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Consistent estimation of species abundance from a presence-absence map

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Abstract

The estimation of the abundance of a species using the presence or absence of the species over a grid of cells simplifies data collection but the resulting statistical analysis is challenging. Several estimators have been proposed but their properties are unknown. Here we consider a generalized gamma-Poisson model which allows dependencies across the grid and develop a new estimator for this model. It is shown that this estimator is consistent, allowing us to conclude that it is indeed possible to estimate abundance from presence-absence maps.

Keywords: Presence Absence Map; Consistency; Gamma-Poisson Model.

1 Introduction

In ecological studies where the interest is in estimating the number of individuals of a certain species in a given area it is often too laborious to count all individuals in an area. A more labor effective procedure is to divide the area in M cells and to determine for each cell whether the species is present or not. The data may be thought of as a presence-absence map that identifies the occupied cells and it is often represented by a $I \times J$ -matrix $(y_{ij})_{i=1,\dots,I, j=1,\dots,J}$ with $M = I \cdot J$ cells, where $y_{ij} = 1$ if the i, j cell is unoccupied cell and 0 otherwise. Then the question is whether it is possible to estimate the total number N of the species in the area from these data in general. The analysis of these data presents an interesting statistical challenge (Kunin 1998, He and Gaston 2000, Kunin et al. 2000, Conlinsk et al. 2007, He and Gaston 2007).

The resolution of the problem requires distributional assumptions on the unknown numbers X_{ij} of individuals in each cell. Suppose that there are N individuals so that $N = \sum_{i=1}^I \sum_{j=1}^J X_{ij}$. The simplest approach is to assume that each individual is independent of other individuals and the probability of its occurrence in a given cell is $1/M$. Then $(X_{ij})_{i=1,\dots,I, j=1,\dots,J}$ has a multinomial distribution where each cell has the same probability and the abundance N can be estimated using maximum likelihood. Alternatively, one may suppose that X_{ij} are independently and identically distributed with a common Poisson distribution with mean λ . In this case, the probability a cell is empty is $e^{-\lambda}$. Let $m_0 = \sum_{i=1}^I \sum_{j=1}^J y_{ij}$ be the number of empty (unoccupied) cells. Then by the method of moments, we may estimate λ by $\log(M/m_0)$ so that an estimate of N is $M \log(M/m_0)$. However, since the assumption of independently and identically distributed random variables does not reflect the spatial aggregation (clustering) typically observed in nature, this method often underestimates the true abundance (He and Gaston 2000) and hence is not recommended in literature.

In order to reflect the spatial aggregation, He and Gaston (2000) used the negative binomial distribution, i.e., they assumed that the X_{ij} follow the negative binomial distribution. Although it is well known that the negative binomial distribution is more appropriate than the Poisson distribution for modeling aggregated organisms and populations (Pielou 1977), estimation is more difficult as there are two parameters in the model and they are not identifiable from a presence-absence map. To circumvent this difficulty, He and Gaston (2000) generate a coarser map, where adjacent pairs of cells are merged from the base map and impose a constraint on the clumping parameter of the coarser map. The resulting estimator for the species abundance was shown to outperform other methods in several empirical studies (He and Gaston 2000, 2007). Nevertheless, He and Gaston (2000, 2007) do not provide theoretical justification for the constraint and this weakness is criticized by Conlisk et al. (2007). By noting that the negative binomial distribution arises from a gamma-Poisson mixture model, Hwang and He (2011) proposed a generalized gamma-Poisson mixture model. In this model, a gamma random variable is first associated with each cell, as in the negative binomial model, but correlations between the cells are modelled by allowing the conditional mean of the Poisson counts to be a weighted sum of all the gamma random variables. Hwang and He (2011) used some gamma approximations to find an estimator. When applied to forest census data (Condit et al., 1996), they showed that the estimator based on the generalized gamma Poisson model could improve estimation of abundance from occurrence data. To date, however, no statistical properties of the estimators of He and Gaston (2000) and Hwang and He (2011) are known. Even consistency has not been proven.

In this study, we develop a new consistent estimator under the generalised gamma-Poisson model. In Section 2 we reformulate the model of He and Gaston (2000) using the gamma-Poisson framework and then describe the generalised gamma-Poisson model. In Section 3 we derive the proposed estimator for the generalised gamma-Poisson model whose consistency is shown in Section 4. This consistency proof is not straightforward and needs some tricky arguments which are given in the Appendix. The estimator is applied to real data in Section 5. The results are discussed in Section 6.

2 Model

For simplicity, assume that I and J are even. Let λ_{ij} be the mean abundance at the cell (i, j) . This mean may depend on environmental characteristics, such as the slope of the land, the soil content and the humidity gradient. Given the means λ_{ij} , we assume that the abundances X_{ij} follow Poisson distributions and they are independent of each other. If the λ_{ij} are independent and identically distributed gamma distributed random variables with parameters k and β so that $E(\lambda_{ij}) = k\beta$ and $\text{var}(\lambda_{ij}) = k\beta^2$, then X_{ij} are independent and each X_{ij} has a negative binomial distribution. The parameter k is called the clumping parameter (Pielou, 1977) as it corresponds to the degree of aggregation of individuals in a cell. It is easy to see the absence probability $P(X_{ij} = 0)$ is $1/(1 + \beta)^k$. As a consequence, the likelihood function based on the observed y_{ij} is

$$\left\{ \frac{1}{(1 + \beta)^k} \right\}^{m_0} \left\{ 1 - \frac{1}{(1 + \beta)^k} \right\}^{M - m_0}.$$

The likelihood is not identifiable as there are two unknown parameters but only a single statistic m_0 . In order to estimate the parameters, He and Gaston (2000) create a coarser map where adjacent cells in the original base map are merged, collapsing a 2×2 cell to a single cell. That is, let $X_{(2i-1)(2j-1)}^q = X_{(2i-1)(2j-1)} + X_{(2i-1)(2j)} + X_{(2i)(2j-1)} + X_{(2i)(2j)}$, for $i = 1, \dots, I/2$ and $j = 1, \dots, J/2$, which is the number of individuals in the merged quadruple cells. The coarser quad map then indicates whether the $X_{(2i-1)(2j-1)}^q$ is larger than zero or not. Denote by m_0^q the number of empty cells in the coarser map. Noting $N = Mk\beta$, He and Gaston (2000) solved the equations $m_0/M = P(X_{ij} = 0)$ and $4m_0^q/M = P(X_{ij}^q = 0)$ to estimate N and k . Simple algebra shows that $X_{(2i-1)(2j-1)}^q$ has a negative binomial distribution with the clumping parameter $4k$. Nevertheless, He and Gaston (2000) suppose that $X_{(2i-1)(2j-1)}^q$ has the same distribution as X_{ij} , which has clumping parameter k and mean $k\beta$. This presumption is mainly based on empirical evidence and obviously contradicts the theoretical calculations. In addition, Conlisk (2007) showed that there is no solution from the negative binomial model under the correct distribution. To overcome the weaknesses of this approach, Hwang and He (2011) proposed an extension in the gamma-Poisson model framework. This extension yields a generalized gamma-Poisson mixtured model that has meaning in the ecological context.

To develop this model, recall that the mean level λ_{ij} represents the environmental characteristics of the immediate area of study, hence the variables λ_{ij} would tend to vary smoothly over nearby cells. Hence, it is unrealistic to assume independence of the λ_{ij} . It is more appropriate to consider the spatial correlation of the λ_{ij} over the cells so that the λ_{ij} in neighboring cells are more similar than between those that are further apart. This observation motivated Hwang and He (2011) to model the mean levels as a gamma random field. Formally, let V_{st} , $s, t \in \mathbb{Z} = \{0, -1, 1, -2, 2, \dots\}$, be independent and identically gamma distributed random variables with parameters a and b (so that $E(V_{st}) = ab$ and $\text{var}(V_{st}) = ab^2$). Given $0 \leq \rho < 1$, we define the mean levels

$$\lambda_{ij} = \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} \rho^{|i-s|+|j-t|} V_{st},$$

which is a spatial moving average of the V_{st} . The parameter ρ reflects the strength of the spatial correlation of the mean levels λ_{ij} . It is then assumed that the conditional distributions of the X_{ij} given $V_{st} = v_{st}$ for all $s, t \in \mathbb{Z}$ are Poisson with mean λ_{ij} and that the X_{ij} and X_{kl} are conditionally independent given $V_{st} = v_{st}$, $s, t \in \mathbb{Z}$ for $(i, j) \neq (k, l)$. As the joint distribution of X_{ij} reduces to the usual independent negative binomial model when $\rho = 0$, it is called the generalized gamma-Poisson mixture model. Hwang and He (2011) used gamma approximations to find an estimator, however, the consistency of this estimator remains an open question. Here we propose a new moment estimator whose consistency can be established.

3 Estimation

Since the conditional distribution of X_{ij} given $V_{st} = v_{st}$, $s, t \in \mathbb{Z}$, is Poisson with parameter $\lambda_{ij} = \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} \rho^{|i-s|+|j-t|} v_{st}$, we see that

$$E(X_{ij}) = \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} \rho^{|i-s|+|j-t|} E(V_{st}) = \sum_{s=-\infty}^{\infty} \rho^{|i-s|} \sum_{t=-\infty}^{\infty} \rho^{|j-t|} ab = \left(\frac{1+\rho}{1-\rho} \right)^2 ab,$$

and thus

$$E(N) = \sum_{i=1}^I \sum_{j=1}^J E(X_{ij}) = IJ \left(\frac{1+\rho}{1-\rho} \right)^2 ab. \quad (1)$$

Once we have estimators $\hat{a}$, $\hat{b}$, $\hat{\rho}$ of a , b , and ρ this yields the estimator

$$\hat{N} = IJ \left(\frac{1+\hat{\rho}}{1-\hat{\rho}} \right)^2 \hat{a}\hat{b}, \quad (2)$$

and our main task is to estimate these parameters using presence-absence data.

Let $Y_{ij} = 1_{\{0\}}(X_{ij})$ be an indicator of whether the cell (i, j) is empty or not. The variables $Y_{i(2j-1)}^d = Y_{i(2j-1)} Y_{i(2j)}$ and $Y_{(2i-1)(2j-1)}^q = Y_{(2i-1)(2j-1)} Y_{(2i-1)(2j)} Y_{(2i)(2j-1)} Y_{(2i)(2j)}$ indicate whether the double and quadruple cells are empty. Our approach is based on the method of moments and to this end we need:

Proposition 1

$$\begin{aligned} a) \quad E(Y_{ij}) &= \prod_{s=-\infty}^{\infty} \prod_{t=-\infty}^{\infty} (1 + b \rho^{|i-s|+|j-t|})^{-a}, \\ b) \quad E(Y_{ij}^d) &= \prod_{s=-\infty}^{\infty} \prod_{t=-\infty}^{\infty} (1 + b \rho^{|i-s|} (\rho^{|j-t|} + \rho^{|j+1-t|}))^{-a}, \\ c) \quad E(Y_{ij}^q) &= \prod_{s=-\infty}^{\infty} \prod_{t=-\infty}^{\infty} (1 + b (\rho^{|i-s|} + \rho^{|i+1-s|}) (\rho^{|j-t|} + \rho^{|j+1-t|}))^{-a}. \end{aligned}$$

Define $M_0 = \sum_{i=1}^I \sum_{j=1}^J Y_{ij}$, $M_0^d = \sum_{i=1}^I \sum_{j=1}^{J/2} Y_{i(2j-1)}^d$ and $M_0^q = \sum_{i=1}^{I/2} \sum_{j=1}^{J/2} Y_{(2i-1)(2j-1)}^q$ with realizations m_0 , m_0^d , and m_0^q . Then, a , b , and ρ can be estimated by solving

$$\begin{aligned} \frac{1}{IJ} m_0 &= E_{(a,b,\rho)} \left(\frac{1}{IJ} M_0 \right) = \prod_{s=-\infty}^{\infty} \prod_{t=-\infty}^{\infty} (1 + b \rho^{|s|+|t|})^{-a}, \\ \frac{2}{IJ} m_0^d &= E_{(a,b,\rho)} \left(\frac{2}{IJ} M_0^d \right) = \prod_{s=-\infty}^{\infty} \prod_{t=-\infty}^{\infty} (1 + b \rho^{|s|} (\rho^{|t|} + \rho^{|t+1|}))^{-a}, \\ \frac{4}{IJ} m_0^q &= E_{(a,b,\rho)} \left(\frac{4}{IJ} M_0^q \right) = \prod_{s=-\infty}^{\infty} \prod_{t=-\infty}^{\infty} (1 + b (\rho^{|s|} + \rho^{|s+1|}) (\rho^{|t|} + \rho^{|t+1|}))^{-a}. \end{aligned} \quad (3)$$

4 Consistency

To establish consistency we let $IJ \rightarrow \infty$ so that the number of cells observed increases. As we suppose the occupancy probabilities do not depend on I or J this implies that $N \rightarrow \infty$. This corresponds to increasing the size of the map.

Theorem 1 *If $(a, b, \rho) \in (0, \infty) \times (0, \infty) \times [0, 1)$, then $(\frac{1}{IJ} M_0, \frac{2}{IJ} M_0^d, \frac{4}{IJ} M_0^q)^\top$ is a weakly consistent estimator of $(E_{(a,b,\rho)}(\frac{1}{IJ} M_0), E_{(a,b,\rho)}(\frac{2}{IJ} M_0^d), E_{(a,b,\rho)}(\frac{4}{IJ} M_0^q))^\top$ as $IJ \rightarrow \infty$.*

The consistency of the estimators $(\hat{a}, \hat{b}, \hat{\rho})$ and thus of $\hat{N}$ follows from the following general proposition, which is similar to those of Amemiya (1973, Lemma 3), White (1980, Lemma 2.2) and White (1981, Theorem 2.1).

Proposition 2 *Let be $\Theta \subset \mathbb{R}^L$ compact, $g : \Theta \rightarrow \mathbb{R}^L$ continuous satisfying $g(\theta) \neq g(\theta')$ for all $\theta, \theta' \in \Theta$ with $\theta \neq \theta'$, and X_N an L dimensional random vector. If X_N is a weakly consistent estimator for $g(\theta)$, then*

$$\hat{\theta}_N \in \arg \min_{\theta \in \Theta} \|X_N - g(\theta)\|$$

is a weakly consistent estimator of θ .

Since $E(N)$ defined in (1) is a continuous function of (a, b, ρ) , Theorem 1 implies at once:

Theorem 2 *The abundance estimator $\hat{N}$ given by (2), where $(a, b, \rho) \in \Theta$ for a compact parameter space $\Theta \subset (0, \infty) \times (0, \infty) \times (0, 1)$ are solutions of (3), satisfies $\hat{N} - E(N) = o_P(IJ)$.*

Hence we need only to prove Theorem 1. This proof is not straightforward because of the dependence structure and therefore given together with the proof of Proposition 1 in the Appendix.

5 Application to real data

We consider data from a complete census of a tropical rain forest in Barro Colorado Island (BCI), Panama (Condit et al. 1996). The study area was a 50-hectare (500×1000 m) rectangle plot referred to as the BCI plot. The tree plot was established in 1980 and has been surveyed six times so far. The data from the 1985 census are used in this study. The data records the exact location of species for each free-standing tree with diameter at breast height (dbh) at least 1 cm in the plot. In the 1985 census, there are 238,018 trees representing 299 species. We illustrate the computation of the estimator on the species *Faramaea occidentalis*. Here $N = 25094$, $m_0 = 7824$, $m_0^d = 1994$ and $m_0^a = 357$. There were $20000 - 7824 = 12176$ occupied cells, giving a lower bound on N . We solved the estimating equations (3) with the products from negative to positive infinity replaced by products from $-T$ to T by minimizing an objective function consisting of the sums of squares of the differences. We considered $\rho = 0.05, 0.06, \dots, 0.99$ and for each value of ρ we found values of a and b to minimize the objective function. In Figure 1 we plot the value of the minimum and the estimated $\hat{N}$ as functions of ρ for $T = 60$. The minimum was quite flat as a function of ρ near the minimum, however, a plot of the logarithm of the minimum, not given here, did show a distinct minimum. At this minimum, $\hat{a} = 0.00066$, $\hat{b} = 3.27$, and $\hat{\rho} = 0.92$ and the estimated abundance was $\hat{N} = 24774$. In the second part of the figure we see that the estimated value of N is also quite flat and is close to the true value for a range of values of ρ .

Figure 1: The minimum value of the sums of squared differences of (3) and $\hat{N}$ as functions of $\rho \in (0, 1)$ for *Faramaea occidentalis*. In the second plot, the horizontal line gives the known value of N (solid line) and the lower bound on N (dashed line).

6 Discussion

We have demonstrated analytically that the new estimator is consistent and conclude that for the mixed gamma-Poisson model it is possible to estimate abundance from presence-absence maps in a quite general setting.

The assumption of a compact parameter space is required to prove Proposition 2 and hence Theorem 2, but since the compact parameter space can be arbitrarily large, this is not a strong restriction. We also require $g(\theta) \neq g(\theta')$ for all $\theta, \theta' \in \Theta$ with $\theta \neq \theta'$. This is an identifiability condition which is usually difficult to verify in nonlinear settings with more than two parameters. Here this means that the solutions $\hat{a}, \hat{b}, \hat{\rho}$ of the equations given by (3) are unique, i.e. (a, b, ρ) is identified by $(\frac{1}{IJ} m_0, \frac{2}{IJ} m_0^d, \frac{4}{IJ} m_0^q)^\top$. This is not the case for $\rho = 0$ or $a = 0$ or $b = 0$, but we conjecture that identifiability holds for $\rho > 0, a > 0, b > 0$. Since $\rho = 0$ does not satisfy the conditions of the Theorem, we have shown consistency only in the case where the cell counts are dependent. Moreover, consistency is obtained as $IJ \rightarrow \infty$ with the cell occupancy probabilities remaining the same so that $N \rightarrow \infty$. An alternate approach not considered here would be to allow the grid to become finer so that N is the number of individuals in a given area. In this case, the large sample properties are problematic as for a fine enough grid, N will be the total number of occupied cells.

We were not concerned with the practical application of the new estimator but have focused on consistency. In particular we have not addressed the goodness of fit of the model. The precision of the estimator may be addressed using bootstrap methods according to Hall 1988.

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A Proofs

To prove Theorem 1, some propositions must be proved. As Proposition 1 and Proposition 3 below are related we combine their proofs.

Proposition 3

$$\begin{aligned}
 a) \quad E(Y_{ij} \cdot Y_{kl}) &= \prod_{s=-\infty}^{\infty} \prod_{t=-\infty}^{\infty} (1 + b [\rho^{|i-s|+|j-t|} + \rho^{|k-s|+|l-t|}])^{-a}, \\
 b) \quad E(Y_{ij}^d \cdot Y_{kl}^d) &= \prod_{s=-\infty}^{\infty} \prod_{t=-\infty}^{\infty} (1 + b [\rho^{|i-s|}(\rho^{|j-t|} + \rho^{|j+1-t|}) + \rho^{|k-s|}(\rho^{|l-t|} + \rho^{|l+1-t|})])^{-a}, \\
 c) \quad E(Y_{ij}^q \cdot Y_{kl}^q) &= \prod_{s=-\infty}^{\infty} \prod_{t=-\infty}^{\infty} (1 + b [(\rho^{|i-s|} + \rho^{|i+1-s|})(\rho^{|j-t|} + \rho^{|j+1-t|}) \\
 &\quad + (\rho^{|k-s|} + \rho^{|k+1-s|})(\rho^{|l-t|} + \rho^{|l+1-t|})])^{-a}.
 \end{aligned}$$

Proof of Propositions 1 and 3.

Since a Poisson random variable X with parameter λ satisfies $P(X = 0) = \exp\{-\lambda\}$, using the moment generating function of the gamma distribution and the independence of the V_{st} , we have

$$\begin{aligned}
 E(Y_{ij}) &= P(X_{ij} = 0) \\
 &= \prod_{s=-\infty}^{\infty} \prod_{t=-\infty}^{\infty} E(\exp\{-\rho^{|i-s|+|j-t|} V_{st}\}) = \prod_{s=-\infty}^{\infty} \prod_{t=-\infty}^{\infty} (1 + b \rho^{|i-s|+|j-t|})^{-a}
 \end{aligned}$$

which is Proposition 1 a). Since X_{ij} and X_{kl} are conditionally independent given V_{st} with $s, t \in \mathbb{Z}$, Proposition 1 b) and Proposition 3 a) follow from

$$\begin{aligned}
 E(Y_{ij} \cdot Y_{kl}) &= P(X_{ij} = 0, X_{kl} = 0) \\
 &= E\left(\exp\left\{-\sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} (\rho^{|i-s|+|j-t|} + \rho^{|k-s|+|l-t|}) V_{st}\right\}\right).
 \end{aligned}$$

Proposition 1 c) is a special case of Proposition 3 b). Since b) and c) of Proposition 3 can be proved similarly, only the proof of c) is shown here:

$$\begin{aligned}
 & E(Y_{ij}^q \cdot Y_{kl}^q) \\
 &= P(X_{ij} = 0, X_{i(j+1)} = 0, X_{(i+1)j} = 0, X_{(i+1)(j+1)} = 0, \\
 &\quad X_{kl} = 0, X_{k(l+1)} = 0, X_{(k+1)l} = 0, X_{(k+1)(l+1)} = 0) \\
 &= E \left(\exp \left[- \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} (\rho^{|i-s|+|j-t|} + \rho^{|i-s|+|j+1-t|} + \rho^{|i+1-s|+|j-t|} + \rho^{|i+1-s|+|j+1-t|} \right. \right. \\
 &\quad \left. \left. + \rho^{|k-s|+|l-t|} + \rho^{|k-s|+|l+1-t|} + \rho^{|k+1-s|+|l-t|} + \rho^{|k+1-s|+|l+1-t|}) V_{st} \right] \right) . \square
 \end{aligned}$$

Proposition 4 Let be $(a, b, \rho) \in (0, \infty) \times (0, \infty) \times [0, 1)$. Then there exists a constant c only depending on a, b, ρ such that

$$\begin{aligned}
 & \sum_{i=1}^I \sum_{i=1}^I \sum_{j=1}^J \sum_{j=1}^J \ln \left(\frac{E(Y_{ij} Y_{kl})}{E(Y_{ij}) E(Y_{kl})} \right) \leq c I J, \\
 & \sum_{i=1}^I \sum_{i=1}^I \sum_{j=1}^{J/2} \sum_{j=1}^{J/2} \ln \left(\frac{E(Y_{i(2j-1)}^d Y_{k(2l-1)}^d)}{E(Y_{i(2j-1)}^d) E(Y_{k(2l-1)}^d)} \right) \leq c I J, \\
 & \sum_{i=1}^{I/2} \sum_{i=1}^{I/2} \sum_{j=1}^{J/2} \sum_{j=1}^{J/2} \ln \left(\frac{E(Y_{(2i-1)(2j-1)}^q Y_{(2k-1)(2l-1)}^q)}{E(Y_{(2i-1)(2j-1)}^q) E(Y_{(2k-1)(2l-1)}^q)} \right) \leq c I J.
 \end{aligned}$$

To prove Proposition 4, we need

Lemma 1 Let $\alpha_i(s) = \rho^{|i-s|} + \rho^{|i+1-s|}$ for $i = 1, \dots, I$. Then

$$\sum_{i=1}^I \sum_{k=1}^I \sum_{s=-\infty}^{\infty} \alpha_i(s) \alpha_k(s) \leq \begin{cases} 12 I, & \text{for } \rho = 0, \\ \frac{1}{(1-\rho^2)(1-\rho)^4} \left(4 \cdot 42 I + \frac{4}{\rho} 20 I \right), & \text{for } \rho \in (0, 1), \end{cases}$$

for all $I \in \mathbb{N}$.

Proof.

If $\rho \in (0, 1)$, then $\alpha_i(s) = \rho^{i-s} + \rho^{i+1-s} = \rho^{i-s}(1 + \rho) \leq 2\rho^{i-s}$ for $s \leq i$ and $\alpha_i(s) = \rho^{s-i} + \rho^{s-(i+1)} = \rho^{s-i-1}(\rho + 1) \leq 2\rho^{s-i-1}$ for $s \geq i + 1$. Analogous bounds hold for $\alpha_k(s)$. Let $i \leq k$. Then

$$\begin{aligned}
 & \sum_{s=-\infty}^{\infty} \alpha_i(s) \alpha_k(s) \\
 & \leq \sum_{s=-\infty}^i (2\rho^{i-s}) (2\rho^{k-s}) + \sum_{s=i+1}^k (2\rho^{s-i-1}) (2\rho^{k-s}) + \sum_{s=k+1}^{\infty} (2\rho^{s-i-1}) (2\rho^{s-k-1}) \\
 & = 4 \left[\rho^{k-i} \sum_{s=0}^{\infty} (\rho^2)^s + \rho^{k-i-1} (k-i) + \rho^{k-i} \sum_{s=0}^{\infty} (\rho^2)^s \right] \\
 & = 4 \left[\rho^{k-i} \frac{1}{1-\rho^2} + \rho^{k-i-1} (k-i) + \rho^{k-i} \frac{1}{1-\rho^2} \right] \\
 & \leq 4 \frac{\rho^{|k-i|-1}}{1-\rho^2} (2\rho + |k-i|).
 \end{aligned}$$

Similarly, the same upper bound holds for $i \geq k$. As

$$\begin{aligned} \sum_{i=1}^I \sum_{k=1}^I \rho^{|k-i|} |k-i| &= \frac{2}{(1-\rho)^4} ((I-1)\rho - 2I\rho^2 + (I+1)\rho^3 + (I+1)\rho^{I+1} \\ &\quad - 2I\rho^{I+2} + (I-1)\rho^{I+3}) \leq \frac{2}{(1-\rho)^4} 10I, \end{aligned}$$

we see that

$$\begin{aligned} &\sum_{i=1}^I \sum_{k=1}^I \sum_{s=-\infty}^{\infty} \alpha_i(s) \alpha_k(s) \\ &\leq \sum_{i=1}^I \sum_{k=1}^I 4 \frac{\rho^{|k-i|-1}}{1-\rho^2} (2\rho + |k-i|) \\ &\leq \frac{4}{1-\rho^2} \left(2 \sum_{i=1}^I \sum_{k=1}^I \rho^{|k-i|} |k-i| + 2I + \frac{1}{\rho} \sum_{i=1}^I \sum_{k=1}^I \rho^{|k-i|} |k-i| \right) \\ &\leq \frac{4}{1-\rho^2} \left(2 \frac{2}{(1-\rho)^4} 10I + 2I + \frac{1}{\rho} \frac{2}{(1-\rho)^4} 10I \right) \\ &\leq \frac{1}{(1-\rho^2)(1-\rho)^4} \left(4 \cdot 42I + \frac{4}{\rho} 20I \right). \end{aligned}$$

If $\rho = 0$, we obtain using $0^0 = 1$

$$\sum_{s=-\infty}^{\infty} \alpha_i(s) \alpha_k(s) \leq \begin{cases} 4, & \text{for } |k-i| \leq 1, \\ 0, & \text{for } |k-i| > 1, \end{cases}$$

as required. $\square$

Proof of Proposition 4.

To complete the proof of Proposition 4, let

$$\begin{aligned} \alpha_i^0(s) &= \rho^{|i-s|}, & \alpha_i(s) &= \rho^{|i-s|} + \rho^{|i+1-s|}, \\ \beta_j^0(t) &= \rho^{|j-t|}, & \beta_j(t) &= \rho^{|j-t|} + \rho^{|j+1-t|}. \end{aligned}$$

Setting

$$\begin{aligned} \alpha_i^1(s) &= \alpha_i^0(s), & \beta_j^1(t) &= \beta_j^0(t), \\ \alpha_i^2(s) &= \alpha_i^0(s), & \beta_j^2(t) &= \beta_j(t), \\ \alpha_i^3(s) &= \alpha_i(s), & \beta_j^3(t) &= \beta_j(t), \end{aligned}$$

and

$$Y_{ij}^1 = Y_{ij}, \quad Y_{ij}^2 = Y_{ij}^d, \quad Y_{ij}^3 = Y_{ij}^q,$$

Propositions 1 and 3 provide for $m = 1, 2, 3$

$$\begin{aligned} E(Y_{ij}^m) &= \prod_{s=-\infty}^{\infty} \prod_{t=-\infty}^{\infty} (1 + b \alpha_i^m(s) \beta_j^m(t))^{-a}, \\ E(Y_{ij}^m Y_{kl}^m) &= \prod_{s=-\infty}^{\infty} \prod_{t=-\infty}^{\infty} (1 + b [\alpha_i^m(s) \beta_j^m(t) + \alpha_k^m(s) \beta_l^m(t)])^{-a}. \end{aligned}$$

The inequality $\ln(1+x) \leq x$ for all $x \geq 0$ yields

$$\begin{aligned}
 & \ln \left(\frac{E(Y_{ij}^m Y_{kl}^m)}{E(Y_{ij}^m) E(Y_{kl}^m)} \right) \\
 &= \ln \left(\prod_{s=-\infty}^{\infty} \prod_{t=-\infty}^{\infty} \frac{(1+b [\alpha_i^m(s) \beta_j^m(t) + \alpha_k^m(s) \beta_l^m(t)])^{-a}}{(1+b \alpha_i^m(s) \beta_j^m(t))^{-a} (1+b \alpha_k^m(s) \beta_l^m(t))^{-a}} \right) \\
 &= a \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} \ln \left(\frac{(1+b \alpha_i^m(s) \beta_j^m(t)) (1+b \alpha_k^m(s) \beta_l^m(t))}{1+b [\alpha_i^m(s) \beta_j^m(t) + \alpha_k^m(s) \beta_l^m(t)]} \right) \\
 &= a \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} \ln \left(1 + \frac{b^2 \alpha_i^m(s) \beta_j^m(t) \alpha_k^m(s) \beta_l^m(t)}{1+b [\alpha_i^m(s) \beta_j^m(t) + \alpha_k^m(s) \beta_l^m(t)]} \right) \\
 &\stackrel{a>0, b>0, \rho \geq 0}{\leq} a \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} \ln (1 + b^2 \alpha_i^m(s) \beta_j^m(t) \alpha_k^m(s) \beta_l^m(t)) \\
 &\leq a \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} b^2 \alpha_i^m(s) \beta_j^m(t) \alpha_k^m(s) \beta_l^m(t).
 \end{aligned}$$

Let

$$\begin{aligned}
 \mathcal{I}^1 &= \{i; i = 1, \dots, I\}, \quad \mathcal{J}^1 = \{j; j = 1, \dots, J\}, \\
 \mathcal{I}^2 &= \{i; i = 1, \dots, I\}, \quad \mathcal{J}^2 = \{j; j = 2n-1 \text{ with } n = 1, \dots, J/2\}, \\
 \mathcal{I}^3 &= \{i; i = 2n-1 \text{ with } n = 1, \dots, I/2\}, \\
 \mathcal{J}^3 &= \{j; j = 2n-1 \text{ with } n = 1, \dots, J/2\}.
 \end{aligned}$$

Then for $m = 1, 2, 3$ using Lemma 1, and noting $\alpha_i^m(s) \leq \alpha_i(s)$ for $i = 1, \dots, I$ and $\beta_j^m(t) \leq \beta_j(t)$ for $j = 1, \dots, J$ we see that

$$\begin{aligned}
 & \sum_{i \in \mathcal{I}^m} \sum_{j \in \mathcal{J}^m} \sum_{k \in \mathcal{I}^m} \sum_{l \in \mathcal{J}^m} \ln \left(\frac{E(Y_{ij}^m Y_{kl}^m)}{E(Y_{ij}^m) E(Y_{kl}^m)} \right) \\
 &\leq a \sum_{i \in \mathcal{I}^m} \sum_{j \in \mathcal{J}^m} \sum_{k \in \mathcal{I}^m} \sum_{l \in \mathcal{J}^m} \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} b^2 \alpha_i^m(s) \beta_j^m(t) \alpha_k^m(s) \beta_l^m(t) \\
 &\leq a \sum_{i=1}^I \sum_{j=1}^J \sum_{k=1}^I \sum_{l=1}^J \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} b^2 \alpha_i(s) \beta_j(t) \alpha_k(s) \beta_l(t) \\
 &= a b^2 \sum_{i=1}^I \sum_{k=1}^I \sum_{s=-\infty}^{\infty} \alpha_i(s) \alpha_k(s) \sum_{j=1}^J \sum_{l=1}^J \sum_{t=-\infty}^{\infty} \beta_j(t) \beta_l(t) \\
 &\leq \begin{cases} a b^2 12^2 I J, & \text{for } \rho = 0 \\ a b^2 \left(\frac{1}{(1-\rho^2)(1-\rho)^4} \left(4 \cdot 42 + \frac{4}{\rho} \cdot 20 \right) \right)^2 I J, & \text{for } \rho \in (0, 1) \end{cases} \\
 &= c I J,
 \end{aligned}$$

where the constant c depends only on a, b, ρ . $\square$

Proposition 5 For $(a, b, \rho) \in (0, \infty) \times (0, \infty) \times [0, 1)$

- a) $\text{var}(M_0) \leq c I J,$
- b) $\text{var}(M_0^d) \leq c I J,$
- c) $\text{var}(M_0^q) \leq c I J,$

where c is a constant only depending on $a, b,$ and ρ .

Proof of Proposition 5.

Using the same notation as in the proof of Proposition 4, we have

$$M_0 = \sum_{i \in \mathcal{I}^1} \sum_{j \in \mathcal{J}^1} Y_{ij}^1, \quad M_0^d = \sum_{i \in \mathcal{I}^2} \sum_{j \in \mathcal{J}^2} Y_{ij}^2, \quad M_0^q = \sum_{i \in \mathcal{I}^3} \sum_{j \in \mathcal{J}^3} Y_{ij}^3.$$

Hence for $m = 1, 2, 3$ we have only to prove

$$\text{var} \left(\sum_{i \in \mathcal{I}^m} \sum_{j \in \mathcal{J}^m} Y_{ij}^m \right) \leq c I J.$$

First note that

$$E(Y_{ij}^m) = E(Y_{kl}^m) \quad \text{and} \quad \text{cov}(Y_{ij}^m, Y_{kl}^m) \leq \text{var}(Y_{ij}^m) = \text{var}(Y_{kl}^m)$$

for all $i, k = 1, \dots, I, j, l = 1, \dots, J,$ and $m = 1, 2, 3.$ This implies

$$\begin{aligned} \frac{\text{var}(Y_{11}^m)}{E(Y_{11}^m)^2} &= \frac{\text{var}(Y_{ij}^m)}{E(Y_{ij}^m)^2} \geq \frac{\text{cov}(Y_{ij}^m, Y_{kl}^m)}{E(Y_{ij}^m)^2} \\ &= \frac{E(Y_{ij}^m Y_{kl}^m) - E(Y_{ij}^m) E(Y_{kl}^m)}{E(Y_{ij}^m) E(Y_{kl}^m)} = \frac{E(Y_{ij}^m Y_{kl}^m)}{E(Y_{ij}^m) E(Y_{kl}^m)} - 1, \end{aligned}$$

so that

$$\frac{E(Y_{ij}^m Y_{kl}^m)}{E(Y_{ij}^m) E(Y_{kl}^m)} \leq \frac{\text{var}(Y_{11}^m)}{E(Y_{11}^m)^2} + 1 =: c_1 \quad (4)$$

for all $i, k = 1, \dots, I, j, l = 1, \dots, J,$ and $m = 1, 2, 3.$ The mean value theorem provides

$$\begin{aligned} \frac{E(Y_{ij}^m Y_{kl}^m)}{E(Y_{ij}^m) E(Y_{kl}^m)} - 1 &= \exp \left(\ln \left(\frac{E(Y_{ij}^m Y_{kl}^m)}{E(Y_{ij}^m) E(Y_{kl}^m)} \right) \right) - \exp(0) \\ &= \exp(\theta_{ijkl}) \left(\ln \left(\frac{E(Y_{ij}^m Y_{kl}^m)}{E(Y_{ij}^m) E(Y_{kl}^m)} \right) - 0 \right) \end{aligned}$$

with

$$\theta_{ijkl} \in \left[0, \ln \left(\frac{E(Y_{ij}^m Y_{kl}^m)}{E(Y_{ij}^m) E(Y_{kl}^m)} \right) \right] \quad \text{if} \quad \ln \left(\frac{E(Y_{ij}^m Y_{kl}^m)}{E(Y_{ij}^m) E(Y_{kl}^m)} \right) \geq 0$$

and

$$\theta_{ijkl} \in \left[\ln \left(\frac{E(Y_{ij}^m Y_{kl}^m)}{E(Y_{ij}^m) E(Y_{kl}^m)} \right), 0 \right] \quad \text{if} \quad \ln \left(\frac{E(Y_{ij}^m Y_{kl}^m)}{E(Y_{ij}^m) E(Y_{kl}^m)} \right) < 0.$$

In particular, θ_{ijkl} satisfies because of (4)

$$\theta_{ijkl} \leq \ln \left(\frac{\text{var}(Y_{11}^m)}{E(Y_{11}^m)^2} + 1 \right) = \ln(c_1)$$

so that

$$\frac{E(Y_{ij}^m Y_{kl}^m)}{E(Y_{ij}^m) E(Y_{kl}^m)} - 1 \leq \exp(\ln(c_1)) \ln \left(\frac{E(Y_{ij}^m Y_{kl}^m)}{E(Y_{ij}^m) E(Y_{kl}^m)} \right) = c_1 \ln \left(\frac{E(Y_{ij}^m Y_{kl}^m)}{E(Y_{ij}^m) E(Y_{kl}^m)} \right).$$

Then Proposition 4 provides

$$\begin{aligned} & \text{var} \left(\sum_{i \in \mathcal{I}^m} \sum_{j \in \mathcal{J}^m} Y_{ij}^m \right) \\ &= \sum_{i \in \mathcal{I}^m} \sum_{j \in \mathcal{J}^m} \sum_{k \in \mathcal{I}^m} \sum_{l \in \mathcal{J}^m} \text{cov}(Y_{ij}^m, Y_{kl}^m) \\ &= \sum_{i \in \mathcal{I}^m} \sum_{j \in \mathcal{J}^m} \sum_{k \in \mathcal{I}^m} \sum_{l \in \mathcal{J}^m} \frac{E(Y_{ij}^m Y_{kl}^m) - E(Y_{ij}^m) E(Y_{kl}^m)}{E(Y_{ij}^m) E(Y_{kl}^m)} E(Y_{ij}^m) E(Y_{kl}^m) \\ &= \sum_{i \in \mathcal{I}^m} \sum_{j \in \mathcal{J}^m} \sum_{k \in \mathcal{I}^m} \sum_{l \in \mathcal{J}^m} \left(\frac{E(Y_{ij}^m Y_{kl}^m)}{E(Y_{ij}^m) E(Y_{kl}^m)} - 1 \right) E(Y_{11}^m)^2 \\ &\leq \sum_{i \in \mathcal{I}^m} \sum_{j \in \mathcal{J}^m} \sum_{k \in \mathcal{I}^m} \sum_{l \in \mathcal{J}^m} c_1 \ln \left(\frac{E(Y_{ij}^m Y_{kl}^m)}{E(Y_{ij}^m) E(Y_{kl}^m)} \right) E(Y_{11}^m)^2 \\ &= c_1 E(Y_{11}^m)^2 \sum_{i \in \mathcal{I}^m} \sum_{j \in \mathcal{J}^m} \sum_{k \in \mathcal{I}^m} \sum_{l \in \mathcal{J}^m} \ln \left(\frac{E(Y_{ij}^m Y_{kl}^m)}{E(Y_{ij}^m) E(Y_{kl}^m)} \right) \\ &\leq c_1 E(Y_{11}^m)^2 c_2 I J \\ &= c I J, \end{aligned}$$

where as c_1 , $E(Y_{11}^m)^2$, and c_2 only depend on a , b , and ρ so does c . $\square$

Proof of Theorem 1.

Theorem 1 now follows from Proposition 5 a), b) and c) and Chebyshev's inequality.

$\square$