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Abstract. A classical result of Miyanishi-Sugie and Keel-M^{c}kernan asserts that for smooth affine surfaces, $\mathbb{A}^1$-uniruledness is equivalent to $\mathbb{A}^1$-ruledness, both properties being in fact equivalent to the negativity of the logarithmic Kodaira dimension. Here we show in contrast that starting from dimension three, there exists smooth affine varieties which are $\mathbb{A}^1$-uniruled but not $\mathbb{A}^1$-ruled.

Introduction

Complex uniruled projective varieties are nowadays considered through the Mori Minimal Model Program (MMP) as the natural generalization to higher dimensions of birationaly ruled surfaces (see e.g. [12]). In particular, as in the case of ruled surfaces, these are the varieties for which the program does not yield a minimal model, but a Mori fiber space. These varieties are also conjectured to be the natural generalization to higher dimensions of surfaces of negative Kodaira dimension, the conjecture being in fact established as long as the abundance conjecture holds true [11], hence in particular for smooth threefolds. During the past decades, the systematic study of the geometry of rational curves on these varieties has been the source of many progress in the structure theory for higher dimensional, possibly singular, projective varieties to which the MMP can be applied. The situation is much less clear for non complete varieties, in particular affine ones.

The natural analogue of ruledness in this context is the notion of $\mathbb{A}^1$-ruledness, a variety X being called $\mathbb{A}^1$-*ruled* if it contains a Zariski dense open subset U of the form $U \simeq \mathbb{A}^1 \times Y$ for a suitable quasi-projective variety Y. A landmark result of Miyanishi-Sugie [14] asserts that a smooth affine surface is $\mathbb{A}^1$-ruled if and only if it has negative logarithmic Kodaira dimension [6]. For such surfaces, the projection $\mathrm{pr}_Y : U \simeq \mathbb{A}^1 \times Y \to Y$ always extends to a fibration $p : X \to Z$ with general fibers isomorphic to the affine line $\mathbb{A}^1$ over an open subset of the smooth projective model of Y, providing the affine counterpart of the fact that a smooth birationaly ruled projective surface has the structure of a fibration with general fibers isomorphic to $\mathbb{P}^1$ over a smooth projective curve. This result, together with the description of the geometry of degenerate fibers of these fibrations, has been one of the cornerstones of the structure theory of smooth affine surfaces developed during the past decades. But in contrast, the foundations for a systematic study of $\mathbb{A}^1$-ruled affine threefolds have been only laid recently in [5]. On the other hand, from the point of view of logarithmic Kodaira dimension, the appropriate counterpart of the notion of uniruledness for a non necessarily complete variety X is to require that X is generically covered by images of the affine line $\mathbb{A}^1$, in the sense that the set of points $x \in X$ with the property that there exists a non constant morphism $f = f_x : \mathbb{A}^1 \to X$ such that $x \in f_x(\mathbb{A}^1)$ is dense in X with respect to the Zariski topology. Such varieties are called $\mathbb{A}^1$-*uniruled*, or *log-uniruled* after Keel and McKernan [8], and can be equivalently characterized by the property that they admit an open embedding into a complete variety $\overline{X}$ which is covered by proper rational curves meeting the boundary $\overline{X} \setminus X$ in at most one point. In particular, a smooth $\mathbb{A}^1$-uniruled quasi-projective variety X has negative logarithmic Kodaira dimension. It is conjectured that the converse holds true in any dimension, but so far, the conjecture has been only established in the case of surfaces by Keel and McKernan [8].

It follows in particular from these results that for smooth affine surfaces the notions of $\mathbb{A}^1$-ruledness and $\mathbb{A}^1$-uniruledness coincide. Pursuing further the analogy with the classical projective notions, it seems then natural to expect that these do no longer coincide for higher dimensional affine varieties. Our main result confirms that this is indeed the case. Namely, we establish the following:

Theorem. *For every $n \geq 3$, the complement of a smooth hypersurface Q_n of degree n in $\mathbb{P}^n$ is $\mathbb{A}^1$-uniruled but not $\mathbb{A}^1$-ruled.*

The anti-ampleness of the divisor $K_{\mathbb{P}^n} + Q_n$ enables to easily deduce the $\mathbb{A}^1$-uniruledness of affine varieties of the form $\mathbb{P}^n \setminus Q_n$ from the general log-deformation theory for rational curves developed by Keel and McKernan [8]. The failure of $\mathbb{A}^1$-ruledness is then obtained in a more indirect fashion. Indeed, it turns out that for the varieties under consideration, $\mathbb{A}^1$-ruledness is equivalent to the stronger property that they admit a non trivial action of the additive group $\mathbb{G}_a$. We then exploit two deep results of projective geometry, namely the non rationality of the smooth cubic threefold in $\mathbb{P}^4$ in the case $n = 3$ and the birational super-rigidity of smooth hypersurfaces $Q_n \subset \mathbb{P}^n$ if $n \geq 4$, to exclude the existence of such non trivial actions.

In the last section, we consider more closely the case of complements of smooth cubic surfaces in $\mathbb{P}^3$ which provides a good illustration of the subtle but crucial difference between the two notions of $\mathbb{A}^1$-ruledness and $\mathbb{A}^1$-uniruledness in higher dimension. We show in particular that even though such complements are not $\mathbb{A}^1$-ruled they admit natural fibrations by $\mathbb{A}^1$-ruled affine surfaces. We study automorphisms of such fibrations in relation with the problem of deciding whether every automorphism of the complement of a smooth cubic surface is induced by a linear transformation of the ambient space $\mathbb{P}^3$.

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1. Recollection on affine ruled and affine uniruled varieties

1.1. **Affine ruledness and algebraic $\mathbb{G}_a$-actions.** Here we review general properties of affine ruled varieties, with a particular focus on the interplay between $\mathbb{A}^1$-ruledness of an affine variety and the existence of non trivial algebraic actions of the additive group $\mathbb{G}_a$ on it. We refer the reader to [4] for basic properties of the correspondence between such actions on affine varieties and their algebraic counterpart, the so-called locally nilpotent derivations of their coordinate rings.

Definition 1. A quasi-projective variety X is called *affine ruled* or $\mathbb{A}^1$*-ruled* if it contains a Zariski dense open subset U of the form $U \simeq \mathbb{A}^1 \times Y$ for a suitable quasi-projective variety Y. An open subset of this type is called an $\mathbb{A}^1$*-cylindrical open subset* of X.

1.1.1. An $\mathbb{A}^1$-ruled variety X is in particular ruled in the usual sense, the inclusion $\mathbb{A}^1 \times Y \hookrightarrow X$ of an $\mathbb{A}^1$-cylindrical open subset of X inducing a dominant birational map $\mathbb{P}^1 \times Y \dashrightarrow X$. The converse is not true since for instance the product of the punctured affine line $\mathbb{A}^1_* = \mathbb{A}^1 \setminus \{0\}$ with a smooth projective curve of positive genus is ruled but does not contain any $\mathbb{A}^1$-cylinder. In particular, in contrast with ruledness, the property of being $\mathbb{A}^1$-ruled is not invariant under birational equivalence. Note similarly that the existence of an $\mathbb{A}^1$-cylindrical open subset of X is a stronger requirement than that of a dominant birational morphism $\mathbb{A}^1 \times Y \to X$ for a suitable variety Y: for instance, the affine cone $X \subseteq \mathbb{A}^3$ over a projective plane curve $C \subseteq \mathbb{P}^2$ of positive genus does not contain $\mathbb{A}^1$-cylinders but on the other hand, blowing-up the vertex of the cone X yields a smooth quasi-projective variety $\sigma : \tilde{X} \to X$ with the structure of a locally trivial $\mathbb{A}^1$-bundle $\rho : \tilde{X} \to C$ hence, restricting to a subset Y of C over which ρ is trivial, a dominant birational morphism $\sigma\mid_{\rho^{-1}(Y)}: \rho^{-1}(Y) \simeq \mathbb{A}^1 \times Y \to X$.

1.1.2. Typical examples of $\mathbb{A}^1$-ruled varieties are normal affine varieties $X = \mathrm{Spec}(A)$ admitting a nontrivial algebraic action of the additive group $\mathbb{G}_a$. Indeed, if X is equipped with such an action induced by a locally nilpotent $\mathbb{C}$-derivation ∂ of A, then for every local slice $f \in \mathrm{Ker}\partial^2 \setminus \mathrm{Ker}\partial$, the principal open subset $X_{\partial f} = \mathrm{Spec}(A_{\partial f})$, where $A_{\partial f}$ denotes the localization $A[(\partial f)^{-1}]$ of A, is $\mathbb{G}_a$-invariant and the morphism $\mathbb{G}_a \times (V(f) \cap X_{\partial f}) \to X_{\partial f}$ induced by the $\mathbb{G}_a$-action on X is an isomorphism. In particular, X_{∂_f} is a principal $\mathbb{A}^1$-cylindrical open subset of X. Note on the contrary that the existence of an $\mathbb{A}^1$-cylindrical open subset $U \simeq \mathbb{A}^1 \times Y$ of an affine variety $X = \mathrm{Spec}(A)$ is in general not enough to guarantee that X can be equipped with a $\mathbb{G}_a$-action whose general orbits coincide with the general fibers of the projection $\mathrm{pr}_Y : U \simeq \mathbb{A}^1 \times Y \to Y$. For instance, the general fibers of the projection $\mathrm{pr}_1 : X = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta \to \mathbb{P}^1$, where $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ denotes the diagonal, cannot coincides with general orbits of an algebraic $\mathbb{G}_a$-action on X. Indeed, otherwise every invariant function on X would descend to a regular function on $\mathbb{P}^1$ and hence would be constant, in contradiction with the fact that for an affine variety X, the field $\mathrm{Frac}(\Gamma(X, \mathcal{O}_X)^{\mathbb{G}_a})$ has transcendence degree $\dim X - 1$ over $\mathbb{C}$.

In fact, it is classically known that the general fibers of an $\mathbb{A}^1$-cylindrical affine open subset $U \simeq \mathbb{A}^1 \times Y$ of a normal affine variety $X = \mathrm{Spec}(A)$ coincide with the general orbits of an algebraic $\mathbb{G}_a$-action on X if and only if U is a principal affine open subset of X (see e.g. [9, Proposition 3.1.5]). Let us briefly recall the argument for the convenience of the reader: the existence of a principal $\mathbb{A}^1$-cylinder in X is equivalent to the existence of an element $a \in A \setminus \{0\}$ and an isomorphism of $\mathbb{C}$-algebra $\varphi : A_a \stackrel{\sim}{\to} B[t]$, where t is transcendental over $\mathrm{Frac}(B)$. The derivation $\partial_0 = \varphi^* \frac{\partial}{\partial t}$ of A_a is then locally nilpotent and since A is a finitely generated algebra, there exists $n \geq 0$ such that $a^n \partial_0$ induces a $\mathbb{C}$-derivation ∂ of A. Noting that being invertible in A_a, a is necessarily contained in the kernel of ∂_0, we conclude that ∂ is a locally nilpotent derivation of A, defining a $\mathbb{G}_a$-action on X whose restriction to the principal invariant open subset $U = \mathrm{Spec}(A_a) \simeq \mathrm{Spec}(B) \times \mathbb{A}^1$ is by construction equivariantly isomorphic to that by translations on the second factor.

It follows in particular that for a normal affine variety X whose divisor class group $\mathrm{Cl}(X)$ is torsion, the existence of $\mathbb{A}^1$-cylindrical open subsets is essentially equivalent to that of nontrivial $\mathbb{G}_a$-actions. More precisely, we have the following criterion:

Proposition 2. *Let $X = \mathrm{Spec}(A)$ be a normal affine variety such that $\mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. Then for every $\mathbb{A}^1$-cylindrical open subset $U \simeq \mathbb{A}^1 \times Y$ of X there exists an action of $\mathbb{G}_a$ on X whose general orbits coincide with the general fibers of the projection $\mathrm{pr}_Y : U \to Y$. In particular, X admits an $\mathbb{A}^1$-cylinder open subset if and only if it admits a nontrivial $\mathbb{G}_a$-action.*

Proof. By replacing Y by an affine open subset of it, we may assume that U is affine whence that its complement $D = X \setminus U$ has pure codimension one in X as X itself is affine. The hypothesis guarantees precisely that D is the support of a principal divisor and hence the assertions follow immediately from the discussion above. □

1.2. **Basic facts on log uniruled quasi-projective varieties.**

Definition 3. A quasi-projective variety X is called $\mathbb{A}^1$*-uniruled*, or (*properly*) *log-uniruled* after [8] if it contains a Zariski dense open subset U with the property that for every $x \in U$, there exists a non constant morphism $f_x : \mathbb{A}^1 \to X$ such that $x \in f_x(\mathbb{A}^1)$.

1.2.1. Equivalently, X is $\mathbb{A}^1$-uniruled if through a general point there exists a maximally affine rational curve, that is, an affine rational curve whose normalization is isomorphic to the affine line $\mathbb{A}^1$. An $\mathbb{A}^1$-uniruled variety is in particular uniruled in the usual sense, i.e. there exists a dominant, generically finite rational map $\mathbb{P}^1 \times Y \dashrightarrow X$ for a suitable quasi-projective variety Y. More precisely, letting (V, D) be a pair consisting of a projective model V of X and a, possibly empty, boundary divisor D such that V is smooth along D, it follows from [8, 5.1] that there exists a closed sub-scheme Y of $\mathrm{Mor}(\mathbb{P}^1, V)$ consisting of morphisms $f : \mathbb{P}^1 \to V$ with the property that $f^{-1}(f(\mathbb{P}^1) \cap D)$ consists of at most one point and on which the restriction of the canonical evaluation morphism $\mathrm{ev} : \mathbb{P}^1 \times \mathrm{Mor}(\mathbb{P}^1, V) \to V$ induces a dominant morphism $\mathbb{P}^1 \times Y \to V$.

1.2.2. It is not difficult to check that a smooth $\mathbb{A}^1$-uniruled quasi-projective variety X has logarithmic Kodaira dimension $\overline{\kappa}(X)=-\infty$ (see e.g. [8, 5.11]). The converse holds true in dimension ≤ 2 thanks to the successive works of Miyanishi-Sugie [14] and Keel-McKernan [8] and in fact, for a smooth affine surface X, the three conditions $\overline{\kappa}(X)=-\infty$, X is $\mathbb{A}^1$-uniruled and X is $\mathbb{A}^1$-ruled turn out to be equivalent to each other. Furthermore, the surface X has then the stronger property that every of its point belongs to a maximally affine rational curve. This follows from the fact that the projection $\mathrm{pr}_Y : U \simeq \mathbb{A}^1\times Y \to Y$ from an $\mathbb{A}^1$-cylindrical open subset U extends to a fibration $p : X \to Z$ over an open subset of a smooth projective model of Y, with general fibers isomorphic to $\mathbb{A}^1$ and whose degenerate fibers consist of disjoint unions of affine lines (see e.g. [13]). Meanwhile, much less is known in higher dimension where one lacks in particular a good logarithmic analogue of Mori's Bend-and-Break techniques. Nevertheless, the following simple criterion due to Keel and McKernan [8, Corollary 5.4] enables to easily confirm $\mathbb{A}^1$-ruledness of certain smooth affine varieties and will be enough for our purpose:

Theorem 4. *Let X be a smooth affine variety and let $X \hookrightarrow (V,D)$ be a projective completion where V is a smooth projective variety and $D=V\setminus X$ a reduced divisor on V. If $-(K_V+D)$ is ample then X is $\mathbb{A}^1$-uniruled.*

Example 5. The above criterion guarantees for instance that for every $n\geq 1$, the complement X of a hypersurface $D\subseteq\mathbb{P}^n$ of degree $d\leq n$ is $\mathbb{A}^1$-uniruled. In dimension 2, we recover in particular the fact that the complement of smooth conic Q_2 in $\mathbb{P}^2$ is $\mathbb{A}^1$-uniruled, actually $\mathbb{A}^1$-ruled say for instance by the restriction to $\mathbb{P}^2\setminus Q_2$ of the rational pencil generated by Q_2 and two times its tangent line at a given point. On the other hand, even though the criterion does not require D to be an SNC divisor, it does not enable to deduce the fact that the complement of a cuspidal cubic D is $\mathbb{A}^1$-uniruled, in fact again $\mathbb{A}^1$-ruled, namely via the restriction of the rational pencil on $\mathbb{P}^2$ generated by D and three times its tangent line at the singular point.

2. Complement of smooth hypersurfaces of degree n in $\mathbb{P}^n$, $n\geq 3$

According to Theorem 4, the complement X of a hypersurface Q_n of degree n in $\mathbb{P}^n$, $n\geq 2$, is a smooth $\mathbb{A}^1$-uniruled affine variety. If $n=2$, then Q_2 is **a** smooth conic whose complement is in fact $\mathbb{A}^1$-ruled (see Example 5). Here we show in contrast that for every $n\geq 3$, the complement of a smooth hypersurface $Q_n\subseteq\mathbb{P}^n$ is not $\mathbb{A}^1$-ruled. It is worthwhile to note that the strategy to establish this fact in case of $n=3$ is quite different from that of the case of $n\geq 4$.

2.1. **The case** $n\geq 4$**.** Here we will exploit the birational super-rigidity of smooth hypersurfaces $Q_n\subset\mathbb{P}^n$ of degree $n\geq 4$ to deduce that the corresponding affine varieties $X=\mathbb{P}^n\setminus Q_n$ are not $\mathbb{A}^1$-ruled. Let us first briefly recall the notion of birational super-rigidity for a class of variety which is enough for our needs (see e.g. [17] for the general definition).

Definition 6. A smooth Fano variety V with $\mathrm{Pic}(V)\simeq\mathbb{Z}$ is called *birationaly super-rigid* if the following two conditions hold:

a) The variety V is not birational to a fibration $V'\to S$ onto a variety of $\dim S>0$ whose general fibers are smooth varieties with Kodaira dimension $-\infty$,

b) Any birational map from V to another smooth Fano variety V' with $\mathrm{Pic}(V')\simeq\mathbb{Z}$ is a biregular isomorphism.

The essential ingredient we will use is the following result, first established by Pukhlikov [17] under suitable generality assumptions and recently extended to arbitrary smooth hypersurfaces by De Fernex [3]:

Theorem 7. *Let $n\geq 4$ and let $Q_n\subseteq\mathbb{P}^n$ be a smooth hypersurface of degree n. Then Q_n is birationaly super-rigid.*

Remark 8. A noteworthy consequence of the above result is that every birational map $Q_n \dashrightarrow Q'_n\subset\mathbb{P}^n$ between smooth hypersurfaces of degree n is in fact a biregular isomorphism which has the additional property to be induced by the restriction of a linear transformation of the ambient space $\mathbb{P}^n$. This follows from the fact that anti-canonical divisors on Q_n coincide with hyperplane sections of Q_n. In particular, the group $\mathrm{Bir}(Q_n)$ of birational automorphisms of Q_n coincides with that of projective automorphisms $\mathrm{Lin}(Q_n)$ and the latter is a finite group.

Now we are ready to prove the following:

Proposition 9. *The complement of every smooth hypersurface $Q_n\subseteq\mathbb{P}^n$ of degree $n\geq 4$ is $\mathbb{A}^1$-uniruled but not $\mathbb{A}^1$-ruled.*

Proof. The $\mathbb{A}^1$-uniruledness of $X=\mathbb{P}^n\setminus Q_n$ is an immediate consequence of Theorem 4. Since $\mathrm{Cl}(X)=\mathbb{Z}/n\mathbb{Z}$, it follows from Proposition 2 that the $\mathbb{A}^1$-ruledness of X is equivalent to the existence of a non trivial algebraic $\mathbb{G}_a$-action on it. Every given $\mathbb{G}_a$-action on X induces a one parameter family of automorphisms $\{\varphi_t\}_{t\in\mathbb{G}_a}$ of X that we interpret through the open inclusion $X=\mathbb{P}^n\setminus Q_n\hookrightarrow\mathbb{P}^n$ as birational automorphisms $\Phi_t:\mathbb{P}^n\dashrightarrow\mathbb{P}^n$, $t\in\mathbb{G}_a$. Since Q_n is birationally super-rigid by virtue of Theorem 7 above, it follows that every Φ_t is in fact a biregular automorphism. Indeed, otherwise, noting that Φ_t cannot be an isomorphism in codimension one (see e.g. [2]) and letting $\mathbb{P}^n\stackrel{p}{\leftarrow}V\stackrel{q}{\to}\mathbb{P}^n$ be a resolution of it, where p consists of successive blow-ups of smooth centers, it would follow that the proper transform $q_*^{-1}Q_n$ of Q_n in V is an exceptional divisor of p whence is birationally ruled in contradiction with condition a) in Definition 6. Therefore the $\mathbb{G}_a$-action on X extends to a linear one on $\mathbb{P}^n$ which leaves X whence Q_n invariant. Since the automorphism group $\mathrm{Aut}(Q_n)$ is finite (see Remark 8 above), the induced $\mathbb{G}_a$-action on Q_n is the trivial one and so, being linear, the $\mathbb{G}_a$-action on $\mathbb{P}^n$ extending that on X is trivial on the linear span of Q_n. But Q_n is obviously not contained in any hyperplane of $\mathbb{P}^n$ and hence the initial $\mathbb{G}_a$-action on X is necessarily trivial, which completes the proof. ☐

2.2. **The case $n = 3$: complements of smooth cubic surfaces in $\mathbb{P}^3$.** Since a smooth cubic surface Q_3 in $\mathbb{P}^3$ is rational, the same argument as in the previous section depending on birational super-rigidity is no longer applicable to deduce the non $\mathbb{A}^1$-ruledness of its complement $X = \mathbb{P}^3 \setminus Q_3$. Instead, we will derive it from the unirationality but non rationality of smooth cubic threefolds in $\mathbb{P}^4$ together with a suitable finite étale covering trick.

The divisor class group of the complement X of a smooth cubic surface $Q_3 \subseteq \mathbb{P}^3$ being isomorphic to $\mathbb{Z}/3\mathbb{Z}$ generated by the class of a hyperplane section, it follows again from Proposition 2 that X is $\mathbb{A}^1$-ruled if and only if it admits a non trivial algebraic $\mathbb{G}_a$-action. On the other hand, since algebraic $\mathbb{G}_a$-actions lift under finite étale covers (see e.g. [18]), to establish the non $\mathbb{A}^1$-ruledness of X it is enough to exhibit a finite étale cover $\pi : \tilde{X} \to X$ of X with the property that $\tilde{X}$ does not admit any non trivial algebraic $\mathbb{G}_a$-action.

Proposition 10. *Let $X = \mathbb{P}^3 \setminus Q_3$ be the complement of a smooth cubic surface and let $\pi : \tilde{X} \to X$ be the canonical étale Galois cover of degree three associated with the canonical sheaf $\omega_X \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \mid_X$ of X. Then $\tilde{X}$ is a smooth affine threefold which does not admit any non trivial algebraic $\mathbb{G}_a$-action. Consequently, X is not $\mathbb{A}^1$-ruled.*

Proof. Letting $Q_3 \subseteq \mathbb{P}^3 = \mathrm{Proj}(\mathbb{C}[x, y, z, u])$ be defined as the zero locus of a homogeneous polynomial $F(x, y, z, u) \in \mathbb{C}[x, y, z, u]$ of degree three, it is straightforward to check that the canonical triple étale covering $\tilde{X}$ of X is isomorphic to the affine variety $\mathbb{A}^4 = \mathrm{Spec}(\mathbb{C}[x, y, z, u])$ defined by the equation $F(x, y, z, u) = 1$, and that the morphism $\pi : \tilde{X} \to X$ coincides with the restriction of the natural morphism $\mathbb{A}^4 \setminus \{0\} \to \mathbb{P}^3$, $(x, y, z, u) \mapsto [x : y : z : u]$. Note that by construction, $\tilde{X}$ is an open subset of the smooth cubic threefold $V \subseteq \mathrm{Proj}(\mathbb{C}[x, y, z, u, v])$ with equation $F(x, y, z, u) - v^3 = 0$, hence is unirational but not rational by virtue of a famous result of Clemens-Griffiths [1]. Now suppose that there exists a non trivial algebraic $\mathbb{G}_a$-action on $\tilde{X}$. Then by virtue of [19], the algebra of invariants $\Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}})^{\mathbb{G}_a}$ is a finitely generated integrally closed domain of transcendence degree two over $\mathbb{C}$. Letting $q : \tilde{X} \to \tilde{Z} = \mathrm{Spec}(\Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}})^{\mathbb{G}_a})$ be the corresponding quotient morphism, the unirationality of $\tilde{X}$ implies that of $\tilde{Z}$ whence its rationality since these two notions coincide in dimension two. But then the existence of a principal affine open subset $\tilde{Z}_a$ of $\tilde{Z}$ such that $q^{-1}(\tilde{Z}_a) \simeq \mathbb{A}^1 \times \tilde{Z}_a$ (see § 1.1.2 above) would imply in turn that $\tilde{X}$ itself is rational, a contradiction. □

Remark 11. Complements of singular cubic surfaces $Q_3 \subseteq \mathbb{P}^3$ may turn out to be $\mathbb{A}^1$-ruled. For instance, it is shown in [10] that if Q_3 has Du Val singularities worse than A_2 then $\mathbb{P}^3 \setminus Q_3$ admits a non trivial algebraic $\mathbb{G}_a$-action. It may even happen that $\mathbb{P}^3 \setminus Q_3$ contains an $\mathbb{A}^2$-cylinder, i.e., an open subset of the form $\mathbb{A}^2 \times Y$ for a smooth curve Y. This holds for instance for the complement of the cubic surface $Q \subseteq \mathbb{P}^3 = \mathrm{Proj}(\mathbb{C}[x, y, z, u])$ defined as the vanishing locus of the homogeneous polynomial $F(x, y, z, u) = yu^2 + z(xz + y^2)$ which has a unique isolated singularity $[1 : 0 : 0 : 0]$ of type D_5. Indeed, noting that $f(x, y, z) = F(x, y, z, 1)$ is a component of the Nagata automorphism [15], it follows that the general fibers of the morphism $\rho : \mathbb{P}^3 \setminus Q \to \mathbb{A}^1$ induced by the restriction of the rational pencil on $\mathbb{P}^3$ generated by Q and three times the hyperplane $\{u = 0\}$ are isomorphic to $\mathbb{A}^2$ hence, by virtue of [7], that there exists an open subset $Y \subseteq \mathbb{A}^1$ such that $\rho^{-1}(Y) \simeq \mathbb{A}^2 \times Y$. A similar construction holds for all normal cubic surfaces listed in [16] which arise as closures in $\mathbb{P}^3$ of zero sets of components of automorphisms of $\mathbb{A}^3$.

3. Rigid rational fibrations on complements of smooth cubic surfaces in $\mathbb{P}^3$

In contrast with the higher dimensional case where a similar argument as the one used in the proof of Proposition 9 shows that for every smooth hypersurface Q_n of the degree n in $\mathbb{P}^n$, $n \geq 4$, the group $\mathrm{Aut}(\mathbb{P}^n \setminus Q_n)$ is embeded as a subgroup of $\mathrm{Aut}(Q_n)$ hence consists of a finite group of linear transformations, it is not known whether every automorphism of the complement of a smooth cubic surface $Q = Q_3$ in $\mathbb{P}^3$ is induced by a linear transformation of $\mathbb{P}^3$.

Even though $\mathbb{P}^3 \setminus Q$ is not $\mathbb{A}^1$-ruled (cf. Proposition 10), it turns out that it is fibered in a natural way by $\mathbb{A}^1$-ruled affine surfaces whose general closures in $\mathbb{P}^3$ are smooth cubic surfaces. Since $\mathbb{A}^1$-ruled affine surfaces are usually good candidates for having non trivial automorphisms, one can expect that these fibrations have interesting automorphisms, in particular automorphisms induced by strictly birational transformations of the ambient space $\mathbb{P}^3$.

In this section, we first review the construction of these fibrations $\rho : \mathbb{P}^3 \setminus Q \to \mathbb{A}^1$ by $\mathbb{A}^1$-ruled affine surfaces. We describe the automorphism groups of their general fibers and we check in particular that for a general smooth cubic hypersurface $Q \subseteq \mathbb{P}^3$, these fibers do indeed carry interesting automorphisms induced by strictly birational Geiser involutions of their projective closures. We show in contrast that every automorphism of the full fibration comes as the restriction of a linear transformation of $\mathbb{P}^3$.

3.1. **Special rational pencils on the complement of a smooth cubic surface.** Given a smooth cubic surface $Q \subseteq \mathbb{P}^3$ and a line L on it, the restriction to Q of the rational pencil $\mathcal{H}_L = |\mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathcal{I}_L|$ on $\mathbb{P}^3$ generated by planes containing L can be decomposed as $\mathcal{H}_L \mid_Q = \mathcal{L} + L$ where $\mathcal{L}$ is a base point free pencil defining a conic bundle $\Phi_{\mathcal{L}} : Q \to \mathbb{P}^1$ with five degenerate fibers, each consisting of the union of two (-1)-curves intersecting transversally. The restriction $\Phi_{\mathcal{L}} \mid_L : L \to \mathbb{P}^1$ is a double cover and for every branch point $x \in \mathbb{P}^1$ of $\Phi_{\mathcal{L}} \mid_L$, the intersection of Q with the corresponding hyperplane $H_x \in \mathcal{H}_L$ consists either of a smooth conic tangent to L or of two distinct lines intersecting L in a same point, which is called an Eckardt point of Q. Letting say $H_0(L)$ and $H_\infty(L)$ be the planes in $\mathcal{H}_L$ such that $\Phi_{\mathcal{L}} \mid_L$ is ramified over the points $\Phi_{\mathcal{L}}(H_t(L) \mid_Q - L)$, $t = 0, \infty$, the divisors Q and $3H_t(L)$ generate a pencil of cubic surfaces $\overline{\rho}_t(L) : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ with a unique multiple member $3H_t(L)$ and whose general members are smooth cubic surfaces.

3.1.1. From now on we consider a pencil $\overline{\rho} = \overline{\rho}_t(L)$ as above associated to a fixed line $L \subseteq Q$ and a fixed distinguished plane $H \in \mathcal{H}_L$ intersecting Q either along the union $L \cup C$ of a line and conic intersecting each other in a single point p or along the union $L \cup L_1 \cup L_2$ of three lines meeting at an Eckardt point p of Q. We denote by $\rho : \mathbb{P}^3 \setminus Q \to \mathbb{A}^1$ the morphism induced by the restriction of $\overline{\rho}$ to the complement of Q. By construction, the closure in $\mathbb{P}^3$ of a general fiber S of ρ is a smooth cubic surface V such that the reduced divisor $D = H \cap V$ has either the form $D = L + C$ or $D = L + L_1 + L_2$. In each case we denote by $\alpha : W \to V$ the blow-up of p, with exceptional divisor E. If p is an Eckardt point, then we let $\tilde{V} = W$ and we denote by $\tilde{D} \subseteq \tilde{V}$ the reduced divisor $\alpha^{-1}(D)_{\mathrm{red}} = L + L_1 + L_2 + E$. Otherwise, if p is not an Eckardt point, then we let $\tilde{V}$ be the variety obtained from W by blowing-up further the intersection point of E and of the proper transform of C, say with exceptional divisor E_2, and we let $\tilde{D} \subseteq \tilde{V}$ be the reduced divisor $\tilde{D} = L + C + E + E_2$. By construction, $\tilde{D}$ is an SNC divisor and the induced birational morphism $\sigma : (\tilde{V}, \tilde{D}) \to (V, D)$ provides a minimal log-resolution of the pair (V, D).

The following lemma summarizes basic properties of the general fibers of fibrations of the form $\rho_t(L) : \mathbb{P}^3 \setminus Q \to \mathbb{A}^1$, $t = 0, \infty$.

Lemma 12. *For a general fiber $S = V \setminus D$ of $\rho_t(L) : \mathbb{P}^3 \setminus Q \to \mathbb{A}^1$, the following holds:*

a) S is a smooth affine surface with a trivial canonical sheaf and logarithmic Kodaira dimension $\overline{\kappa}(S) = -\infty$,

b) If $D = L + C$ (resp. $D = L + L_1 + L_2$) then the Picard group of S is isomorphic to $\mathbb{Z}^5$ (resp. $\mathbb{Z}^4$).

Proof. The affineness of S is clear as D is a hyperplane section of V. Since V is a smooth cubic surface, it follows from adjunction formula that $\omega_V \simeq \omega_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(3) \mid_V \simeq \mathcal{O}_{\mathbb{P}^3}(-1) \mid_V \simeq \mathcal{O}_V(-D)$. This implies in turn the triviality of the canonical sheaf of S as $\omega_S \simeq \omega_V \mid_S \simeq \mathcal{O}_V(-D) \mid_S \simeq \mathcal{O}_S$. We may identify S with $\tilde{V} \setminus \tilde{D}$ via the birational morphism $\sigma : (\tilde{V}, \tilde{D}) \to (V, D)$ constructed above. Since $\tilde{D}$ is an SNC divisor by construction, we have $\overline{\kappa}(S) = \kappa(\tilde{V}, K_{\tilde{V}} + \tilde{D})$. Using the fact that $-D$ is a canonical divisor on V, we deduce from the logarithmic ramification formula for σ that

$$\begin{cases} K_{\tilde{V}} + \tilde{D} = -E & \text{if } D = L + L_1 + L_2 \\ K_{\tilde{V}} + \tilde{D} = -E_2 & \text{if } D = L + C. \end{cases}$$

Since E and E_2 are exceptional divisors, one has $H^0(\tilde{V}, m(K_{\tilde{V}} + \tilde{D})) = 0$ for every $m > 0$ and hence $\overline{\kappa}(S) = -\infty$.

To determine the Picard group of S, we will exploit the conic bundle structure $\Phi_{\mathcal{L}} : V \to \mathbb{P}^1$ described in the beginning of 3.1. By contracting a suitable irreducible component of each of the five degenerate fibers of $\Phi_{\mathcal{L}} : V \to \mathbb{P}^1$, we obtain a birational morphism $\tau : V \to \mathbb{P}^1 \times \mathbb{P}^1$ fitting into a commutative diagram

$$\begin{array}{ccc} V & \stackrel{\tau}{\to} & \mathbb{P}^1 \times \mathbb{P}^1 \\ \Phi_{\mathcal{L}} \downarrow & & \downarrow \mathrm{pr}_1 \\ \mathbb{P}^1 & = & \mathbb{P}^1 \end{array}$$

and such that the proper transform of L is a smooth 2-section of pr_1 with self-intersection 4, whence a section of the second projection $\mathrm{pr}_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. We can then identify the divisor class group $\mathrm{Cl}(V) \simeq \mathbb{Z}^7$ of V with the group generated by the proper transforms of a fiber of pr_1, a fiber of pr_2, and the five exceptional divisors of τ. The proper transform $\tau_*(D)$ of D in $\mathbb{P}^1 \times \mathbb{P}^1$ is the union of the proper transform of L and of a fiber of pr_1. Thus if $D = L + C$ then L and C together with the five exceptional divisors of τ generate $\mathrm{Cl}(V)$ and hence $\mathrm{Cl}(S) \simeq \mathrm{Pic}(S)$ is isomorphic to $\mathbb{Z}^5$ generated by the classes of the intersections of these exceptional divisors with S. Similarly, if $D = L + L_1 + L_2$ then L_1 or L_2, say L_2, is an exceptional divisor of τ and so $\mathrm{Cl}(V)$ is generated by L, L_1, L_2 and the four other exceptional divisors of τ. This implies in turn that $\mathrm{Cl}(S) \simeq \mathrm{Pic}(S)$ is isomorphic to $\mathbb{Z}^4$ generated by the classes of the intersections of these four exceptional divisors with S. ☐

3.1.2. Since a general fiber $S = V \setminus D$ of $\rho_t(L) : \mathbb{P}^3 \setminus Q \to \mathbb{A}^1$ has logarithmic Kodaira dimension $-\infty$ by Lemma 12, it is $\mathbb{A}^1$-ruled and hence, by virtue of [13], it admits an $\mathbb{A}^1$-fibration $q : S \to C$ over a smooth curve. Let us briefly explain how to construct such fibrations using the birational morphism $\tau : V \to \mathbb{P}^1 \times \mathbb{P}^1$ which contracts an irreducible component of each of the five degenerate fibers of $\Phi_{\mathcal{L}} : V \to \mathbb{P}^1$ as described in the proof of the previous Lemma. According to the configuration of D, we have the following:

a) If $D = L + C$ where C is a smooth conic, then each of the points blown-up by τ is the intersection point of the proper transform of L with a fiber of $\mathrm{pr}_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. The proper transforms $F_1, \ldots, F_5 \subseteq V$ of these fibers are disjoint (-1)-curves which do not intersect L. Let $\mu : V \to \mathbb{P}^2$ be the contraction of L and $F_1, \ldots, F_5$. Since each F_i, $i = 1, \ldots, 5$ intersects C transversally and L is tangent to C, the image of C in $\mathbb{P}^2$ is a cuspidal cubic. The rational pencil on $\mathbb{P}^2$ generated by the image of C and three times its tangent line T at its unique singular point lifts to a rational pencil $\overline{q} : V \dashrightarrow \mathbb{P}^1$ having the divisors $C + \sum_{i=1}^{5} F_i$ and $3T + L$ as singular members. Its restriction to $S = V \setminus D$ is an $\mathbb{A}^1$-fibration $q : S \to \mathbb{P}^1$ with two degenerate fibers: one is irreducible of multiplicity three consisting of the intersection of the proper transform of T with S and the other is reduced, consisting of the disjoint union of the curves $F_i \cap S \simeq \mathbb{A}^1$, $i = 1, \ldots, 5$.

b) The construction in the case where $D = L + L_1 + L_2$ is very similar: we may suppose that L_1 is contracted by τ so that contracting the same irreducible components $F_1, \ldots, F_4$ of the degenerate fibers of $\Phi_{\mathcal{L}} : V \to \mathbb{P}^1$ but L_2 instead of L_1, we obtain a birational morphism $\tau_1 : V \to \mathbb{F}_1$. The image of L in $\mathbb{F}_1$ is a smooth 2-section of the $\mathbb{P}^1$-bundle structure $\pi_1 : \mathbb{F}_1 \to \mathbb{P}^1$ with self-intersection 4. Letting C_0 be the exceptional section of π_1 with self-intersection (-1), one has necessarily $L \sim 2C_0 + 2F$ where F is a fiber of π_1. So L does not intersect C_0 and its image in $\mathbb{P}^2$ by the morphism $q : \mathbb{F}_1 \to \mathbb{P}^2$ contracting C_0 is a smooth conic. Since L_1 is tangent to L in $\mathbb{F}_1$, its image in $\mathbb{P}^2$ is the tangent to the image of C at a point distinct from the one blown-up by q. The rational pencil on $\mathbb{P}^2$ generated by the image of L and two times that of L_1 lifts to a rational pencil $\overline{\rho} : V \dashrightarrow \mathbb{P}^1$ having $2L_1 + L_2 + 2C_0$ and $L + \sum_{i=1}^{4} F_i$ as degenerate members. Its restriction to $S = V \setminus D$ yields an $\mathbb{A}^1$-fibration $\rho : S \to \mathbb{P}^1$ with two degenerate fibers : one is irreducible of multiplicity two consisting of the intersection of the proper transform of C_0 with S and the other is reduced, consisting of the disjoint union of the curves $F_i \cap S \simeq \mathbb{A}^1$, $i = 1, \ldots, 4$.

Remark 13. It follows from Corollary 15 below that the automorphism group of a smooth affine surface $S = V \setminus D$ as above is finite. In particular, such surfaces are $\mathbb{A}^1$-ruled but do not admit non trivial $\mathbb{G}_a$-actions and so the complement of a smooth cubic surface $Q \subseteq \mathbb{P}^3$ is a smooth affine $\mathbb{A}^1$-uniruled threefold without non trivial $\mathbb{G}_a$-action which has the structure of a family $\rho_t(L) : \mathbb{P}^3 \setminus Q \to \mathbb{A}^1$ of $\mathbb{A}^1$-ruled affine surfaces without non trivial $\mathbb{G}_a$-actions. In contrast, the total space of a family of affine varieties with non trivial $\mathbb{G}_a$-actions often admits itself a non trivial $\mathbb{G}_a$-action. For instance, suppose that $\varphi : X = \mathrm{Spec}(B) \to S = \mathrm{Spec}(A)$ is a dominant morphism between complex affine varieties and that $\partial : B \to B$ is an A-derivation of B which is locally nilpotent in restriction to general closed fibers of φ, defining non trivial $\mathbb{G}_a$-actions on these fibers. Then ∂ is a locally nilpotent A-derivation of B and hence X carries a non trivial $\mathbb{G}_a$-action. Indeed, up to shrinking S if necessary, we may assume that for every maximal ideal $\mathfrak{m} \in \mathrm{Specmax}(A)$, the $A/\mathfrak{m}$-derivation $\partial_{\mathfrak{m}} : B/\mathfrak{m}B \to B/\mathfrak{m}B$ induced by ∂ is locally nilpotent. Given an element $f \in B$ and an integer $n \geq 0$, denote by $K^n(f)$ the closed sub-variety of S whose points are the maximal ideals $\mathfrak{m} \in \mathrm{Specmax(A)}$ for which the residue class of f in $B/\mathfrak{m}B$ belongs to $\mathrm{Ker}\partial_{\mathfrak{m}}^n$. The hypothesis implies that S is equal to the increasing union of its closed sub-variety $K^n(f)$, $n \geq 0$, and so this sequence must stabilizes as the base field $\mathbb{C}$ is uncountable. So there exists $n_0 = n_0(f)$ such that the restriction of $\partial^{n_0} f$ to every closed fiber of $\varphi : X \to S$ is the zero function, and hence $\partial^{n_0} f = 0$ since these fibers form a dense subset of X.

3.2. **Automorphisms of general fibers of special rational pencils.** This sub-section is devoted to the study of automorphisms of general fibers of the rational fibrations $\rho : \mathbb{P}^3 \setminus Q \to \mathbb{A}^1$ constructed as above. We show in particular that such general fibers admit biregular involutions induced by Geiser involutions of their projective closures.

3.2.1. To simplify the discussion, let us call a pair (V, D) as in 3.1.1 *special* if V is a smooth cubic hypersurface in $\mathbb{P}^3$ and D is composed of three concurrent lines $L + L_1 + L_2$ meeting in an Eckardt point p of V, or of the union of L and a smooth conic C intersecting L with multiplicity two in a single point p. Similarly as in § 3.1.1, we denote by $\alpha : W \to V$ the blow-up of p and we denote by E its exceptional divisor.

By construction, W is a weak del Pezzo surface (i.e., $-K_W$ is nef and big) of degree 2 on which the anticanonical linear system $|-K_W|$ defines a morphism $\theta : W \to \mathbb{P}^2$. The latter factors into a birational morphism $W \to Y$ contracting L (resp. L, L_1 and L_2) if $D = L + C$ (resp. if $D = L + L_1 + L_2$) followed by a Galois double covering $Y \to \mathbb{P}^2$ ramified over a quartic curve Δ with a unique double point (resp. three double points) located at the image of L (resp. at the images of L, L_1 and L_2). The non trivial involution of the double covering $Y \to \mathbb{P}^2$ induces an involution $G_W : W \to W$ fixing L and exchanging the proper transform of D and the exceptional divisor E (resp. fixing L, L_1 and L_2). The latter descends either to birational involution $G_{V,p} : V \dashrightarrow V$ if $D = L + C$, or to a biregular involution $G_{V,p} : V \to V$ having p as an isolated fixed point if $D = L + L_1 + L_2$.

In each case, we say that $G_{V,p}$ is the *Geiser involution of* V *with center at* p. By construction, $G_{V,p}$ restricts further to a biregular involution $j_{G_{V,p}}$ of the affine surface $S = V \setminus D$ which we call the *affine Geiser involution of* S *with center at* p.

Proposition 14. *Let* $\varphi : S = V \setminus D \xrightarrow{\sim} S' = V' \setminus D'$ *be an isomorphism between affine surfaces associated to special pairs* (V, D) *and* (V', D') *respectively. Then the following assertions hold:*

a) If either D *or* D' *consists of three concurrent lines then* φ *extends to an isomorphism of pairs* $\overline{\varphi} : (V, D) \xrightarrow{\sim} (V', D')$.

b) Otherwise, if both $D = L+C$ *and* $D' = L'+C'$ *consist of the union of a line and a conic, then the induced birational map* $\overline{\varphi} : V \dashrightarrow V'$ *is either an isomorphism of pairs, or it can be decomposed as* $\overline{\varphi} = \Psi \circ G_{V,p}$ *or* $\overline{\varphi} = G_{V',p'} \circ \Psi'$ *where* Ψ, Ψ' *are isomorphisms of pairs and where* $G_{V,p}$ *and* $G_{V',p'}$ *denote the Geiser involutions of* V *and* V' *with centers are* $p = L \cap C$ *and* $p' = L' \cap C'$ *respectively. Furthermore, every birational map* $V \dashrightarrow V'$ *of the form* $\Psi \circ G_{V,p}$ (resp. $G_{V',p'} \circ \Psi'$) can be uniquely re-written in the form $G_{V',p'} \circ \Psi'$ (resp. $\Psi \circ G_{V,p}$). *In particular, if there exists an isomorphism* $\varphi : S = V \setminus D \xrightarrow{\sim} S' = V' \setminus D'$ *then* V *and* V' *are isomorphic smooth cubic surfaces.*

Proof. Let $\sigma : (\tilde{V}, \tilde{D}) \to (V, D)$ and $\sigma' : (\tilde{V}', \tilde{D}') \to (V', D')$ be the minimal log-resolutions of the pairs (V, D) and (V', D') respectively described in § 3.1.1. Since $V \setminus D \simeq \tilde{V} \setminus \tilde{D}$ and $V' \setminus D' \simeq \tilde{V}' \setminus \tilde{D}'$ by construction, every isomorphism $\varphi : S \to S'$ extends in a natural way to a birational map $\tilde{\varphi} : \tilde{V} \dashrightarrow \tilde{V}'$ restricting to φ on S. We claim that $\tilde{\varphi}$ is in fact a biregular isomorphism of pairs. Indeed, suppose on the contrary that $\tilde{\varphi}$ is strictly birational and let $\tilde{V} \xleftarrow{\beta} X \xrightarrow{\beta'} \tilde{V}'$ be its minimal resolution. Recall that the minimality of the resolution implies in particular that there is no (-1)-curve in X which is exceptional for β and β' simultaneously. Since $\tilde{V}$ is smooth and $\tilde{D}$ is an SNC divisor, β' decomposes into a finite sequence of blow-downs of successive (-1)-curves supported on the boundary $B = \beta^{-1}(\tilde{D})_{\mathrm{red}} = (\beta')^{-1}(\tilde{D})_{\mathrm{red}}$. In view of the structure of $\tilde{D}$, the only possible (-1)-curve in B which is not exceptional for β is the proper transform of E if $D = L + L_1 + L_2$ or the proper transform of E_2 if $D = L + C$. But after the contraction of these curves, the boundary would no longer be an SNC divisor, a contradiction. This implies that $\tilde{\varphi} : \tilde{V} \to \tilde{V}'$ is a morphism and the same argument shows that it does not contract any curve in the boundary $\tilde{D}$. So $\tilde{\varphi} : \tilde{V} \to \tilde{V}'$ is an isomorphism restricting to an isomorphism between $\tilde{V} \setminus \tilde{D}$ and $\tilde{V}' \setminus \tilde{D}'$ whence an isomorphism between the pairs $(\tilde{V}, \tilde{D})$ and $(\tilde{V}', \tilde{D}')$. It follows in particular that the intersection matrices of the divisors $\tilde{D}$ and $\tilde{D}'$ must be the same up to a permutation. In view of the description given in § 3.1.1, we conclude that either D and D' simultaneously consist of three concurrent lines or of the union of a line and a smooth conic.

In the first case, since the exceptional divisors E of the blow-up of V at p and E' of the blow-up of V' at p' are the only (-1)-curves in $\tilde{D}$ and $\tilde{D}'$ respectively, the biregular isomorphism $\tilde{\varphi} : \tilde{V} \to \tilde{V}'$ extending φ necessarily maps E isomorphically onto E'. So $\tilde{\varphi}$ descends to an isomorphism of pairs $\overline{\varphi} : (V, D) \to (V', D')$ which gives a).

In the second case, letting $D = L + C$ and $D' = L' + C'$, a similar argument implies that the isomorphism $\tilde{\varphi} : \tilde{V} \to \tilde{V}'$ maps E_2 onto E_2' and the proper transform of L onto that of L'. This implies in turn that $\tilde{\varphi}$ descends to an isomorphism $\tilde{\varphi}_1 : W \to W'$ between the surfaces obtained from V and V' by blowing-up the points $p = L \cap C$ and $p' = L' \cap C'$ with respective exceptional divisors E and E'. Now we have the following alternative: either $\tilde{\varphi}_1$ maps E and C onto E' and C' respectively, and then it descends to a biregular isomorphism of pairs $\overline{\varphi}:(V,D)\to(V',D')$, or $\tilde{\varphi}_1$ maps E and C onto C' and E' respectively. In the second case, by composing $\overline{\varphi}:V\dashrightarrow V'$ either by $G_{V,p}$ on the left or by $G_{V',p'}$ on the right, we get a new birational maps $\Psi,\Psi':V\dashrightarrow V'$ which lift to a birational map $\Psi_1:W\dashrightarrow W'$ mapping L, E and C isomorphically onto L', E' and C' respectively. The previous discussion implies that Ψ and Ψ' are isomorphism of pairs and so we get the desired decompositions $\overline{\varphi}=\Psi\circ G_{V,p}^{-1}=\Psi\circ G_{V,p}$ and $\overline{\varphi}=G_{V',p'}^{-1}\circ\Psi'=G_{V',p'}\circ\Psi'$. The last assertion about the uniqueness of the re-writing is clear by construction. □

Corollary 15. *Let (V,D) be a special pair and let $S=V\setminus D$ be the corresponding affine surface. Then the following holds:*

a) If D consists of three concurrent lines then $\mathrm{Aut}(S)$ is a nontrivial finite subgroup of $\mathrm{PGL}(4,\mathbb{C})$ which coincides with the automorphism group $\mathrm{Aut}(V,D)$ of the pair (V,D).

b) Otherwise, if D consists of a line and a smooth conic, then we have an exact sequence

$$0\to\mathrm{Aut}(V,D)\to\mathrm{Aut}(S)\to\mathbb{Z}_2\cdot j_{G_{V,p}}\to 0$$

and an isomorphism $\mathrm{Aut}(S)=\mathrm{Aut}(V,D)\rtimes\mathbb{Z}_2\cdot j_{G_{V,p}}$, where the affine Geiser involution $j_{G_{V,p}}$ acts on $\mathrm{Aut}(V,D)$ by $j_{G_{V,p}}\cdot\Psi=\Psi'$ where for every $\Psi\in\mathrm{Aut}(V,D)$, Ψ' is the unique automorphism of the pair (V,D) such that $\Psi\circ G_{V,p}=G_{V,p}\circ\Psi'$.

Proof. Since $V\subseteq\mathbb{P}^3$ is embedded by its anti-canonical linear system $|-K_V|$, every automorphism of V is induced by a linear transformation of $\mathbb{P}^3$. Furthermore, since D is a hyperplane section of V, $\mathrm{Aut}(V,D)$ is an algebraic sub-group of the group of linear transformation of $\mathbb{P}^3$ preserving globally the corresponding hyperplane. The automorphism group of a smooth cubic surface in $\mathbb{P}^3$ being finite, it follows that $\mathrm{Aut}(V,D)$ is a finite algebraic group. If D consists of three concurrent lines, then their common point p is an Eckardt point of V which is always the isolated fixed point of a biregular involution of V preserving D. This shows that $\mathrm{Aut}(V,D)$ is never trivial in this case. In case b), where $D=L+C$ is the union of a line and smooth conic, the assertion is an immediate consequence of the previous Proposition 14. □

Remark 16. If $D=L+C$ then every automorphism Ψ of the pair (V,D) preserves the rational pencil $\overline{q}:V\dashrightarrow\mathbb{P}^1$ constructed in § 3.1.2. Indeed such an automorphism certainly preserves the conic C and it maps the five (-1)-curves $F_1,\ldots,F_5$ to five other (-1)-curves each intersecting C transversally. But it follows from the construction of V from $\mathbb{P}^2$ that $F_1,\ldots,F_5$ are the only five (-1)-curves in V with this property. Similarly, since L and $p=L\cap C$ are Ψ-invariant, the image of T by Ψ must be a smooth rational curve intersecting L and C transversally at p but the construction of V again implies that T is the only curve in V with this property. It follows that the divisors $C+\sum_{i=1}^5 F_i$ and $3T+L$ which generate the pencil $\overline{q}:V\dashrightarrow\mathbb{P}^1$ are Ψ-invariant whence that $\overline{q}$ is globally preserved by Ψ as claimed. This implies in turn that every automorphism φ of S which extends to a biregular automorphism of the pair (V,D) fits into a commutative diagram

$$\begin{array}{ccc} S & \overset{\varphi}{\to} & S \\ q\downarrow & & \downarrow q \\ \mathbb{P}^1 & \overset{\xi}{\to} & \mathbb{P}^1, \end{array}$$

where $\xi\in\mathrm{PG}L(2,\mathbb{C})$ fixes the two points of $\mathbb{P}^1$ corresponding to the non isomorphic degenerate fibers $3T\cap S$ and $\sum_{i=1}^5 F_i\cap S$ of q. In contrast, it is straightforward to check from the construction that $q:S\to\mathbb{P}^1$ is not globally preserved by the affine Geiser involution $j_{G_{V,p}}$ of S with center at $p=D\cap L$ and hence that S carries a second $\mathbb{A}^1$-fibration $q\circ j_{G_{V,p}}:S\to\mathbb{P}^1$ whose general fibers are distinct from that of q. In particular, if V is chosen generally so that $\mathrm{Aut}(V,D)$ is trivial, then $\mathrm{Aut}(S)$ is isomorphic to $\mathbb{Z}_2$, generated by the affine Geiser involution $j_{G_{V,p}}$ (cf. Corollary 15, (b)), and S carries two conjugated $\mathbb{A}^1$-fibrations $q:S\to\mathbb{P}^1$ and $q\circ j_{G_{V,p}}:S\to\mathbb{P}^1$, each of these having no non trivial automorphism.

3.3. **Automorphisms of special rational pencils.**

This subsection is devoted to the proof of the following result:

Proposition 17. *Let $Q\subseteq\mathbb{P}^3$ be a smooth cubic surface and let $\overline{\rho}:\mathbb{P}^3\dashrightarrow\mathbb{P}^1$ be the special pencil associated to a line $L\subseteq Q$ and a distinguished hyperplane $H\in|\mathcal{O}_{\mathbb{P}^3}(1)\otimes\mathcal{I}_L|$ as in § 3.1.1. Then every automorphism of the induced rational fibration $\rho:\mathbb{P}^3\setminus Q\to\mathbb{A}^1$ is the restriction of an automorphism of $\mathbb{P}^3$.*

Proof. Let us denote by $\mathrm{Aut}(\mathbb{P}^3\setminus Q,\rho)$ the subgroup of $\mathrm{Aut}(\mathbb{P}^3\setminus Q)$ consisting of automorphisms preserving the fibration $\rho:\mathbb{P}^3\setminus Q\to\mathbb{A}^1=\mathrm{Spec}(\mathbb{C}[\lambda])$ fiberwise. Taking restriction over the generic point η of $\mathbb{A}^1$ induces an injective homomorphism from $\mathrm{Aut}(\mathbb{P}^3\setminus Q,\rho)$ to the group $\mathrm{Aut}(S_\eta)$ of automorphisms of the generic fiber S_η of ρ, and we identify from now on $\mathrm{Aut}(\mathbb{P}^3\setminus Q,\rho)$ with its image in $\mathrm{Aut}(S_\eta)$. By definition, S_η is a nonsingular affine surface defined over the field $\mathbb{C}(\lambda)$ whose closure in $\mathbb{P}^3_{\mathbb{C}(\lambda)}$ is a nonsingular cubic surface V_η such that $D_\eta=V_\eta\setminus S_\eta$ consists of either three lines L_η, $L_{\eta,1}$ and $L_{\eta,2}$ meeting a unique $\mathbb{C}(\lambda)$-rational point or the union of a line L_η and nonsingular conic C_η intersecting L_η at a unique $\mathbb{C}(\lambda)$-rational point p. The same argument as in the proofs of Proposition 14 and its Corollary 15 implies that $\mathrm{Aut}(S_\eta)=\mathrm{Aut}(V_\eta,D_\eta)$ in the first case whereas $\mathrm{Aut}(S_\eta)=\mathrm{Aut}(V_\eta,D_\eta)\rtimes\mathbb{Z}_2\cdot j_{G_{V_\eta,p}}$, where $j_{G_{V_\eta,p}}$ denotes the affine Geiser involution with the center at p, in the second case. In the first case, we conclude that every automorphism of $\mathrm{Aut}(\mathbb{P}^3\setminus Q,\rho)$ is generically of degree one when considered as a birational self-map of $\mathbb{P}^3$ and hence is the restriction of an automorphism of $\mathbb{P}^3$. So it remains to show that in the second case, one has $\mathrm{Aut}(\mathbb{P}^3\setminus Q,\rho)\subseteq\mathrm{Aut}(V_\eta,D_\eta)$ necessarily. Suppose on the contrary that $\mathrm{Aut}(\mathbb{P}^3\setminus Q,\rho)\not\subseteq\mathrm{Aut}(V_\eta,D_\eta)$. Then since we have an extension

$$0\to\mathrm{Aut}(V_\eta,D_\eta)\to\mathrm{Aut}(S_\eta)\to\mathbb{Z}_2\cdot j_{G_{V_\eta,p}}\to 0,$$

it would follow that $j_{G_{V_{\eta,p}}} \in \mathrm{Aut}(\mathbb{P}^3 \setminus Q, \rho)$. On the other hand, letting $\pi_p : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be the projection from the point p, the rational map $(\overline{\rho}, \pi_p) : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ is generically of degree 2, contracting the plane H to a line $\pi_p(H) \in \overline{\rho}(H) \times \mathbb{P}^2 \simeq \mathbb{P}^2$ and inducing a Galois double covering

$$\mathbb{P}^3 \setminus (Q \cup H) \to \mathbb{P}^1 \setminus \{\overline{\rho}(Q) \cup \overline{\rho}(H)\} \times \mathbb{P}^2 \setminus (\pi_p(Q) \cup \pi_p(H)) \simeq \mathbb{A}^1_* \times \mathbb{A}^2 \setminus (\pi_p(Q)).$$

By definition, the affine Geiser involution $j_{G_{V_{\eta,p}}}$ of S_η is simply the restriction to the generic fiber of ρ of the nontrivial involution J of this covering. So $j_{G_{V_{\eta,p}}} \in \mathrm{Aut}(\mathbb{P}^3 \setminus Q, \rho)$ considered as a birational self-map of $\mathbb{P}^3$ would coincide with J which is absurd since the latter contracts H whence the fiber $\rho^{-1}(\overline{\rho}(H))$ of ρ. Thus $\mathrm{Aut}(\mathbb{P}^3 \setminus Q, \rho) \subsetneq \mathrm{Aut}(V_\eta, D_\eta)$ as desired. □

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