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Instrumental variable methods for identifying partial differential equation models of distributed parameter systems

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Abstract: This paper presents instrumental variable methods for identifying partial differential equation models of distributed parameter systems in presence of output measurement noise. Two instrumental variable-based techniques are proposed to handle this continuous-time model identification problem: a basic one using input-only instruments and a more sophisticated refined instrumental variable method. Numerical examples are presented to illustrate and compare the performances of the proposed approaches.

1. INTRODUCTION

A large variety of natural, industrial, and environmental systems involve phenomena that are continuous functions not only of time, but also of other independent variables, such as space coordinates. Typical examples are transportation phenomena of mass or energy, such as heat transmission and/or exchange, humidity diffusion or concentration distributions. These systems are intrinsically distributed parameter systems (DPS) whose description usually requires the introduction of partial differential equations (PDE). When performing simulations based on a PDE model, it is assumed that all involved parameters are known. However, beforehand these parameters have to be determined. Since they are frequently hardly (or even not at all) a priori known, they have to be fitted from measurements. This amounts to the problem of PDE parameter estimation. There are different ways to identify distributed parameter systems. Three main families of methods can be distinguished (Kubrusly [1977]):

- methods where optimization techniques are used to directly estimate the PDE model parameters;
- methods which aim at approximating the DPS (described by a PDE) to a continuous-time (CT) or discrete-time (DT) lumped parameter system (LPS) (described by ordinary differential or difference equation);
- methods which reduce the partial differential equation to an algebraic equation.

Many methods of the three families have been developed. In Liu [2005] an approach is proposed to reduce the PDE models described by two or more independent variables to ordinary differential equations (ODE) models described by only one independent variable. PDE model identification based on continuous space-time separation of the signals is also proposed in Li and Qi [2010]. The input/output signals are in this case expressed in a basis of polynomial functions such as wavelet (Sadabadi et al. [2008a]), Walsh series (Tzafestas [1978]) or Fourier series (Sadabadi et al. [2008b]). It is also possible to pass through a filtering method involving integrals (Sagara et al. [2000]). The problem to estimate the partial derivatives is then got around, the derivatives are no longer approached but integrals are obtained by approximation. In Guo and Billings [2006], an approach which combines implicit Adams integration and an orthogonal least squares is developed. In Ali et al. [2011] an identification technique is introduced for discretized PDE models. After discretization of the PDE by finite differences, the parameters are estimated by using techniques developed for discrete-time lumped model identification.

The last decade has witnessed a resurgence of interest devoted to identifying parameters in CT models from DT data. Without relying any longer on analogue computers, the present techniques exploit the power of the digital tools. The developed methods have proven successful in many practical applications and are also available in user-friendly toolboxes for Matlab (see e.g. H. Garnier and L. Wang (Eds.) [2008]). Amongst the available identification approaches for CT input-output models, the interest for instrumental variable (IV) methods has been growing in the last years (Söderström and Stoica [1983], Young [2011], Young et al. [2008], L. Wang and H. Garnier (Eds.) [2012]). The main reason of this increasing interest is that IV methods offer similar performance as extended least squares (LS) methods or other prediction-error-minimization (PEM) methods and provide consistent results even for an imperfect noise structure which is the case in most practical applications. Amongst the available IV estimators, the Simplified Refined Instrumental Variable for Continuous-time systems, denoted by SRIVC from hereon (Young and Jakeman [1980]), is, in the authors’ experience, the most efficient. It is known to yield asymptotically efficient parameter estimates of CT ordinary differential equation in the presence of white output measurement noise.
The main objective in this paper is to develop a SRIVC version for identifying CT partial differential equation from sampled data.

The paper is organized as follows. Section 2 states the identification problem. In Section 3, two methods based on the simple least squares and a basic instrumental variable estimators are recalled. Both are associated to the state-variable filtering method to handle the partial derivative measurement problem. Section 4 presents the main contribution of the paper, that is the extension of the SRIVC method to identify CT PDE models directly from the sampled data. Section 5 provides numerical simulations in order to illustrate and compare the effectiveness of the proposed methods.

2. PROBLEM FORMULATION

Consider a two-dimensional linear continuous-time system defined by partial differential equations in the single-input, single-output situation as follows:

\[
\sum_{i_x=0}^{n_x} \sum_{i_y=0}^{n_y} a_{i_x,i_y} p_x^{i_x} p_y^{i_y} \dot{y}(x,t) = \sum_{i_x=0}^{m_x} b_{i_x} p_x^{i_x} u(x,t)
\]

(1)

where \( u(x,t) \) denotes the input variable observed at space point \( x \) and at time \( t \) and \( \dot{y}(x,t) \) the noise-free output. \((n_x,n_y)\) are order of the system (assumed to be known), \( n_x \geq m_x, m_t \) and \( n_y \geq m_x, m_t \) and \( a_{i_x,i_y} = 1 \). \( p_x \) and \( p_t \) denote the differential operator, respectively

\[
p_x = \frac{\partial}{\partial x}, \quad p_t = \frac{\partial}{\partial t}
\]

(2)

This PDE is subjected to initial and boundary conditions. Equation (1) can also be written in the following compact transfer function (TF) form,

\[
\dot{y}(x,t) = G(p_x,p_t)u(x,t)
\]

(3)

where

\[
G(p_x,p_t) = \frac{B(p_x,p_t)}{A(p_x,p_t)}
\]

(4)

with

\[
A(p_x,p_t) = \sum_{i_x=0}^{n_x} \sum_{i_y=0}^{n_y} a_{i_x,i_y} p_x^{i_x} p_y^{i_y}
\]

(5)

\[
B(p_x,p_t) = \sum_{i_x=0}^{m_x} b_{i_x} p_x^{i_x} p_y^{i_y}
\]

(6)

Let \( h \) be the spatial sampling distance between two observation points, and \( T_s \) the time sampling period for the system signals. Denote \( L \) as the number of observation points, and \( N \) the number of time instants. The sampled signals will be denoted as \( u(x,t_k) \) and \( y(x,t_k) \) and the output observation equation then takes the form,

\[
\begin{align*}
S & : \left\{ \\
\dot{y}(x,t) &= G(p_x,p_t)u(x,t) \\
y(x,t_k) &= \hat{y}(x,t_k) + e(x,t_k)
\right. 
\end{align*}
\]

(7)

where \( \dot{y}(x,t_k) \) is the sampled value of the unobserved, noise-free output \( \dot{y}(x,t) \) and \( e(x,t_k) \) is a zero-mean, DT

Fig. 1. Data-generating system

white noise source which is assumed to be uncorrelated with the input \( u(x,t_k) \) (Fig. 1).

The model is given as follows

\[
G(p_x,p_t,\theta) = \frac{B(p_x,p_t,\theta)}{A(p_x,p_t,\theta)} = \sum_{i_x=0}^{m_x} \sum_{i_y=0}^{m_y} b_{i_x,i_y} p_x^{i_x} p_y^{i_y}
\]

(8)

where

\[
\theta = [a_{00} \cdots a_{m_x,m_y-1} \cdots a_{m_x,0} b_{00} \cdots b_{m_x,m_y}]^T
\]

(9)

is the parameter vector that characterises the model \( (\theta \in \mathbb{R}^{n_x+m_x+m_y}) \).

Finally, the identification model can be written in the form

\[
\mathcal{M} : \left\{ \begin{align*}
\dot{y}(x,t,\theta) &= G(p_x,p_t,\theta)u(x,t) \\
y(x,t_k,\theta) &= \hat{y}(x,t_k,\theta) + e(x,t_k)
\end{align*} \right. 
\]

(10)

The objective is then to estimate the parameters (9) of the model (10) based on the sampled input and output data \( Z_L^N = \{u(x,t_k),y(x,t_k)\}_{i=1,k=1}^{L,N} \).

3. BASIC LS AND IV-BASED SVF ESTIMATORS

In comparison with discretised PDE model identification, direct identification of PDE model raises several technical issues. Unlike the discretized PDE model, where only sampled input and output appear, the CT PDE contains partial derivative terms that are required, which are usually not available from measurement. This problem is well-known and various methods have been devised to handle the time-derivatives in the case of ordinary differential equation models (Garnier et al. [2003]). One traditional approach that dates from the days of analogue computer is known as the state variable filtering (SVF) method. The main idea is to pre-filter the PDE model so that the pre-filtered partial derivatives can then be exploited for model parameter estimation. Basic LS and instrumental variable (IV) solutions are described in this section.

3.1 The traditional LS-based SVF estimator

Let us first consider the identification problem in the noise-free case. Equation (3) can be written in the form

\[
A(p_x,p_t)\dot{y}(x,t) = B(p_x,p_t)u(x,t)
\]

(11)
Let $F(p_x, p_t)$ be a filter applied to (11),
\[ A(p_x, p_t)F(p_x, p_t)\hat{y}(x, t) = B(p_x, p_t)F(p_x, p_t)u(x, t) \]  
(12)
The minimum-order SVF filter is typically chosen to have the following form
\[ F(p_x, p_t) = \frac{1}{(p_x + \lambda_x)^{n_x}(p_t + \lambda_t)^{n_t}} \]  
(13)
where $\lambda_x$ and $\lambda_t$ are breakpoint frequencies. This filter is $n_x$-order in space and $n_t$-order in time. Let $F_{i_i}^n(p_x, p_t)$ for $i = 0, 1, \cdots, n_x$ and $j = 0, 1, \cdots, n_t$ be a set of filters defined as
\[ F_{i_i}^n(p_x, p_t) = \frac{p_x^{i_i} p_t^n}{(p_x + \lambda_x)^{n_x}(p_t + \lambda_t)^{n_t}} \]  
(14)
when the initial and boundary conditions are both null, by using the filters defined in (14), (12) can be rewritten as
\[ \left\{ \sum_{i_x=0}^{n_x} \sum_{i_t=0}^{n_t} a_{i_x, i_t} F_{i_x}^{i_t}(p_x, p_t) \right\} \hat{y}(x, t) = \left\{ \sum_{i_x=0}^{n_x} \sum_{i_t=0}^{n_t} b_{i_x, i_t} F_{i_x}^{i_t}(p_x, p_t) \right\} u(x, t) \]  
(15)
or
\[ \sum_{i_x=0}^{n_x} \sum_{i_t=0}^{n_t} a_{i_x, i_t} \hat{y}_{i_x}^{i_t}(x, t) = \sum_{i_x=0}^{n_x} \sum_{i_t=0}^{n_t} b_{i_x, i_t} u_{i_x}^{i_t}(x, t) \]  
(16)
where
\[ \hat{y}_{i_x}^{i_t}(x, t) = F_{i_x}^{i_t}(p_x, p_t)\hat{y}(x, t) \]
\[ u_{i_x}^{i_t}(x, t) = F_{i_x}^{i_t}(p_x, p_t)u(x, t) \]  
(17)
At time-instant $t = t_k$ and for a space point $x = x_k$, considering now the situation where there is an additive noise on the output measurement, equation (15) can be rewritten as
\[ y_{n_x}^{n_t}(x_k, t_k) = \varphi^T(x_k, t_k)\theta + \epsilon(x_k, t_k) \]  
(18)
with
\[ \varphi(x_k, t_k) = [-y_{n_x}^{n_t-1}(x_k, t_k) \cdots - y_0^{n_t}(x_k, t_k) - y_{n_x}^{n_t-1}(x_k, t_k) \cdots - y_0^{n_t}(x_k, t_k) u_{n_x}^{n_t}(x_k, t_k) \cdots u_0^{n_t}(x_k, t_k)] \]  
(19)
The LS-based SVF parameter estimates are given by
\[ \hat{\theta}_{LSFV} = \left[ \sum_{i=1}^{L} \sum_{k=1}^{N} \varphi(x_i, t_k)\varphi^T(x_i, t_k) \right]^{-1} \]  
\[ \cdot \left[ \sum_{i=1}^{L} \sum_{k=1}^{N} \varphi(x_i, t_k)y_{n_x}^{n_t}(x_i, t_k) \right] \]  
(20)
provided that the inverse exists.

3.2 Input-only IV-based SVF estimator

Although simple, the LS-based SVF estimates are always biased when the observed data are corrupted by noise. A solution to obtain consistent estimates is to use an instrumental variable method (Söderström and Stoica [1983], Young [2011]). The instrumental variable estimator is a classical variant of least squares method. The principle of this estimator is to introduce a new vector such as its components called instruments or instrumental variables denoted here as $\zeta(x_k, t_k)$ are correlated with the regression variables $\varphi(x_i, t_k)$ but uncorrelated with the noise.
\[ \begin{align*}
E[\zeta(x_k, t_k)\varphi^T(x_i, t_k)] & \neq 0 \\
E[\zeta(x_k, t_k)\epsilon(x_i, t_k)] & = 0
\end{align*} \]  
(21a)
(21b)
where $E(.)$ denotes the mathematical expectation.
The common IV-SVF based parameter estimate is then given by
\[ \hat{\theta}_{IVSVF} = \left[ \sum_{i=1}^{L} \sum_{k=1}^{N} \zeta(x_i, t_k)\varphi^T(x_i, t_k) \right]^{-1} \]  
\[ \cdot \left[ \sum_{i=1}^{L} \sum_{k=1}^{N} \zeta(x_i, t_k)y_{n_x}^{n_t}(x_i, t_k) \right] \]  
(22)
Different choice for the instruments have been developed. A first and simple solution was presented in Sagara et al. [1991]. It involves the use of filtered partial derivatives of the input as instruments. The so-called input-only instrument is defined by
\[ \zeta^T(x_k, t_k) = [-u_{n_x}^{n_t-1}(x_k, t_k) \cdots - u_0^{n_t}(x_k, t_k) - u_{n_x}^{n_t-1}(x_k, t_k) \cdots - u_0^{n_t}(x_k, t_k) u_{n_x}^{n_t}(x_k, t_k) \cdots u_0^{n_t}(x_k, t_k)] \]  
(23)
Since the instrument is built up from filtered input signals assumed to be uncorrelated with the disturbance, the condition (21b) is satisfied. However the choice of the input type might lead to parameter estimates with quite poor accuracy as illustrated in Section 5.

Comments

(1) The SVF is taken for a large observation time $T$ and a space interval $[0, X]$, the terms related to the initial and boundary conditions decay exponentially provided the system is stable and become negligible quite quick. Thus these terms may be neglected after a time $t_0 = k_0 T$, and a space point $x_0 = k_0 h$. The estimation algorithm is then applied over $[x_0, X] \times [t_0, T]$, where $t_0$ and $x_0$ have to be chosen comparable to the settling time of the SVF filters. The number of parameters to be estimated can in this way be reduced substantially and this is surely advantageous with regard to computation efforts and numerical properties.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{estimation_area.png}
\caption{Estimation area}
\end{figure}

(2) An another approach to get rid off the initial and boundary condition problem consists in using a modulating function based method (see e.g. Sadabadi
et al. [2008b]). A clear advantage of the modulating function methods is that because of their properties, the effects of initial and boundary conditions disappear in the estimation model equation.

(3) The user must provide a priori the breakpoint frequencies $\lambda_x$ and $\lambda_t$ of the SVF (13). Intuitively, this can be chosen in order to emphasize the frequency band of interest and generally, they should be chosen equal to, or larger than, the bandwidth of the system to be identified. The a priori choice can affect the variance of the parameter estimates.

(4) A second IV method consists in using an instrumental variable built up from an auxiliary model. The instrument is then given by

$$
\zeta^T(x,t,k) = \left[ -y_{n_1}^m(x,t_k) \cdots -y_{n_1}^m(x,t) 
- y_{n_2}^{m-1}(x,t_k) \cdots - y_{n_2}^m(x,t) 
\right. 
\left. u_{m_1}^{i_1}(x,t_k) \cdots u_{m_0}^{i_0}(x,t_k) \right] 
$$

where $\hat{y}(x,t_k)$ is the output of the auxiliary model obtained from the LS-SVF estimate (20). This method, as the previously presented input-only IV gives unbiased parameter estimates, but not minimum variance.

(5) The digital implementation of the various continuous filtering operations can have an important impact on the quality of the parameter estimates. A possibility consists in discretizing the continuous-time filters by the bilinear method as proposed in Sagara et al. [1991].

4. REFINED IV ESTIMATOR

For ordinary differential equations, one particularly successful IV-based stochastic identification method is the iterative Simplified Refined Instrumental Variable (SRIVC) method for continuous-time model identification. This method of adaptive prefiltering is based on a quasi-optimal statistical solution to the problem when the additive noise is white. SRIVC is a logical extension of the more heuristically defined SVF (see e.g. Garnier and Young [2004], Young [2011] and the references therein). This technique presents also the advantage of not requiring manual specification of prefilter parameters. In the following, the SRIVC method is extended to handle the case of PDE.

Following the usual prediction error minimization (PEM) approach, a suitable error function $\epsilon(x,t)$ is given as

$$
\epsilon(x,t) = y(x,t) - \frac{B(p_x,p_t)}{A(p_x,p_t)} u(x,t). 
$$

Minimization of a least squares criterion function in $\epsilon(x,t)$ measured at the sampling instants provides the basis for the output error estimation methods. However $\epsilon(x,t)$ can also be rewritten as

$$
\epsilon(x,t) = \frac{1}{A(p_x,p_t)} (A(p_x,p_t) y(x,t) - B(p_x,p_t) u(x,t)) \tag{26}
$$

Since the operators commute in this linear case, the filter

$$
F(p_x,p_t) = \frac{1}{A(p_x,p_t)} 
$$

can be taken inside the brackets to yield

$$
\epsilon(x,t) = A(p_x,p_t) F(p_x,p_t) y(x,t) 
- B(p_x,p_t) F(p_x,p_t) u(x,t) \tag{28}
$$

or,

$$
\epsilon(x,t) = \sum_{i=0}^{n_x} \sum_{i=0}^{n_t} a_{i_x,i_t} y_{i_x}^m(x,t) - \sum_{i=0}^{n_x} \sum_{i=0}^{n_t} b_{i_x,i_t} u_{i_t}^m(x,t) \tag{29}
$$

where the set of filters now takes the form

$$
F_{i_x}^{i_t}(p_x,p_t) = \frac{p_x^{i_x} p_t^{i_t}}{A(p_x,p_t)} \tag{30}
$$

The associated estimation model can be written at time-instant $t = t_k$ and at space point $x = x_i$ in the form

$$
y_{i_x}^m(x,t_k) = \varphi^T(x_i,t_k) \theta + \epsilon(x_i,t_k) \tag{31}
$$

where $\varphi^T(x_i,t_k)$ and $\theta$ are defined as in (19) and (9) respectively with $F_{i_x}^{i_t}(p_x,p_t)$ now defined in (30).

The proposed SRIVC algorithm for PDE model identification is summarized below. It involves an iterative procedure in which, at each iteration, the auxiliary model used to generate the instruments, as well as the associated pre-filter, are updated, based on the parameter estimates obtained at the previous iteration.

- **Step 1 Initialisation**
  Generate an initial estimate of the TF model parameter vector $\hat{\theta}^0$ using for instance the LSSVF estimator and use this to define the initial prefilter $F(p_x,p_t,\hat{\theta}^0) = \frac{1}{A(p_x,p_t,\hat{\theta}^0)} \eta = 1$.

- **Step 2**
  Compute an estimate of the noise-free output $\hat{y}(x,t)$ by simulating the auxiliary model

$$
\hat{y}(x,t) = \frac{B(p_x,p_t,\hat{\theta}^{iter-1})}{A(p_x,p_t,\hat{\theta}^{iter-1})} u(x,t) \tag{32}
$$

based on the estimated parameters $\hat{\theta}^{iter-1}$.

- **Step 3**
  Compute

$$
F_{i_x}^{i_t}(p_x,p_t,\hat{\theta}^{iter-1}) = \frac{p_x^{i_x} p_t^{i_t}}{A(p_x,p_t,\hat{\theta}^{iter-1})} \tag{33}
$$

Prefilter the input $u(x_i,t_k)$, the output $y(x_i,t_k)$ and auxiliary model output $\hat{y}(x_i,t_k)$ by the filter $F_{i_x}^{i_t}$.

- **Step 4**
  Form the filtered estimated regressor as

$$
\varphi^T(x_i,t_k) = \left[ -y_{i_x}^m(x_i,t_k) \cdots - y_{i_x}^0(x_i,t_k) 
- y_{i_x}^{m-1}(x_i,t_k) \cdots - y_{i_x}^m(x_i,t_k) 
\right. 
\left. u_{m_1}^{i_1}(x_i,t_k) \cdots u_{m_0}^{i_0}(x_i,t_k) \right] 
$$

with the instrument

$$
\zeta^T(x_i,t_k) = \left[ -y_{n_1}^m(x_i,t_k) \cdots - y_{n_1}^0(x_i,t_k) 
- y_{n_2}^{m-1}(x_i,t_k) \cdots - y_{n_2}^m(x_i,t_k) 
\right. 
\left. u_{m_1}^{i_1}(x_i,t_k) \cdots u_{m_0}^{i_0}(x_i,t_k) \right] 
$$

- **Step 5**
  Compute the IV estimate

$$
\hat{\theta}_{SRIVC}^{iter} = \left[ \sum_{i=1}^{L} \sum_{k=1}^{N} \zeta(x_i,t_k) \varphi^T(x_i,t_k) \right]^{-1} \cdot 
\left[ \sum_{i=1}^{L} \sum_{k=1}^{N} \zeta(x_i,t_k) y_{i_x}^m(x_i,t_k) \right] 
$$

where $\hat{\theta}_{SRIVC}^{iter}$ is the SRIVC estimate at iteration $iter$ based on filtered data. If convergence occurs or the
maximum number of iterations is reached then stop, else set \( \text{iter} = \text{iter} + 1 \) and go to Step 2.

Comments

(1) The SRIVC method is known to be asymptotically unbiased and optimal if the noise on the output is white in the case of ODE. These properties are therefore expected in the proposed version PDE and will be illustrated in the next section.

(2) The digital implementation of the analog filtering operations can be achieved, as for the SVF-based estimates, by the bilinear method or by using traditional PDE discretization techniques (Claes [1987]). Filter stability is then provided while the PDE discretization scheme is stable.

5. NUMERICAL EXAMPLES

In this section, several simulated examples are considered to illustrate and compare the performances of the proposed approaches. In these examples, the unknown parameters are estimated by three methods, i.e. the least squares based SVF (denoted as LSSVF), the input-only IV-based SVF (denoted as IVSVF) and the Refined IV (denoted as SRIVC). The breakpoint frequencies are taken as \( \lambda_s = 10 \) and \( \lambda_t = 10 \) (see Sagara et al. [1991]).

5.1 Example 1

This example comes from Sagara et al. [1991]. It is used here to evaluate the effectiveness of the input-only IVSVF method. The same simulation conditions are used.

\[
\begin{align*}
\frac{\partial^2 y(x,t)}{\partial x^2} - 2 \frac{\partial^2 y(x,t)}{\partial x \partial t} &= u(x,t) \\
0 \leq t \leq 10 & \quad 0 \leq x \leq 2 \\
y(x,t) &= \hat{y}(x,t_k) + e(x,t_k)
\end{align*}
\]

The sampling intervals are taken to be \( h = 0.25, \quad T_s = 0.01 \) (39)

The input and the noise-free output are

\[
\begin{align*}
u(x,t) &= -4(x^2 + 4)\cos(t) \\
y(x,t) &= 4x^2\cos(t)
\end{align*}
\]

The input (Fig. 3) and the output are sampled at nine space points and 1000 times instants. The measured output (Fig. 4) is corrupted by a white noise with zero-mean and space-dependent variance \( \sigma_x^2 \) such that (Sagara et al. [1991])

\[
\sigma_x = 0.1x^2
\]

Monte Carlo (MC) simulation results of 100 runs are presented in Table 1 where the mean and standard deviation of the estimated LSSVF and IVSVF parameters are displayed. It can be seen that the LSSVF estimates are, as expected, biased while the IVSVF method delivers unbiased estimates of the model parameters with reasonable standard deviation in this situation where the input is well-correlated with the output. With the simulation conditions chosen, the input is well correlated with the system output which gives unbiased estimates for the IVSVF. However it must be noted that the choice of the breakpoint frequencies can have a considerable effect on the quality of the parameter estimates. Furthermore as illustrated in the following example, the results deteriorate when the input is less correlated with the output.

![Fig. 3. Input (in Example 1)](image1)

![Fig. 4. Noisy output (in Example 1)](image2)

<table>
<thead>
<tr>
<th>Method</th>
<th>True value</th>
<th>( \hat{a}_{21} )</th>
<th>( \hat{a}_{20} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>LSSVF</td>
<td>mean</td>
<td>0.0509</td>
<td>-2.1475</td>
</tr>
<tr>
<td></td>
<td>std</td>
<td>0.0166</td>
<td>0.0085</td>
</tr>
<tr>
<td>IVSVF</td>
<td>mean</td>
<td>1.0535</td>
<td>-1.9964</td>
</tr>
<tr>
<td></td>
<td>std</td>
<td>0.3579</td>
<td>0.0331</td>
</tr>
</tbody>
</table>

Let us now consider the following PDE model

\[
\begin{align*}
1 \frac{\partial^2 y(x,t)}{\partial x^2} - 0.5 \frac{\partial^2 y(x,t)}{\partial x \partial t} &= u(x,t) \\
0 \leq t \leq 10 & \quad 0 \leq x \leq 1 \\
y(x,t_k) &= \hat{y}(x,t_k) + e(x,t_k)
\end{align*}
\]

The input and the noise-free output are

\[
\begin{align*}
u(x,t) &= 6x(1-x)t - t^3 \\
y(x,t) &= x(1-x)t^3
\end{align*}
\]

In this example, \( T_s = 0.01s \) and \( h = 0.01 \). The noise is taken as a 2-D zero-mean normally distributed white noise. The results are shown in Table 2 and were obtained from a MC simulation of 200 runs. In this situation, the input is not well-correlated with the output. The LSSVF estimates are again biased since the output is noisy. Moreover, the IVSVF method leads to poor results in this case. This can be explained from the conditioning of the matrix to be inverted (see eq 21a). Indeed with the chosen input, the condition number of the matrix to be inverted is much poorer (1.7 \( \times \) 10\(^7\)) than in example 1 where the condition number was 152. The matrix to be inverted becomes then ill-conditioned which leads to erroneous parameter estimates. Unlike the IVSVF method, the proposed SRIVC method does not suffer from this drawback and gives unbiased parameter estimates.
Table 2. MC simulation results (example 2)

<table>
<thead>
<tr>
<th>Method</th>
<th>True value</th>
<th>$a_{02}$</th>
<th>$a_{20}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LSSVF</td>
<td>mean</td>
<td>0.5824</td>
<td>-1.1202</td>
</tr>
<tr>
<td></td>
<td>std</td>
<td>0.1648</td>
<td>0.2449</td>
</tr>
<tr>
<td>IVSVF</td>
<td>mean</td>
<td>0.6467</td>
<td>-1.1216</td>
</tr>
<tr>
<td></td>
<td>std</td>
<td>-3.6778</td>
<td>6.3423</td>
</tr>
<tr>
<td>SRIVC</td>
<td>mean</td>
<td>1.0418</td>
<td>-0.4983</td>
</tr>
<tr>
<td></td>
<td>std</td>
<td>0.2698</td>
<td>0.2284</td>
</tr>
</tbody>
</table>

5.3 Example 3

To study further the effects of the input type on the performance of the 3 estimators, another system is also considered.

\[
\begin{aligned}
1 \frac{\partial^2 \hat{y}(x,t)}{\partial t^2} - 2 \frac{\partial^2 \hat{y}(x,t)}{\partial x^2} = u(x,t) \\
0 \leq t \leq 100 & \quad 0 \leq x \leq 1 \\
y(x_0,t_k) = y(x_k,t_k) + e(x_k,t_k)
\end{aligned}
\]  

(43)

The input and the noise are taken as 2-D zero-mean normally distributed white noise and the output is obtained here from the discretization of the partial differential equation by a finite difference method. The data set used for identification involves $100 \times 10000$ samples for $h = 0.01$ and $T_s = 0.01$. Table 3 shows the results of a MC simulation of 200 runs. As in example 2, it can be noticed that both LSSVF and IVSVF estimates are again not acceptable, contrary to the SRIVC estimates which are unbiased with low standard deviation.

Table 3. MC simulation results (example 3)

<table>
<thead>
<tr>
<th>Method</th>
<th>True value</th>
<th>$a_{02}$</th>
<th>$a_{20}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LSSVF</td>
<td>mean</td>
<td>0.14231</td>
<td>-0.13226</td>
</tr>
<tr>
<td></td>
<td>std</td>
<td>0.0009</td>
<td>0.0007</td>
</tr>
<tr>
<td>IVSVF</td>
<td>mean</td>
<td>1.0422</td>
<td>-0.37014</td>
</tr>
<tr>
<td></td>
<td>std</td>
<td>0.0668</td>
<td>0.0034</td>
</tr>
<tr>
<td>SRIVC</td>
<td>mean</td>
<td>0.99614</td>
<td>-1.0943</td>
</tr>
<tr>
<td></td>
<td>std</td>
<td>0.0152</td>
<td>0.0243</td>
</tr>
</tbody>
</table>

6. CONCLUSION

In this paper, a refined IV-type method has been proposed to directly estimate asymptotically efficient parameters in continuous-time partial derivative equation models from sampled data, when the additive measurement noise is white. One of the main advantages of this method is that it provides partial differential equation models whose parameters can be interpreted immediately in physically meaningful terms.

REFERENCES


