \delta k^j_t - a^j_t f^j(k^j_t, \ell^j_t) + \frac{\phi}{2} k^j_t \left(\frac{i^j_t}{k^j_t} - \delta \right)^2 \right]}{\sum_{j=1}^{N} \left(c^j_t + i^j_t - \delta k^j_t - a^j_t f^j(k^j_t, \ell^j_t) \right)}
$$

for $j = 1, \ldots, N$, where $k_{t+1} = F(k_t, a_t)$, $(c_t, \ell_t, i_t, \lambda_t) = G(k_t, a_t)$ and $(c_{t+1}, \ell_{t+1}, i_{t+1}, \lambda_{t+1}) = G(F(k_t, a_t), a_{t+1})$.

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By definition, the true policy function (F, G) is such that $\mathcal{R}_{(F,G)}(k_t, a_t) = 0$ for all points of the state space.

Note that, from a computational point of view, evaluating $\mathcal{R}_{(\hat{F},\hat{G})}(k_t, a_t)$ given $\hat{F}$ and $\hat{G}$ is straightforward, except for the approximation error of the Euler equation where an expectation term enters in; this requires the use of a numerical integration method, over the distribution of shocks which enter equation (11), as we describe it below.

3.2 Accuracy on a sphere (Test 1)

The first test compares the approximation errors of the participants' solutions on a hypersphere around the steady state of the model.

More formally, we denote $x_t = (k_t, a_t)$ a point in the state space, $\bar{x} = (\bar{k}, \bar{a}) = (1, 1)$ the steady state and S_r the hypersphere of radius r around the steady state given by $S_r = \{x_t \in \mathbb{R}^{2N}; \|x_t - \bar{x}\| = r\}$ (where $\| \|$ is the Euclidian norm).

For the three values $r = 0.01$, $r = 0.1$ and $r = 0.3$, we draw 1,000 points in S_r, and for every approximated policy function $(\hat{F}, \hat{G})$, we compute the approximation errors $\mathcal{R}_{(\hat{F},\hat{G})}$ on these points, and report the maximum errors (for each equation separately, and the aggregate across equations). This is the way errors are calculated in Judd (1992).

The purpose of this test is to quantify the loss of accuracy of a solution when going away from the steady state. It is expected that projection methods will perform better than perturbation methods for large values of the radius r, but this test will indicate to which extent this is true, and whether this is counterbalanced by a greater accuracy of perturbation methods at points very close to the steady state.

3.3 Accuracy along a simulated path (Test 2)

The second test compares the approximation errors along a simulated path, starting from the steady state of the model, and generating random draws of the shocks.

For a given approximated policy function $(\hat{F}, \hat{G})$, we generate a series of points in the state space $x_0, \ldots, x_T$ where $x_t = (k_t, a_t)$ and T=10,200. The initial state is the steady state $x_0 = (\bar{k}, \bar{a})$, and the subsequent states are given by $k_{t+1} = F(x_t)$ and the law of motion of a_t (equation (11)).

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The series of shocks $(e_t^1, \ldots, e_t^N, e_t)$ for $t = 1, \ldots, T$ are generated with a pseudo-random number generator.

The first 200 points in the generated series are discarded, so that the 10,000 remaining points can be considered as representative of the ergodic distribution of the model variables. On these points, we compute the approximation errors $\mathcal{R}_{(\hat{F}, \hat{G})}$, and report the mean and maximum errors (for each equation separately, and for the aggregate across equations). This is the way errors are calculated in Jin and Judd (2002).

3.4   Den Haan-Marcet statistics (Test 3)

It was initially planned to use the Den Haan and Marcet (1994) statistics as a third testing device for participants' solutions.

The test statistics checks the orthogonality of the error residuals (as specified by $\mathcal{R}$) with some arbitrary function of the state variables (called *instruments*), along a simulated path (like those used in accuracy test 2). The instruments used here are a constant and the first and second order monomials of the state variables. For each approximated policy function, this exercise is repeated a given number of times, in order to obtain an empirical distribution of the test statistics. For the true policy function, the test statistics is distributed according to a χ^2 distribution. The accuracy criterion for an approximated policy function is therefore the closeness of the empirical distribution of the statistics to the χ^2 distribution.

It was finally decided to drop this test in the comparison project for the following two reasons:

- The computational time for the test is extremely large for high values of N (number of countries) and reasonable simulation lengths (T=10,000) when all the instruments are used. Computations are only feasible with a small number of countries, smaller simulation lengths or restricted sets of instruments.

- More importantly, the test seems unable to usefully discriminate between the participants' solutions. For the exercises which were performed, the distribution of the test statistics was very close across all solutions: they were either all very close to the χ^2 distribution, or all very far from it, depending on the simulation length and the set of instruments. Since this property was not extensively tested, it cannot

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be considered as a more general result, but nevertheless this apparent lack of discriminative power of the Den Haan-Marcet statistics for this exercise was deemed a sufficient reason to drop it.

4   Test bench

In order to test the participants' solutions on an equal basis, we wrote a testing program which performs the accuracy tests on all the solutions. With that test bench, one can be sure that:

- the random elements of the tests are the same across all solutions: for test 1, these random elements are the points on the sphere around the steady state; for test 2, they are the series of random shocks used to generate the simulated paths;

- when computing the approximation error of the Euler equation, the numerical integration method is the same across all solutions;

- the numerical precision used in the calculations is the same across all solutions.

Here we shortly describe some algorithmic choices made in the test bench, and then give an overview of the implementation.

4.1   Algorithmic choices

Three numerical integration methods are implemented in the test bench:

- the product four-point Gauss-Hermite quadrature, as described in Abramowitz and Stegun (1964, p. 890, eq. 25.4.46);

- the degree 5 monomial formula described in Judd (1998, p. 275, eq. 7.5.11);

- quasi-Monte Carlo integration, using Niederreiter quasi-random low-discrepancy sequence (see Bratley et al., 1992).

The default choice of the test bench is to use the Gauss-Hermite formula up to dimension 6 (*i.e.* for $N \leq 5$ since there are $N+1$ random shocks), and the monomial formula for higher dimensional problems (*i.e.* for $N \geq 6$).

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For test 1, the test bench generates uniformly distributed points on a hypersphere of dimension $2N$ using the simple algorithm described by Muller (1959). The idea is to draw points from a multivariate standard Gaussian distribution of dimension $2N$, and to divide these points by their Euclidian norm: it is easy to see that the resulting points are uniformly distributed over the unit hypersphere. Points uniformly distributed over any hypersphere can immediately be generated by rescaling and translating. The Gaussian draws are generated using draws from the Sobol quasi-random low-discrepancy sequences (see Antonov and Saleev, 1979) which are transformed in Gaussian vectors by applying the reciprocal of the Gaussian cumulative distribution function.

For test 2, the random draws for shocks are generated using a Mersenne-Twister random number generator (see Matsumoto and Nishimura, 1998).

4.2   Implementation choices

The test bench is a standalone program written in the C++ programming language, and relies on the GNU Scientific library[5] for linear algebra computations and random number generation. The source code of the program is freely available on the web.[6]

The program is made of two main modules, plus one module per solution to be tested. The first module implements the 30 specifications and computes the residuals.[7] The second one implements the accuracy tests.[8] Then for each of the six participants' solutions (PER1, PER2, MRGAL, SMOL, CGA and SSA1, using the notations of Kollmann et al., 2010), a module implements the corresponding approximated policy function.[9] Note that the test bench does not contain the code to fully recompute each solution:[10] it is only able to simulate the solution, using the coefficients of the

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[5]See Galassi et al. (2003) or http://www.gnu.org/software/gsl.

[6]The source code that was used for generating the comparison results presented in Kollmann et al. (2010) is available as supplemental material to this paper, and is downloadable at: ....   Future versions of the software will be available at: http://www.dynare.org/JedcTestsuiteWiki/TestBenchProblemA.

[7]See class ModelSpec and its subclasses ModelSpecA1 to ModelSpecA8.

[8]See class SolutionTester and main file tester.cc.

[9]Note that SMOL, CGA and SSA1 approximated policy functions do not return a value for the Lagrange multiplier λ_t. The workaround is the following: for SMOL solution, the value of $\tau^1 u_c^1(c_t^1, \ell_t^1)$ is used as a replacement for λ_t; for CGA and SSA, the arithmetic mean of $\tau^j u_c^j(c_t^j, \ell_t^j)$ over $j = 1, \ldots, N$ is used as a replacement for λ_t.

[10]The source code for recomputing the various solutions is available as supplemental

parameterized functional form used for approximating the true solution.

By default the program computes test 1 and 2 for all the specifications (symmetric and asymmetric ones), and uses the same settings as those used in the comparison paper of Kollmann et al. (2010). It is however possible to change the tasks to be performed (computing the Den Haan-Marcet statistics, restricting to only some solutions or model specifications), or to change some parameters of the tests (the number of points, the seed of the random number generators, the numerical integration method, or the normalization of the residual for the aggregate resource constraint equation).

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