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PROPAGATION OF CHAOS FOR THE 2D VISCOUS VORTEX MODEL.

NICOLAS FOURNIER, MAXIME HAURAY, STÉPHANE MISCHLER

Abstract. We consider a stochastic system of N particles, usually called vortices in that setting, approximating the 2D Navier-Stokes equation written in vorticity. Assuming that the initial distribution of the position and circulation of the vortices has finite (partial) entropy and a finite moment of positive order, we show that the empirical measure of the particle system converges in law to the unique (under suitable a priori estimates) solution of the 2D Navier-Stokes equation. We actually prove a slightly stronger result : the propagation of chaos of the stochastic paths towards the solution of the expected nonlinear stochastic differential equation. Moreover, the convergence holds in a strong sense, usually called entropic (there is no loss of entropy in the limit). The result holds without restriction (but positivity) on the viscosity parameter.

The main difficulty is the presence of the singular Biot-Savart kernel in the equation. To overcome this problem, we use the dissipation of entropy which provides some (uniform in N) bound on the Fisher information of the particle system, and then use extensively that bound together with classical and new properties of the Fisher information.

1. Introduction

The subject of this paper is the convergence of a stochastic vortices system to the $2D$ Navier-Stokes equation written in vorticity without restriction (but positivity) on the viscosity parameter.

The particle system. We consider a system of N vortices labeled by an index $1 \le i \le N$, each one being fully described by its position $\mathcal{X}_i^N \in \mathbb{R}^2$ and its circulation $\frac{1}{N}\mathcal{M}_i^N \in \mathbb{R}$ which measures the "strength" of the vortices. We use what is called a *mean-field* scaling : the factor $\frac{1}{N}$ is there in order to keep the global circulation (or also total vorticity) bounded. The case $\mathcal{M}_i^N > 0$ corresponds to a vortex which turns round in the direct (trigonometric) sense while the case $\mathcal{M}_i^N < 0$ corresponds to a vortex which turns in the reverse sense. We assume that the system evolves stochastically according to the following system of $\mathbb{R}^2$-valued S.D.E.s on the vortices positions

$$\forall i = 1, \dots, N, \quad \mathcal{X}_i^N(t) = \mathcal{X}_i^N(0) + \frac{1}{N}\sum_{j\neq i}\mathcal{M}_j^N \int_0^t K(\mathcal{X}_i^N(s) - \mathcal{X}_j^N(s))ds + \sigma\mathcal{B}_i(t), \tag{1.1}$$

where $((\mathcal{B}_i(t))_{t\ge0})_{i=1,\dots,N}$ stands for an independent family of $2D$ standard Brownian motions, $\sigma > 0$ is a parameter linked to the viscosity and $K : \mathbb{R}^2 \to \mathbb{R}^2$ is the Biot-Savart kernel defined by

$$\forall\, x = (x_1, x_2) \in \mathbb{R}^2, \quad K(x) = \frac{x^\perp}{|x|^2} = \Big(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\Big) = \nabla^\perp \log|x|,$$

It is worth emphasizing that we assume here that any vortex keeps its initial circulation, so that the $\mathcal{M}_i^N$ are time-independent and act like fixed parameters in (1.1). Each vortex's position moves

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randomly according to a Brownian noise as well as deterministically according to a vector field generated by all the other vortices through the Biot-Savart kernel. The singularity of K makes difficult the study of the particle system (1.1). However, Osada [42] and others have shown that the particles a.s. never encounter, so that the singularity of K is a.s. never visited and (1.1) is well-posed. See Theorem 2.10 and Section 2.5 below for more details.

The vorticity equation. As is now well-known, the dynamics of such models is linked to the $2D$ Navier-Stokes equation written in vorticity formulation which will be called later the *vorticity equation* with a viscosity $\nu = \sigma^2/2$

$$\partial_t w_t(x) = (K \star w_t)(x) \cdot \nabla_x w_t(x) + \nu \Delta_x w_t(x), \tag{1.2}$$

where $w : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ is the vorticity function and the initial vorticity $w_0 : \mathbb{R}^2 \to \mathbb{R}$ is given. It is worth emphasizing again that we do not assume here that w is non-negative and this is of course related to the fact that the circulations in the N-vortex system may be positive and negative. There is a huge literature on that model. Our analysis will be based on the well-posedness of (1.2) in a L^1-framework as developed in Ben-Artzi in [3] and slightly improved in [7]. We also refer to Gallagher-Gallay [20] and the references therein for more recent well-posedness results.

Origin of the model. The deterministic N-particle system (with $\sigma = 0$) was originally introduced by Helmholtz in 1858 [25] and later studied by Kirchhoff [28] and many others. It is sometimes quoted as the Helmholtz-Kirchhoff (HK in short) system. In the non-viscous case $\sigma = 0$, the empirical measure associated to the finite particle system (1.1) solves *exactly* the non viscous vorticity equation (i.e. (1.2) with $\sigma = 0$) if the self interaction is neglected. This is not the case when $\sigma > 0$, where the Navier-Stokes equation is expected to be solved only in the limit $N \to \infty$. Thus for a fixed N, addition of noise on the position of the vortices as in (1.1) is not the most relevant idea. Physicists prefer to introduce damped vortices, known as Oseen vortices. The dynamics of Oseen vortices is still driven by the a variant of the deterministic HK system, where K now varies with time, in order to take into account the dissipation. Both systems (vortices driven by (1.1) or Oseen vortices) are interesting because they approximate the dynamics of real vortices, that appear, for instance, in geostrophic or atmospheric flows, and are remarkably stable. See the works of Marchioro [32] and Gallay [21] for a justification of the approximation, and the one of Gallay-Wayne [22] for a precise mathematical result on the stability of Oseen vortices.

Interest of the limit in large number of vortices. Later, Onsager [40] was the first to see the interest of the statistical properties of the N vortices system to distinguish which among the numerous stationary solutions of the vorticity equation are physically relevant. His heuristic ideas where made more rigorous by Caglioti, Lions, Marchioro and Pulvirenti in [9]. After that, the question of the convergence of the HK model towards Euler and Navier-Stokes equations was studied by several authors. In the deterministic case Schochet [47] proved the convergence towards solutions of the Euler equation. But since such solutions seem to be numerous under weak a priori conditions, his results does not implies propagation of chaos.

As mentioned before, the stochastic model (1.1) becomes much more relevant when N is large. However, in that case the use of independent noise on each vortex is not motivated by the underlying physics. Since vortices are not real *particles* but rather small structures appearing in fluid models, a noise acting on them should depend on their relative positions: the noise of two close vortices should be quite correlated. We refer to the work of Flandoli, Gubinelli and Priola [18] for such more realistic models with a fixed number of vortices. But that kind of noise is much more difficult to handle in the limit of large number of vortices. Thus in the sequel, we will use independent Brownian motion, despite this shortcoming.

A second interest of the stochastic vortices system is that it may be seen as a companion model for stochastic particle systems with positions and velocities, interacting trough a two-body force and with velocities excited by independent brownian motions. The case of a smooth interaction force has been extensively studied since the work of McKean [35], but there is only few references in the case of singular interactions. Rather strangely, the deterministic case is better known since there exist some convergence results for not too singular interaction without cut-off [23]. However, we do not know any result valid for singular interaction in the stochastic case. In fact such models seem tougher than the vortices one, since the diffusion does not act on the full position-velocity variables.

Main result. In order to connect the sequence of solutions $\mathcal{Z}_t^N = (\mathcal{M}_1, \mathcal{X}_1^N(t), ..., \mathcal{M}_N, \mathcal{X}_N^N(t))$, to the random particle system (1.1) with a solution to the vorticity equation (1.2), we introduce the vorticity empirical measure

$$\mathcal{W}_t^N(dx) := \frac{1}{N}\sum_{i=1}^N \mathcal{M}_i^N \, \delta_{\mathcal{X}_i^N(t)}(dx)$$

which a.s. takes values in the space of bounded measures on $\mathbb{R}^2$, as well as the typical vorticity defined from the law of $(\mathcal{M}_1^N, \mathcal{X}_1^N(t))$ in the following way

$$w_1^N(t,x) := \int_{\mathbb{R}} m \, \mathscr{L}(\mathcal{M}_1^N, \mathcal{X}_1^N(t))(dm, x).$$

Then, under suitable (chaos) hypothesis on the initial conditions $\mathcal{Z}_0^N$ we shall show that for any positive time

$$\mathcal{W}_t^N \;\Rightarrow\; w_t \quad \text{in law as} \quad N \to \infty, \tag{1.3}$$

$$w_1^N(t) \to w_t \quad \text{strongly in } L^1(\mathbb{R}^2) \text{ as } N \to \infty, \tag{1.4}$$

where w is the unique solution to the vorticity equation (1.2) with appropriate initial datum w_0.

In the other way round and in particular, for any initial (Lebesgue measurable) vorticity function $w_0 : \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\int_{\mathbb{R}^2} |w_0| \, (1 + |x|^k + |\log |w_0||) \, dx < \infty, \quad \text{for some} \quad k \in (0,2), \tag{1.5}$$

we can build a sequence of initial conditions $\mathcal{Z}_0^N$ and then define the family of solutions $\mathcal{Z}_t^N$ to the N-particle vortex system (1.1) so that the vorticity empirical measure $\mathcal{W}_t^N$ and the typical vorticity $w_1^N(t)$ converge to the solution w_t as stated in (1.3) and (1.4), where w is the unique solution to the vorticity equation (1.2) with initial datum w_0.

The construction of the initial conditions is not very elaborated, but requires some notation that will be introduced in Section 2. Let us just mention that in the case of a non-negative initial vorticity, we can assume up to some scaling that ω_0 is a probability, and then the choice $\mathcal{M}_i^N = 1$ for any $1 \le i \le N$ and $(\mathcal{X}_i^N(0))_{1 \le i \le N}$ i.i.d. with law ω_0 will do the job.

Chaos and limit trajectories. The solution w of the vorticity equation is thus obtained as the limit in a kind of law of large numbers from the N-vortex system. However, the picture is not that simple.

It is indeed rather reasonable to assume that the initial positions and circulations of the vortices are (at least asymptotically) independent. Then, as time passes, vortices interact and that create correlations so that vortices are never any longer independent. We may expect that these correlations vanish asymptotically because the interactions between pairs of vortices in (1.1) tends to zero. For a smooth (say Lipschitz) interaction force field K, such a result is well-known since the pioneer work by McKean [35], and it is related to the notions of *"chaos"* and *"propagation of chaos"* as introduced by Kac in [27]. Here, the Biot-Savard force field kernel is singular which leads to additional mathematical difficulties. Nevertheless, we are able to handle them and we prove that still for the Biot-Savard kernel correlations vanish asymptotically.

We establish this asymptotic independence and the convergence (1.3) as a consequence of a stronger *"trajectorial chaos"* that we briefly describe now. The method we follow is closely related to the strategy introduced by Sznitman in [49] which consists in showing that the sequence of empirical trajectories $\mu^N_{\mathcal{Z}^N}$ converges to some stochastic process which is a solution to a nonlinear martingale problem.

Let us first notice that if we accept that correlations asymptotically disappear, then the trajectories (and circulations) $(\mathcal{M}^N_i, (\mathcal{X}^N_i(t))_{t\geq 0})_{i\leq N}$ of the vortices must behave asymptotically like N independent copies of the same process, $(\mathcal{M}, (\mathcal{X}(t))_{t\geq 0}$, solution to the nonlinear stochastic differential equation

$$\mathcal{X}(t) = \mathcal{X}(0) + \int_0^t \int_{\mathbb{R}^2} K(\mathcal{X}(s) - x) w_s(dx) ds + \sigma \mathcal{B}_t, \tag{1.6}$$

where $w_t(dx) = \int_{\mathbb{R}\times B} m\, g_t(dm, dx)$ with $g_t = \mathcal{L}(\mathcal{M}, \mathcal{X}(t))$. It is important to stress that w_t solves necessarily the vorticity equation (1.2) if $(\mathcal{M}, (\mathcal{X}(t))_{t\geq 0}$ is a solution to (1.6).

As a matter of fact, we will prove that under appropriate hypothesis on the initial law $\mathcal{L}(\mathcal{Z}^N_0)$ (which includes chaos type assumption and bound on the entropy and on some moment), the N-vorticity system enjoys a chaos property at the level of the trajectories, namely,

$$\mu^N_{\mathcal{Z}^N} := \frac{1}{N} \sum_{i=1}^N \delta_{(\mathcal{M}^N_i, (\mathcal{X}^N_i(t))_{t\geq 0})} \;\Rightarrow\; g \quad \text{in law as} \quad N \to \infty, \tag{1.7}$$

where g is the law of the nonlinear process $(\mathcal{M}, (\mathcal{X}(t))_{t\geq 0}$ defined in (1.6). This convergence at the level of trajectories implies (1.3). Moreover, using a trick introduced in [38] which consists in carefully estimating what happens for the dissipation of the entropy, we deduce (1.4).

An overview of the proof. Let us briefly describe our method, which relies on compactness/consistency/uniqueness as in Sznitman in [49] who was studying the homogeneous Boltzmann equation. As already mentioned, the main difficulty comes from the fact that the kernel K is singular so that the drift in (1.1) may be very large when two particles are very close. Using standard dissipation of entropy estimates, we obtain uniform bounds on the Fisher information of the time marginal of the law (of the positions) of the N vortices. These uniform bounds provide enough regularity to

(1) prove that *close encounters* of particles are rare, from which we deduce the tightness of the law of the trajectories of the N vortices system (compactness),
(2) prove that the possible limit are made of solutions of the nonlinear SDE (consistency), which satisfies some appropriate additional a priori bounds,
(3) prove the uniqueness of the above limit stochastic process.

We may also remark that for the two first points (tightness and consistency), the singularity of the kernel in $1/|x|$ is not the critical one. Everything would work for divergence free kernels with singularity behaving like $1/|x|^\alpha$, with $\alpha \in (0, 2)$. But, it is critical for the question of uniqueness of the limit stochastic process and the Navier-Stokes equation.

We also emphasize the following important point. In order to get enough *a priori* bounds on the possible limit, we use a new result: the fact that the Fisher information (properly rescaled) can only decrease when we perform the many-particle limit. This is a consequence of the fact that the (so called) level 3 Fisher information is linear on mixed states. That property, known from the work of Robinson and Ruelle for the entropy [46] was proved for the Fisher information by the two last authors in [24], with the help of the first author. The precise result is properly stated and explained in Section 4.

We can also remark that our method is interesting only in low dimension (of space). The extensive use of the Fisher information is very interesting in this case, since it provides rather strong regularity. But, if we increase the dimension, the regularity obtained from the Fisher information gets weaker and weaker and we will not be able to treat interesting singularities. As a matter of fact, it can be checked that our method is valid for a divergence free kernel, with a singularity at most like $1/|x|$ (included) near 0 in any dimension d. Again, the limitation will come from the uniqueness part, the tightness and consistency will also hold for singularity up to $1/|x|^2$, not included.

Already known results. If we replace the singular kernel K by a regularized one K_ε, the result of propagation of chaos is well-known. The more standard strategy is due to McKean [35] and applies when the interaction is Lipschitz. It relies on a coupling argument between the solution of the N-vortices stochastic system and N independent copies of the solution of the nonlinear SDE. But since the result gives a quantitative estimate of convergence, an optimization may lead to a similar result valid for a regularization parameter ε going to 0 with N. That approach or some variants was performed by Marchioro and Pulvirenti in [33], for bounded initial vorticity. For that regularity on the initial condition, there is a good well-posedness theory even in the non-viscous case, so that their method also applies if $\nu = 0$. The drawback is that the speed of convergence of the regularization parameter is very slow : $\varepsilon(N) \sim \log(N)^{-1}$. See also Méléard [37] for a similar result for more general initial data.

It is worth emphasizing that the convergences (1.3), (1.4), (1.7) are proved without any rate. This is a consequence of the compactness method we use. In particular, we were not able to implement the coupling method popularized by Sznitman [50] and revisited by Malrieu [31], see also [4] and the references therein for recent developments, nor the quantitative Grunbaum's duality method elaborated in [39].

In a series of papers, Osada proves the convergence of the particle system (1.1) to the vorticity equation (1.2): the case of a large viscosity is studied in [44] and the case of any positive viscosity is discussed in [45]. In this last paper, the pathwise convergence is not obtained (while it is checked in [44] when σ is large enough). His strategy relies strongly on a deep result obtained by himself in [41]: estimates *à la Nash* for convection-diffusion equation, with divergence free and very singular drift. This last result is also a key argument in most works about existence and uniqueness for the $2D$ Navier-Stokes equation, with the exception of the work of Ben-Artzi [3] that we use here.

Let us finally mention the result of Cepa and Lepingle [12] about Coulomb gas models in dimension one. Their models are very similar to ours, but their singularity is repulsive and strong since it behaves like $1/|x|$ (far above the singularity of the Coulomb law since we are in dimension one). However, their technics are limited to dimension one.

The present paper improves on preceding results in several directions. It does not require that the viscosity coefficient is large as in Osada [43], nor to cutoff the interaction kernel in the particle system as in [33] or [37]. Moreover, in the two above mentioned previous works of Osada, the convergence (1.4) was only established in the weak sense (of measures) and only for non-negative vorticity. Moreover, the results of Osada basically apply when $w_0 \in L^\infty$, while we allow any $w_0 \in L^1(\mathbb{R}^2)$ with a finite entropy and moment of positive order. Last but not least, our proof seems simpler than the one of Osada in [43], which uses very technical estimates.

The case of bounded domains. In the case of general bounded domains Ω with boundaries, the problem is more delicate. The first difficulty is that the vorticity formulation of the Navier-Stokes equation does not behave well with the boundaries conditions. In fact, vorticity is created at the boundary. However, it is still possible to imagine branching processes of interacting particles that will take the possible creation and annihilation of vortices at the boundary, as is done by Benachour, Roynette and Vallois in [2], but the analysis of such systems seems much more difficult.

However, if we move to some periodic and bounded setting, $\Omega = \mathbb{T}^2$, then our results will apply with small modifications. All we have to do is to replace the Biot-Savard K by its periodization

$$K_{per}(x) := \frac{x^\perp}{|x|^2} + g_\infty(x)$$

where g_∞ is some C^∞ function. The singularity is exactly the same, and the addition of a smooth function g_∞ in the kernel does not raise any difficulty. As a consequence, our result will apply to that case with the appropriate modifications.

2. Statement of the main results

2.1. **Notation.** For any Polish space E, we denote by $\mathbf{P}(E)$ the set of probability measures on E and by $\mathbf{M}(E)$ the set of finite signed measures on E. Both are endowed with the topology of weak convergence defined by duality against functions of $C_b(E)$. For $N \geq 2$, we denote by $\mathbf{P}_{sym}(E^N)$ the set of symmetric probability measures F on E^N (i.e. such that F is the law of an exchangeable E^N-valued random variable $(\mathcal{Y}_1, \dots, \mathcal{Y}_N)$).

In the whole paper, when $f \in \mathbf{M}(\mathbb{R}^d)$ has a density, we also denote by $f \in L^1(\mathbb{R}^d)$ its density.

For $x \in \mathbb{R}^2$, we introduce $\langle x \rangle := (1 + |x|^2)^{1/2}$. For $k \in (0, 1]$ and $N \geq 1$ we set

$$\forall\, X = (x_1, ..., x_N) \in (\mathbb{R}^2)^N, \quad \langle X \rangle^k := \frac{1}{N} \sum_{i=1}^N \langle x_i \rangle^k.$$

For $F \in \mathbf{P}((\mathbb{R}^2)^N)$, we define

$$M_k(F) := \int_{(\mathbb{R}^2)^N} \langle X \rangle^k \, F(dX).$$

We also introduce

$$\mathbf{P}_k((\mathbb{R}^2)^N) := \{F \in \mathbf{P}((\mathbb{R}^2)^N) \ : \ M_k(F) < \infty\}.$$

For $F \in \mathbf{P}((\mathbb{R}^2)^N)$ with a density (and a finite moment of positive order for the entropy), we introduce the Boltzmann entropy and the Fisher information of F defined as

$$H(F) := \frac{1}{N} \int_{(\mathbb{R}^2)^N} F(X) \log(F(X)) \, dX \quad \text{and} \quad I(F) := \frac{1}{N} \int_{(\mathbb{R}^2)^N} \frac{|\nabla F(X)|^2}{F(X)} \, dX.$$

If $F \in \mathbf{P}((\mathbb{R}^2)^N)$ has no density, we simply put $H(F) = +\infty$ and $I(F) = +\infty$. The somewhat unusual normalization by $1/N$ is made in order that for any $f \in \mathbf{P}(\mathbb{R}^2)$,

$$H(f^{\otimes N}) = H(f) \quad \text{and} \quad I(f^{\otimes N}) = I(f).$$

We will often deal here with probability measures on $(\mathbb{R}\times\mathbb{R}^2)^N$, representing the circulations and positions of N vortices. But the circulations only act like parameters. We thus adapt all the previous notation by a simple integration. For $G \in \mathbf{P}((\mathbb{R}\times\mathbb{R}^2)^N)$, write the disintegration $G(dM,dX) = R(dM)F^M(dX)$, where $R \in \mathbf{P}(\mathbb{R}^N)$ and for each $M \in \mathbb{R}^N$, $F^M \in \mathbf{P}((\mathbb{R}^2)^N)$ and define partial moment, entropy and Fisher information by

$$\tilde M_k(G) := \int_{\mathbb{R}^N} M_k(F^M)\, R(dM) = \int_{(\mathbb{R}\times\mathbb{R}^2)^N} \langle X\rangle^k\, G(dM,dX) \tag{2.1}$$

$$\tilde H(G) := \int_{\mathbb{R}^N} H(F^M)\, R(dM) \tag{2.2}$$

$$\tilde I(F) := \int_{\mathbb{R}^N} I(F^M)\, R(dM). \tag{2.3}$$

To understand these objects, let us make a few observations. When G has a density on $(\mathbb{R}\times\mathbb{R}^2)^N$,

$$\tilde I(F) = N^{-1}\int_{(\mathbb{R}\times\mathbb{R}^2)^N} \frac{|\nabla_X G(M,X)|^2}{G(M,X)} dM dX.$$

When G has a finite (classical) entropy, we can write

$$\tilde H(G) = \int_{(\mathbb{R}\times\mathbb{R}^2)^N} G(M,X)\log G(M,X) dM dX - \int_{\mathbb{R}^N} R(M)\log R(M) dM = H(G) - H(R).$$

We finally introduce

$$\mathbf{P}_k((\mathbb{R}\times\mathbb{R}^2)^N) := \{G \in \mathbf{P}((\mathbb{R}\times\mathbb{R}^2)^N);\ \tilde M_k(G) < \infty\}.$$

2.2. **Notions of chaos.** In this subsection, E will stand for an abstract polish space.

Definition 2.1 (Chaos for probability measures)**.** *A sequence (F^N) of symmetric probability measures on E^N is said to be f-chaotic, for a probability measure f on E if one of three following equivalent conditions is satisfied:*

(i) the sequence of second marginals $F_2^N \rightharpoonup f\otimes f$ as $N\to+\infty$;

(ii) for all $j\geq 1$, the sequence of j-th marginals $F_j^N \rightharpoonup f^{\otimes j}$ as $N\to+\infty$;

(iii) the law $\hat F^N$ of the empirical measure (under F^N) converges towards δ_f in $\mathbf{P}(\mathbf{P}(E))$ as $N\to\infty$.

This definition translates into an equivalent definition in terms of random variables.

Definition 2.2 (Chaos for random variables)**.** *A sequence $(\mathcal{Y}_1^N,\dots,\mathcal{Y}_N^N)$ of exchangeable E-valued random variables is said to be $\mathcal{Y}$-chaotic for some E-valued random variable $\mathcal{Y}$ if the sequence of laws $\mathcal{L}(\mathcal{Y}_1^N,\dots,\mathcal{Y}_N^N)$ is $\mathcal{L}(\mathcal{Y})$-chaotic, in other words, if one of three following equivalent condition is satisfied:*

(i) $(\mathcal{Y}_1^N,\mathcal{Y}_2^N)$ goes in law to 2 independent copies of $\mathcal{Y}$ as $N\to\infty$;

(ii) for all $j\geq 1$, $(\mathcal{Y}_1^N,\dots,\mathcal{Y}_j^N)$ goes in law to j independent copies of $\mathcal{Y}$ as $N\to\infty$;

(iii) the empirical measures $\mu^N_{\mathcal{Y}^N} = \frac{1}{N}\sum_1^N \delta_{\mathcal{Y}_i^N} \in \mathbf{P}(E)$ go in law to the constant $\mathcal{L}(\mathcal{Y})$ as $N\to\infty$.

We refer for instance to the lecture of Sznitman [50] for the equivalence of the three conditions, as well as [24, Theorem 1.2] where that equivalence is established in a *quantitative way*. Let us only mention that exchangeability is very important in order to understand point (i).

Propagation of chaos in the sense of Sznitman holds for a system N exchangeable particles evolving in time (for instance the system (1.1)) if when the initial conditions $(\mathcal{Y}_1^N(0), \dots, \mathcal{Y}_N^N(0))$ are $\mathcal{Y}(0)$-chaotic, the trajectories $((\mathcal{Y}_1^N(t))_{t\geq 0}, \dots, (\mathcal{Y}_N^N(t))_{t\geq 0})$ are $(\mathcal{Y}(t))_{t\geq 0}$-chaotic, where $(\mathcal{Y}(t))_{t\geq 0}$ is the (unique) solution of the expected (one-particle) limit model (here the nonlinear SDE (1.6)).

Another (stronger) sense of chaos has been developped: the entropic chaos. It goes back to a celebrated work of Kac [27] and was formalized recently in [11, 24] (see also [24] for a notion of Fisher information chaos).

Definition 2.3 (Entropic chaos)**.** *A sequence (F^N) of symmetric probability measures on E^N is said to be entropically f-chaotic, for a probability measure f on E, if*

$$F_1^N \to f \quad \textit{weakly in } \mathbf{P}(E) \quad \textit{and} \quad H(F^N) \to H(f)$$

as $N \to \infty$, where F_1^N stands for the first marginal of F^N.

It is shown in [24] that this is in fact a stronger notion than propagation of chaos. Actually, it is known that the entropy can only decrease if a sequence F^N is f chaotic: we say that the entropy is Γ-lower semi continuous. With our normalization, it writes

$$H(f) \leq \liminf_{N\to\infty} H(F^N).$$

Since the entropy is convex, $\lim H(F^N) = H(f)$ is a stronger notion of convergence, which implies that for all $j \geq 1$, the density of the law of $(\mathcal{Y}_1^N, \dots, \mathcal{Y}_j^N)$ goes to $f^{\otimes j}$ strongly in L^1.

Here, we will have to modify slightly this notion, replacing the use of H by that of $\tilde{H}$, since the circulations of the vortices only act like parameters.

2.3. **The Navier-Stokes equation.**

Definition 2.4. *We say that $w = (w_t)_{t\geq 0} \in C([0,\infty), \mathbf{M}(\mathbb{R}^2))$ is a weak solution to (1.2) if*

$$\forall\, T > 0, \quad \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |K(x-y)|\, |w_s|(dx)\, |w_s|(dx)\, ds < \infty \tag{2.4}$$

and if for all $\varphi \in C_b^2(\mathbb{R}^2)$, all $t \geq 0$,

$$\begin{aligned}\int_{\mathbb{R}^2} \varphi(x) w_t(dx) = \int_{\mathbb{R}^2} \varphi(x) w_0(dx) + \int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} K(x-y)\cdot\nabla\varphi(x)\, w_s(dy)\, w_s(dx)\, ds \\ + \nu \int_0^t \int_{\mathbb{R}^2} \Delta\varphi(x) w_s(dx)\, ds.\end{aligned} \tag{2.5}$$

We will establish the following extension of [3, 7] which is well adapted to our purpose.

Theorem 2.5. *Assume that $w_0 \in L^1(\mathbb{R}^2)$ satisfies (1.5). There exists a unique weak solution w to (1.2) such that*

$$\nabla_x w \in L^{2q/(3q-2)}(0,T,L^q(\mathbb{R}^2)) \quad \forall\, q \in [1,2), \quad \forall\, T > 0. \tag{2.6}$$

This solution furthermore satisfies

$$w \in C([0,\infty); L^1(\mathbb{R}^2)) \cap C((0,\infty); L^\infty(\mathbb{R}^2)) \tag{2.7}$$

and

$$\partial_t \beta(w) = (K * w)\cdot\nabla_x\beta(w) + \nu\Delta\beta(w) - \nu\beta''(w)\,|\nabla w|^2 \quad \textit{on} \quad [0,\infty)\times\mathbb{R}^2 \tag{2.8}$$

in the distributional sense, for any $\beta \in C^1(\mathbb{R}) \cap W^{2,\infty}_{loc}(\mathbb{R})$ such that β'' is piecewise continuous and vanishes outside of a compact set.

As we will see in the proof of the above result, thanks to the Sobolev embedding and the Hardy-Littlewood-Sobolev inequality one can show that for $w \in C([0,\infty), \mathbf{M}(\mathbb{R}^2))$, (2.6) implies (2.4). The proof of (2.8) is classical. When $\nu = 0$ such a result has been proved in [16, Theorem II.2] while the case $\nu > 0$ can be obtained by adapting a result from [15, Section III]. For the sake of completeness, we will however sketch the proof of (2.8) in Section 7.

2.4. **Stochastic paths associated to the Navier-Stokes equation.** Since a solution $(w_t)_{t\geq 0}$ to the vorticity equation (1.2) does not take values in probability measures on $\mathbb{R}^2$, a few work is needed to find some related stochastic paths: roughly, we write the initial vorticity w_0 as some partial information of the law g_0 of circulations and positions of the vortices.

We consider $g_0 \in \mathbf{P}(\mathbb{R}\times\mathbb{R}^2)$ satisfying, for some $A \in (0,\infty)$ and some $k \in (0,1]$,

$$\text{Supp}\, g_0 \subset \mathcal{A}\times\mathbb{R}^2, \;\; \mathcal{A} = [-A, A], \;\; \tilde M_k(g_0) < \infty \;\text{ and }\; \tilde H(g_0) < \infty. \tag{2.9}$$

Remark 2.6. *(i) Let $g_0 \in \mathbf{P}(\mathbb{R}\times\mathbb{R}^2)$ satisfying (2.9) and define $w_0 \in \mathbf{M}(\mathbb{R}^2)$ by*

$$\forall\, B \in \mathcal{B}(\mathbb{R}^2), \quad w_0(B) = \int_{\mathbb{R}\times B} m g_0(dm, dx). \tag{2.10}$$

Then $w_0 \in L^1(\mathbb{R}^2)$ and satisfies (1.5).

(ii) For $w_0 \in L^1(\mathbb{R}^2)$ satisfy (1.5), it is possible to find a probability measure g_0 on $\mathbb{R}\times\mathbb{R}^2$ satisfying (2.9) and such that (2.10) holds true.

Proof. We first check (i). First, $|w_0|(dx) \leq A\int_{m\in\mathbb{R}} g_0(dm, dx)$, so that $|w_0|(\mathbb{R}) \leq A$ and w_0 is a finite measure. Next, there holds $\int_{\mathbb{R}^2}\langle x\rangle^k |w_0|(dx) \leq A\int_{\mathbb{R}\times\mathbb{R}^2}\langle x\rangle^k g_0(dm,dx) < \infty$. Finally, to prove that $|w_0|$ has a density satisfying $\int_{\mathbb{R}^2}|w_0(x)|\log(|w_0(x)|)dx < \infty$, it obviously suffices to check that $\kappa(dx) := \int_{m\in\mathbb{R}} g_0(dm,dx)$ has a finite entropy, since $|w_0| \leq A\kappa$. We thus disintegrate $g_0(dm) = r_0(dm) f_0^m(dx)$ and use the convexity of the entropy functional to get

$$H(\kappa) = H\left(\int_{m\in\mathbb{R}} r_0(dm) f_0^m(dx)\right) \leq \int_{\mathbb{R}} r_0(dm) H(f_0^m) = \tilde H(g_0) < \infty \quad \text{by assumption.}$$

To verify (ii), write $w_0 = w_0^+ - w_0^-$, for two non-negative functions with disjoint supports w_0^+ and w_0^-, put $a := \int_{\mathbb{R}^2}|w_0(x)|dx$ and set (for example)

$$g_0(dm,dx) = \frac{1}{a}\delta_a(dm) w_0^+(x)dx + \frac{1}{a}\delta_{-a}(dm) w_0^-(x)dx.$$

Then (2.10) holds true and (2.9) is easily deduce from (1.5). This is the most simple possibility, but many other exist. In general, g may be seen as a Young measure associated to w, and it may be of physical interest to introduce Young measures in the context of the Euler equation, see for instance [5]. □

We can now introduce some (stochastic) paths associated to the vorticity equation.

Definition 2.7. *Let g_0 be a probability measure on $\mathbb{R}\times\mathbb{R}^2$ and consider a g_0-distributed random variable $(\mathcal{M}, \mathcal{X}(0))$ independent of a 2D-Brownian motion $(\mathcal{B}_t)_{t\geq 0}$. We say that a $\mathbb{R}^2$-valued process $(\mathcal{X}(t))_{t\geq 0}$ solves the nonlinear SDE* (1.6) *if for all $t \geq 0$,*

$$\mathcal{X}(t) = \mathcal{X}(0) + \int_0^t \int_{\mathbb{R}^2} K(\mathcal{X}(s) - x)\, w_s(dx) ds + \sigma\mathcal{B}_t, \tag{2.11}$$

where w_t is the measure on $\mathbb{R}^2$ defined by

$$\forall\, B \in \mathcal{B}(\mathbb{R}^2), \quad w_t(B) = \mathbb{E}[\mathcal{M}\,\mathbb{1}_{\{\mathcal{X}(t)\in B\}}] = \int_{\mathbb{R}\times B} m\, g_t(dm, dx) \tag{2.12}$$

where $g_t = \mathcal{L}(\mathcal{M}, \mathcal{X}(t))$*, and if* $(w_t)_{t\geq 0}$ *satisfies (2.4).*

Roughly, $\mathcal{M}$ represents the circulation of a typical vortex and $(\mathcal{X}(t))_{t\geq 0}$ its path, in an infinite vortices system subjected to the vorticity equation. The rigorous link is the following.

Remark 2.8. *For* $(\mathcal{X}(t))_{t\geq 0}$ *a solution to* (1.6)*,* $(w_t)_{t\geq 0}$ *is a weak solution to (1.2).*

Proof. This can be checked by an application of the Itô formula: for $\varphi \in C_b^2(\mathbb{R}^2)$, we have

$$\begin{aligned}\mathcal{M}\varphi(\mathcal{X}(t)) \quad &= \quad \mathcal{M}\varphi(\mathcal{X}(0)) + \int_0^t \int_{\mathbb{R}^2} \mathcal{M}\nabla\varphi(\mathcal{X}(s)) \cdot K(\mathcal{X}(s) - x) w_s(dx) ds \\ &\quad + \nu \int_0^t \mathcal{M}\Delta\varphi(\mathcal{X}(s)) ds + \sigma \int_0^t \mathcal{M}\nabla\varphi(\mathcal{X}(s)) d\mathcal{B}_s,\end{aligned}$$

where we recall that $\nu := \sigma^2/2$. Taking expectations and using that the last term is a martingale with mean 0, we find (2.5). □

We will check the following consequence of Theorem 2.5.

Theorem 2.9. *Let* g_0 *be a probability measure on* $\mathbb{R} \times \mathbb{R}^2$ *satisfying (2.9). There exists a unique strong solution* $(\mathcal{X}(t))_{t\geq 0}$ *to the nonlinear SDE* (1.6) *such that*

$$\forall\, T > 0, \qquad \int_0^T \tilde{I}(g_s) ds < \infty, \tag{2.13}$$

$g_t \in \mathbf{P}(\mathbb{R} \times \mathbb{R}^2)$ *being the law of* $(\mathcal{M}, \mathcal{X}(t))$*. Furthermore, its associated vorticity function* $(w_t)_{t\geq 0}$ *satisfies* (2.6) *and* $(g_t)_{t\geq 0}$ *satisfies the entropy equation*

$$\tilde{H}(g_t) + \nu \int_0^t \tilde{I}(g_s)\, ds = \tilde{H}(g_0) \qquad \forall\, t > 0. \tag{2.14}$$

2.5. **The stochastic particle system.** As shown by Osada [42] and others, the system (1.1) is well-posed.

Theorem 2.10. *Consider any family* $(\mathcal{M}_i^N, \mathcal{X}_i^N(0))_{i=1,\dots,N}$ *of* $\mathbb{R} \times \mathbb{R}^2$*-valued random variables, independent of a family* $(\mathcal{B}_i(t))_{i=1,\dots,N,t\geq 0}$ *of i.i.d. 2D-Brownian motions and such that a.s.,* $\mathcal{X}_i^N(0) \neq \mathcal{X}_j^N(0)$ *for all* $i \neq j$*. There exists a unique strong solution to (1.1).*

Actually, Osada [42] shows that a.s., for all $t \geq 0$, all $i \neq j$, $\mathcal{X}_i^N(t) \neq \mathcal{X}_j^N(t)$. This implies the well-posedness of (1.1), since the singularity of K is thus a.s. never visited by the system.

Let us give a few more references. When the circulations $\mathcal{M}_i^N$ are positive, Takanobu proved the well-posedness of the system using a martingale argument [51]. Osada extended in [42] his results to arbitrary vorticities using estimates *à la Nash* for fundamental solutions to parabolic equations with divergence free drift [41]. More recently, Fontbona and Martinez adapted in [19] the technique used by Marchioro and Pulvirenti for the deterministic N vortex models [34, Chapter 4.2] to the stochastic case (1.1).

2.6. **The result of propagation of chaos.** To study the many-particle limit of the vortex system (1.1), we have to impose some compactness and consistency properties on the initial system.

Denote by $G_0^N \in \mathbf{P}((\mathbb{R} \times \mathbb{R}^2)^N)$ the law of $(\mathcal{M}_i^N, \mathcal{X}_i^N(0))_{i=1,\dots N}$. We will assume that there are $k \in (0, 1]$, $A \in (0, \infty)$ and $g_0 \in \mathbf{P}(\mathbb{R} \times \mathbb{R}^2)$ supported in $\mathcal{A} \times \mathbb{R}^2$, where $\mathcal{A} = [-A, A]$, such that, setting $r_0(dm) := \int_{x\in\mathbb{R}^2} g_0(dm, dx) \in \mathbf{P}(\mathcal{A})$,

$$
(2.15) \qquad \begin{cases} G_0^N \in \mathbf{P}_{sym}((\mathbb{R}\times\mathbb{R}^2)^N) \text{ is } g_0\text{-chaotic;} \\[2ex] \sup_{N\geq 2} \tilde M_k(G_0^N) < \infty, \quad \sup_{N\geq 2} \tilde H(G_0^N) < \infty; \\[2ex] R_0^N(dm_1,\dots,dm_N) := \int_{(\mathbb{R}^2)^N} G_0^N(dm_1,dx_1,\dots,dm_N,dx_N) = r_0^{\otimes N}(dm_1,\dots,dm_N). \end{cases}
$$

This last condition asserts that $\mathcal{M}_1^N,\dots,\mathcal{M}_N^N$ are i.i.d. and r_0-distributed.

Remark 2.11. *(i) The typical situation is the following: let $g_0 \in \mathbf{P}(\mathbb{R}\times\mathbb{R}^2)$ satisfy* (2.9) *and consider, for $N \geq 2$, an i.i.d. family $(\mathcal{M}_i^N, \mathcal{X}_i^N(0))_{i=1,\dots,N}$ of g_0-distributed random variables. Then $G_0^N = g_0^{\otimes N}$ and* (2.15) *is met.*

(ii) Consider a family $(\mathcal{M}_i^N, \mathcal{X}_i^N(0))_{i=1,\dots N}$ satisfying (2.15) with some $g_0 \in \mathbf{P}(\mathbb{R}\times\mathbb{R}^2)$. Then g_0 automatically satisfies (2.9), so that the nonlinear SDE (1.6) *has a unique solution associated to g_0 by Theorem 2.9. Also, w_0 defined from g_0 as in (2.10) satisfies (1.5) by Remark 2.6-(i), so that Theorem 2.5 implies that (1.2) with w_0 as initial condition is well-posed.*

(iii) Under (2.15)*, we have $\tilde H(G_0^N) < \infty$ for each $N \geq 2$, whence the law of $(\mathcal{X}_1^N(0),\dots,\mathcal{X}_N^N(0))$ has a density on $(\mathbb{R}^2)^N$. In particular, $\mathcal{X}_i^N(0) \neq \mathcal{X}_j^N(0)$ a.s. for all $N \geq 2$, all $i \neq j$, so that for each $N \geq 2$, the particle system* (1.1) *is well-posed by Theorem 2.10.*

Proof. Point (i) is easily checked, using in particular that $\tilde M_k(G_0^N) = \tilde M_k(g_0)$ and $\tilde H(G_0^N) = \tilde H(g_0)$ for all $N \geq 2$. For (ii), we just have to check that g_0 satisfies (2.9). But by exchangeability we have $M_k(G_0^N) = M_k(G_{0,1}^N)$, where $G_{0,1}^N$ denotes the first marginal of G_0^N. Since $G_{0,1}^N \rightharpoonup g_0$ weakly in the sense of measures, we get $\tilde M_k(g_0) \leq \liminf_N \tilde M_k(G_{0,1}^N) < \infty$. Finally, the Γ-sci property $\tilde H(g_0) \leq \liminf_N \tilde H(G_0^N)$ is more difficult to prove but follows from Theorem 4.1 below. Point (iii) is obvious. □

Let us now write down the first part of our main result, concerning the paths of the particles. Here $C([0,\infty),\mathbb{R}^2)$ is endowed with the topology of uniform convergence on compacts.

Theorem 2.12. *Consider, for each $N \geq 2$, a family $(\mathcal{M}_i^N, \mathcal{X}_i^N(0))_{i=1,\dots,N}$ of $\mathbb{R}\times\mathbb{R}^2$-valued random variables. Assume that the initial chaos assumptions* (2.15) *holds true for some g_0. For each $N \geq 2$, consider the unique solution (see Remark 2.11-(iii)) $(\mathcal{X}_i^N(t))_{i=1,\dots,N,t\geq 0}$ to (1.1), and the unique solution $(\mathcal{X}(t))_{t\geq 0}$ to the nonlinear SDE* (1.6) *given by Theorem 2.9 associated to g_0 (see Remark 2.11-(ii)). Then, the sequence $(\mathcal{M}_i^N, (\mathcal{X}_i^N(t))_{t\geq 0})_{i=1\dots,N}$ is $(\mathcal{M}, (\mathcal{X}(t))_{t\geq 0})$-chaotic.*

In particular, it implies that if we set

$$
(2.16) \qquad \mathcal{W}_t^N := \frac{1}{N}\sum_{i=1}^N \mathcal{M}_i^N \delta_{\mathcal{X}_i^N(t)},
$$

then $(\mathcal{W}_t^N)_{t\geq 0}$ goes in probability in $C([0,\infty),\mathbf{M}(\mathbb{R}^2))$, as $N \to \infty$, to the unique weak solution $(w_t)_{t\geq 0}$ given by Theorem 2.5 to the vorticity equation (1.2) starting from w_0 (see Remark 2.11-(ii)).

Our last result deals with entropic chaos.

Theorem 2.13. *Adopt the same notation and assumptions as in Theorem 2.12 and assume furthermore that $\lim_n \tilde H(G_0^N) = \tilde H(g_0)$ (which is the case if $G_0^N = g^{\otimes N}$). For $t \geq 0$, denote by $g_t \in \mathbf{P}(\mathbb{R}\times\mathbb{R}^2)$ the law of $(\mathcal{M}, \mathcal{X}(t))$.*

(i) For all $t \geq 0$, $((\mathcal{M}_i^N, \mathcal{X}_i^N(t))_{i=1,\dots,N}$ is g_t-entropically chaotic in the sense that, denoting by $G_t^N \in \mathbf{P}((\mathbb{R} \times \mathbb{R}^2)^N)$ its law,

$$(\mathcal{M}_1^N, \mathcal{X}_1^N(t)) \to g_t \quad \textit{in law} \quad \textit{and} \quad \tilde{H}(G_t^N) \to \tilde{H}(g_t) \quad \textit{as} \quad N \to \infty.$$

(ii) For $j = 1, \dots, N$, define the j-particle vorticity w_{jt}^N as the measure on $(\mathbb{R}^2)^j$:

$$w_{jt}^N(dx_1, \dots, dx_j) := \int_{m_1,\dots,m_N \in \mathbb{R},\; x_{j+1},\dots,x_N \in \mathbb{R}^2} m_1 \dots m_j \, G_t^N(dm_1, dx_1, \dots, dm_N, dx_N). \tag{2.17}$$

This measure has a density and for all fixed $t \geq 0$, all fixed $j \geq 1$,

$$w_{jt}^N \to w_t^{\otimes j} \quad \textit{strongly in } L^1((\mathbb{R}^2)^j) \textit{ as } N \to \infty. \tag{2.18}$$

2.7. **Plan of the proof.** In Section 3, we prove various functional inequalities, showing in particular that a Fisher information estimate for the N-particles distribution allow us to control the *close encounters* between particles. Section 4 is dedicated to a result in the spirit of Robinson-Ruelle [46]: the partial entropy $\tilde{H}$ and partial Fisher information $\tilde{I}$ are affine on mixed states, which implies the Γ-lower semi continuity of both functionals. Precisely, that was proved in [24] for the full entropy and Fisher information and here we only present the adaptation necessary in the partial case. In Section 5, we prove our main estimate: denoting by $G_t^N = \mathscr{L}((\mathcal{M}_1^N, \mathcal{X}_1^N(t)), \dots, (\mathcal{M}_N^N, \mathcal{X}_N^N(t)))$,

$$\forall\, T > 0, \quad \sup_{N \geq 2} \left\{ \sup_{[0,T]} [\tilde{H}(G_t^N) + \tilde{M}_k(G_t^N)] + \int_0^T \tilde{I}(G_t^N)\, dt \right\} < \infty$$

and deduce the tightness of our system. We then show that any limit point solves the nonlinear S.D.E. in Section 6, and satisfies the a priori condition of Theorem 2.9. We prove our uniqueness results (Theorems 2.5 and 2.9) in Section 7 and conclude the proofs of Theorems 2.12 and 2.13 in Section 8.

We close that section with a convention that we shall use in all the sequel. We write C for a (large) finite constant and c for a positive constant depending only on σ and on all the bounds assumed in (1.5), (2.9) and (2.15). Their values may change from line to line. All other dependence will be indicated in subscript.

3. Entropy and Fisher information

In this section, we present a series of results involving the Boltzmann entropy H, the Fisher information I and their modified versions $\tilde{H}, \tilde{I}$. In the sequel of the article, they will provide key estimates in order to exploit the regularity of the objects we will deal with.

The following very classical estimate will be useful in order to get bounds on the system of particles in the next section. It also explains why the entropy is well-defined from $\mathbf{P}_k((\mathbb{R}^2)^N)$ into $\mathbb{R} \cup \{+\infty\}$. See the comments before [24, Lemma 3.1] for the proof.

Lemma 3.1. *For any $k, \lambda \in (0, \infty)$, there is a constant $C_{k,\lambda} \in \mathbb{R}$ such that for any $N \geq 1$, any $F \in \mathbf{P}_k((\mathbb{R}^2)^N)$*

$$H(F) \geq -C_{k,\lambda} - \lambda\, M_k(F).$$

We next establish some kind of Gagliardo-Nirenberg-Sobolev inequality involving the Fisher information.

Lemma 3.2. *For any $f \in \mathbf{P}(\mathbb{R}^2)$ with finite Fisher information, there holds*

$$\forall\, p \in [1,\infty), \quad \|f\|_{L^p(\mathbb{R}^2)} \le C_p\, I(f)^{1-1/p}, \tag{3.1}$$

$$\forall\, q \in [1,2), \quad \|\nabla f\|_{L^q(\mathbb{R}^2)} \le C_q\, I(f)^{3/2-1/q}. \tag{3.2}$$

Proof. We start with (3.2). Let $q \in [1,2)$ and use the Hölder inequality:

$$\|\nabla f\|^q_{L^q} = \int \left|\frac{\nabla f}{\sqrt{f}}\right|^q f^{q/2} \le \left(\int \frac{|\nabla f|^2}{f}\right)^{q/2} \left(\int f^{q/(2-q)}\right)^{(2-q)/2} = I(f)^{q/2}\, \|f\|^{q/2}_{L^{q/(2-q)}}.$$

Denoting by $q^* = 2q/(2-q) \in [2,\infty)$ the Sobolev exponent associated to q, we have, thanks to a standard interpolation inequality and to the Sobolev inequality,

$$\|f\|_{L^{q/(2-q)}} = \|f\|_{L^{q^*/2}} \le \|f\|^{1/(q^*-1)}_{L^1}\, \|f\|^{(q^*-2)/(q^*-1)}_{L^{q^*}} \le C_q\, \|f\|^{1/(q^*-1)}_{L^1}\, \|\nabla f\|^{(q^*-2)/(q^*-1)}_{L^q}. \tag{3.3}$$

Gathering these two inequalities, it comes

$$\|\nabla f\|_{L^q} \le C_q\, I(f)^{1/2}\, \|f\|^{1/(2(q^*-1))}_{L^1}\, \|\nabla f\|^{(q^*-2)/(2(q^*-1))}_{L^q},$$

from which we easily deduce (3.2) using that $f \in \mathbf{P}(\mathbb{R}^2)$.

We now verify (3.1). For $p \in [1,\infty)$, write $p = q^*/2 = q/(2-q)$ with $q := 2p/(1+p) \in [1,2)$ and use (3.3) and (3.2):

$$||f||_{L^p} \le C_p ||f||^{1/(q^*-1)}_{L^1} I(f)^{(3/2-1/q)(q^*-2)/(q^*-1)},$$

from which one easily concludes since $f \in \mathbf{P}(\mathbb{R}^2)$. □

As a first consequence, we deduce that pairs of particles which law has finite Fisher information are not too close in the following sense.

Lemma 3.3. *Consider $F \in \mathbf{P}(\mathbb{R}^2 \times \mathbb{R}^2)$ with finite Fisher information and $(\mathcal{X}_1, \mathcal{X}_2)$ a random variable with law F. Then for any $\gamma \in (0,2)$ and any $\beta > \gamma/2$ there exists $C_{\gamma,\beta}$ so that*

$$\mathbb{E}(|\mathcal{X}_1 - \mathcal{X}_2|^{-\gamma}) = \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{F(x_1,x_2)}{|x_1 - x_2|^\gamma}\, dx_1 dx_2 \le C_{\gamma,\beta}\, (I(F)^\beta + 1). \tag{3.4}$$

Proof. We introduce the unitary linear transformation

$$\forall\, (x_1,x_2) \in \mathbb{R}^2 \quad \Phi(x_1,x_2) = \frac{1}{\sqrt{2}}\big(x_1 - x_2, x_1 + x_2\big) =: (y_1,y_2).$$

Defining $\tilde F := F \circ \Phi^{-1}$ which is nothing but the law of $\frac{1}{\sqrt{2}}\big(\mathcal{X}_1 - \mathcal{X}_2, \mathcal{X}_1 + \mathcal{X}_2\big)$ and $\tilde f$ as the first marginal of $\tilde F$ (the law of $\frac{1}{\sqrt{2}}(\mathcal{X}_1 - \mathcal{X}_2)$). A simple substitution shows that $I(\tilde F) = I(F)$. Furthermore, the super-additivity property of Fisher's information proved in [10, Theorem 3] (the factor 2 below is due to our normalized definition of the Fisher information), see also [24, Lemma 3.7], implies that

$$I(\tilde f) \le 2\, I(\tilde F) = 2\, I(F). \tag{3.5}$$

Let $\beta \in (\gamma/2, 1)$ be fixed (the case $\beta \geq 1$ will then follow immediately). We have

$$\begin{aligned}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{F(x_1,x_2)}{|x_1-x_2|^\gamma}\,dx_1dx_2 &= 2^{\gamma/2}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\tilde F(y_1,y_2)}{|y_1|^\gamma}\,dy_1dy_2 \\ &= 2^{\gamma/2}\int_{\mathbb{R}^2} \frac{\tilde f(y)}{|y|^\gamma}\,dy \\ &\leq 2^{\gamma/2} + 2^{\gamma/2}\int_{|y|\leq 1} \frac{\tilde f(y)}{|y|^\gamma}\,dy.\end{aligned}$$

Using the Hölder inequality, that $\gamma/\beta < 2$ and (3.1), we deduce that

$$\begin{aligned}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{F(x_1,x_2)}{|x_1-x_2|^\gamma}\,dx_1dx_2 &\leq 2^{\gamma/2} + 2^{\gamma/2}\left[\int_{|y|\leq 1} |y|^{-\gamma/\beta}dx\right]^\beta \|\tilde f\|_{L^{1/(1-\beta)}(\mathbb{R}^2)} \\ &\leq C_{\gamma,\beta}(1+I(\tilde f)^\beta).\end{aligned}$$

We conclude thanks to (3.5). $\square$

We also need something similar to Lemma 3.2 that can be applied to vorticity measures.

Lemma 3.4. *Consider a probability measure g on $\mathbb{R}\times\mathbb{R}^2$ with* $\operatorname{Supp} g \subset [-A,A]\times\mathbb{R}^2$ *and define the probability measure v and the (signed) measure w on $\mathbb{R}^2$ by*

$$v(B) = \int_{\mathbb{R}\times B} g(dx,dm), \quad w(B) = \int_{\mathbb{R}\times B} m g(dx,dm), \qquad \forall B \in \mathcal{B}(\mathbb{R}^2).$$

We have

$$\forall\, p \in [1,\infty), \quad \|v\|_{L^p(\mathbb{R}^2)} + \|w\|_{L^p(\mathbb{R}^2)} \leq C_{p,A}\tilde I(g)^{1-1/p}, \tag{3.6}$$

$$\forall\, q \in [1,2), \quad \|\nabla v\|_{L^q(\mathbb{R}^2)} + \|\nabla w\|_{L^q(\mathbb{R}^2)} \leq C_{q,A}\tilde I(g)^{3/2-1/q}. \tag{3.7}$$

Proof. We disintegrate $g(dm,dx) = r(dm)f^m(dx)$. For $p \in [1,\infty)$, using the support condition on g, that the L^p-norm is convex and (3.1) (since $f^m \in \mathbf{P}(\mathbb{R}^2)$ for each $m \in \mathbb{R}$), we get

$$||w||_{L^p} = \left\|\int_{\mathbb{R}} m\, r(dm) f^m(x)\right\|_{L^p} \leq A\int_{\mathbb{R}} r(dm)\|f^m\|_{L^p} \leq AC_p\int_{\mathbb{R}} r(dm) I(f^m)^{1-1/p}.$$

Since $r \in \mathbf{P}(\mathbb{R})$, the Jensen inequality leads us to

$$||w||_{L^p} \leq C_{p,A}\left(\int_{\mathbb{R}} r(dm)I(f^m)\right)^{1-1/p} = C_{p,A}\tilde I(g)^{1-1/p}.$$

The same proof works for v. Finally, (3.7) is shown similarly, using (3.2) instead of (3.1). $\square$

We end this section with some easy functional estimates.

Lemma 3.5. *Let $(w_t)_{t\geq 0} \in C([0,\infty), \mathbf{M}(\mathbb{R}^2))$ satisfy (2.6). Then*

$$\forall\, T > 0, \quad \forall\, p \in (1,\infty), \quad w \in L^{p/(p-1)}([0,T], L^p(\mathbb{R}^2)) \tag{3.8}$$

and

$$\forall\, T > 0, \quad \forall\, \gamma \in (0,2), \quad \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |x-y|^{-\gamma}|w_s(x)||w_s(y)|dydxds < \infty. \tag{3.9}$$

Proof. The first estimate follows from (3.3) (applied with $q = 2p/(1+p)$ and thus $q^* = 2p$) and the fact that $w \in L^\infty(0,T;L^1(\mathbb{R}^2))$ (because $w \in C([0,\infty), \mathbf{M}(\mathbb{R}^2))$).

To check the second estimate, consider $p > 2/(2-\gamma)$ and observe that for any $x \in \mathbb{R}^2$, by the Hölder inequality, since $\gamma p/(p-1) < 2$,

$$\int_{\{|x-y|\le 1\}} |x-y|^{-\gamma} |w_s(y)| dy \le \|w_s\|_{L^p} \left(\int_{\{|x-y|\le 1\}} |x-y|^{-\gamma p/(p-1)} dy \right)^{1-1/p} \le C_{\gamma,p} \|w_s\|_{L^p}.$$

Consequently,

$$\begin{aligned} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |x-y|^{-\gamma}|w_s(x)||w_s(y)|dydx &\le \int_{\{|x-y|\ge 1\}}\int_{\mathbb{R}^2} |w_s(x)||w_s(y)|dydx \\ &\quad + \int_{\{|x-y|<1\}} |x-y|^{-\gamma}|w_s(x)||w_s(y)|dydx \\ &\le \|w_s\|_{L^1}^2 + C_{\gamma,p}\|w_s\|_{L^p}\|w_s\|_{L^1}. \end{aligned}$$

We easily conclude using that $w \in L^\infty(0,T;L^1(\mathbb{R}^2)) \cap L^{p/(p-1)}([0,T], L^p(\mathbb{R}^2))$. □

4. Many-particle entropy and Fisher information

We will need a result showing that if the particle distribution of the N-particle system has a uniformly bounded entropy and Fisher information, then any limit point of the associated empirical measure has finite entropy and Fisher information. As we will see, such a result is a consequence of representation identities for *level-3 functionals* as first proven by Robinson and Ruelle in [46] for the entropy in a somewhat different setting. Recently in [24], that kind of representation identity has been extended to the Fisher information. The proof is mainly based on the De Finetti-Hewitt-Savage representation theorem [26, 14] (see also [24] and the references therein) together with convexity tricks for the entropy, and concentration for the Fisher information. Unfortunately, we cannot apply directly the result of [24] due to the additional variable corresponding to the circulations of vortices. But the result still holds true and will be stated in the next theorem after some necessary definitions.

For a given $r \in \mathbf{P}(\mathbb{R})$, we define $\mathscr{E}_N(r)$ as the set of probability measures $G^N \in \mathbf{P}_{sym}((\mathbb{R}\times\mathbb{R}^2)^N)$ such that $\int_{(\mathbb{R}^2)^N} G^N(dm_1, dx_1, \ldots, dm_N, dx_N) = r^{\otimes N}(dm_1, \ldots, dm_N)$. We also denote by $\mathscr{E}_\infty(r)$ the set of probability measures $\pi \in \mathbf{P}(\mathbf{P}(\mathbb{R}\times\mathbb{R}^2))$ supported in $\{g \in \mathbf{P}(\mathbb{R}\times\mathbb{R}^2) : \int_{x\in\mathbb{R}^2} g(dm,dx) = r(dm)\}$. In addition, $\mathbf{P}_k(\mathbf{P}(\mathbb{R}\times\mathbb{R}^2))$ will denote the set of probabilities measure π with finite moment $\tilde{M}_k(\pi) := \int_{\mathbf{P}(\mathbb{R}\times\mathbb{R}^2)} \tilde{M}_k(g)\pi(dg) = \tilde{M}_k(\pi_1)$, where $\pi_1 := \int_{\mathbf{P}(\mathbb{R}\times\mathbb{R}^2)} g\, \pi(dg) \in \mathbf{P}(\mathbb{R}\times\mathbb{R}^2)$.

Theorem 4.1. *Let $k > 0$ and $r \in \mathbf{P}(\mathbb{R})$ supported in $\mathcal{A} = [-A, A]$ for some $A > 0$. Consider, for each $N \ge 2$, a probability measure $G^N \in \mathscr{E}_N(r)$. For $j \ge 1$, denote by $G_j^N \in \mathbf{P}((\mathbb{R}\times\mathbb{R}^2)^j)$ the j-th marginal of G^N. Assume that $\sup_N \tilde{M}_k(G_1^N) < \infty$ and that there exists a compatible sequence (π_j) of symmetric probability measures on $(\mathbb{R}\times\mathbb{R}^2)^j$ so that $G_j^N \to \pi_j$ in the weak sense of measures in $\mathbf{P}((\mathbb{R}\times\mathbb{R}^2)^j)$. Denoting by $\pi \in \mathbf{P}_k(\mathbf{P}(\mathbb{R}\times\mathbb{R}^2))$ the probability measure associated to the sequence (π_j) thanks to the De Finetti-Hewitt-Savage theorem, there holds*

$$\int_{\mathbf{P}(\mathbb{R}\times\mathbb{R}^2)} \tilde{H}(g)\,\pi(dg) = \sup_{j\ge 1} \tilde{H}(\pi_j) \le \liminf_{N\to\infty} \tilde{H}(G^N), \tag{4.1}$$

as well as

$$\int_{\mathbf{P}(\mathbb{R}\times\mathbb{R}^2)} \tilde{I}(g)\,\pi(dg) = \sup_{j\ge 1} \tilde{I}(\pi_j) \le \liminf_{N\to\infty} \tilde{I}(G^N). \tag{4.2}$$

The De Finetti-Hewitt-Savage theorem asserts that for a sequence (π_j) of symmetric probability on E^j, compatible in the sense that the k-th marginal of π_j is π_k for all $1 \le k \le j$, there exists a unique probability measure $\pi \in \mathbf{P}(\mathbf{P}(E))$ such that $\pi_j = \int_{\mathbf{P}(E)} g^{\otimes j} \pi(dg)$. See for instance [24, Theorem 5.1].

Theorem 4.1 is an immediate consequence of [24, Lemma 5.6] and of the series of properties on the partial entropy and Fisher information functionals that we state in the following lemma.

Lemma 4.2. *Let $k > 0$ and $r \in \mathbf{P}(\mathbb{R})$ supported in $\mathcal{A} = [-A, A]$ for some $A > 0$. The partial entropy and Fisher information functionals satisfy, with the common notation $\tilde J = \tilde H$ or $\tilde I$, $\tilde{\mathcal{J}} = \tilde{\mathcal{H}}$ or $\tilde{\mathcal{I}}$, the following properties*

(i1) For any $j \ge 1$, $\tilde I : \mathbf{P}((\mathcal{A} \times \mathbb{R}^2)^j) \to \mathbb{R} \cup \{+\infty\}$ is non-negative, convex, proper and lower semi-continuous for the weak convergence.

(i2) For any $j \ge 1$, $\tilde H : \mathbf{P}_k((\mathcal{A} \times \mathbb{R}^2)^j) \to \mathbb{R} \cup \{+\infty\}$ is convex, proper, lower semi-continuous for the weak convergence and there exists some constant $C_k \in \mathbb{R}$ such that

$$\mathbf{P}((\mathcal{A} \times \mathbb{R}^2)^j) \to \mathbb{R} \cup \{+\infty\}, \quad G \mapsto \tilde H(G) + \tilde M_{k/2}(G) + C_k$$

is lower semi-continuous for the weak convergence and non-negative.

(ii) For all $j \ge 1$, all $g \in \mathbf{P}_k(\mathcal{A} \times \mathbb{R}^2)$, $\tilde J(g^{\otimes j}) = \tilde J(g)$.

(iii) For all $G \in \mathbf{P}_k((\mathcal{A} \times \mathbb{R}^2)^j)$, all ℓ, n with $j = \ell + n$, there holds $j\, \tilde J(G) \ge \ell\, \tilde J(G_\ell) + n\, \tilde J(G_n)$, where $G_\ell \in \mathbf{P}_k((\mathcal{A} \times \mathbb{R}^2)^\ell)$ stands for the ℓ-marginal of G.

(iv) The functional $\tilde{\mathcal{J}}' : \mathbf{P}_k(\mathbf{P}(\mathcal{A} \times \mathbb{R}^2)) \cap \mathscr{E}_\infty(r) \to \mathbb{R} \cup \{\infty\}$ defined by

$$\tilde{\mathcal{J}}'(\pi) := \sup_{j \ge 1} \tilde J(\pi_j) \quad \text{where} \quad \pi_j := \int_{\mathbf{P}(\mathcal{A} \times \mathbb{R}^2)} g^{\otimes j} \pi(dg)$$

is affine in the following sense. For any $\pi \in \mathbf{P}_k(\mathbf{P}(\mathcal{A} \times \mathbb{R}^2))$ and any partition of $\mathbf{P}_k(\mathcal{A} \times \mathbb{R}^2)$ by some sets ω_i, $1 \le i \le M$, such that ω_i is an open set in $(\mathcal{A} \times \mathbb{R}^2) \backslash (\omega_1 \cup \ldots \cup \omega_{i-1})$ for any $1 \le i \le M-1$ and $\pi(\omega_i) > 0$ for any $1 \le i \le M$, defining

$$\alpha_i := \pi(\omega_i) \quad \text{and} \quad \gamma^i := \frac{1}{\alpha_i} \mathbf{1}_{\omega_i} \pi \in \mathbf{P}_k(\mathbf{P}(\mathcal{A} \times \mathbb{R}^2))$$

so that

$$\pi = \alpha_1 \gamma^1 + ... + \alpha_M \gamma^M \quad \text{and} \quad \alpha_1 + ... + \alpha_M = 1,$$

there holds

$$\mathcal{J}'(\pi) = \alpha_1 \mathcal{J}'(\gamma^1) + \ldots + \alpha_M \mathcal{J}'(\gamma^M).$$

Proof of Lemma 4.2. We only sketch the proof, which is roughly an adaptation to the partial case of the proofs of [24, Lemma 5.5] and [24, Lemma 5.10].

Step 1. We first prove point (i).

Let us first present alternative expressions of the entropy and the Fisher information. For $G \in \mathbf{P}_k((\mathcal{A} \times \mathbb{R}^2)^j)$, it holds that

$$\tilde H(G) \quad = \quad \frac{1}{j} \sup_{\phi \in C_c((\mathcal{A} \times \mathbb{R}^2)^j)} \Big\langle G, \phi - H^*(\phi) + \log \theta^j \Big\rangle \tag{4.3}$$

with $\theta^j(M, X) := \theta^j(X) = c^j \exp(-|x_1|^k - ... - |x_j|^k)$, where c chosen so that θ^j is a probability, where

$$H^*(\phi)(M) := \int_{\mathbb{R}^{2j}} h^*(\phi(M, X))\, \theta^j(X)\, dX$$

and where $h^*(t) := e^t - 1$ is the Legendre transform of $h(s) := s\,\log s - s + 1$. The RHS term is well-defined in $\mathbb{R} \cup \{+\infty\}$ because the function $\phi - H^*(\phi) + \log \theta^j$ is continuous and bounded by $-C\,\langle X\rangle^k$ by below for any $\phi \in C_c((\mathcal{A}\times\mathbb{R}^2)^j)$.

We also have, for $G \in \mathbf{P}((\mathcal{A}\times\mathbb{R}^2)^j)$,

$$\tilde{I}(G) := \frac{1}{j} \sup_{\psi \in C_c^1((\mathcal{A}\times\mathbb{R}^2)^j)^{2j}} \Big\langle G, -\frac{|\psi|^2}{4} - \operatorname{div}_X \psi \Big\rangle. \tag{4.4}$$

Again, the RHS term is well defined in $\mathbb{R}$ because the function $|\psi|^2/4 - \operatorname{div}_X \psi$ is continuous and bounded for any $\psi \in C_c^1((\mathcal{A}\times\mathbb{R}^2)^j)^{2j}$.

As a consequence of the representation formulas (4.3) and (4.4) we immediately conclude that $\tilde{H}$ and $\tilde{I}$ are convex, lower semi-continuous and proper so that point (i1) and the first part of (i2) hold. The lower bound expressed in (i2) is nothing but the result stated in Lemma 3.1.

Step 2. Point (ii) is obvious from (2.2)-(2.3).

Step 3. We now prove (iii). For the partial entropy, we define

$$\tilde{h}_i := i\,\tilde{H}(G_i)$$

for any $G \in \mathbf{P}_k((\mathcal{A}\times\mathbb{R}^2)^j)$ and $1 \le i \le j$, and we just write $G_i = R_i\,F_i$ instead of the more explicit expression $G_i(dM, dX) = R_i(dM)\,F_i^M(X)\,dX$. We then compute

$$\begin{aligned}
\tilde{h}_j - \tilde{h}_i - \tilde{h}_{j-i} &= \int_{(\mathcal{A}\times\mathbb{R}^2)^j} G_j \log F_j - \int_{(\mathcal{A}\times\mathbb{R}^2)^j} G_i \log F_i - \int_{(\mathcal{A}\times\mathbb{R}^2)^{j-i}} G_{j-i} \log F_{j-i} \\
&= \int_{(\mathcal{A}\times\mathbb{R}^2)^j} G_j \,[\log F_j - \log F_i \otimes F_{j-i}] \\
&= \int_{\mathcal{A}^j} R_j \int_{(\mathbb{R}^2)^j} F_j \,[\log F_j - \log F_i \otimes F_{j-i}] \\
&= \int_{\mathcal{A}^j} R_j \int_{(\mathbb{R}^2)^j} F_i \otimes F_{j-i} \,[u\,\log u - u + 1] \ge 0,
\end{aligned}$$

where we have set $u := F_j/(F_i \otimes F_{j-i})$ and we have used that $F_j, F_i \otimes F_{j-i} \in \mathbf{P}((\mathbb{R}^2)^j)$ for any given $M \in \mathbb{R}^j$.

For the partial Fisher information we reproduce the proof of the same super-additivity property established for the usual Fisher information in [24, Lemma 3.7]. We define for any $i \le j$

$$\tilde{\iota}_i := i\,\tilde{I}(G_i) = \sup_{\psi \in C_c^1((\mathcal{A}\times\mathbb{R}^2)^i)^{2i}} \int_{(\mathcal{A}\times\mathbb{R}^2)^i} \Big(\nabla_X G_i \cdot \psi - G_i \,\frac{|\psi|^2}{4}\Big)$$

where the sup is taken on all $\psi = (\psi_1, \dots, \psi_i)$, with all $\psi_\ell : (\mathcal{A}\times\mathbb{R}^2)^i \to \mathbb{R}^2$. We then write the previous equality for ι_j and restrict the supremum over all ψ such that for some $i \le j$:

- the i first ψ_ℓ depend only on $(x_1, \dots, x_i)$, with the notation $\psi^i = (\psi_1, \dots, \psi_i)$,
- the $j - i$ last ψ_ℓ depend only on $(x_{i+1}, \dots, x_j)$, with the notation $\psi^{j-i} = (\psi_{i+1}, \dots, \psi_j)$.

We then have the inequality

$$
\begin{aligned}
\tilde\iota_j &\geq \sup_{\psi^i,\,\psi^{j-i}} \int_{(\mathcal{A}\times\mathbb{R}^2)^j} [\nabla_i G\cdot\psi^i + \nabla_{j-i} G\cdot \psi^{j-i} - G\,\frac{|\psi^i|^2+|\psi^{j-i}|^2}{4}] \\
&= \sup_{\psi^i\in C^1_c((\mathcal{A}\times\mathbb{R}^2)^i)^{2i}} \int_{(\mathcal{A}\times\mathbb{R}^2)^i} [\nabla_i G_i\cdot\psi^i - G_i\,\frac{|\psi^i|^2}{4}] \\
&\qquad\qquad + \sup_{\psi^{j-i}\in C^1_c((\mathcal{A}\times\mathbb{R}^2)^{j-i})^{2(j-i)}} \int_{(\mathcal{A}\times\mathbb{R}^2)^{j-i}} [\nabla_{j-i} G_{j-i}\cdot\psi^{j-i} - G_{j-i}\,\frac{|\psi^{j-i}|^2}{4}] \\
&= \tilde\iota_i + \tilde\iota_{j-i},
\end{aligned}
$$

where all the gradients appearing are only gradients in the X variables.

Step 3. We first note that as a consequence of (iii) we have (see [24, Lemma 5.5] for details) for any $\pi\in\mathbf{P}_k(\mathbf{P}(\mathcal{A}\times\mathbb{R}^2))$

$$
\tilde{\mathcal{J}}'(\pi) := \sup_{j\geq 1}\tilde J(\pi_j) = \lim_{j\to\infty}\tilde J(\pi_j). \tag{4.5}
$$

We now prove the affine caracter (iv) for the partial entropy $\tilde{\mathcal{H}}'$, considering only the case $M=2$ for simplicity. Let us consider $A,B\in\mathbf{P}_k(\mathbf{P}(\mathcal{A}\times\mathbb{R}^2))\cap\mathscr{E}_\infty(r)$, $\theta\in(0,1)$, and let us introduce the disintegration $A_j = R_j\alpha_j$, $B_j = R_j\beta_j$, with $R_j = r^{\otimes j}$ (because both A and B belong to $\mathscr{E}_\infty(r)$). Using that $s\mapsto\log s$ is an increasing function and that $s\mapsto s\,\log s$ is a convex function, we have

$$
\begin{aligned}
\tilde H(\theta\,A_j + (1-\theta)\,B_j) &= \frac{1}{j}\int_{(\mathcal{A}\times\mathbb{R}^2)^j} R_j\,(\theta\,\alpha_j + (1-\theta)\,\beta_j)\,\log(\theta\,\alpha_j+(1-\theta)\,\beta_j) \\
&\geq \frac{1}{j}\int_{(\mathcal{A}\times\mathbb{R}^2)^j} R_j\,\{\theta\,\alpha_j\,\log(\theta\,\alpha_j) + (1-\theta)\,\beta_j\,\log((1-\theta)\,\beta_j)\} \\
&= \theta\,\tilde H(A_j) + (1-\theta)\,\tilde H(B_j) + \frac{1}{j}[\theta\,\log\theta + (1-\theta)\,\log(1-\theta)] \\
&\geq \tilde H(\theta\,A_j + (1-\theta)\,B_j) + \frac{1}{j}[\theta\,\log\theta + (1-\theta)\,\log(1-\theta)].
\end{aligned}
$$

Passing to the limit $j\to\infty$ in the two preceding inequalities and using (4.5), we get

$$
\tilde{\mathcal{H}}'(\theta\,A + (1-\theta)\,B) \geq \theta\,\tilde{\mathcal{H}}'(A) + (1-\theta)\,\tilde{\mathcal{H}}'(B) \geq \tilde{\mathcal{H}}'(\theta\,A+(1-\theta)\,B),
$$

from which the announced affine caracter follows.

We next prove the affine caracter (iv) for the partial Fisher information. For the sake of simplicity we only consider the case when $M=2$ and ω_1 is a ball. The case when ω_1 is a general open set can be handled in a similar way and the case when $M\geq 3$ can be deduced by an iterative argument. For some given $\pi\in\mathbf{P}_k(\mathbf{P}(\mathbb{R}\times\mathbb{R}^2))\cap\mathscr{E}_\infty(r)$ which is not a Dirac mass, some $f_1\in\mathbf{P}_k(\mathbb{R}\times\mathbb{R}^2)$ and some $\eta\in(0,\infty)$ so that

$$
\theta := \pi(B_\eta)\in(0,1),\quad B_\eta = B(f_1,\eta) := \{\rho, W_1(\rho,f_1)<\eta\},
$$

we define

$$
A := \frac{1}{\theta}\mathbb{1}_{B_\eta}\pi,\quad B := \frac{1}{1-\theta}\mathbb{1}_{B^c_\eta}\pi
$$

so that

$$
A,B\in\mathbf{P}_k(\mathbf{P}(\mathcal{A}\times\mathbb{R}^2))\cap\mathscr{E}_\infty(r)\quad\text{and}\quad \pi = \theta A + (1-\theta)B,
$$

and we have to prove that

$$\tilde{\mathcal{I}}'(\pi) = \theta\,\tilde{\mathcal{I}}'(A) + (1-\theta)\,\tilde{\mathcal{I}}'(B). \tag{4.6}$$

We claim that proceeding as in the proof of [24, Lemma 5.10] we may assume, up to regularization by convolution in the X variables, that

$$\sup_{j,M,X} \Big(|\nabla_1 \log \pi_j| + |\nabla_1 \log A_j| + |\nabla_1 \log B_j| \Big) \le C < \infty,$$

where the ∇_1 stands for the gradient in the first position variable only. For any given $j \ge 1$, we define

$$Z_j := \theta\,\tilde{I}(A_j) + (1-\theta)\,\tilde{I}(A_j) - \tilde{I}(\theta\,A_j + (1-\theta)\,B_j),$$

and after some calculations, we obtain, using the disintegrations $A_j = R_j \alpha_j$ and $B_j = R_j \beta_j$ as previously,

$$\begin{aligned} Z_j &= \theta(1-\theta) \int R_j\, \frac{\alpha_j \beta_j}{(1-\theta)\alpha_j + \theta\beta_j} \left| \nabla_1 \log \frac{\alpha_j}{\beta_j} \right|^2 \\ &\le 2\,\theta(1-\theta) \int R_j\, \frac{\alpha_j \beta_j}{(1-\theta)\alpha_j + \theta\beta_j} \Big(\big|\nabla_1 \log \beta_j\big|^2 + \big|\nabla_1 \log \alpha_j\big|^2 \Big) \\ &\le 4\theta(1-\theta)\,C \int R_j\, \frac{\alpha_j \beta_j}{(1-\theta)\alpha_j + \theta\beta_j} = 4\theta(1-\theta)\,C \int \frac{A_j B_j}{(1-\theta)A_j + \theta B_j}. \end{aligned}$$

At this stage, the proof follows exactly the one done in [24, Lemma 5.10] which states that the same property holds for the full Fisher information. Let us introduce, for any $s \in (0,\eta)$ the two measures on $\mathbf{P}(\mathcal{A} \times \mathbb{R}^2)$ (which are not necessarily probability measures)

$$A' := \mathbb{1}_{B_s} A = \frac{1}{\theta} \mathbb{1}_{B_s} \pi, \qquad A'' := \mathbb{1}_{B_\eta \setminus B_s} A,$$

and let us observe that

$$\lim_{s \to \eta} \int A''(d\rho) = 0,$$

by Lebesgue's dominated convergence theorem. By construction, there holds $A' + A'' = A$ as well as for any $j \ge 1$ there holds $A'_j + A''_j = A_j$ with $A''_j \ge 0$, so that we may write for any $\varepsilon > 0$

$$Z_j \;\le\; 4\theta(1-\theta)\,C \int_{\mathbf{P}(\mathcal{A}\times\mathbb{R}^2)} \frac{B_j A'_j}{(1-\theta)B_j + \theta A'_j} + \varepsilon$$

taking s close enough to r (independently of j). We introduce the notation $y = (m,x)$ for the couple circulation-position, define the distance $d(y,y') := \min(|y-y'|,1)$ on $\mathbb{R} \times \mathbb{R}^2$ and the Monge-Kantorovitch-Wasserstein distance W_1 defined on $\mathbf{P}(\mathbb{R} \times \mathbb{R}^2)$ according to distance d.

We introduce the real numbers $u = \frac{\eta+s}{2}$ and $\delta = \frac{\eta-s}{2}$, as well as the set

$$\tilde{B}_u := \{Y^j = (y_1,\dots,y_j)\ ,\ W_1(\mu^j_{Y^j}, f_1) < u\} \subset (\mathbb{R} \times \mathbb{R}^2)^j$$

which is nothing but the reciprocal image of the ball $B_u \subset \mathbf{P}(\mathbb{R} \times \mathbb{R}^2)$ by the empirical measure map. Using that

$$\frac{B_j A'_j}{(1-\theta)B_j + \theta A'_j} \le \frac{1}{\theta} B_j\, \mathbf{1}_{\tilde{B}_u} + \frac{1}{1-\theta} A'_j\, \mathbf{1}_{\tilde{B}_u^c},$$

we get

$$Z_j \le 4\,C \left((1-\theta) \int_{\tilde{B}_u} B_j + \theta \int_{\tilde{B}_u^c} A'_j \right) + \varepsilon. \tag{4.7}$$

Using concentration of empirical measures exactly as in Step 3 of the proof of [24, Lemma 5.11] we deduce that

$$Z_j \leq \frac{4\,C\,[A^k + \tilde M_k(\pi)]^{1/k}}{\delta j^\gamma} + \varepsilon,$$

with $\gamma := 1/(5+3/k)$. Remark that we have the bound $A^k + \tilde M_k(\pi)$ for the full moment in $z = (m,x)$ of the probability π, since the m variable always belongs to $\mathcal{A} = [-A, A]$. Using the above estimate and the convexity estimate $Z_j \geq 0$, we obtain that

$$\lim_{j\to 0} Z_j = 0$$

from which we conclude using (4.5). □

As a consequence of Lemma 4.2, we also have some super-additivity inequalities as well as some weak lower semi-continuity properties that we will frequently use.

Corollary 4.3. *Let $k > 0$ and $r \in \mathbf{P}(\mathbb{R})$ supported in $\mathcal{A} = [-A, A]$ for some $A > 0$.*

(i) Let $G \in \mathscr{E}_N(r)$ for some $N \geq 2$. For any $1 \leq j \leq N$, denoting by G_j the j-marginal of G and introducing the Euclidian decomposition $N = n\,j + \ell$, $0 \leq \ell \leq j-1$, there holds

$$\tilde I(G_j) \leq (1 + \frac{\ell}{nj})\,\tilde I(G) \leq 2\,\tilde I(G) \quad \text{and} \quad \tilde H(G_j) \leq (1+\frac{\ell}{nj})\,\tilde H(G) + \frac{\ell}{nj}\,(C_k + \tilde M_{k/2}(G_\ell)). \tag{4.8}$$

(ii) Let $j \geq 1$ be fixed and $\pi_j \in \mathscr{E}_j(r)$. Consider a sequence $G^N \in \mathscr{E}_N(r)$ such that $G_j^N \rightharpoonup \pi_j$ weakly in $\mathbf{P}((\mathbb{R}\times\mathbb{R}^2)^j)$ as $N \to \infty$ and $\sup_N \tilde M_k(G_1^N) < \infty$. Then

$$\tilde H(\pi_j) \leq \liminf_{N\to\infty} \tilde H(G^N) \quad \text{and} \quad \tilde I(\pi_j) \leq \liminf_{N\to\infty} \tilde I(G^N).$$

Proof of Corollary 4.3. We start with point (i). Iterating the super-additivity property expressed in Lemma 4.2-(iii) tells us that

$$\ell \tilde J(G_\ell) + nj\tilde J(G_j) \leq N \tilde J(G), \tag{4.9}$$

for $\tilde J = \tilde H$ and $\tilde J = \tilde I$. In the case of the Fisher information (which is non-negative), we deduce that $nj\tilde I(G_j) \leq N\tilde I(G)$ which implies the first assertion in (4.8). For the entropy, (4.9) together with the non-negativity property $\ell\,\tilde H(G_\ell) + \ell\,(\tilde M_{k/2}(G_\ell) + C_k) \geq 0$ established in Lemma 4.2-(i2) imply the last assertion in (4.8).

We next check (ii). The lower semi-continuity stated in Lemma 4.2-(i) implies that

$$\tilde J(\pi_j) \leq \liminf_{N\to\infty} \tilde J(G_j^N)$$

for $\tilde J = \tilde H$ and $\tilde J = \tilde I$. We conclude using point (i). □

5. Main estimates and tightness

In the whole section, a family $G_0^N \in \mathbf{P}_{sym}((\mathbb{R}\times\mathbb{R}^2)^N)$ satisfying (2.15) for some $k \in (0,1]$, some $A \in (0,\infty)$ and some $g_0 \in \mathbf{P}(\mathbb{R}\times\mathbb{R}^2)$ is fixed. The following estimate is central in our proof.

Proposition 5.1. *For $N \geq 2$, let $(\mathcal{M}_i^N, \mathcal{X}_i^N(0))_{i=1,\dots,N}$ be G_0^N-distributed and consider the unique solution $(\mathcal{X}_i^N(t))_{i=1,\dots,N,t\geq 0}$ to (1.1). For $t \geq 0$, denote by $G_t^N \in \mathbf{P}_{sym}((\mathbb{R}\times\mathbb{R}^2)^N)$ the law of $(\mathcal{M}_i^N, \mathcal{X}_i^N(t))_{i=1,\dots,N}$. There is a constant $C = C(\sigma, k, A)$ such that for all $t \geq 0$,*

$$\tilde H(G_t^N) + \tilde M_k(G_t^N) + \frac{\nu}{2}\int_0^t \tilde I(G_s^N)ds \leq \tilde H(G_0^N) + \tilde M_k(G_0^N) + C\,t. \tag{5.1}$$

As a consequence, there exists a constant C which depends furthermore on an upper bound on $\tilde H(G_0^N)$ and $\tilde M_k(G_0^N)$ so that for all $N \ge 2$, all $t \ge 0$,

$$\tilde H(G_t^N) \le C(1+t), \quad \tilde M_k(G_t^N) \le C(1+t) \quad \text{and} \quad \int_0^t \tilde I(G_s^N) ds \le C(1+t). \tag{5.2}$$

Proof. The computations below are formal. To handle a rigorous proof, it suffices to approximate the singular kernel K by a smoothed kernel K_ε enjoying the properties that div $K_\varepsilon = 0$ and that $K_\varepsilon(x) = K(x)$ for all $|x| \ge \varepsilon$, which makes all the computations below rigorous. Since (1.1) is well-posed thanks to Osada [42] (see Theorem 2.10) and since the functionals $\tilde M_k$, $\tilde H$ and $\tilde I$ are lower semi-continuous for the weak convergence, it is not hard to conclude to (5.1) (with only an inequality now).

Step 1. Denoting by $X = (x_1, \dots, x_N)$ and $M = (m_1, \dots, m_N)$, we disintegrate $G_t^N(dM, dX)$ as $R_t^N(dM) F_t^{N,M}(dX)$ and we observe that $F_t^{N,M}$ is nothing but the conditional law of $(\mathcal X_i^N(t))_{i=1,\dots,N}$ knowing that $(\mathcal M_i^N)_{i=1,\dots,N} = M$. We also observe that $R_t^N(dM) = R_0^N(dM)$, because the circulations $\mathcal M_i^N$ do not depend on time. Conditionally on $(\mathcal M_i^N)_{i=1,\dots,N} = M$, $(\mathcal X_1^N(t), \dots, \mathcal X_N^N(t))$ solves

$$\forall i = 1, \dots, N, \quad \mathcal X_i^N(t) = \mathcal X_i(0) + \frac{1}{N} \sum_{j \ne i} \int_0^t m_j K(\mathcal X_i^N(s) - \mathcal X_j^N(s)) ds + \sigma \mathcal B_i(t). \tag{5.3}$$

Applying the Itô formula to compute the conditional expectation of $\varphi(\mathcal X_1^N(t), \dots, \mathcal X_N^N(t))$ knowing that $(\mathcal M_i^N)_{i=1,\dots,N} = M$, we get, for any $\varphi \in C_b^2((\mathbb R^2)^N)$, any $t \ge 0$,

$$\begin{aligned} \frac{d}{dt} \int_{(\mathbb R^2)^N} \varphi(X) F_t^{N,M}(dX) = & \int_{(\mathbb R^2)^N} \left[\frac{1}{N} \sum_{i \ne j} m_j K(x_i - x_j) \cdot \nabla_{x_i} \varphi(X) \right] F_t^{N,M}(dX) \\ & + \nu \int_{(\mathbb R^2)^N} \Delta_X \varphi(X). \end{aligned} \tag{5.4}$$

Consequently (recall that div $K = 0$), $F^{N,M}$ is a weak solution to

$$\partial_t F_t^{N,M}(X) + \frac{1}{N} \sum_{i \ne j} m_j K(x_i - x_j) \cdot \nabla_{x_i} F_t^{N,M}(X) = \nu \Delta_X F_t^{N,M}(X). \tag{5.5}$$

Step 2. We easily compute the evolution of the entropy (in the space variable):

$$\begin{aligned} \frac{d}{dt} H(F_t^{N,M}) &= \frac{1}{N} \int_{(\mathbb R^2)^N} (\partial_t F_t^{N,M}(X))(1 + \log(F_t^{N,M}(X)))\, dX \\ &= -\frac{1}{N^2} \sum_{i \ne j} m_j \int_{(\mathbb R^2)^N} K(x_i - x_j).\nabla_{x_i} F_t^{N,M}(X)(1 + \log(F_t^{N,M}(X)))\, dX \\ &\quad + \frac{\nu}{N} \int_{(\mathbb R^2)^N} \Delta_X F_t^{N,M}(X)\, (1 + \log(F_t^{N,M}(X)))\, dX. \end{aligned}$$

Observing that the first term vanishes (because div $K = 0$) and performing an integration by parts on the second term, we immediately and classically deduce that

$$H(F_t^{N,M}) + \nu \int_0^t I(F_s^{N,M})\, ds = H(F_0^{N,M}), \quad \forall\, t \ge 0. \tag{5.6}$$

Integrating this equality against $R_0(dM)$, we finally get

$$\tilde H(G_t^N) + \nu \int_0^t \tilde I(G_s^N)ds = \tilde H(G_0^N), \quad \forall\, t \ge 0. \tag{5.7}$$

Step 3. Applying (5.4) with $\varphi(X) = \langle X\rangle^k$ (for which $|\nabla_{x_i}\varphi| \le C/N$ and $|\Delta_X \varphi| \le C$ because $k \in (0,1]$) and integrating against $R_0^N(dM)$, we get

$$\begin{aligned}
\frac{d}{dt}\tilde M_k(G_t^N) &\le \frac{C}{N^2}\int_{\mathbb{R}^N} R_0^N(dM)\int_{(\mathbb{R}^2)^N} F_t^{N,M}(dX)\sum_{i\ne j}|m_j||K(x_i-x_j)| \\
&\quad + C\int_{\mathbb{R}^N} R_0^N(dM)\int_{(\mathbb{R}^2)^N} F_t^{N,M}(dX) \\
&\le CA\int_{(\mathbb{R}\times\mathbb{R}^2)^N} G_t^N(dM,dX)|K(x_1-x_2)| + C.
\end{aligned}$$

For the last inequality, we used that $R_0^N(dM)F_t^{N,M}(dX) = G_t^N(dM,dX)$ is a symmetric probability measure supported in $([-A,A]\times\mathbb{R}^2)^N$. Denoting by G_{t2}^N the two-marginal of G_t^N, disintegrating $G_{t2}^N(dm_1,dx_1,dm_2,dx_2) = r_t^N(dm_1,dm_2) f_t^{N,m_1,m_2}(dx_1,dx_2)$ and using Lemma 3.3 with $\gamma = 1$ and $\beta = 2/3$, then the Jensen inequality (r_t^N is a probability measure) and finally the definition of $\tilde I$, we find

$$\begin{aligned}
\int_{(\mathbb{R}\times\mathbb{R}^2)^N} |K(x_1-x_2)|\, G_t^N(dM,dX) &= \int_{(\mathbb{R}\times\mathbb{R}^2)^2} \frac{1}{|x_1-x_2|} G_{t2}^N(dm_1,dm_2,dx_1,dx_2) \\
&= \int_{\mathbb{R}^2} r_t^N(dm_1,dm_2)\int_{(\mathbb{R}^2)^2}\frac{1}{|x_1-x_2|} f_t^{N,m_1,m_2}(dx_1,dx_2) \\
&\le C\int_{\mathbb{R}^2} r_t^N(dm_1,dm_2)\left(1 + I(f_t^{N,m_1,m_2})^{2/3}\right) \\
&\le C + C\left(\int_{\mathbb{R}^2} r_t^N(dm_1,dm_2) I(f_t^{N,m_1,m_2})\right)^{2/3} \\
&\le C + C\tilde I(G_{t2}^N)^{2/3}.
\end{aligned}$$

Finally, using Corollary 4.3, we have $\tilde I(G_{t2}^N) \le 2\tilde I(G_t^N)$, so that

$$\frac{d}{dt}\tilde M_k(G_t^N) \le C + C\tilde I(G_t^N)^{2/3} \le C + \frac{\nu}{2}\tilde I(G_t^N).$$

For the last inequality, we recall that the value of C is allowed to change and we mention that we used the inequality $Cx^{2/3} \le C' + (\nu/2)x$ for all $x \ge 0$. Integrating in time, we thus get

$$\tilde M_k(G_t^N) \le Ct + \tilde M_k(G_0^N) + \frac{\nu}{2}\int_0^t \tilde I(G_s^N)ds. \tag{5.8}$$

Step 4. Summing (5.7) and (5.8), we thus find (5.1). This implies the first inequality in (5.2) by positivity of $\tilde M_k$ and $\tilde I$. Finally, we write

$$\tilde M_k(G_t^N) + \frac{\nu}{2}\int_0^t \tilde I(G_s^N)ds \le C(1+t) - \tilde H(G_t^N) \le C(1+t) + \tilde M_k(G_t^N)/2$$

by Lemma 3.1 (with the choice $\lambda = 1/2$). Thus $\tilde M_k(G_t^N) + \nu\int_0^t \tilde I(G_s^N)ds \le C(1+t)$ which implies the second and third inequalities in (5.2) by positivity of $\tilde M_k$ and $\tilde I$ again. $\square$

We can now easily prove the tightness of our particle system.

Lemma 5.2. *For each $N \geq 2$, recall that $(\mathcal{M}_i^N, \mathcal{X}_i^N(0))_{i=1,\dots,N}$ is G_0^N-distributed and consider the unique solution $(\mathcal{X}_i^N(t))_{i=1,\dots,N,t\geq 0}$ to (1.1) . We also set $\mathcal{Q}^N := N^{-1}\sum_{i=1}^N \delta_{(\mathcal{M}_i^N,(\mathcal{X}_i^N(t))_{t\geq 0})}$.*

(i) The family $\{\mathcal{L}((\mathcal{X}_1^N(t))_{t\geq 0}), N\geq 2\}$ is tight in $\mathbf{P}(C([0,\infty),\mathbb{R}^2))$.

(ii) The family $\{\mathcal{L}(\mathcal{Q}^N), N\geq 2\}$ is tight in $\mathbf{P}(\mathbf{P}(\mathbb{R}\times C([0,\infty),\mathbb{R}^2)))$.

Proof. First, point (ii) follows from point (i). Indeed, $\mathcal{M}_1^N$ takes values in the compact set $[-A,A]$, so that we deduce from (i) that the family $\{\mathcal{L}(\mathcal{M}_1^N,(\mathcal{X}_1^N(t))_{t\geq 0}), N\geq 2\}$ is tight in $\mathbf{P}(\mathbb{R}\times C([0,\infty),\mathbb{R}^2))$. Then (ii) follows from the exchangeability of the system, see [50, Proposition 2.2] or [36, Lemma 4.5].

To prove (i), we have to check that for all $\eta>0$ and all $T>0$, we can find a compact subset $\mathcal{K}_{\eta,T}$ of $C([0,T],\mathbb{R}^2)$ such that $\sup_N \mathbb{P}[(\mathcal{X}_1^N(t))_{t\in[0,T]}\notin\mathcal{K}_{\eta,T}]\leq\eta$. Let thus $\eta>0$ be fixed. We introduce the random variable $Z_T := \sup_{0<s<t<T}|\sigma\mathcal{B}_1(t)-\sigma\mathcal{B}_1(s)|/|t-s|^{1/3}$ which is a.s. finite, since the paths of $\mathcal{B}_1$ are a.s. Hölder continuous with index $1/3$. Note also that the law of Z_T does not depend on N. Next, we use the Hölder inequality and the fact that a.s., $|\mathcal{M}_i^N|\leq A$ for all i (recall (2.15)) to get, for all $0<s<t<T$,

$$\begin{aligned}
\Big|\frac{1}{N}\int_s^t\sum_{j\neq 1}\mathcal{M}_j^N K(\mathcal{X}_1^N(u)-\mathcal{X}_j^N(u))du\Big| \leq & \frac{A}{N}\int_s^t\sum_{j\neq 1}|\mathcal{X}_1^N(u)-\mathcal{X}_j^N(u)|^{-1}du\\
\leq & \frac{A}{N}(t-s)^{1/3}\sum_{j\neq 1}\left[\int_0^T|\mathcal{X}_1^N(u)-\mathcal{X}_j^N(u)|^{-3/2}du\right]^{2/3}\\
\leq & (t-s)^{1/3}\left[A+\frac{A}{N}\sum_{j\neq 1}\int_0^T|\mathcal{X}_1^N(u)-\mathcal{X}_j^N(u)|^{-3/2}du\right]\\
=: & U_T^N(t-s)^{1/3}.
\end{aligned}$$

All this yields that for all $0<s<t<T$ (recall that $\mathcal{X}_1^N$ satisfies the first equation of (1.1)),

$$|\mathcal{X}_1^N(t)-\mathcal{X}_1^N(s)|\leq (Z_T+U_T^N)(t-s)^{1/3}.$$

By exchangeability and using Lemma 3.3,

$$\mathbb{E}[U_T^N]=A+A\frac{N-1}{N}\int_0^T\mathbb{E}[|\mathcal{X}_1^N(u)-\mathcal{X}_2^N(u)|^{-3/2}]du\leq A+A\int_0^T\tilde{I}(G_{u2}^N)du,$$

where G_{u2}^N is the two-marginal of G_u^N. But $\tilde{I}(G_{u2}^N)\leq 2\tilde{I}(G_u^N)$ (by Corollary 4.3), so that using finally Proposition 5.1, $\mathbb{E}[U_T^N]\leq A+CA(1+T)$.

Thus $\sup_{N\geq 2}\mathbb{E}[U_T^N]<\infty$ and since Z_T is a.s. finite, we can clearly find $R>0$ such that $\mathbb{P}[Z_T+U_T^N>R]\leq\eta/2$ for all $N\geq 2$. We also know by (2.15) that $\sup_{N\geq 2}\mathbb{E}[\langle\mathcal{X}_1^N(0)\rangle^k]=\sup_{N\geq 2}\tilde{M}_k(G_0^N)<\infty$, so that there is $a>0$ such that $\sup_{N\geq 2}\mathbb{P}[|\mathcal{X}_1^N(0)|>a]\leq\eta/2$. Let now $\mathcal{K}_{\eta,T}$ be the set of all continuous functions $f:[0,T]\mapsto\mathbb{R}^2$ with $|f(0)|\leq a$ and $|f(t)-f(s)|\leq R(t-s)^{1/3}$ for all $0<s<t<T$. For all $N\geq 2$, we have $\mathbb{P}[(\mathcal{X}_1^N(t))_{t\in[0,T]}\notin\mathcal{K}_{\eta,T}]\leq\mathbb{P}[|\mathcal{X}_1^N(0)|>a]+\mathbb{P}[Z_T+U_T^N>R]\leq\eta$. Since $\mathcal{K}_{\eta,T}$ is a compact subset of $C([0,T],\mathbb{R}^2)$, this ends the proof. □

6. Consistency

In the whole section, we assume (2.15) for some $k\in(0,1]$, some $A>0$ and some $g_0\in\mathbf{P}(\mathbb{R}\times\mathbb{R}^2)$. We define $\mathcal{S}$ as the set of all probability measures $g\in\mathbf{P}(\mathbb{R}\times C([0,\infty),\mathbb{R}^2))$ such that g is the law of $(\mathcal{M},(\mathcal{X}(t))_{t\ge 0})$ with $(\mathcal{X}(t))_{t\ge 0}$ solution to the nonlinear SDE (1.6) associated with g_0 and satisfying (2.13): for $g_t \in \mathbf{P}(\mathbb{R}\times\mathbb{R}^2)$ the law of $(\mathcal{M},\mathcal{X}(t))$,

$$\forall\, T>0, \quad \int_0^T \tilde I(g_s)ds < \infty.$$

Proposition 6.1. *For each $N \ge 2$, let $(\mathcal{M}_i^N,\mathcal{X}_i^N(0))_{i=1,\dots,N}$ be G_0^N-distributed and consider the unique solution $(\mathcal{X}_i^N(t))_{i=1,\dots,N,t\ge0}$ to (1.1). Assume that there is a subsequence of $\mathcal{Q}^N := N^{-1}\sum_{i=1}^N \delta_{(\mathcal{M}_i^N,(\mathcal{X}_i^N(t))_{t\ge0}}$ going in law to some $\mathbf{P}(\mathbb{R}\times C([0,\infty),\mathbb{R}^2))$-valued random variable $\mathcal{Q}$. Then $\mathcal{Q}$ a.s. belongs to $\mathcal{S}$.*

Proof. We consider a (not relabelled) subsequence of $\mathcal{Q}^N$ going in law to some $\mathcal{Q}$. We adopt in this proof the convention that $K(0)=0$.

Step 1. Consider the identity maps $m:\mathbb{R}\mapsto\mathbb{R}$ and $\gamma: C([0,\infty),\mathbb{R}^2)\mapsto C([0,\infty),\mathbb{R}^2)$. Using the classical theory of martingale problems, we realize that g belongs to $\mathcal{S}$ as soon as

(a) $g\circ(m,\gamma(0))^{-1}=g_0$;

(b) setting $g_t = g\circ(m,\gamma(t))^{-1}$, (2.13) holds true;

(c) for all $0<t_1<\dots<t_k<s<t$, all $\psi\in C_b(\mathbb{R})$, all $\varphi_1,\dots,\varphi_k\in C_b(\mathbb{R}^2)$, all $\varphi\in C_b^2(\mathbb{R}^2)$,

$$\mathcal{F}(g) := \iint g(dm,d\gamma)g(d\tilde m,d\tilde\gamma)\psi(m)\varphi_1(\gamma_{t_1})\dots\varphi_k(\gamma_{t_k}) \\ \left[\varphi(\gamma_t)-\varphi(\gamma_s)-\int_s^t \tilde m K(\gamma_u-\tilde\gamma_u)\cdot\nabla\varphi(\gamma_u)du - \nu\int_s^t \Delta\varphi(\gamma_u)du\right]=0.$$

Indeed, let $(\mathcal{M},(\mathcal{X}(t))_{t\ge0})$ be g-distributed. Then (a) implies that $(\mathcal{M},\mathcal{X}(0))$ is g_0-distributed and (b) says that the requirement (2.13) is fulfilled. Moreover, defining the vorticity $w_t(B) := \int_{\mathbb{R}\times\mathbb{R}^2} m\mathbb{1}_B(x)g_t(dm,dx)$ for all $B\in\mathcal{B}(\mathbb{R})$, we see from to (2.13) and (3.7) that $(w_t)_{t\ge0}$ satisfies (2.6), which implies (2.4) by Lemma 3.5. Finally, point (c) tells us that for all $\varphi\in C_b^2(\mathbb{R}^2)$,

$$\varphi(\mathcal{X}(t))-\varphi(\mathcal{X}(0))-\int_0^t\int \tilde m K(\mathcal{X}(s)-\tilde\gamma_s)\cdot\nabla\varphi(\mathcal{X}(s))g(d\tilde m,d\tilde\gamma)ds-\nu\int_0^t\Delta\varphi(\mathcal{X}(s))ds$$

is a martingale. This classically implies the existence of a $2D$-Brownian motion $(\mathcal{B}(t))_{t\ge0}$ such that

$$\mathcal{X}(t)=\mathcal{X}(0)+\int_0^t\int \tilde m K(\mathcal{X}(s)-\tilde\gamma_s)g(d\tilde m,d\tilde\gamma)ds+\sigma\mathcal{B}(t).$$

¿From the definition of w_t, we see that $\int \tilde m K(\mathcal{X}(s)-\tilde\gamma_s)g(d\tilde m,d\tilde\gamma)$ is nothing but $\int_{\mathbb{R}^2}K(\mathcal{X}(s)-x)w_s(dx)$. Hence $(\mathcal{X}(t))_{t\ge0}$ solves (1.6) as desired.

We thus only have to prove that $\mathcal{Q}$ a.s. satisfies points (a), (b) and (c). For each $t\ge0$, we set $\mathcal{Q}_t=\mathcal{Q}\circ(m,\gamma(t))^{-1}$.

Step 2. We know from (2.15) that the sequence G_0^N is g_0-chaotic, which implies that $\mathcal{Q}_0^N = \mathcal{Q}^N\circ(m,\gamma(0))^{-1}$ goes weakly to g_0 in law (and thus in probability since g_0 is deterministic), whence $\mathcal{Q}_0=g_0$ a.s. Hence $\mathcal{Q}$ satisfies (a).

Step 3. Point (b) follows from Theorem 4.1 and Proposition 5.1. Indeed, recall that G_t^N is the law of $(\mathcal{M}_i^N,\mathcal{X}_i^N(t))_{i=1,\dots,N}$. Since the $\mathcal{M}_i^N$ are i.i.d. and r_0-distributed by assumption (2.15) and since the system is exchangeable, it holds that $G_t^N\in\mathscr{E}_n(r_0)$ for all $t\ge0$ and r_0 is supported in $[-A,A]$ still by (2.15). Next, Proposition 5.1 implies that $\sup_{N\ge2}\tilde M_k(G_t^N)<\infty$, which is equivalent, by exchangeability, to $\sup_{N\ge2}\tilde M_k(G_{t1}^N)<\infty$. Finally, we know that $N^{-1}\sum_{i=1}^N\delta_{(\mathcal{M}_i^N,\mathcal{X}_i^N(t))}$ goes weakly to $\mathcal{Q}_t$ in law (by hypothesis), which classically implies (see e.g. Sznitman [50]) that for all $j \geq 1$, G^N_{tj} goes weakly to π_{tj}, where $\pi_t := \mathcal{L}(\mathcal{Q}_t)$ and where $\pi_{tj} = \int_{\mathbf{P}(\mathbb{R}\times\mathbb{R}^2)} g^{\otimes j}\pi_t(dg)$. We thus may apply Theorem 4.1 (for each $t \geq 0$) and deduce that $\int_{\mathbf{P}(\mathbb{R}\times\mathbb{R}^2)} \tilde{I}(g)\pi_t(dg) \leq \liminf_N \tilde{I}(G^N_t)$. By the Fatou Lemma and by definition of π_t, this yields

$$\mathbb{E}\left[\int_0^T \tilde{I}(\mathcal{Q}_s)ds\right] = \int_0^T \int_{\mathbf{P}(\mathbb{R}\times\mathbb{R}^2)} \tilde{H}(g)\pi_t(dg)dt \leq \int_0^T \liminf_N \tilde{I}(G^N_s)dt \leq \liminf_N \int_0^T \tilde{I}(G^N_s)dt.$$

This last quantity is finite by Proposition 5.1, so that $\int_0^T \tilde{I}(\mathcal{Q}_s)ds < \infty$ a.s.

Step 4. From now on, we consider some fixed $\mathcal{F} : \mathbf{P}(\mathbb{R}\times C([0,\infty),\mathbb{R}^2)) \mapsto \mathbb{R}$ as in point (c). We will check that $\mathcal{F}(\mathcal{Q}) = 0$ a.s. and this will end the proof.

Step 4.1. Here we prove that for all $N \geq 2$,

$$\mathbb{E}\left[(\mathcal{F}(\mathcal{Q}^N))^2\right] \leq \frac{C_{\mathcal{F}}}{N}. \tag{6.1}$$

To this end, we recall that $\varphi \in C^2_b(\mathbb{R}^2)$ is fixed and we apply the Itô formula to (1.1): for all $i = 1,\dots,N$, (here we use the convention that $K(0) = 0$)

$$\begin{aligned}O^N_i(t) :=&\varphi(\mathcal{X}^N_i(t)) - \frac{1}{N}\sum_j \mathcal{M}^N_j \int_0^t \nabla\varphi(\mathcal{X}^N_i(s))\cdot K(\mathcal{X}^N_i(s) - \mathcal{X}^N_j(s))ds - \frac{\sigma^2}{2}\int_0^t \Delta\varphi(\mathcal{X}^N_i(s))ds\\ =&\varphi(\mathcal{X}^N_i(0)) + \sigma\int_0^t \nabla\varphi(\mathcal{X}^N_i(s))d\mathcal{B}^i_s.\end{aligned}$$

But one easily get convinced that

$$\begin{aligned}\mathcal{F}(\mathcal{Q}^N) =&\frac{1}{N}\sum_{i=1}^N \psi(\mathcal{M}^N_i)\varphi_1(\mathcal{X}^N_i(t_1))\dots\varphi_k(\mathcal{X}^N_i(t_k))[O^N_i(t) - O^N_i(s)]\\ =&\frac{\sigma}{N}\sum_{i=1}^N \psi(\mathcal{M}^N_i)\varphi_1(\mathcal{X}^N_i(t_1))\dots\varphi_k(\mathcal{X}^N_i(t_k))\int_s^t \nabla\varphi(\mathcal{X}^N_i(u))d\mathcal{B}^i_u.\end{aligned}$$

Then (6.1) follows from some classical stochastic calculus, using that $0 < t_1 < \dots < t_k < s < t$, that $\psi, \varphi_1, \dots, \varphi_k, \nabla\varphi$ are bounded and that the Brownian motions $\mathcal{B}^1,\dots,\mathcal{B}^N$ are independent.

Step 4.2. Next we introduce, for $\varepsilon \in (0,1)$, the smoothed kernel $K_\varepsilon : \mathbb{R}^2 \mapsto \mathbb{R}$ defined by $K_\varepsilon(x) = K(x\max(|x|,\varepsilon)/|x|)$. This kernel is continuous, bounded, verifies $K_\varepsilon(x) = K(x)$ as soon as $|x| \geq \varepsilon$ and $|K_\varepsilon(x)| \leq |K(x)| = |x|^{-1}$. We also introduce $\mathcal{F}_\varepsilon$ defined as $\mathcal{F}$ with K replaced by K_ε. Then one easily checks that $g \mapsto \mathcal{F}_\varepsilon(g)$ is continuous and bounded from $\mathbf{P}(\mathbb{R}\times C([0,\infty),\mathbb{R}^2))$ to $\mathbb{R}$. Since $\mathcal{Q}^N$ goes in law to $\mathcal{Q}$, we deduce that for any $\varepsilon \in (0,1)$,

$$\mathbb{E}[|\mathcal{F}_\varepsilon(\mathcal{Q})|] = \lim_N \mathbb{E}[|\mathcal{F}_\varepsilon(\mathcal{Q}^N)|].$$

Step 4.3. We now prove that for all $N \geq 2$, all $\varepsilon \in (0,1)$,

$$\mathbb{E}[|\mathcal{F}(\mathcal{Q}^N) - \mathcal{F}_\varepsilon(\mathcal{Q}^N)|] \leq C_{\mathcal{F}}\sqrt{\varepsilon}.$$

Using that all the functions (including the derivatives) involved in $\mathcal{F}$ are bounded and that $|K_\varepsilon(x)-K(x)| \le |x|^{-1}\mathbb{1}_{\{0<|x|<\varepsilon\}}$, we get

$$
\begin{aligned}
(6.2) \quad |\mathcal{F}(g)-\mathcal{F}_\varepsilon(g)| \le & C_\mathcal{F} \int\int |\tilde{m}| \int_0^t |\gamma(u)-\tilde{\gamma}(u)|^{-1}\mathbb{1}_{\{0<|\gamma(u)-\tilde{\gamma}(u)|<\varepsilon\}} du g(d\tilde{m},d\tilde{\gamma})g(dm,d\gamma)\\
\le & C_\mathcal{F}\sqrt{\varepsilon} \int\int |\tilde{m}| \int_0^t |\gamma(u)-\tilde{\gamma}(u)|^{-3/2}\mathbb{1}_{\{\gamma(u)\neq\tilde{\gamma}(u)\}} du g(d\tilde{m},d\tilde{\gamma})g(dm,d\gamma).
\end{aligned}
$$

Thus

$$
|\mathcal{F}(\mathcal{Q}^N)-\mathcal{F}_\varepsilon(\mathcal{Q}^N)| \le C_\mathcal{F}\frac{\sqrt{\varepsilon}}{N^2}\sum_{i\neq j}|\mathcal{M}_j^N|\int_0^t |\mathcal{X}_i^N(u)-\mathcal{X}_j^N(u)|^{-3/2}du,
$$

whence by exchangeability (and since $|\mathcal{M}_j^N| \le A$ a.s. for all j by (2.15)),

$$
\mathbb{E}[|\mathcal{F}(\mathcal{Q}^N)-\mathcal{F}_\varepsilon(\mathcal{Q}^N)|] \le C_\mathcal{F}\sqrt{\varepsilon}\int_0^t \mathbb{E}[|\mathcal{X}_1^N(u)-\mathcal{X}_2^N(u)|^{-3/2}]du.
$$

Denoting by G^N_{u2} the two-marginal of G^N_u and using Lemma 3.3 with $\gamma = 3/2$ and $\beta = 1$, we get

$$
\mathbb{E}[|\mathcal{F}(\mathcal{Q}^N)-\mathcal{F}_\varepsilon(\mathcal{Q}^N)|] \le C_\mathcal{F}\sqrt{\varepsilon}\int_0^t \tilde{I}(G^N_{u2})du.
$$

We conclude using Proposition 5.1 and that $\tilde{I}(G^N_{u2}) \le 2\tilde{I}(G^N_u)$, see Corollary 4.3.

Step 4.4. We next check that a.s.,

$$
\lim_{\varepsilon\to 0}|\mathcal{F}(\mathcal{Q})-\mathcal{F}_\varepsilon(\mathcal{Q})| = 0.
$$

Since $\mathcal{Q}$ is the limit in law of $\mathcal{Q}^N$ by assumption and since Supp $\mathcal{Q}^N \subset [-A,A]\times C([0,T],\mathbb{R}^2)$ a.s. thanks to (2.15), we deduce that Supp $\mathcal{Q} \subset [-A,A]\times C([0,T],\mathbb{R}^2)$ a.s. Hence Supp $\mathcal{Q}_s \subset [-A,A]\times\mathbb{R}^2$ a.s. for each $s \ge 0$. Denote by $v_s(dx) := \int_\mathbb{R} \mathcal{Q}_s(dm,dx)$, we have from Step 3 and Lemma 3.4 that $\nabla v \in L^{2q/(3q-2)}([0,T],L^q(\mathbb{R}^2))$ for all $q\in[1,2)$ a.s., whence

$$
\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|x-y|^{-3/2}v_s(dx)v_s(dy)ds < \infty
$$

a.s. by Lemma 3.5. Using now (6.2), we deduce that

$$
\begin{aligned}
|\mathcal{F}(\mathcal{Q})-\mathcal{F}_\varepsilon(\mathcal{Q})| \le & C_\mathcal{F}A\sqrt{\varepsilon}\int\int\int_0^t |x(s)-\tilde{x}(s)|^{-3/2}ds\mathcal{Q}(d\tilde{m},d\tilde{x})\mathcal{Q}(dm,dx)\\
= & C_\mathcal{F}A\sqrt{\varepsilon}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|x-y|^{-3/2}v_s(dx)v_s(dy)ds.
\end{aligned}
$$

The conclusion follows.

Step 4.5. We finally conclude: for any $\varepsilon\in(0,1)$, we write, using Steps 4.1, 4.2 and 4.3,

$$
\begin{aligned}
\mathbb{E}[|\mathcal{F}(\mathcal{Q})|\wedge 1] \le & \mathbb{E}[|\mathcal{F}_\varepsilon(\mathcal{Q})|] + \mathbb{E}[|\mathcal{F}(\mathcal{Q})-\mathcal{F}_\varepsilon(\mathcal{Q})|\wedge 1]\\
= & \lim_N \mathbb{E}[|\mathcal{F}_\varepsilon(\mathcal{Q}^N)|] + \mathbb{E}[|\mathcal{F}(\mathcal{Q})-\mathcal{F}_\varepsilon(\mathcal{Q})|\wedge 1]\\
\le & \limsup_N \mathbb{E}[|\mathcal{F}(\mathcal{Q}^N)|] + \limsup_N \mathbb{E}[|\mathcal{F}(\mathcal{Q}^N)-\mathcal{F}_\varepsilon(\mathcal{Q}^N)|] + \mathbb{E}[|\mathcal{F}(\mathcal{Q})-\mathcal{F}_\varepsilon(\mathcal{Q})|\wedge 1]\\
\le & C_\mathcal{F}\sqrt{\varepsilon} + \mathbb{E}[|\mathcal{F}(\mathcal{Q})-\mathcal{F}_\varepsilon(\mathcal{Q})|\wedge 1].
\end{aligned}
$$

We now make tend $\varepsilon\to 0$ and use that $\lim_\varepsilon \mathbb{E}[|\mathcal{F}(\mathcal{Q})-\mathcal{F}_\varepsilon(\mathcal{Q})|\wedge 1] = 0$ thanks to Step 4.4 by dominated convergence. Consequently, $\mathbb{E}[|\mathcal{F}(\mathcal{Q})|\wedge 1] = 0$, whence $\mathcal{F}(\mathcal{Q}) = 0$ a.s. as desired. □

7. Well-posedness for the limit equation and its stochastic paths

We first give the

Proof of Theorem 2.5. First, existence follows from Proposition 6.1. Let w_0 satisfy (1.5), introduce g_0 satisfying (2.9) such that (2.10) holds true as in Remark 2.6-(ii) and finally consider set $G_0^N := g_0^{\otimes N}$, which satisfies (2.15). Then Proposition 6.1 implies the existence (in law) of a solution to the nonlinear SDE 1.6 associated to g_0 and such that (2.13) holds true. Defining $(w_t)_{t\geq 0}$ by (2.12), Remark 2.8 implies that $(w_t)_{t\geq 0}$ is a weak solution starting from w_0 to (1.2). Furthermore, we have seen in the proof of Proposition 6.1, Step 1, that $(w_t)_{t\geq 0}$ satisfies (2.6).

We now turn to uniqueness and renormalization, which we prove in several steps. We consider a weak solution $(w_t)_{t\geq 0}$ of (1.2) satisfying (2.6) and we put $\bar K(t,x) := (K \star w_t)(x)$.

Step 1. First Estimates. Because of (2.6), we know that a.e. in time, w_s is a measurable function, and thanks to the $\mathbf{M}([0,T],\mathbb{R}^2)$-weak continuity assumption, we deduce that

$$w \in L^\infty(0,T;L^1(\mathbb{R}^2)) \quad \forall\, T>0. \tag{7.1}$$

Also observe that (2.6) and (7.1) imply, thanks to Lemma 3.5, that

$$w \in L^{p/(p-1)}(0,T;L^p(\mathbb{R}^2)) \quad \forall\, p \in (1,\infty), \forall\, T>0, \tag{7.2}$$

By definition of K and by the Hardy-Littlewood-Sobolev inequality (of which a particular case is $|| \int_{\mathbb{R}^2} |.-y|^{-1} f(y)dy||_{L^{2p/(2-p)}} \leq C_p ||f||_{L^p}$ for all $p\in(1,2)$, see e.g. [30, Theorem 4.3]), we thus get

$$\bar K \in L^{p/(p-1)}(0,T;L^{2p/(2-p)}(\mathbb{R}^2)) \quad \forall\, p \in (1,2), \forall\, T>0. \tag{7.3}$$

Similarly, (2.6) and the Hardy-Littlewood-Sobolev inequality imply that

$$\nabla_x \bar K = K * (\nabla_x w) \in L^{p/(p-1)}(0,T;L^p(\mathbb{R}^2)) \quad \forall\, p\in(2,\infty), \quad \forall\, T>0. \tag{7.4}$$

Step 2. Continuity. Consider a mollifier sequence (ρ_n) on $\mathbb{R}^2$ and introduce the mollified function $w_t^n := w_t * \rho_n$. Clearly, $w^n \in C([0,\infty), L^1(\mathbb{R}^2))$. Using (7.2) and (7.4), a variant of the commutation Lemma [16, Lemma II.1 and Remark 4] tells us that

$$\partial_t w^n - \bar K \cdot \nabla_x w^n - \nu \Delta_x w^n = r^n, \tag{7.5}$$

with

$$r^n := (\bar K \cdot \nabla_x w) * \rho_n - \bar K \cdot \nabla_x w^n \to 0 \quad \text{in} \quad L^1(0,T;L^1_{loc}(\mathbb{R}^2)).$$

The important point here is that $|\nabla_x \bar K|\,|\omega| \in L^1((0,T)\times\mathbb{R}^2)$, thanks to (7.4) and (7.2). Remark that the singularity of the Biot-Savard kernel is sharp for that property : it will no longer be true if we increase the singularity. It is the first time that this happens, all we have done before remains valid for a singularity like $|x|^{-\gamma}$ with $\gamma\in(0,2)$.

As a consequence, the chain rule applied to the smooth w^n reads

$$\partial_t \beta(w^n) = \bar K \cdot \nabla_x \beta(w^n) + \nu \Delta_x \beta(w^n) - \nu \beta''(w^n)\,|\nabla_x w^n|^2 + \beta'(w^n)\, r^n, \tag{7.6}$$

for any $\beta \in C^1(\mathbb{R}) \cap W^{2,\infty}_{loc}(\mathbb{R})$ such that β'' is piecewise continuous and vanishes outside of a compact set. Because the equation (7.5) with $\bar K$ fixed is linear, the difference $w^{n,k} := w^n - w^k$ satisfies (7.5) with r^n replaced by $r^{n,k} := r^n - r^k \to 0$ in $L^1(0,T;L^1_{loc}(\mathbb{R}^2)$ and then also (7.6) (with again w^n and r^n changed in $w^{n,k}$ and $r^{n,k}$). In that last equation, we choose $\beta(s) = \beta_1(s)$ where

$\beta_M(s) = s^2/2$ for $|s| \le M$, $\beta_M(s) = M\,|s| - M^2/2$ for $|s| \ge M$ and we obtain for any non-negative $\chi \in C^2_c(\mathbb{R}^d)$,

$$\begin{aligned}\int_{\mathbb{R}^2} \beta_1(w^{n,k}(t,x))\,\chi(x)\,dx \quad &\le \quad \int_{\mathbb{R}^2} \beta_1(w^{n,k}(0,x))\,\chi(x)\,dx + \int_0^t\int_{\mathbb{R}^2} |r^{n,k}(s,x)|\,\chi(x)\,dxds \\ &\quad + \int_0^t\int_{\mathbb{R}^2} \beta_1(w^{n,k}(s,x))\,\Big(\nu\Delta\chi(x) - \bar K(s,x)\cdot\nabla\chi(x)\Big)\,dxds\end{aligned}$$

where we have used that $\mathrm{div}_x\,\bar K = 0$, that $|\beta_1'| \le 1$ and that $\beta_1'' \ge 0$. Because $w_0 \in L^1$, we have $w^{n,k}(0) \to 0$ in $L^1(\mathbb{R}^2)$, and we deduce from the previous inequality, the convergence $r^{n,k} \to 0$ in $L^1(0,T;L^1_{loc}(\mathbb{R}^2))$, the convergence $\beta_1(w^{n,k})\bar K \to 0$ in $L^1(0,T;L^1_{loc}(\mathbb{R}^2))$ (because $\beta_1(s) \le |s|$, because $w^{n,k} \to 0$ in $L^3(0,T,L^{3/2}(\mathbb{R}^2))$ by (7.2) with $p = 3/2$ and since $\bar K \in L^6(0,T;L^3(\mathbb{R}^2)) \subset L^{3/2}(0,T;L^3(\mathbb{R}^2))$ by (7.3) with $p = 6/5$), that

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^2} \beta_1(w^{n,k}(t,x))\,\chi(x)\,dx \underset{n,k\to\infty}{\longrightarrow} 0.$$

Since χ is arbitrary, we deduce that there exists $\bar w \in C([0,\infty);L^1_{loc}(\mathbb{R}^2))$ so that $w^n \to \bar w$ in $C([0,\infty);L^1_{loc}(\mathbb{R}^2))$, with the topology of uniform convergence on any compact subset in time. Together with the convergence $w^n \to w$ in $C([0,\infty);\mathbf{M}(\mathbb{R}^2))$ we deduce that $w = \bar w$ and with the same convention for the notion of convergence on $[0,\infty)$

$$w^n \to w \quad \text{in} \quad C([0,\infty);L^1(\mathbb{R}^2)). \tag{7.7}$$

Step 3. Additional estimates. We come back to (7.6), which implies, for all $0 < t_0 < t_1$, all $\chi \in C^2_c(\mathbb{R}^2)$,

$$\begin{aligned}\int_{\mathbb{R}^2} \beta(w^n_{t_1})\,\chi\,dx + \nu\int_{t_0}^{t_1}\int_{\mathbb{R}^2} \beta''(w^n_s)\,|\nabla_x w^n_s|^2\,\chi\,dxds = \int_{\mathbb{R}^2} \beta(w^n_{t_0})\,\chi\,dx \\ + \int_{t_0}^{t_1}\int_{\mathbb{R}^2} \Big\{\beta'(w^n_s)\,r^n\,\chi + \beta(w^n_s)\,\nu\Delta\chi - \beta(w^n_s)\,\bar K\cdot\nabla\chi\Big\}\,dxds.\end{aligned} \tag{7.8}$$

Choosing $0 \le \chi \in C^2_c(\mathbb{R}^2)$ and $\beta \in C^1(\mathbb{R}) \cap W^{2,\infty}_{loc}(\mathbb{R})$ such that β'' is non-negative and vanishes outside of a compact set, and passing to the limit as $n \to \infty$ (see Step 2 for the details of a similar convergence), we get

$$\int_{\mathbb{R}^2} \beta(w_{t_1})\,\chi\,dx \le \int_{\mathbb{R}^2} \beta(w_{t_0})\,\chi\,dx + \int_{t_0}^{t_1}\int_{\mathbb{R}^2} \beta(w_s)\,\Big\{\nu\Delta\chi - \bar K\cdot\nabla\chi\Big\}\,dxds.$$

It is not hard to deduce, by approximating $\chi \equiv 1$ by a well-chosen sequence χ_R, using that $\int_{t_0}^{t_1}\int_{\mathbb{R}^2} |w_s(x)|\mathbb{1}_{\{|x|\ge R\}}dx$ clearly tends to 0 as $R \to \infty$ and that β is sublinear, that

$$\int_{\mathbb{R}^2} \beta(w_t)\,dx \le \int_{\mathbb{R}^2} \beta(w_{t_0})\,dx \quad \forall\, t \ge t_0 \ge 0.$$

Finally, letting $\beta(s) \to |s|^p/p$ and then $p \to \infty$, we get

$$\|w(t,.)\|_{L^p} \le \|w(t_0,.)\|_{L^p}, \quad \forall\, p \in [1,\infty],\ \forall\, t \ge t_0 \ge 0. \tag{7.9}$$

Taking now $\beta = \beta_M$ in (7.8), we have

$$\begin{aligned}\int_{\mathbb{R}^2} \beta_M(w^n_{t_1})\,\chi\,dx + \nu\int_{t_0}^{t_1}\int_{\mathbb{R}^2} \mathbb{1}_{\{|w^n_s|\le M\}}|\nabla_x w^n_s|^2\,\chi\,dxds = \int_{\mathbb{R}^2} \beta_M(w^n_{t_0})\,\chi\,dx \\ + \int_{t_0}^{t_1}\int_{\mathbb{R}^2} \Big\{\beta_M'(w^n_s)\,r^n\ \chi + \beta_M(w^n_s)\,\nu\Delta\,\chi - \beta_M(w^n_s)\,\bar K\cdot\nabla\,\chi\Big\}\,dxds,\end{aligned}$$

Similarly as above we first make tend $n \to \infty$, then we approximate $\chi \equiv 1$ by a well-chosen sequence χ_R and make tend $R \to \infty$, and finally we take the limit as $M \to \infty$: this yields

$$\int_{\mathbb{R}^2} w_{t_1}^2 \, dx + \nu \int_{t_0}^{t_1} \int_{\mathbb{R}^2} |\nabla_x w_s|^2 \, dxds \le \int_{\mathbb{R}^2} w_{t_0}^2 \, dx \quad \forall \, t_1 \ge t_0 \ge 0. \tag{7.10}$$

Using (7.2), (7.9) and (7.10), we deduce that for all $0 < t_0 < T$,

$$\forall \, p \in [1,\infty), \;\; w \in L^\infty(t_0,T;L^p(\mathbb{R}^2)) \quad \text{and} \quad \nabla_x w \in L^2((t_0,T) \times \mathbb{R}^2). \tag{7.11}$$

It is easily checked, using the Hölder inequality, that $||\bar{K}_t||_{L^\infty} \le C(||w_t||_{L^1} + ||w_t||_{L^3})$. Hence, $\bar{K} \in L^\infty(t_0,T;L^\infty(\mathbb{R}^2))$. We thus have

$$\partial_t w + \Delta_x w = \bar{K} \cdot \nabla_x w \in L^2((t_0,T) \times \mathbb{R}^2), \quad \forall \, t_0 > 0 \tag{7.12}$$

so that the maximal regularity of the heat equation in L^2-spaces (see Theorem X.11 stated in [6] and the quoted reference) provides the bound

$$w \in L^2(t_0,T;H^2(\mathbb{R}^2)) \cap L^\infty(t_0,T;H^1(\mathbb{R}^2)), \quad \forall \, t_0 > 0. \tag{7.13}$$

We emphasize that starting form the bound $\bar{K} \cdot \nabla w \in L^2(L^2((t_0,T) \times \mathbb{R}^2)$ and when $w_{t_0} \in H^1$, the maximal regularity implies the above bound on the time interval $[t_0,\infty)$. But thanks to (7.11), we can find t_0 arbitrarily close to 0 such that $w_{t_0/2} \in H^1$, and this implies that (7.13) is correct for any $t_0 > 0$.

Thanks to (7.13), an interpolation inequality and the Sobolev inequality, we deduce that $\nabla_x w \in L^p((t_0,T) \times \mathbb{R}^2)$ for any $1 < p < \infty$, whence $\bar{K} \cdot \nabla_x w \in L^p((t_0,T) \times \mathbb{R}^2)$, for all $t_0 > 0$. Then the maximal regularity of the heat equation in L^p-spaces (see Theorem X.12 stated in [6] and the quoted references) provides the bound

$$\partial_t w, \nabla_x w \in L^p((t_0,T) \times \mathbb{R}^2), \quad \forall \, t_0 > 0 \tag{7.14}$$

and then the Morrey inequality implies $w \in C^{0,\alpha}((t_0,T) \times \mathbb{R}^2)$ for any $0 < \alpha < 1$, and any $t_0 > 0$. All together we conclude with

$$w \in C([0,T);L^1(\mathbb{R}^2)) \cap C((0,T);L^\infty(\mathbb{R}^2)),$$

which is nothing but (2.7).

Step 4. Uniqueness. At this stage, we thus have shown that any weak solution to (1.2) satisfying (2.6) meets the assumptions of [7] (which improves, thanks to very quick but smart arguments, the uniqueness result stated in [3, Theorem B]). Such a solution is thus unique.

Step 5. Renormalization. We end the proof by showing (2.8). Let thus $\beta \in C^1(\mathbb{R}) \cap W^{2,\infty}_{loc}(\mathbb{R})$ such that β'' is piecewise continuous and vanishes outside of a compact set. Thanks to (7.11), we can pass to the limit in the similar identity as (7.8) obtained for time dependent test functions $\chi \in C_c^2([0,\infty) \times \mathbb{R}^2)$ and we get

$$\begin{aligned} \nu \int_{t_0}^\infty \int_{\mathbb{R}^2} \beta''(w_s) \, |\nabla_x w_s|^2 \, \chi \, dxds &= \int_{\mathbb{R}^2} \beta(w_{t_0}) \, \chi \, dx \\ &+ \int_{t_0}^\infty \int_{\mathbb{R}^2} \beta(w_s) \left\{ \nu \Delta \chi - \bar{K} \cdot \nabla \chi - \partial_t \chi \right\} dxds. \end{aligned} \tag{7.15}$$

When moreover $\chi \ge 0$ and $\beta'' \ge 0$, we can pass to the limit $t_0 \to 0$ thanks to monotonous convergence in the first term, the continuity property (7.7) in the second term and the Lebesgue dominated convergence theorem in the third term (recall that β is sublinear and that $|w|(1+$

$|\bar K|)$ belongs to $L^1(0,T;L^1(\mathbb{R}))$ because $w \in L^3(0,T;L^{3/2}(\mathbb{R}^2))$ by (7.2) with $p = 3/2$ and $\bar K \in L^6(0,T;L^3(\mathbb{R}^2)) \subset L^{3/2}(0,T;L^3(\mathbb{R}^2))$ by (7.3) with $p = 6/5$) and we get

$$\begin{aligned}\nu \int_0^\infty \int_{\mathbb{R}^2} \beta''(w_s)\, |\nabla_x w_s|^2 \, \chi \, dxds = \int_{\mathbb{R}^2} \beta(w_0)\, \chi \, dx \\ + \int_0^\infty \int_{\mathbb{R}^2} \beta(w_s) \Big\{ \nu \Delta \chi - \bar K \cdot \nabla \chi - \partial_t \chi \Big\} \, dxds.\end{aligned} \tag{7.16}$$

With the new bound on the first term provided by (7.16), we can pass to the limit as $t_0 \to 0$ in (7.15) and get (7.16) for arbitrary test functions χ and renormalizing functions β (i.e. without the assumptions that χ and β'' are non-negative). This is nothing but (2.8) in the distributional sense. □

We now turn to the well-posedness of the nonlinear SDE (1.6).

Proof of Theorem 2.9. Let thus g_0 satisfy (2.9). Here again, Proposition 6.1 (e.g. with the choice $G_0^N = g_0^{\otimes N}$) shows the existence (in law) of a solution to the nonlinear SDE (1.6) such that (2.13) holds true. Defining $(w_t)_{t\geq 0}$ by (2.12), Remark 2.8 implies that $(w_t)_{t\geq 0}$ is a weak solution to (1.2). Furthermore, we have seen in the proof of Proposition 6.1, Step 1, that $(w_t)_{t\geq 0}$ satisfies (2.6). Hence $(w_t)_{t\geq 0}$ is uniquely determined by Theorem 2.5. We will check below the pathwise uniqueness for the linear equation

$$\mathcal{X}(t) = \mathcal{X}(0) + \int_0^t \bar K_s(\mathcal{X}(s))\, ds + \sigma \mathcal{B}_t, \tag{7.17}$$

where $\bar K_s = K * w_s$. This will end the proof. Indeed, pathwise uniqueness for (1.6) will immediately follow (consider two solutions $\mathcal{X}, \mathcal{Y}$ to (1.6) associated to the same Brownian motion $\mathcal{B}$ and the same $(\mathcal{M}, \mathcal{X}(0))$, observe that both satisfy (7.17) with the same Brownian motion, so that they coincide). Now existence in law and pathwise uniqueness classically imply strong existence by the Yamada-Watanabe theorem [52].

For the weak uniqueness to (7.17), we might refer to [17] which assume that $\bar K \in L^2_{t,x,loc}$. For the pathwise uniqueness to (7.17), we might use [29], who assume that $\bar K \in L^1([0,T], W^{1,1}(\mathbb{R}^2))$. But we shall give here an alternative proof for pathwise uniqueness, which is well-suited to our initial (entropic) distribution. We adapt to our context the method of [13] concerning deterministic ODEs with low regularity vector-field.

We thus assume that we have two solutions $\mathcal{X}$ and $\mathcal{Y}$ to (7.17) with the same Brownian motion $\mathcal{B}$, the same value of $(\mathcal{M}, \mathcal{X}(0))$ and the same vector-field $\bar K$. Then, obviously,

$$\mathcal{X}(t) - \mathcal{Y}(t) = \int_0^t (\bar K_s(\mathcal{X}(s)) - \bar K_s(\mathcal{Y}(s)))\, ds,$$

so that for any $\delta > 0$,

$$\log(\delta + |\mathcal{X}(t) - \mathcal{Y}(t)|) \leq \log \delta + \int_0^t \frac{|\bar K_s(\mathcal{X}(s)) - \bar K_s(\mathcal{Y}(s))|}{\delta + |\mathcal{X}(s) - \mathcal{Y}(s)|}\, ds$$

and thus

$$\mathbb{E}\left[\log(\delta + \sup_{0\leq s\leq t} |\mathcal{X}(s) - \mathcal{Y}(s)|)\right] \leq \log \delta + \int_0^t \mathbb{E}\left[\frac{|\bar K_s(\mathcal{X}(s)) - \bar K_s(\mathcal{Y}(s))|}{\delta + |\mathcal{X}(s) - \mathcal{Y}(s)|}\right] ds.$$

We will use the following facts: for a measurable function f on $\mathbb{R}^2$, define the Hardy-Littlewood maximal function $Mf(x) = \sup_{r>0} |B_r(x)|^{-1} \int_{B_r(x)} |f(y)| dy$, where $B_r(x)$ is the ball centered at x

with radius r. Then, for a.e. $x, y \in \mathbb{R}^2$, see [1, Corollary 4.3 with $\alpha = 0$]

$$|f(x) - f(y)| \leq C \left[M\nabla f(x) + M\nabla f(y)\right] |x - y| \tag{7.18}$$

and for all $p \in [1, \infty]$, see [48, Theorem 1 in Chapter 1],

$$\|Mf\|_{L^p} \leq C_p \|f\|_p, \qquad \forall p \in [1, \infty]. \tag{7.19}$$

Using (7.18), we obtain

$$\mathbb{E}\left[\log(\delta + \sup_{0\leq s\leq t} |\mathcal{X}(s) - \mathcal{Y}(s)|)\right] \leq \log\delta + C\int_0^t \mathbb{E}\left[|M\nabla_x \bar{K}_s(\mathcal{X}(s)) + M\nabla_x \bar{K}_s(\mathcal{Y}(s))|\right]\, ds.$$

Denoting now by v^1 (resp. v^2) the law of $\mathcal{X}(t)$ (resp. $\mathcal{Y}(t)$), we remember that (2.13) (which is assumed for both solutions) and Lemma 3.4 imply that for $i = 1, 2$

$$\forall p \in [1, +\infty), \;\; v_t^i \in L^{p/(p-1)}([0,T], L^p(\mathbb{R}^2)). \tag{7.20}$$

Using (7.20) with $p = 3/2$, (7.19) and the estimate (7.4) with $p = 3$,

$$\begin{aligned}
\int_0^t \mathbb{E}\left[M\nabla_x \bar{K}_s(\mathcal{X}(s))\right]\, ds &= \int_0^t \int_{\mathbb{R}^2} M\nabla_x \bar{K}_s(x) v_s^1(x) dx \\
&\leq \int_0^t \|M\nabla_x \bar{K}_s\|_{L^3} \|v_s^1\|_{L^{3/2}}\, ds \\
&\leq C\int_0^t \|\nabla_x \bar{K}_s\|_{L^3} \|v_s^1\|_{L^{3/2}}\, ds \\
&\leq C\|\nabla_x \bar{K}\|_{L^{3/2}([0,t],L^3(\mathbb{R}^2))} \|v^1\|_{L^3([0,t],L^{3/2}(\mathbb{R}^2))} < \infty.
\end{aligned}$$

Handling the same computation for $\mathcal{Y}$, we get that

$$\mathbb{E}\left[\log(\delta + \sup_{0\leq s\leq t} |\mathcal{X}(s) - \mathcal{Y}(s)|)\right] \leq \log\delta + C_t,$$

where the constant C_t is independent of δ. From that and the fact that $u \mapsto \log u$ is increasing, setting $\mathcal{Z}_t := \sup_{0\leq s\leq t} |\mathcal{X}(s) - \mathcal{Y}(s)|$, we can estimate for any $\varepsilon > 0$

$$\begin{aligned}
\mathbb{P}(\mathcal{Z}_t > \varepsilon)\, \log(1 + \varepsilon\,\delta^{-1}) + \log\delta &= \mathbb{P}(\mathcal{Z}_t \leq \varepsilon)\, \log\delta + \mathbb{P}(\mathcal{Z}_t > \varepsilon)\, \log(\delta + \varepsilon) \\
&= \mathbb{E}\Big(\mathbf{1}_{\{\mathcal{Z}_t \leq \varepsilon\}}\, \log\delta + \mathbf{1}_{\{\mathcal{Z}_t > \varepsilon\}}\, \log(\delta + \varepsilon)\Big) \\
&\leq \mathbb{E}\Big(\log(\delta + \mathcal{Z}_t)\Big) \\
&\leq \log\delta + C_t.
\end{aligned}$$

We have proved

$$\mathbb{P}(\sup_{0\leq s\leq t} |\mathcal{X}(s) - \mathcal{Y}(s)| \geq \varepsilon) \leq \frac{C_t}{\log(1 + \varepsilon\delta^{-1})}.$$

Letting $\delta \to 0$, we obtain $\mathbb{P}(\sup_{0\leq s\leq t} |\mathcal{X}(s) - \mathcal{Y}(s)| \geq \varepsilon) = 0$. Pathwise uniqueness is proved.

It remains to prove (2.14). We denote by $g_t \in \mathbf{P}(\mathbb{R}\times\mathbb{R}^2)$ the law of $(\mathcal{M}, \mathcal{X}(t))$ and by $w_t \in \mathbf{M}(\mathbb{R}^2)$ the associated vorticity, see (2.12). Since $(\mathcal{M}, (\mathcal{X}(t))_{t\geq 0})$ has been obtained by passing to the limit in the particle system (1.1), we deduce from Theorem 2.12, Theorem 4.1 and Lemma 5.1 that

$$\sup_{[0,T]} \tilde{H}(g_t) < \infty, \qquad \sup_{t\in[0,T]} \tilde{M}_k(g_t) < \infty, \qquad \int_0^T \tilde{I}(g_s)\, ds < \infty. \tag{7.21}$$

We call $r_0 \in \mathbf{P}(\mathbb{R})$ the law of $\mathcal{M}$ and for $m \in \mathbb{R}$, we denote by f_t^m the law of $\mathcal{X}(t)$ knowing that $\mathcal{M} = m$. We then have $g_t(dm, dx) = r_0(dm) f_t^m(dx)$ for all $t \ge 0$. Thanks to Itô calculus, $(f_t^m)_{t\ge 0}$ clearly belongs to $C([0,T]; \mathbf{P}(\mathbb{R}^2))$ (because $t \mapsto \mathcal{X}(t)$ is a.s. continuous) and is a weak solution, for $m \in \mathbb{R}$ fixed, to

$$\partial_t f^m = \nu \Delta_x f^m + \bar{K} \cdot \nabla_x f^m \tag{7.22}$$

where $\bar{K}_t = K \star w_t$. Using the definitions of $\tilde{H}, \tilde{M}_k, \tilde{I}$, we deduce from (7.21) that for r_0-almost every $m \in \mathbb{R}$, for all $t \ge 0$,

$$H(f_t^m) < \infty, \quad M_k(f_t^m) < \infty, \quad \int_0^t I(f_s^m)\, ds < \infty. \tag{7.23}$$

The Fisher information bound in (7.23) implies, by Lemma 3.2, that

$$\nabla_x f^m \in L^{2q/(3q-2)}(0, T, L^q(\mathbb{R}^2)) \quad \forall\, q \in [1,2), \quad \forall\, T > 0.$$

Then we use the same arguments as in the proof of (2.8) in Theorem 2.5 (which was entirely based on such an estimate plus an estimate saying that $(f_t^m)_{t\ge0}$ belongs to $C([0,T]; \mathbf{P}(\mathbb{R}^2))$): for any $t > 0$, any $\beta \in C^1(\mathbb{R}) \cap W^{2,\infty}_{loc}(\mathbb{R})$ such that β'' is piecewise continuous and vanishes outside of a compact set and any $\chi \in C_c^2(\mathbb{R}^2)$,

$$\begin{aligned} &\int_{\mathbb{R}^2} \beta(f_t^m) \chi\, dx + \nu \int_0^t \int_{\mathbb{R}^2} \beta''(f_s^m)\, |\nabla f_s^m|^2 \chi\, dx ds \\ = \;& \int_{\mathbb{R}^2} \beta(f_0^m) \chi\, dx + \int_0^t \int_{\mathbb{R}^2} \beta(f_s^m) [\nu \Delta \chi - \bar{K}_s . \nabla \chi] dx ds. \end{aligned}$$

Assume now additionally that $\beta'' \ge 0$ and that $\beta(0) = 0$. Considering an increasing sequence of uniformly bounded non-negative functions $\chi_k \in C_c^2(\mathbb{R}^2)$ so that $\chi_k(x) = 1$ for $|x| \le k$, it is not hard to deduce that (use the monotonous convergence theorem for the second term, the dominated convergence theorem and that $|\beta(f_t^m)| + |\beta(f_0^m)| \le C(f_t^m + f_0^m) \in L^1(\mathbb{R}^2)$ for the first and third terms and finally, for the last term, the dominated convergence theorem and the fact that $|\beta(f^m)|(1 + |\bar{K}|) \in L^1([0,T] \times \mathbb{R}^2)$ because $\bar{K} \in L^6(0,T; L^3(\mathbb{R}^2)) \subset L^{3/2}(0,T; L^3(\mathbb{R}^2))$ by (7.3) with $p = 6/5$ and because $f^m \in L^3(0,T; L^{3/2}(\mathbb{R}^2))$ thanks to the Fisher information estimate in (7.23) and Lemma 3.4 with $p = 3/2$),

$$\int_{\mathbb{R}^2} \beta(f_t^m)\, dx + \nu \int_0^t \int_{\mathbb{R}^2} \beta''(f_s^m)\, |\nabla f_s^m|^2 dx ds = \int_{\mathbb{R}^2} \beta(f_0^m)\, dx.$$

We apply this with $\beta_p : \mathbb{R}_+ \to \mathbb{R}$ defined by $\beta_p''(s) := (1/s)\, \mathbf{1}_{\{s \in [1/p, p]\}}$, $\beta_p(0) = \beta_p(1) = 0$ and let $p \to \infty$. The second term tends to $\nu \int_0^t I(f_s^m) ds$ as $p \to \infty$ by monotonous convergence. The first and third terms tend to $H(f_t^m)$ and $H(f_0^m)$ by monotonous convergence, because $0 \le -\beta_p(s) \mathbb{1}_{s \in [0,1]}$ increases to $-s \log s \mathbb{1}_{s\in[0,1]}$ while $0 \le \beta_p(s) \mathbb{1}_{s \in [1,\infty)}$ increases to $s \log s \mathbb{1}_{s \in [1,\infty)}$. In fact, it can be checked that $\beta_p(s) = s \ln s + (1-s)/p$ if $s \in [1/p, p]$. We finally get

$$H(f_t^m) + \nu \int_0^t I(f_s^m)\, ds = H(f_0^m).$$

Integrating this equality against $r_0(dm)$ leads us to (2.14). $\square$

8. Conclusion

It only remains to put together all the intermediate results.

Proof of Theorem 2.12. Let us consider, for each $N \geq 2$, a family $(\mathcal{M}_i^N, \mathcal{X}_i^N(0))_{i=1,\dots,N}$ of $\mathbb{R}\times\mathbb{R}^2$-valued random variables. Assume that (2.15) holds true for some g_0. For each $N \geq 2$, consider the unique solution $(\mathcal{X}_i^N(t))_{i=1,\dots,N,t\geq 0}$ to (1.1) and define $\mathcal{Q}^N := \frac{1}{N}\sum_{i=1}^N \delta_{(\mathcal{M}_i^N,(\mathcal{X}_i^N(t))_{t\geq 0})}$. As shown in Lemma 5.2, the family $\{\mathscr{L}(\mathcal{Q}^N), N\geq 2\}$ is tight in $\mathbf{P}(\mathbf{P}(\mathbb{R}\times C([0,\infty),\mathbb{R}^2)))$. Proposition 6.1 shows that any (random) limit point $\mathcal{Q}$ of this sequence belongs a.s. to $\mathcal{S}$, the set of all probability measures $g \in \mathbf{P}(\mathbb{R}\times C([0,\infty),\mathbb{R}^2))$ such that g is the law of $(\mathcal{M},(\mathcal{X}(t))_{t\geq 0})$ with $(\mathcal{X}(t))_{t\geq 0}$ solution to the nonlinear SDE (1.6) satisfying that, denoting by $g_t \in \mathbf{P}(\mathbb{R}\times\mathbb{R}^2)$ the law of $(\mathcal{M},\mathcal{X}(t))$, (2.13) holds true. But Theorem 2.9 implies that $\mathcal{S}$ is reduced to one point $\mathcal{S}=\{g\}$. All this implies that $\mathcal{Q}^N$ tends in law to g as $N\to\infty$: the sequence $(\mathcal{M}_i^N,(\mathcal{X}_i^N(t))_{t\geq 0})$ is $(\mathcal{M},(\mathcal{X}(t))_{t\geq 0})$ -chaotic.

The last point follows thanks to the fact that all the circulations are bounded by A: we know that $\mathcal{Q}^N$ goes in probability to g, in $\mathbf{P}(\mathbb{R}\times C([0,\infty),\mathbb{R}^2))$. We also know that $\mathcal{W}^N = \Phi(\mathcal{Q}^N)$ and $w=\Phi(g)$, where $\Phi : \mathbf{P}(\mathbb{R}\times C([0,\infty),\mathbb{R}^2)) \mapsto C([0,\infty),\mathbf{M}(\mathbb{R}^2))$ is defined by $(\phi(q))_t(B) = \int_{\mathbb{R}\times C([0,\infty),\mathbb{R}^2)} m \mathbb{1}_{\{\gamma(t)\in B\}} q(dm,d\gamma)$ for all $B\in\mathcal{B}(\mathbb{R}^2)$. A slightly tedious but straightforward study shows that this map is continuous on the subset of all $q\in\mathbf{P}(\mathbb{R}\times C([0,\infty),\mathbb{R}^2))$ such that supp $q\subset[-A,A]\times C([0,\infty),\mathbb{R}^2)$. The conclusion follows, since both $\mathcal{Q}^N$ and g a.s. belong to this subset by (2.15). □

Finally, we give the proof of Theorem 2.13 on entropy chaos and strong convergence by adapting a trick introduced in [38] for the Boltzmann equation.

Proof of Theorem 2.13. Recall that G_t^N stands for the law of $(\mathcal{M}_i^N,\mathcal{X}_i^N(t))_{i=1,\dots,N}$, that g_t is the law of $(\mathcal{M},\mathcal{X}(t))$, that we assume (2.15) and additionally that $\lim_N \tilde H(G_0^N) = \tilde H(g_0)$.

Point (i). It readily follows from Theorem 2.12 that for each $t\geq 0$, G_t^N is g_t-chaotic (in the sense of Kac) so that in particular, $(\mathcal{M}_1^N,\mathcal{X}_1^N(t))$ goes in law to g_t. It remains to prove that $\lim_N \tilde H(G_t^N) = \tilde H(g_t)$. We first recall that from (5.7) and the remark at the beginning of the proof of Proposition 5.1

$$\forall\, t\geq 0, \qquad \tilde H(G_t^N) + \nu\int_0^t \tilde I(G_s^N)\,ds \leq \tilde H(G_0^N),$$

whence

$$\limsup_N \{\tilde H(G_t^N) + \nu\int_0^t \tilde I(G_s^N)\,ds\} \leq \limsup_N \tilde H(G_0^N) = \tilde H(g_0).$$

On the other hand, applying Theorem 4.1 (see Step 3 of the proof of Proposition 6.1 for similar considerations), we get

$$\liminf_N \tilde H(G_t^N) \geq \tilde H(g_t), \quad \liminf_N \int_0^t \tilde I(G_s^N)\,ds \geq \int_0^t \tilde I(g_s)\,ds.$$

Using that $\tilde H(g_t) + \nu\int_0^t \tilde I(g_s)\,ds = \tilde H(g_0)$ by (2.14), we easily conclude that for all $t\geq 0$,

$$\lim_N \tilde H(G_t^N) = \tilde H(g_t), \quad \lim_N \int_0^t \tilde I(G_s^N)\,ds = \int_0^t \tilde I(g_s)\,ds$$

as desired.

Point (ii). Denote by r_0 the law of $\mathcal{M}$, recall that r_0 is supported in $\mathcal{A}=[-A,A]$ and that the $\mathcal{M}_i^N$ are i.i.d. and r_0-distributed. For $j=1,\dots,N$, we denote by G_{tj}^N the j-th marginal of G_t^N

(that is, the law of $(\mathcal{M}_i^N, \mathcal{X}_i^N(t))_{i=1,\dots,j}$), and by $F_{tj}^{N,M}$ the law of $(\mathcal{X}_i^N(t))_{i=1,\dots,j}$ knowing that $(\mathcal{M}_i^N)_{i=1,\dots,j} = M$ for any given $M \in \mathcal{A}^j$. Then we have the disintegration formula $G_{tj}^N(dM, dX) = r_0^{\otimes j}(dM) F_{tj}^{N,M}(dX)$. We also disintegrate $g_t(dm, dx) = r_0(dm) f_t^m(dx)$.

Using first Corollary 4.3 (since $G_{tj}^N \to g_t^{\otimes j}$ weakly as $N \to \infty$ because G_t^N is g_t-chaotic and since $\sup_N \tilde{M}_k(G_t^N) < \infty$) and then that $\lim_N \tilde{H}(G_t^N) = \tilde{H}(g_t)$ by Step 1, we have, for any $j \geq 1$,

$$\tilde{H}(g_t^{\otimes j}) \leq \liminf_N \tilde{H}(G_t^N) \limsup_N \tilde{H}(G_t^N) = \tilde{H}(g_t) = \tilde{H}(g_t^{\otimes j}),$$

so that, for any $j \geq 1$, $\tilde{H}(G_{tj}^N) \to \tilde{H}(g_t^{\otimes j})$.

Introducing artificially

$$Q_{tj}^N(dM, dX) = r_0^{\otimes j}(dM) \left(\frac{1}{2} F_{tj}^{N,M}(dX) + \frac{1}{2} \prod_{i=1}^j f_t^{m_i}(dx_i) \right) = \frac{1}{2} G_{tj}^N(dM, dX) + \frac{1}{2} g_t^{\otimes j}(dM, dX)$$

(here we use the notation $X = (x_1, \dots, x_j)$ and $M = (m_1, \dots, m_j)$), it obviously holds that Q_{tj}^N goes weakly to $g_t^{\otimes j}$ so that by lower semi-continuity, $\liminf_N \tilde{H}(Q_{tj}^N) \geq \tilde{H}(g_t^{\otimes j})$. We deduce that $\limsup_N \left[\frac{1}{2} \tilde{H}(G_{tj}^N) + \frac{1}{2} \tilde{H}(g_t^{\otimes j}) - \tilde{H}(Q_{tj}^N) \right] \leq 0$, whence, by convexity of $\tilde{H}$,

$$\limsup_N \left[\frac{1}{2} \tilde{H}(G_{tj}^N) + \frac{1}{2} \tilde{H}(g_t^{\otimes j}) - \tilde{H}(Q_{tj}^N) \right] = 0.$$

Using the disintegration formulae and the definition of $\tilde{H}$, this rewrites

$$\lim_N \int_{\mathbb{R}^j} r_0^{\otimes j}(dM) \left\{ \frac{1}{2} H\big(F_{tj}^{N,M}\big) + \frac{1}{2} H\Big(\prod_{i=1}^j f_t^{m_i}\Big) - H\Big(\frac{1}{2} F_{tj}^{N,M} + \frac{1}{2} \prod_{i=1}^N f_t^{m_i}\Big) \right\} = 0.$$

By the strict convexity of $s \mapsto s \log s$, this classically implies, see for instance [8] (all this can be rewritten as the integral against $r_0^{\otimes j}(dM) dX$ of a non-negative function), that from any (not relabelled) subsequence we can extract a (not relabelled) such that

$$F_{tj}^{N,M}(X) \to \prod_{i=1}^j f_t^{m_i}(x_i) \quad \text{for} \quad r_0^{\otimes j}\text{-a.e.}\, M \in \mathbb{R}^j, \ \text{Lebesgue-a.e.}\, X \in (\mathbb{R}^2)^j. \tag{8.1}$$

On the other hand, the estimate established in Proposition 5.1 together with Lemma 3.1 and Corollary 4.3 imply that $C_k + \tilde{M}_k(G_{tj}^N) + \tilde{H}(G_{tj}^N) \leq 2\,(C_k + \tilde{M}_k(G_t^N) + \tilde{H}(G_t^N)) \leq C$, which rewrites

$$\forall\, N \geq 1 \qquad \int_{(\mathcal{A} \times \mathbb{R}^2)^j} (\langle X \rangle^k + \log F_{tj}^{N,M}(X))\, F_{tj}^{N,M}(X)\, r_0^{\otimes j}(dM) dX \leq C.$$

The Dunford-Pettis theorem thus implies that

$$F_{tj}^{N,M}(X) \quad \text{is weakly compact in} \quad L^1((\mathcal{A} \times \mathbb{R}^2)^j; r_0^{\otimes j}(dM) dX). \tag{8.2}$$

It is then a well-known application of the Egorov theorem that (8.1) and (8.2) imply that

$$F_{tj}^{N,M}(X) \to \prod_{i=1}^j f_t^{m_i}(x_i) \quad \text{strongly in} \quad L^1((\mathcal{A} \times \mathbb{R}^2)^j; r_0^{\otimes j}(dM) dX).$$

We immediately deduce that $w_{tj}^N(X) = \int_{R^j} m_1 \dots m_j F_{tj}^{N,M}(X) r_0^{\otimes j}(dM)$ goes strongly in $L^1((\mathbb{R}^2)^j)$ to $w_t^{\otimes j}(X) = \int_{R^j} m_1 \dots m_j (\prod_{i=1}^j f_t^{m_i}(x_i)) r_0^{\otimes j}(dM)$, since $\mathcal{A}$ is compact. $\square$

References

[1] Aalto, D. and Kinnunen, J. Maximal functions in Sobolev spaces. *Sobolev spaces in mathematics. I,* 25–67, Int. Math. Ser. (N. Y.), 8, Springer, New York, 2009.

[2] Benachour, S., Roynette, B. and Vallois, P. Branching process associated with 2d-Navier Stokes equation. *Rev. Mat. Iberoamericana 17*, 2 (2001), 331–373.

[3] Ben-Artzi, M. Global solutions of two-dimensional Navier-Stokes and Euler equations. *Arch. Rational Mech. Anal. 128*, 4 (1994), 329–358.

[4] Bolley, F., Gentil, I. and Guillin, A. Uniform convergence to equilibrium for granular media. To appear in *Arch. Rational Mech. Anal.*

[5] Bouchet,F and Sommeria, J. Statistical mechanics for geophysical flows. From Jovian atmosphere modelization to the parameterization of small scale turbulence. World Scientific, Hackensack, 2012.

[6] Brezis, H. *Analyse fonctionnelle*. Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[7] Brezis, H. Remarks on the preceding paper by M. Ben-Artzi: "Global solutions of two-dimensional Navier-Stokes and Euler equations" [Arch. Rational Mech. Anal. **128** (1994), no. 4, 329–358]. *Arch. Rational Mech. Anal. 128*, 4 (1994), 359–360.

[8] Brezis, H. Convergence in $\mathcal{D}'$ and in L^1 under strict convexity. In *Boundary value problems for partial differential equations and applications*, vol. 29 of *RMA Res. Notes Appl. Math.* Masson, Paris, 1993, 43–52.

[9] Caglioti, E., Lions, P.-L., Marchioro, C. and Pulvirenti, M. A special class of stationary flows for two - dimensional Euler equation: a statistical mechanics description. *Comm. Math. Phys. 143*, (1992), 501–525.

[10] Carlen, E. A. Superadditivity of Fisher's information and logarithmic Sobolev inequalities. *J. Funct. Anal. 101*, 1 (1991), 194–211.

[11] Carlen, E., Carvalho, M., Le Roux, J., Loss, M., and Villani, C. Entropy and Chaos in the Kac's model. *Kinet. Relat. Models 3*, 1 (2010), 85–122.

[12] Cépa, E. and Lépingle, D. Diffusing particles with electrostatic repulsion. *Probab. Theory Related Fields 107*, 4 (1997), 429–449.

[13] Crippa, G., and De Lellis, C. Estimates and regularity results for the DiPerna-Lions flow. *J. Reine Angew. Math. 616* (2008), 15–46.

[14] de Finetti, B. La prévision : ses lois logiques, ses sources subjectives. *Ann. Inst. H. Poincaré 7*, 1 (1937), 1–68.

[15] DiPerna, R. J., and Lions, P.-L. On the Fokker-Planck-Boltzmann equation. *Comm. Math. Phys. 120*, 1 (1988), 1–23.

[16] DiPerna, R. J., and Lions, P.-L. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math. 98*, 3 (1989), 511–547.

[17] Figalli, A. Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients. *J. Funct. Anal. 254*, 1 (2008), 109–153.

[18] Flandoli, F., Gubinelli, M. and Priola, E. Full well-posedness of point vortex dynamics corresponding to stochastic 2D Euler equations. *Stochastic Process. Appl. 121* (2011), 1445–1463.

[19] Fontbona, J. and Martinez, M. Paths clustering and an existence result for stochastic vortex systems. *J. Stat. Phys. 128*, 3 (2007), 699–719.

[20] Gallagher, I., and Gallay, T. Uniqueness for the two-dimensional Navier-Stokes equation with a measure as initial vorticity. *Math. Ann. 332*, 2 (2005), 287–327.

[21] Gallay, Th. Interaction of vortices in weakly viscous planar flows. *Archive Rat. Mech. Anal. 200* (2011), 445–490.

[22] Gallay, Th. and Wayne, C.E. Global stability of vortex solutions of the two-dimensional Navier-Stokes equation. *Commun. Math. Phys. 255,* (2005), 97–129.

[23] Hauray, M. and Jabin, P.-E. Propagation of chaos for particles approximations of Vlasov equations with singular forces. http://hal.archives-ouvertes.fr/hal-00609453.

[24] Hauray, M. and Mischler, S. On Kac's chaos and related problems. http://hal.archives-ouvertes.fr/hal-00682782.

[25] Helmholtz, H., Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen. *J. Reine Angew. Math. 55* (1858), 25–55.

[26] Hewitt, E., and Savage, L. J. Symmetric measures on Cartesian products. *Trans. Amer. Math. Soc. 80* (1955), 470–501.

[27] Kac, M. Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. III* (Berkeley and Los Angeles, 1956), University of California Press, 171–197.
[28] Kirchhoff, G. Vorlesungen Über Math. Phys. *Teuber*, Leipzig (1876).
[29] Le Bris, C., and Lions, P.-L. Existence and uniqueness of solutions to Fokker-Planck type equations with irregular coefficients. *Comm. Partial Differential Equations 33*, 7-9 (2008), 1272–1317.
[30] Lieb, E. and Loss, M. Analysis. *Graduate Studies in Mathematics, 14*, AMS, Providence, 1997.
[31] Malrieu, F. Convergence to equilibrium for granular media equations and their Euler schemes. *Ann. Appl. Probab.13,* 2 (2003), 540–560.
[32] Marchioro, C. On the inviscid limit for a fluid with a concentrated vorticity. *Comm. Math. Phys. 196* (1998), 53–65.
[33] Marchioro, C. and Pulvirenti, M. Hydrodynamics in two dimensions and vortex theory. *Comm. Math. Phys. 84*, 4 (1982), 483–503.
[34] Marchioro, C. and Pulvirenti, M. Mathematical theory of incompressible nonviscous fluids. *Applied Mathematical Sciences, 96*, Springer-Verlag, New York, 1994.
[35] McKean, Jr., H. P. Propagation of chaos for a class of non-linear parabolic equations. In *Stochastic Differential Equations (Lecture Series in Differential Equations, Session 7, Catholic Univ., 1967).* Air Force Office Sci. Res., Arlington, Va., 1967, 41–57.
[36] Méléard, S. Asymptotic behaviour of some interacting particle systems; McKean-Vlasov and Boltzmann models. In *Probabilistic models for nonlinear partial differential equations (Montecatini Terme, 1995)*, vol. 1627 of *Lecture Notes in Math.* Springer, Berlin, 1996, 42–95.
[37] Méléard, S. Monte-Carlo approximations for 2d Navier-Stokes equations with measure initial data. *Probab. Theory Related Fields* 121 (2001), 367–388.
[38] Mischler, S. and Mouhot, C. Kac's Program in Kinetic Theory. To appear in *Inventiones Mathematicae*
[39] Mischler, S., Mouhot, C. and Wennberg, B. A new approach to quantitative chaos propagation for drift, diffusion and jump processes. arXiv:1101.4727
[40] Onsager, L. Statistical hydrodynamics. *Il Nuovo Cimento 6* (1949), 279–287.
[41] Osada, H. Diffusion processes with generators of generalized divergence form. *J. Math. Kyoto Univ. 27*, 4 (1987), 597–619.
[42] Osada, H. A stochastic differential equation arising from the vortex problem. *Proc. Japan Acad. Ser. A Math. Sci. 61*, 10 (1986), 333–336.
[43] Osada, H. Propagation of chaos for the two-dimensional Navier-Stokes equation. *Proc. Japan Acad. Ser. A Math. Sci. 62*, 1 (1986), 8–11.
[44] Osada, H. Propagation of chaos for the two-dimensional Navier-Stokes equation. *Probabilistic methods in mathematical physics* (Katata/Kyoto, 1985), 303–334, Academic Press, Boston, MA, 1987.
[45] Osada, H. Limit points of empirical distributions of vortices with small viscosity. In *Hydrodynamic behavior and interacting particle systems (Minneapolis, Minn., 1986)*, vol. 9 of *IMA Vol. Math. Appl.* Springer, New York, 1987, 117–126.
[46] Robinson, D. W., and Ruelle, D. Mean entropy of states in classical statistical mechanics. *Comm. Math. Phys. 5* (1967), 288–300.
[47] Schochet, S. The point vortex method for periodic weak solutions of the 2-D Euler equations. *Comm. Pure Appl. Math. 49* (1996) 911–965.
[48] Stein, E. M. *Harmonic Analysis.* Princeton Mathematical Series, No. 43. Princeton University Press, Princeton, N.J., 1993.
[49] Sznitman, A.-S. Équations de type de Boltzmann, spatialement homogènes. *Z. Wahrsch. Verw. Gebiete 66*, 4 (1984), 559–592.
[50] Sznitman, A.-S. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, vol. 1464 of *Lecture Notes in Math.* Springer, Berlin, 1991, 165–251.
[51] Takanobu, S. On the existence and uniqueness of SDE describing an n-particle system interacting via a singular potential. *Proc. Japan Acad. Ser. A Math. Sci. 6* (1985) 287–290.
[52] Yamada, T., and Watanabe, S. On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ. 11* (1971), 155–167.

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