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Multivariate copulas, quasi-copulas, and lattices

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Abstract

We investigate some properties of the partially ordered sets of multivariate copulas and quasi-copulas. Whereas the set of bivariate quasi-copulas is a complete lattice, which is order-isomorphic to the Dedekind-MacNeille completion of the set of bivariate copulas, we show that this is not the case in higher dimensions.

Keywords: Copula, Lattice, Quasi-copula.

1. Introduction

Copulas—multivariate distribution functions with uniform margins—have proven to be remarkably useful in statistical modelling and in the study of dependence and association of random variables. Quasi-copulas, a more general concept, share many properties with copulas. The set of copulas is a proper subset of the set of quasi-copulas, and both sets have a natural partial ordering. The purpose of this paper is to investigate some properties of those partially ordered sets (posets). In particular, the poset of bivariate quasi-copulas is a complete lattice, which is order-isomorphic to

the Dedekind-MacNeille completion of the poset of copulas in the bivariate case (Nelsen and Úbeda-Flores, 2005), but this last is not true in higher dimensions.

2. Preliminaries

Let $n \geq 2$ be an integer. An n -copula is a function $C: \mathbb{I}^n \rightarrow \mathbb{I} (= [0, 1])$ which satisfies the following properties:

- (C1) For $\mathbf{u} = (u_1, u_2, \dots, u_n)$ in $\mathbb{I}^n$, $C(\mathbf{u}) = 0$ if at least one coordinate of $\mathbf{u}$ is 0, and $C(\mathbf{u}) = u_k$ whenever all coordinates of $\mathbf{u}$ are equal to 1 except maybe u_k ; and
- (C2) the C -volume of any n -box $J = \times_{i=1}^n [a_i, b_i] \subset \mathbb{I}^n$ is nonnegative, i.e., $V_C(J) = \sum (-1)^{k(\mathbf{c})} C(\mathbf{c}) \geq 0$, where the sum is taken over all the vertices $\mathbf{c} = (c_1, c_2, \dots, c_n)$ of J (i.e., $c_k = a_k$ or $c_k = b_k$, for all $k = 1, 2, \dots, n$); and $k(\mathbf{c})$ is the number of indices k such that $c_k = a_k$.

The importance of copulas in statistics is described in Sklar's theorem (Sklar, 1959): Let H be a multivariate distribution function with univariate marginal distribution functions $F_1, F_2, \dots, F_n$. Then there exists a copula C (which is uniquely determined on $\times_{i=1}^n \text{Range } F_i$) such that the following equality holds: $H(\mathbf{x}) = C(F_1(x_1), F_2(x_2), \dots, F_n(x_n))$ for all $\mathbf{x} = (x_1, x_2, \dots, x_n) \in [-\infty, \infty]^n$. Thus copulas link joint distribution functions to their univariate margins. For a complete survey on copulas, see Nelsen (2006).

We will use along this paper the fact that set of n -copulas $\mathbf{C}^n$ is compact—see Durante et al. (2011) for a detailed study—and consequently any sequence of n -copulas $\{C_k\}_{k \in \mathbb{N}}$ contains a subsequence $\{C_{k(j)}\}_{j \in \mathbb{N}} \subseteq \{C_k\}_{k \in \mathbb{N}}$ which converges (point-wise) to an n -copula C .

The concept of a quasi-copula is a more general notion than that of a copula, and was introduced by Alsina et al. (1993) in the bivariate case, and Nelsen et al. (1996) for the general case, in order to characterize operations on distribution functions that can or cannot be derived from operations on random variables defined on the same probability space.

An n -quasi-copula is a function $Q: \mathbb{I}^n \rightarrow \mathbb{I}$ which satisfies condition (C1) for n -copulas, but in place of (C2), the weaker conditions:

- (Q1) Q is non-decreasing in each variable; and

(Q2) the *Lipschitz condition* $|Q(\mathbf{v}) - Q(\mathbf{u})| \leq \sum_{i=1}^n |v_i - u_i|$ for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{I}^n$

(see Cuculescu and Theodorescu, 2001). While every copula is a quasi-copula, there exist *proper* quasi-copulas, i.e., quasi-copulas which are not copulas. If $\mathbf{Q}^n$ denotes the set of n -quasi-copulas, $\mathbf{Q}^n \setminus \mathbf{C}^n$ will denote the set of proper n -quasi-copulas. In this note we will also consider Q -volumes when Q is an n -quasi-copula. Similarities and differences between n -copulas and proper n -quasi-copulas can be found, for instance, in De Baets et al. (2007), Nelsen et al. (2002,2010), and Rodríguez-Lallena and Úbeda-Flores (2009).

For any quasi-copula Q we have $W^n(\mathbf{u}) = \max(0, \sum_{i=1}^n u_i - n + 1) \leq Q(\mathbf{u}) \leq \min(u_1, u_2, \dots, u_n) = M^n(\mathbf{u})$ for all $\mathbf{u}$ in $\mathbb{I}^n$. M^n (for every $n \geq 2$) and W^2 are copulas; however, for every $n \geq 3$, W^n is a proper n -quasi-copula. We will use the following notation: Given two n -quasi-copulas Q_1 and Q_2 , $Q_1 \leq Q_2$ denotes the point-wise inequality $Q_1(\mathbf{u}) \leq Q_2(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{I}^n$.

Aggregation of pieces of information coming from different sources is an important task in expert and decision support systems, multi-criteria decision making, and group decision making. *Aggregation operators* (Calvo et al., 2002) are precisely the mathematical objects that allow this type of information fusion. Aggregation operators include copulas, quasi-copulas, *triangular norms* (or *t-norms*)—associative copulas are continuous t-norms—, and *semi-copulas*—a generalization of the concept of t-norm—see Durante and Sempi (2005) and Klement et al. (2000).

We will also need some notions from lattice theory (Davey and Priestley, 2002). Given two elements x and y of a poset $(P, \leq)$, let $x \vee y$ denote the *join* (or the least upper bound) of x and y (when it exists); similarly for $\bigvee S$, where S is a subset of P ; $x \wedge y$ denotes the *meet* (or the greatest lower bound) of x and y (when it exists); and similarly for $\bigwedge S$. In particular, for any pair Q_1 and Q_2 of quasi-copulas (or copulas), $Q_1 \vee Q_2 = \inf\{Q \in \mathbf{Q}^n \mid Q_1 \leq Q, Q_2 \leq Q\}$ and $Q_1 \wedge Q_2 = \sup\{Q \in \mathbf{Q}^n \mid Q \leq Q_1, Q \leq Q_2\}$. If the join or meet is found within a particular poset P , we subscript $\bigvee_P S$. Given two posets A and B , we say that A is *join-dense* (respectively, *meet-dense*) in B if for any D in B , there exists a set $S \subseteq A$ such that $D = \bigvee_B S$ (respectively, $D = \bigwedge_B S$). A poset $P \neq \emptyset$ is a *lattice* if for every x, y in P , $x \vee y$ and $x \wedge y$ are in P ; and P is a *complete lattice* if for every $S \subseteq P$, $\bigvee S$ and $\bigwedge S$ are in P . If $\varphi: P \longrightarrow L$ is an *order-embedding* (i.e., order-preserving injection) of a poset P into a complete lattice L , then we say that L is a *completion* of P . Finally, if φ maps P onto L , φ is an *order-isomorphism* (i.e., order-preserving

bijection).

We also have the following definition (Davey and Priestley, 2002):

Definition 1. A completion $\mathcal{C}$ of a lattice $\mathcal{L}$ is called a Dedekind-MacNeille completion of $\mathcal{L}$ if $\mathcal{C}$ is join-dense and meet-dense in $\mathcal{L}$.

3. The lattice of quasi-copulas

In Nelsen and Úbeda-Flores (2005), the authors show that the set of 2-quasi-copulas is a complete lattice, which is order-isomorphic to the Dedekind-MacNeille completion of the set of 2-copulas (for a study of the lattice-theoretic structure of the sets of triangular norms and semi-copulas, see Durante et al. (2008)). Consequently, any set of 2-copulas sharing a particular statistical property is guaranteed to have pointwise best-possible bounds within the set of quasi-copulas. We now wonder if these results can be extended to higher dimensions. We then prove that the set of n -quasi-copulas is a complete lattice; however it is not order-isomorphic to the Dedekind-MacNeille completion of the set of n -copulas.

We first prove some results of the posets $\mathbf{Q}^n$, $\mathbf{C}^n$, and $\mathbf{Q}^n \setminus \mathbf{C}^n$.

Theorem 2. $\mathbf{Q}^n$ is a complete lattice; however, neither $\mathbf{C}^n$ nor $\mathbf{Q}^n \setminus \mathbf{C}^n$ is a lattice.

Proof. Let S be any set of n -quasi-copulas, and define $\overline{Q}_S(\mathbf{u}) = \sup\{Q(\mathbf{u}) \mid Q \in S\}$ and $\underline{Q}_S(\mathbf{u}) = \inf\{Q(\mathbf{u}) \mid Q \in S\}$ for each $\mathbf{u}$ in $\mathbb{I}^n$. Since $\overline{Q}_S$ and $\underline{Q}_S$ are n -quasi-copulas (Rodríguez-Lallena and Úbeda-Flores, 2004), it now follows that $\vee S (= \overline{Q}_S)$ and $\wedge S (= \underline{Q}_S)$ are in $\mathbf{Q}^n$, hence $\mathbf{Q}^n$ is a complete lattice.

Now suppose that $\mathbf{C}^n$ is a lattice, and consider the following 2-copulas:

$$\begin{aligned} C_1(u, v) &= \min(u, v, \max(0, u - 2/3, v - 1/3, u + v - 1)), \\ C_2(u, v) &= C_1(v, u), \\ C_3(u, v) &= \min(u, v, \max(0, u - 1/3, v - 1/3, u + v - 2/3)), \\ C_4(u, v) &= \min(u, v, \max(1/3, u - 1/3, v - 1/3, u + v - 1)). \end{aligned}$$

We consider the following n -copulas: $C_i^*(\mathbf{u}) = C_i(u_1, u_2) \prod_{k=3}^n u_k$, for $i = 1, 2, 3, 4$, and for all $\mathbf{u} \in \mathbb{I}^n$ (Nelsen et al., 1996). Since $\mathbf{C}^n$ is a lattice, $C = C_1^* \vee C_2^*$ exists and is a copula. Hence $1/3 \geq C(1/3, 2/3, 1, \dots, 1) \geq C_1^*(1/3, 2/3, 1, \dots, 1) = 1/3$, so that $C(1/3, 2/3, 1, \dots, 1) = 1/3$. Similarly (using C_2^*), we have $C(2/3, 1/3, 1, \dots, 1) = 1/3$. Since $C_1^* \leq C_3^*$ and

$C_2^* \leq C_3^*$, $C \leq C_3^*$ and so $C(1/3, 1/3, 1, \dots, 1) \leq C_3^*(1/3, 1/3, 1, \dots, 1) = 0$, thus $C(1/3, 1/3, 1, \dots, 1) = 0$. Similarly, we have $C(2/3, 2/3, 1, \dots, 1) \leq C_4^*(2/3, 2/3, 1, \dots, 1) = 1/3$, so $C(2/3, 2/3, 1, \dots, 1) = 1/3$. Hence, we obtain $V_C([1/3, 2/3]^2 \times \mathbb{I}^{n-2}) = -1/3$, i.e., C is a proper n -quasi-copula, which is a contradiction.

To prove that $\mathbf{Q}^n \setminus \mathbf{C}^n$ is not a lattice, it suffices to exhibit two proper n -quasi-copulas Q_1 and Q_2 whose join (or meet) is an n -copula. Let Q be the proper 2-quasi-copula $C_1 \vee C_2$, and define

$$Q_1(\mathbf{u}) = \begin{cases} Q(2u_1, 2u_2)u_3 \cdots u_n/2, & \mathbf{u} \in [0, 1/2]^2 \times \mathbb{I}^{n-2}, \\ \min(u_1, u_2, \dots, u_n), & \text{elsewhere,} \end{cases}$$

and

$$Q_2(\mathbf{u}) = \begin{cases} (1 + Q(2u_1 - 1, 2u_2 - 1)u_3 \cdots u_n)/2, & \mathbf{u} \in [1/2, 1]^2 \times \mathbb{I}^{n-2}, \\ \min(u_1, u_2, \dots, u_n), & \text{elsewhere.} \end{cases}$$

Note that Q_1 and Q_2 are proper n -quasi-copulas. Finally, since $Q_1 \vee Q_2 = M^n$, which is an n -copula, the proof is done. $\square$

Lemma 3. Let $\mathbf{a} = (a_1, a_2, \dots, a_n) \in (0, 1)^n$, let $\theta \in [W^n(\mathbf{a}), M^n(\mathbf{a})]$, and define $S_{\mathbf{a}, \theta} = \{Q \in \mathbf{Q}^n \mid Q(\mathbf{a}) = \theta\}$. Then $\bigvee S_{\mathbf{a}, \theta}$ and $\bigwedge S_{\mathbf{a}, \theta}$ are the proper n -quasi-copulas (except when $\bigvee S_{\mathbf{a}, \theta} = M^n$) given by $\bigvee S_{\mathbf{a}, \theta}(\mathbf{u}) = \min(u_1, u_2, \dots, u_n, \theta + \sum_{i=1}^n (u_i - a_i)^+)$ and $\bigwedge S_{\mathbf{a}, \theta}(\mathbf{u}) = \max(0, \sum_{i=1}^n u_i - n + 1, \theta + \sum_{i=1}^n (u_i - a_i)^+)$, respectively, where $x^+ = \max(x, 0)$.

Proof. Let Q be any n -quasi-copula. The defining conditions (Q1) and (Q2) for quasi-copulas yield, for all $\mathbf{u} \in \mathbb{I}^n$, the inequalities $-(a_i - u_i)^+ \leq Q(\mathbf{u}) - Q(u_1, \dots, a_i, \dots, u_n) \leq (u_i - a_i)^+$ for all $i = 1, 2, \dots, n$, hence $\theta - \sum_{i=1}^n (a_i - u_i)^+ \leq Q(\mathbf{u}) \leq \theta + \sum_{i=1}^n (u_i - a_i)^+$. Thus $\bigwedge S_{(a,b), \theta} \leq Q \leq \bigvee S_{\mathbf{a}, \theta}$, and these bounds are n -quasi-copulas (Theorem 3.2 in Rodríguez-Lallena and Úbeda-Flores, 2004). $\square$

Unlike the bivariate case, for any integer $n \geq 3$, $\mathbf{C}^n$ is neither join-dense nor meet-dense in $\mathbf{Q}^n$, as the following two examples show —note that W^n is not an n -copula for $n \geq 3$, so it is trivial that there does not exist a set B such that $W^n = \bigvee_{\mathbf{Q}^n}(B)$; however we provide another “non-trivial” example. But before proceeding, we provide the definition of an *ordinal sum* of a family of n -quasi-copulas, which is an n -quasi-copula and a simple generalization of

the definition of ordinal sum of n -copulas —which can be found in Mesiar and Sempi (2010).

Definition 4. Let J be a finite or countable subset of the natural numbers $\mathbb{N}$, let $(a_k, b_k)_{k \in J}$ be a family of sub-intervals of the unit interval $\mathbb{I}$ indexed by J , and let $\{Q_k\}_{k \in J}$ a family of n -quasi-copulas also indexed by J . It is required that any two of the intervals (a_k, b_k) have at most an endpoint in common. Then the *ordinal sum* Q of $\{Q_k\}_{k \in J}$ with respect to the family of intervals (a_k, b_k) is defined, for all $\mathbf{u} \in \mathbb{I}$, by

$$Q(\mathbf{u}) = \begin{cases} a_k + (b_k - a_k)Q_k\left(\frac{\min(u_1, b_k) - a_k}{b_k - a_k}, \dots, \frac{\min(u_n, b_k) - a_k}{b_k - a_k}\right), \\ \min(u_1, u_2, \dots, u_n) \in (a_k, b_k), \text{ for some } k \in J, \\ \min(u_1, u_2, \dots, u_n), & \text{elsewhere.} \end{cases}$$

Example 5. Let $n \geq 3$ be an integer; and D the *ordinal sum* of M^n and W^n with respect to the n -boxes $[0, 1/2]^n$ and $[1/2, 1]^n$, respectively. Since any ordinal sum of n -quasi-copulas is an n -quasi-copula, then D is a proper n -quasi-copula. Now, suppose $\mathbf{C}^n$ is join-dense in $\mathbf{Q}^n$, then there exists a set $S \subseteq \mathbf{C}^n$ such that $D = \bigvee_{\mathbf{Q}^n} S$. Then, we have two cases:

- (a) There exists an n -copula C such that $C(\mathbf{1}/2) = 1/2$.
- (b) Since $\mathbf{C}^n$ is compact, there exists a sequence of n -copulas $\{C_k\}$ (with $C_k \leq D$ for every $k \in \mathbb{N}$) such that $\{C_k(\mathbf{1}/2)\} \rightarrow 1/2$ as $k \rightarrow \infty$. Let C be the n -copula for which $\{C_k\} \rightarrow C$ point-wise; then it is clear that $C(\mathbf{1}/2) = 1/2$.

In both cases, C is an ordinal sum —this is a simple generalization of the result in (Theorem 3.2.1 in Nelsen, 2006). Since both C and D are ordinal sums and $C \leq D$, with respect to the region $[1/2, 1]^n$ there exists an n -copula which is less than W^n ; but this is absurd. We then conclude that $\mathbf{C}^n$ is not join-dense in $\mathbf{Q}^n$.

Example 6. We first look at the case $n = 3$. Let C_1 and C_2 be the 3-copulas whose mass is distributed uniformly along the main diagonals of the dark cubes in Figures 1 and 2, respectively (for more details, see Carley, 2002); and consider the 3-quasi-copula $Q = C_1 \wedge C_2$ —note that, in such a

Figure 1: Blocks used for the construction of the 3-copula C_1 in Example 6.

Figure 2: Blocks used for the construction of the 3-copula C_2 in Example 6.

case, $Q(1/2, 1/2, 1/2) = 1/4$, $Q(1, 1/2, 1/2) = 1/2$, and $Q(1/2, 1, 1/2) = 1/2$. Now, suppose $\mathbf{C}^3$ is meet-dense in $\mathbf{Q}^3$; then there exists a set $S \subseteq \mathbf{C}^3$ such that $Q = \bigwedge_{\mathbf{Q}^3} S$. Thus, for all $\varepsilon > 0$, there exists a sequence of 3-copulas $\{C_k^*\}$ such that $C_k^* \geq Q$ and $C_k^*(1/2, 1/2, 1) < 1/4 + \varepsilon$ for every $k \in \mathbb{N}$. Observe also that $C_k^*(1/2, 1/2, 1/2) < 1/4 + \varepsilon$ and $C_k^*(1, 1/2, 1/2) = C_k^*(1/2, 1, 1/2) = 1/2$. Thus, it must be satisfied $V_{C_k^*}([0, 1/2]^3) \geq 1/4$, $V_{C_k^*}([1/2, 1] \times [0, 1/2]^2) \geq 1/4 - \varepsilon$, and $V_{C_k^*}([0, 1/2] \times [1/2, 1] \times [0, 1/2]) \geq 1/4 - \varepsilon$. However, note that taking $\varepsilon < 1/8$, we have a contradiction. Thus, we have $Q \neq \bigwedge_{\mathbf{Q}^3} S$.

To prove the result for the n -dimensional case, it suffices to take the n -quasi-copula Q^* given by $Q^*(\mathbf{u}) = Q(u_1, u_2, u_3) \prod_{i=4}^n u_i$ for all $\mathbf{u} \in \mathbb{I}^n$ (Nelsen et al., 1996), where Q is the 3-quasi-copula defined above—and similarly the generalization for C_1 and C_2 —, and using similar arguments to those from the trivariate case, we conclude that $Q^* \neq \bigwedge_{\mathbf{Q}^n} T$, where T is a set of n -copulas such that $T \subseteq \mathbf{C}^n$; whence $\mathbf{C}^n$ is not meet-dense in $\mathbf{Q}^n$.

As a consequence of the two previous examples, and taking into account Definition 1, we have the following result:

Theorem 7. For $n \geq 3$, $\mathbf{Q}^n$ is not order-isomorphic to the Dedekind-MacNeille completion of $\mathbf{C}^n$.

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