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# HARDY-LITTLEWOOD SERIES AND EVEN CONTINUED FRACTIONS

TANGUY RIVOAL AND STÉPHANE SEURET

ABSTRACT. For any  $s \in (1/2, 1]$ , the series  $F_s(x) = \sum_{n=1}^{\infty} e^{i\pi n^2 x}/n^s$  converges almost everywhere on  $[-1, 1]$  by a result of Hardy-Littlewood concerning the growth of the sums  $\sum_{n=1}^N e^{i\pi n^2 x}$ , but not everywhere. However, there does not yet exist an intrinsic description of the set of convergence for  $F_s$ . In this paper, we define in terms of even continued fractions a subset of points of  $[-1, 1]$  of full measure where the series converges.

As an intermediate step, we prove that, for  $s > 0$ , the sequence of functions

$$\sum_{n=1}^N \frac{e^{i\pi n^2 x}}{n^s} - e^{\operatorname{sign}(x)i\frac{\pi}{4}} |x|^{s-\frac{1}{2}} \sum_{n=1}^{\lfloor N|x| \rfloor} \frac{e^{-i\pi n^2/x}}{n^s}$$

converges when  $N \rightarrow \infty$  to a function  $\Omega_s$  continuous on  $[-1, 1] \setminus \{0\}$  with (at most) a singularity at  $x = 0$  of type  $x^{\frac{s-1}{2}}$  ( $s \neq 1$ ) or a logarithmic singularity ( $s = 1$ ). We provide an explicit expression for  $\Omega_s$  and the error term.

Finally, we study thoroughly the convergence properties of certain series defined in term of the convergents of the even continued fraction of an irrational number.

## 1. INTRODUCTION

The famous lacunary Fourier series

$$\sum_{k=1}^{\infty} \frac{\sin(\pi k^2 x)}{k^2} \tag{1.1}$$

was proposed by Riemann in the 1850's as an example of continuous but nowhere differentiable function. Since then, this series has drawn much attention from many mathematicians (amongst them, Hardy and Littlewood), and its complete local study was finally achieved by Gerver in [7] and Jaffard in [10]. In particular, its local regularity at a point  $x$  depends on the Diophantine type of  $x$ , and it is differentiable only at rationals  $p/q$  where  $p$  and  $q$  are both odd.

In this article, we study the series defined for  $(x, t) \in \mathbb{R}^2$  and  $s \in \mathbb{R}^+$  by

$$F_s(x, t) = \sum_{k=1}^{\infty} \frac{e^{i\pi k^2 x + 2i\pi kt}}{k^s}$$

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and we denote by  $F_{s,n}(x,t) = \sum_{k=1}^n \frac{e^{i\pi k^2 x + 2i\pi kt}}{k^s}$  its  $n$ -th partial sum. Both are periodic functions of period 2 in  $x$  and 1 in  $t$ . For  $s = 2$  and  $t = 0$  the imaginary part of is (1.1). For any fixed  $t$ , if  $s > 1/2$ ,  $F_s$  is in  $L^2(-1,1)$  and it converges almost everywhere by Carleson's theorem. It is not everywhere convergent however. One of the aim of this paper is to understand better the convergence of  $F_s(x,t)$  especially when  $t = 0$ .

We set  $\rho = \exp(i\pi/4)$ ,  $\sigma(x) = 1$ , resp.  $-1$  if  $x > 0$ , resp.  $x < 0$ , and  $\sigma(0) = 0$ . We define  $\log(z) = \ln|z| + i \arg(z)$  with  $-\pi < \arg(z) \leq \pi$ . We denote by  $[x]$  and  $\{x\}$  the integer part and fractional part respectively of a real number  $x$ . For  $x > 0$ ,  $t \in \mathbb{R}$  and  $s \geq 0$ , we set

$$I_s(x,t) = \int_{1/2-\rho\infty}^{1/2+\rho\infty} \frac{e^{i\pi z^2 x + 2i\pi z\{t\}}}{z^s(1 - e^{2i\pi z})} dz + \rho x^s \int_{-\infty}^{\infty} e^{-\pi x u^2} \left( \sum_{k=1}^{\infty} e^{-i\pi(k-\{t\})^2/x} \left( \frac{1}{(\rho x u + k - \{t\})^s} - \frac{1}{k^s} \right) \right) du. \quad (1.2)$$

This function is well-defined and if  $s = 0$ , the second integral is equal to 0 (because the series vanishes). We then define a function  $\Omega_s(x,t)$  as follows:

$$\Omega_s(x,t) = \begin{cases} I_s(x,t) & \text{when } x > 0, \\ \overline{I_s(-x,-t)} & \text{when } x < 0. \end{cases}$$

For simplicity, given a function  $f(x,t)$ , we will write  $f(x)$  for  $f(x,0)$ . The function  $\Omega_s(x,t)$  will be particularly important in this paper.

**1.1. Statement of our main results.** Our first result is a consequence of the celebrated ‘‘approximate functional equation for the theta series’’ of Hardy and Littlewood (Proposition 3 in Section 3), which corresponds exactly to the case  $s = 0$  in our Theorem 2 below; see [1] where many references and historical notes are given.

**Theorem 1.** *Let  $x$  be an irrational number in  $(0,1)$  whose (regular) continued fraction is denoted by  $(P_k/Q_k)_{k \geq 0}$ , and let  $t \in \mathbb{R}$ .*

(i) *If  $s \in (\frac{1}{2}, 1)$  and*

$$\sum_{k=0}^{\infty} \frac{Q_{k+1}^{\frac{1-s}{2}}}{Q_k^{s/2}} < \infty, \quad (1.3)$$

*then  $F_s(x,t)$  is absolutely convergent.*

(ii) *If  $s = 1$  and*

$$\sum_{k=0}^{\infty} \frac{\log(Q_{k+1})}{\sqrt{Q_k}} < \infty, \quad (1.4)$$

*then  $F_1(x,t)$  is absolutely convergent.*

Conditions (1.3) and (1.4) hold for Lebesgue-almost all  $x$ . It is possible to prove a quantitative version of Theorem 1. We need to introduce more notations. We introduce the two transformations

$$G(x) = \left\{ \frac{1}{x} \right\}, \quad \tilde{G}(x, t) = \left\{ \frac{1}{2} \left[ \frac{1}{x} \right] - \frac{t}{x} \right\}.$$

Then, for all  $x$  satisfying at least the conditions (1.3) or (1.4), it can be proved <sup>(1)</sup> that

$$F_s(x, t) = \sum_{j=0}^{\infty} e^{i\frac{\pi}{8}(1+(-1)^{j-1})+i\pi \sum_{1 \leq \ell \leq j} (-1)^\ell \widehat{G}_\ell(x, t)} (xG(x) \cdots G^{j-1}(x))^{s-\frac{1}{2}} \Omega_{j,s}(G^j(x), \tilde{G}_j(x, t)), \quad (1.5)$$

where  $\Omega_{j,s}(x, t) = \Omega_s(x, t)$  if  $j$  is even,  $\Omega_{j,s}(x, t) = \overline{\Omega_s(x, t)}$  if  $j$  is odd, and

$$\begin{aligned} \tilde{G}_0(x, t) &= x, & \widehat{G}_0(x, t) &= t, & \tilde{G}_1(x, t) &= \tilde{G}(x, t), & \widehat{G}_1(x, t) &= \frac{t^2}{x} \\ \tilde{G}_{j+1}(x, t) &= \tilde{G}_1(T^j(x), \tilde{G}_j(x, t)), & \widehat{G}_{j+1}(x, t) &= \widehat{G}_1(T^j(x), \tilde{G}_j(x, t)) \quad (j \geq 0). \end{aligned}$$

Eq. (1.5) holds very generally, the right-hand side converges quickly and the appearance of Gauss' transform  $G$  is a nice feature. But this is at the cost of the simultaneous appearance of the operator  $\tilde{G}$  and this makes (1.5) looks very complicated, even when  $t = 0$  because  $\tilde{G}_j(x, 0) \neq 0$  in general.

However, the underlying modular nature of  $F_s(x, t)$  implies that the transformation of  $[-1, 1] \setminus \{0\}$  given by

$$T(x) = -\frac{1}{x} \pmod{2}$$

is more natural than Gauss' in this specific study, and in particular it leads to another expression (i.e, (1.26) below) for  $F_s(x, t)$  which is formally similar to (1.5) but simpler. The comparison of both approaches is one of our motivations.

Our next theorem below explains what we mean by “the modular nature of  $F_s(x, t)$ ” and the subsequent theorems are devoted to convergence conditions of  $F_s(x, t)$  (mainly when  $t = 0$ ) in terms of series defined by the operator  $T$ , as well as their relations with Theorem 1.

**Theorem 2.** (i) For any  $x \in [-1, 1] \setminus \{0\}$ ,  $t \in \mathbb{R}$ ,  $s \geq 0$ , we have the estimate

$$\begin{aligned} F_{s,n}(x, t) - e^{\sigma(x)i\frac{\pi}{4}} e^{-i\pi \frac{\{\sigma(x)t\}^2}{x}} |x|^{s-\frac{1}{2}} F_{s, [n|x|]} \left( -\frac{1}{x}, \frac{\{\sigma(x)t\}}{x} \right) \\ = \Omega_s(x, \sigma(x)t) + \mathcal{O} \left( \frac{1}{n^s \sqrt{x}} \right). \end{aligned} \quad (1.6)$$

when  $n$  tends to infinity. The implicit constant depends on  $s$  and  $t$ , but not on  $x$ .

<sup>1</sup>The details will not be given here because the process is similar to that leading to Theorem 3 and this would add nearly ten more pages to the paper.

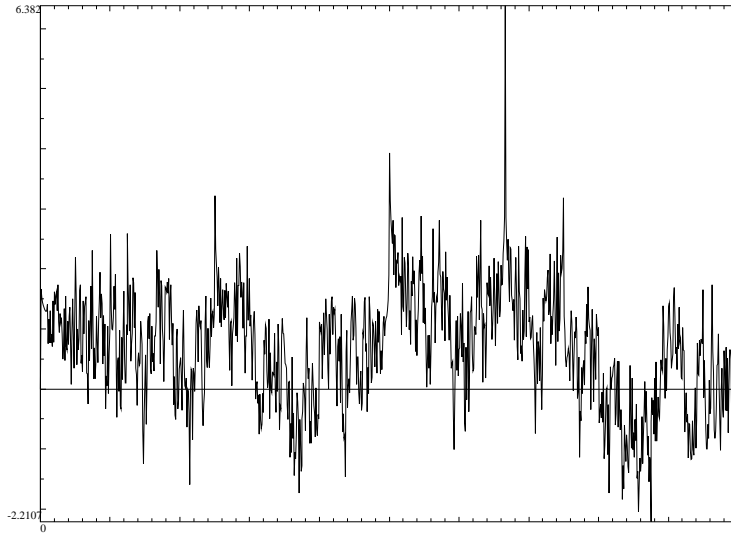


FIGURE 1. Plot of  $\text{Im}(F_{0.7,100}(x))$  on  $[0, 2]$

(ii) When  $0 \leq s \leq 1$ , the function  $\Omega_s(x)$  is continuous on  $\mathbb{R} \setminus \{0\}$ , differentiable at any rational number  $p/q$  with  $p, q$  both odd, and

$$\Omega_s(x) - \frac{\rho^{1-s} \Gamma(\frac{1-s}{2})}{2\pi^{\frac{1-s}{2}}} |x|^{\frac{s-1}{2}} \quad (0 \leq s < 1) \quad \text{and} \quad \Omega_1(x) - \log(1/\sqrt{|x|})$$

are bounded on  $\mathbb{R}$ .

(iii) When  $s > 1$ , the function  $\Omega_s(x)$  is differentiable on  $\mathbb{R} \setminus \{0\}$  and continuous at 0.

*Remark.* The function  $\Omega_0(x)$  is the same as the one used by Cellarosi [3] and Fedotov-Klopp [6]. When  $s > 1$ , the proof of item (ii) yields only that  $\Omega_s$  is bounded around 0.

See Figures 1 and 2 for an illustration of Theorem 2.

As  $n \rightarrow +\infty$ , the left hand side of (1.6) tends to  $\Omega_s(x, t)$  when  $x > 0$ . The resulting “modular” equation

$$F_s(x, t) - e^{i\frac{\pi}{4}} e^{-i\pi\frac{t^2}{x}} x^{s-\frac{1}{2}} F_s\left(-\frac{1}{x}, \frac{t}{x}\right) = \Omega_s(x, t) \quad (1.7)$$

holds a priori at least *almost everywhere* for  $x \in (0, 1)$  for any fixed  $s \geq 0$  and  $t \in [0, 1)$ , and Theorem 2 shows in which sense we can say it holds *everywhere*. For other examples of this phenomenon, see [2, 14] for instance. Of course, if  $s > 1$ , (1.7) holds for all  $x$ .

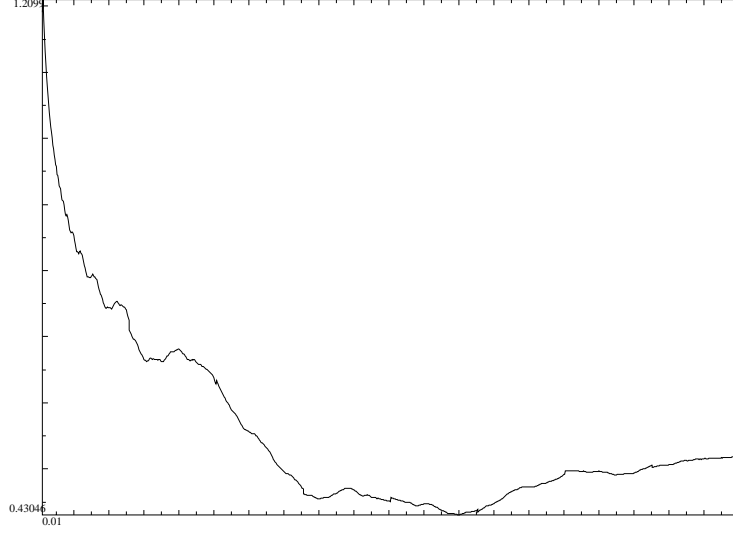


FIGURE 2. Plot of  $\text{Im}(F_{0.7,1000}(x) - e^{i\pi/4}x^{0.2}F_{0.7,[1000x]}(-1/x))$  on  $[0, 2]$

In fact, we will obtain a more precise estimate for the error term in (1.6), uniform in  $x \in [-1, 1] \setminus \{0\}$ ,  $n \geq 0$  and  $t \in \mathbb{R}$ :

$$\begin{aligned} \mathcal{O}\left(\frac{1}{n^s\sqrt{x}}\right) &= \mathcal{O}\left(\frac{|x|^{s-\frac{1}{2}}}{([\lfloor n|x| + \sigma(x)t \rfloor + 1 - \sigma(x)t]^s)}\right) \\ &\quad + \begin{cases} \mathcal{O}\left(\min\left(\frac{1}{(n+1)^s\sqrt{|x|}}, \frac{1}{|x|^{\frac{1-s}{2}}}\right)\right) & (s \neq 1) \\ \mathcal{O}\left(\min\left(\frac{1}{(n+1)\sqrt{|x|}}, 1 + |\log((n+1)\sqrt{|x|})|\right)\right) & (s = 1), \end{cases} \end{aligned} \quad (1.8)$$

where the constants in the  $\mathcal{O}$  on the right-hand side depend now at most on  $s$  and are effective. We need this refinement to prove Theorem 3 below.

Theorem 2 will be used to get informations of the convergence of  $F_s(x)$  in terms of the diophantine properties of  $x$ . In the sequel  $T^m(x)$  denotes the  $m$ -th iterate of  $x$  by  $T$ . By 2-periodicity of  $T$ , Eq. (1.6) can be rewritten as follows (when  $t = 0$ ): for any  $x \in [-1, 1] \setminus \{0\}$ ,  $s > \frac{1}{2}$  and integer  $n \geq 0$ ,

$$\begin{aligned} F_{s,n}(x) &= e^{\sigma(x)i\frac{\pi}{4}}|x|^{s-\frac{1}{2}}F_{s,[n|x|]}(T(x)) + \Omega_s(x) \\ &\quad + \begin{cases} \mathcal{O}\left(\min\left(\frac{1}{(n+1)^s\sqrt{|x|}}, \frac{1}{|x|^{\frac{1-s}{2}}}\right)\right) & (s \neq 1) \\ \mathcal{O}\left(\min\left(\frac{1}{(n+1)\sqrt{|x|}}, 1 + |\log((n+1)\sqrt{|x|})|\right)\right) & (s = 1). \end{cases} \end{aligned} \quad (1.9)$$

(When  $t = 0$ , the error term  $|x|^{s-\frac{1}{2}} \cdot ([\lfloor n|x| + \sigma(x)t \rfloor + 1 - \sigma(x)t]^{-s})$  in (1.8) is absorbed by the error term in (1.9), for some constant that depends only on  $s$ ; this is enough for the

application we have in mind.) The second sum  $F_{s, \lfloor n|x| \rfloor}(T(x))$  in (1.9) involves less terms than the first one for any irrational number in  $(-1, 1)$ , because  $\lfloor n|x| \rfloor < n$ . Hence for any fixed  $n$  and  $x$ , we can iterate (1.9) because  $T(x) \in (-1, 1)$ . After a finite number of steps (say  $\ell$ , which depends on  $x$ ), we get an empty sum  $F_{s,0}(T^\ell(x)) = 0$  on the right hand side together with a finite sum defined in terms of iterates of  $\Omega_s(x)$  and a quantity we expect to be an error term (i.e., that tends to 0 as  $n$  tends to infinity under suitable condition on  $x$ ).

We prove the following result.

**Theorem 3.** *Let  $x \in (-1, 1)$  be an irrational number.*

(i) *If  $s \in (\frac{1}{2}, 1)$  and if*

$$\sum_{j=0}^{\infty} \frac{|xT(x) \cdots T^{j-1}(x)|^{s-\frac{1}{2}}}{|T^j(x)|^{\frac{1-s}{2}}} < \infty, \quad (1.10)$$

*then  $F_s(x)$  is also convergent and the following identity holds:*

$$F_s(x) = \sum_{j=0}^{\infty} e^{i\frac{\pi}{4} \sum_{\ell=0}^{j-1} \sigma(T^\ell x)} |xT(x) \cdots T^{j-1}(x)|^{s-\frac{1}{2}} \Omega_s(T^j(x)). \quad (1.11)$$

(ii) *If*

$$\sum_{j=0}^{\infty} \sqrt{|xT(x) \cdots T^{j-1}(x)|} < \infty \quad (1.12)$$

$$\text{and} \quad \sum_{j=0}^{\infty} \sqrt{|xT(x) \cdots T^{j-1}(x)|} \log \left( \frac{1}{|T^j x|} \right) < \infty, \quad (1.13)$$

*then  $F_1(x)$  is also convergent and the following identity holds:*

$$F_1(x) = \sum_{j=0}^{\infty} e^{i\frac{\pi}{4} \sum_{\ell=0}^{j-1} \sigma(T^\ell x)} \sqrt{|xT(x) \cdots T^{j-1}(x)|} \Omega_1(T^j(x)). \quad (1.14)$$

The series in (1.10), (1.12) and (1.13) do not converge everywhere. It was an open question whether such series converge Lebesgue-almost everywhere. Note that the results in the cited papers of Cellarosi [3], Kraaikamp-Lopes [11], Schweiger [15, 16], and Sinai [17] give estimates for the average behavior of  $|xT(x) \cdots T^{j-1}(x)|$  when  $j$  tends to infinity, but these estimates are not sharp enough <sup>(2)</sup> to guarantee the almost everywhere convergence. It would be interesting to know if  $F_1(x)$  converges under the assumption of convergence of (1.13) only.

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<sup>2</sup>It is known that  $T$  is ergodic with respect to a measure  $\nu$  supported on  $[-1, 1]$  but, in contrast with the ergodic theory of Gauss' transformation  $G$ , the measure  $\nu$  is infinite. As a consequence, the analogue of Birkoff's ergodic theorem is not known and one must content with "convergence in probability" results (see [3]), which are not well suited to our study.

1.2. **More results around Theorem 3.** We now precise the diophantine content of Theorem 3.

**Theorem 4.** Let  $\alpha > 0$ ,  $\beta \geq 0$ , and set  $\beta_\alpha = \frac{\sqrt{\alpha^2 + 4} - 1}{2}$ .

(i) If  $0 \leq \beta < \beta_\alpha$ , then the series

$$\sum_{j=0}^{\infty} \frac{|xT(x) \cdots T^{j-1}(x)|^\alpha}{|T^j(x)|^\beta} \quad (1.15)$$

converges if

$$\sum_{n=1}^{\infty} \frac{Q_{n+1}^{\beta+1}}{Q_n^{\alpha+\beta+1}} < \infty. \quad (1.16)$$

(ii) If  $\beta > \beta_\alpha$ , then the series (1.15) converges if

$$\sum_{n=1}^{\infty} \frac{Q_{n+2}^\beta}{Q_n^{\alpha+\beta}} < \infty \quad (1.17)$$

(iii) If  $\beta = \beta_\alpha$ , then (1.16) and (1.17) impose the convergence of (1.15).

(iv) If  $\alpha \geq 0$  and

$$\sum_{n=1}^{\infty} \left( \frac{\log(Q_{n+1})}{Q_{n-1}^\alpha} + \frac{Q_{n+1} \log(Q_{n+1})^2}{Q_n^{1+\alpha}} \right) < \infty, \quad (1.18)$$

then the series

$$\sum_{j=0}^{\infty} |xT(x) \cdots T^{j-1}(x)|^\alpha \log \left( \frac{1}{T^j(x)} \right)$$

converges.

The condition (1.16) can be simplified according to the values of  $\alpha$  and  $\beta$ , in terms of the irrationality exponent  $\mu(x)$  of an irrational  $x \in \mathbb{R}$ , defined as

$$\mu(x) = \sup \left\{ \mu \geq 1 : \left| x - \frac{p}{q} \right| < \frac{1}{q^\mu} \text{ for infinitely many integers } q \geq 1 \right\}.$$

It is well-known that  $\mu(x) = 2$  for almost every real numbers  $x$ . When  $s = 1$ , choosing  $\alpha = \frac{1}{2}$ ,  $\beta = 0$ , one sees that (1.18) implies (1.16). When  $s \in (\frac{1}{2}, 1)$ , putting  $\alpha = s - \frac{1}{2}$  and  $\beta = \frac{1-s}{2}$ , a simple computation shows that  $\beta < \beta_\alpha$  in Theorem 4.

**Corollary 1.** (i) If  $1/2 < s < 1$  and

$$\sum_{n=1}^{\infty} \frac{Q_{n+1}^{\frac{3-s}{2}}}{Q_n^{1+\frac{s}{2}}} \quad (1.19)$$

is convergent, then the identity (1.11) holds true. The series (1.19) converges for every  $x$  such that  $\mu(x) < 1 + \frac{2+s}{3-s}$ .



(ii) If

$$\sum_{n=1}^{\infty} \left( \frac{\log(Q_{n+1})}{\sqrt{Q_{n-1}}} + \frac{Q_{n+1} \log(Q_{n+1})^2}{Q_n^{3/2}} \right) \quad (1.20)$$

converges, then the equality (1.14) holds true. The series (1.20) converges for every  $x$  such that  $\mu(x) < \frac{5}{2}$ .

Corollary 1 is not entirely satisfying, since the series  $F_s(x)$  converges (even absolutely) when a weaker condition on the standard convergents of  $x$  holds (Theorem 1, conditions (1.3) and (1.4)). The main reason for this discrepancy is the factor  $\exp(i\frac{\pi}{4} \sum_{0 \leq \ell < j} \sigma(T^\ell x))$ : it is present in (1.11) but not in (1.10), respectively in (1.14) but not in (1.12)-(1.13). Our next result shows that the role of this factor is very important, even though its modulus is 1. We explain after the theorem why we are not able to keep track of it in our proof of Theorem 3.

**Theorem 5.** (i) Let  $\Omega$  be a bounded function, differentiable at  $x = 1$  and  $x = -1$  (in particular, if  $\Omega \equiv 1$ ). Then for any  $\alpha > 0$  and any irrational number  $x \in (0, 1)$ , the series

$$\sum_{j=1}^{\infty} e^{i\frac{\pi}{4} \sum_{\ell=0}^{j-1} \sigma(T^\ell x)} |xT(x) \cdots T^{j-1}(x)|^\alpha \Omega(T^j(x)) \quad (1.21)$$

converges.

(ii) For any  $\alpha > 0$ , any  $\beta \in \mathbb{R}$  and any irrational number  $x \in (0, 1)$ , the series

$$\sum_{j=0}^{\infty} e^{i\frac{\pi}{4} \sum_{\ell=0}^{j-1} \sigma(T^\ell x)} \frac{|xT(x) \cdots T^{j-1}(x)|^\alpha}{|T^j(x)|^\beta} \quad (1.22)$$

converges if

$$\sum_{n=1}^{\infty} \frac{Q_{n+1}^\beta}{Q_n^{\alpha+\beta}} < \infty. \quad (1.23)$$

(iii) For any  $\alpha > 0$  and any irrational number  $x \in (0, 1)$ , the series

$$\sum_{j=0}^{\infty} e^{i\frac{\pi}{4} \sum_{\ell=0}^{j-1} \sigma(T^\ell x)} |xT(x) \cdots T^{j-1}(x)|^\alpha \log \left( \frac{1}{|T^j(x)|} \right) \quad (1.24)$$

converges if

$$\sum_{n=1}^{\infty} \frac{\log(Q_{n+1})}{Q_n^\alpha} < \infty. \quad (1.25)$$

These convergence properties are essentially optimal. They are very different from the absolute convergence properties (which are also essentially optimal), as stated in Theorem 3.

In fact, in our proof of Theorem 3, only one technical detail impedes us to prove that formulas (1.11) and (1.14) hold true when conditions (1.3) and (1.4) are satisfied (and not only when the more constraining conditions in (i) and (ii) of Corollary 1 hold). Indeed, we

show that the convergence of the series (1.11) and (1.14) is equivalent to the convergence of three auxiliary simpler series (see equations (6.3) and (6.8)). For two of these series, their convergence follows from Theorem 5, and the convergence conditions are optimal (i.e., when conditions (1.3) and (1.4) are satisfied). For the third series, which contains heuristically a sort of "error" term, we do not have an estimate precise enough to apply Theorem 5, and we can only use Theorem 4 and the conditions ensuring absolute convergence of the series (1.24) and (1.22), which are stronger. Nevertheless, we do believe that conditions (1.3) and (1.4) imply the identities (1.11) and (1.14) respectively.

**1.3. Some further remarks.** For  $t$  not necessarily an integer, the iteration of (1.6) leads to an identity, for which we have to introduce the following sequences of operators:

$$\begin{aligned} \tilde{T}_0(x, t) = x, \quad \widehat{T}_0(x, t) = t, \quad \tilde{T}_1(x, t) = \left\{ \frac{\{\sigma(x)t\}}{x} \right\}, \quad \widehat{T}_1(x, t) = \frac{\{\sigma(x)t\}^2}{x} \\ \tilde{T}_{j+1}(x, t) = \tilde{T}_1(T^j(x), \tilde{T}_j(x, t)), \quad \widehat{T}_{j+1}(x, t) = \widehat{T}_1(T^j(x), \widehat{T}_j(x, t)) \quad (j \geq 0). \end{aligned}$$

Then, for any fixed  $t \in [0, 1]$  and  $s > \frac{1}{2}$ , the following identity <sup>(3)</sup> holds for almost every  $x$ :

$$F_s(x, t) = \sum_{j=0}^{\infty} e^{i\pi \sum_{\ell=0}^{j-1} (\frac{1}{4}\sigma(T^\ell x) - \widehat{T}_{\ell+1}(x, t))} |xT(x) \cdots T^{j-1}(x)|^{s-\frac{1}{2}} \Omega_s(T^j(x), \tilde{T}_j(x, t)). \quad (1.26)$$

This new representation of  $F_s(x, t)$  is similar to Identity (1.5) displayed after Theorem 1. For  $x$  unspecified, Eq. (1.26) is simpler than (1.5) when  $t = 0$ , because  $\tilde{T}_j(x, 0) = \widehat{T}_j(x, 0) = 0$  for all  $j$  while neither  $\tilde{G}_j(x, 0)$  nor  $\widehat{T}_j(x, 0)$  necessarily vanish. Finally, it is easy to see that when  $x$  has *even* partial quotients and  $t = 0$ , then the summands of (1.26) and (1.5) are equal. The simplicity of (1.26) (relatively to (1.5)) when  $t = 0$  was our motivation to make the detailed study of various series defined in term of the operator  $T$ , for which apparently nothing was done in the direction of our results.

Finally, when  $s > 1$ , the series  $F_s(x, t)$  obviously converges absolutely for any real numbers  $x$  and  $t$ . It turns out that Identities (1.5) and (1.26) hold for any  $t$  and any irrational number  $x$ , and with minor modification for any rational number  $x$  as well. On the one hand, this is not difficult to prove for (1.5), whose right hand side converges very quickly. On the other hand, the convergence of the right-hand side of (1.26) for all irrational  $x$  is a consequence of Theorem 5(i) applied with  $\alpha = s - \frac{1}{2}$  because for  $s > 1$ ,  $\Omega_s(x)$  is bounded on  $[-1, 1]$  and differentiable at  $x = \pm 1$  by Theorem 2(iii).

We note that  $T$  is closely related to the Theta group, a subgroup of  $SL_2(\mathbb{Z})$  of the matrices  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with  $a \equiv d \pmod{2}$ ,  $b \equiv c \pmod{2}$ ; see [11]. This relation has been used in the papers [5, 9] to study the (non)-derivability of Riemann series  $\text{Im}(F_2(x))$ , culminating with Jaffard's determination of its spectrum of singularities [10]. It would be very interesting

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<sup>3</sup>This could be precisely described as in Theorem 3 but, again, we skip the details to shorten the length of the paper.

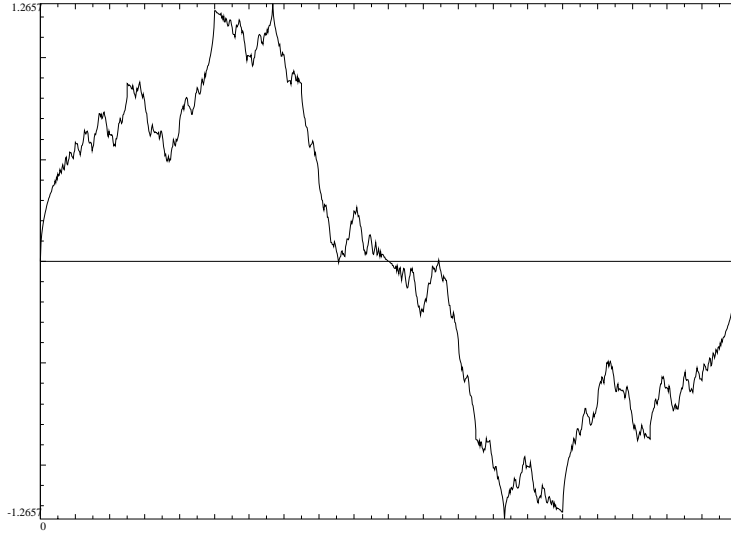


FIGURE 3. The “non-differentiable” Riemann function  $\text{Im}(F_2(x))$  on  $[0, 2]$

to know if Jaffard’s results can be recovered by a direct study of (1.26) in the case  $s = 2$  and  $t = 0$ , which reads

$$F_2(x) = \sum_{k=1}^{\infty} \frac{e^{i\pi k^2 x}}{k^2} = \sum_{j=0}^{\infty} e^{i\frac{\pi}{4} \sum_{\ell=0}^{j-1} \sigma(T^\ell x)} |xT(x) \cdots T^{j-1}(x)|^{\frac{3}{2}} \Omega_2(T^j(x)),$$

where  $\Omega_2(x)$  is differentiable on  $[-1, 1] \setminus \{0\}$ , and continuous at 0.

The paper is organized as follows. We start by recalling some facts on regular and even continued fractions in Section 2. Then, in Section 3, we prove Theorem 1. Section 4, which is rather long, contains the proof of the approximate functional equation for  $F_{s,n}$  (part (i) of Theorem 2). The second part (ii) of Theorem 2, i.e. the regularity properties of the functions  $\Omega_s$ , is dealt with in Section 5. Theorem 3 and the Diophantine identities (1.11) and (1.14) are proven in Section 6. Finally, the standard and absolute convergence properties of the series (1.24) and (1.22) (Theorems 5 and 4) are studied in Sections 7 and 8.

## 2. BASIC PROPERTIES OF REGULAR AND EVEN CONTINUED FRACTIONS

The regular theory of continued fractions is well-known, and is related to Gauss dynamical system  $G : x \in (0, 1] \mapsto 1/x \pmod{1}$ . In the following, we use capital letters when we refer to the regular convergent  $P_n/Q_n$  of an irrational number  $x \in \mathbb{R}$ . We set

$$\frac{P_n}{Q_n} := [x] + \frac{1}{A_1 + \frac{1}{A_2 + \frac{1}{\ddots + \frac{1}{A_n}}}} \quad \text{with} \quad A_n = \left\lfloor \frac{1}{G^n(x)} \right\rfloor, \quad (2.1)$$

where  $G^n$  is the  $n$ -th iterate of  $G$ . We write the RCF (Regular Continued Fraction) of  $x$  as

$$x = [x] + [A_1, A_2, \dots]_R.$$

Recall that one has the recurrence relations

$$P_{n+1} = A_{n+1}P_n + P_{n-1} \quad \text{and} \quad Q_{n+1} = A_{n+1}Q_n + Q_{n-1}.$$

It is also classical that in this case, for every irrational  $x \in [0, 1]$ ,

$$|xG(x)G^2(x) \cdots G^n(x)| \leq |Q_n x - P_n| \leq (Q_{n+1})^{-1}. \quad (2.2)$$

This guarantees the convergence for every  $\alpha > 0$  and  $x \in \mathbb{R}$  of the series

$$\sum_{n \geq 1} |xG(x)G^2(x) \cdots G^n(x)|^\alpha. \quad (2.3)$$

In Theorems 3 and 4, the natural underlying dynamical system is the one generated by the map  $T : [-1, 1] \setminus \{0\} \mapsto [-1, 1]$  defined by  $T(x) = -1/x \pmod{2}$ . As is classical with the Gauss map  $G$ , using this transformation  $T$  one can associate with each irrational real number  $x \in [-1, 1] \setminus \{0\}$  a kind of continued fraction: for every  $j \geq 1$ , denote by  $a_j$  the unique even number such that  $T^j(x) - a_j \in (-1, 1)$ , and set  $e_j = \sigma(T^j(x))$ . Then one has the unique decomposition called the even continued fraction (ECF) expansion (see [11] for instance)

$$x = \frac{e_1}{a_1 + \frac{e_2}{a_2 + \frac{e_3}{a_3 + \dots}}}. \quad (2.4)$$

The difference with the RCF expansion (2.1) is twofold: first, only even integers  $(a_j)_{j \geq 1}$  are allowed in the decomposition, and second the integers  $e_1, e_2$ , etc, may take the values 1 and  $-1$  (not only 1). For compactness, we write for  $x \in (-1, 1)$

$$x = [(e_1, a_1), (e_2, a_2), \dots]_E. \quad (2.5)$$

Now, given the ECF (2.4), we define the  $n$ -th convergent and the  $n$ -th remainder respectively as

$$\frac{p_n}{q_n} := \frac{1}{a_1 + \frac{e_1}{a_2 + \frac{e_2}{\ddots + \frac{e_{n-1}}{a_n}}}} \quad \text{and} \quad x_n := \frac{e_n}{a_{n+1} + \frac{e_{n+1}}{a_{n+2} + \frac{e_{n+2}}{\ddots}}}.$$

We use small letters for ECF, and capital letters for RCF.

ECF expansions are obtained from the RCF expansions via the following iterative method. Observe that for any positive integers  $(A_n, A_{n+1}, A_{n+2})$  and any positive real

number  $\gamma$ , one has

$$A_n + \frac{1}{A_{n+1} + \frac{1}{A_{n+2} + \gamma}} = (A_n + 1) + \frac{-1}{2 + \frac{-1}{2 + \dots + \frac{-1}{2 + \frac{-1}{(A_{n+2} + 1) + \gamma}}}}, \quad (2.6)$$

where the term  $\frac{-1}{2 + \dots}$  appears exactly  $A_{n+1} - 1$  times.

The procedure is then as follows: Write the RCF expansion of a real number  $x = [A_1, A_2, \dots]_R$  as  $x = [(1, A_1), (1, A_2), \dots]_E$ , which looks like an ECF expansion, except that all  $e_n$  are 1 and some integers  $A_n$  may be odd.

If all  $A_n$  are even, then this expansion is indeed the ECF of  $x$ .

Otherwise, consider the smallest index  $n$  such that  $A_n$  is odd, and apply (2.6) to transform

$$x = [(1, A_1), \dots, (1, A_n), (1, A_{n+1}), (1, A_{n+2}), (1, A_{n+3}), \dots]_E$$

into

$$[(1, A_1), \dots, (1, A_n + 1), (-1, 2), \dots, (-1, 2), (-1, A_{n+2} + 1), (1, A_{n+3}), \dots]_E.$$

The odd number  $A_n$  has been removed, and one iterates the procedure with this new ECF-like expansion, whose coefficients before  $A_{n+2} + 1$  are even. By uniqueness of the ECF expansion for irrational numbers, the expansion obtained as a limit is indeed the ECF expansion of  $x$ .

From the above construction, one also derives some useful properties between the even and the regular convergents. Next Proposition is contained in [11].

**Proposition 1.** *Let  $x$  be an irrational number in  $(0, 1)$ .*

(i) *For any  $n \geq 1$ , if the regular convergent  $P_n/Q_n$  is not an even convergent  $p_j/q_j$ , then necessarily  $P_{n+1}/Q_{n+1}$  is an even convergent.*

(ii) *If the regular convergent  $P_n/Q_n$  is equal to the even convergent  $p_j/q_j$  for some  $j \geq 1$ , and if  $P_{n+1}/Q_{n+1}$  is also an even convergent, then  $p_{j+1}/q_{j+1} = P_{n+1}/Q_{n+1}$ .*

(iii) *If the regular convergent  $P_n/Q_n$  is equal to the even convergent  $p_j/q_j$  for some  $j \geq 1$ , and if  $P_{n+1}/Q_{n+1}$  is not an even convergent, then for every  $m \in \{1, \dots, A_{n+2}\}$ ,*

$$\frac{p_{j+m}}{q_{j+m}} = \frac{mP_{n+1} + P_n}{mQ_{n+1} + Q_n}.$$

*In particular,  $\frac{p_{j+A_{n+2}}}{q_{j+A_{n+2}}} = \frac{P_{n+2}}{Q_{n+2}}$ .*

Hence, amongst two consecutive regular convergents, there is at least one even convergent, and an even convergent  $p_j/q_j$  is either a principal or a median convergent of the regular continued fraction. This will be useful to prove Theorem 4.

The next proposition gathers some useful informations about even continued fraction (references include [11, p. 307], [17, p. 2027, eq. (1.13)], and [15, 16]).

**Proposition 2.** *For every irrational  $x \in [0, 1]$  and every  $j \geq 1$ , we have*

$$q_{j+1} > q_j, \quad \lim_{n \rightarrow +\infty} (q_{n+1} - q_n) = +\infty$$

and

$$\frac{1}{2q_{j+1}} \leq |xT(x) \cdots T^j(x)| = \frac{1}{|q_{j+1} + e_{j+1}x_{j+1}q_j|} \leq \frac{1}{q_{j+1} - q_j}. \quad (2.7)$$

(We will freely use these properties without necessarily quoting Proposition 2.) Unfortunately there is no uniform convergence rate for this sequence that guarantees the convergence of series of the form (1.22) for every  $x$ . This is in sharp contrast with the classical continued fractions and equation (2.3). Nevertheless we have found some optimal condition to guarantee the convergence of the sum  $\sum_{j \geq 0} |xT(x) \cdots T^j(x)|^\alpha$ , see Theorem 4.

### 3. PROOF OF THEOREM 1

In this section, we obtain sufficient conditions of convergence of  $F_s(x, t)$  expressed in term of the usual regular continued fraction of  $x$ . These conditions are simple consequences of the following proposition, due to Hardy and Littlewood [8]. At the end of this section, we present an identity which, in principle, would be a qualitative version of Theorem 1.

**Proposition 3.** *For any irrational number  $x$  in  $(0, 1)$  with regular continued fraction  $(P_k/Q_k)_{k \geq 0}$  and any  $t \in \mathbb{R}$ , we have*

$$\sum_{k=1}^N e^{i\pi k^2 x + 2i\pi kt} = \mathcal{O}\left(\frac{N}{\sqrt{Q_r}} + \sqrt{Q_r}\right) \quad (3.1)$$

for any integers  $N, r \geq 0$ , where the implicit constant is absolute.

Eq. (3.1) is a corollary of the approximate functional equation of Hardy-Littlewood for the theta series, which is exactly the case  $s = 0$  of our Theorem 2. In this case, the error term reduces to  $\mathcal{O}(1/\sqrt{x})$  where the constant is absolute. The most precise version of the functional equation is given in [4]. We do not reproduce the proof of (3.1): it is obtained by iteration of (1.6), see [1].

We now prove Theorem 1. Fix an integer  $N \geq 2$ . By Abel summation,

$$F_{s,N}(x, t) = \sum_{k=1}^{N-1} \left( \frac{1}{k^s} - \frac{1}{(k+1)^s} \right) \sum_{j=1}^k e^{i\pi j^2 x + 2i\pi jt} + \frac{1}{N^s} \sum_{k=1}^N e^{i\pi k^2 x + 2i\pi kt}.$$

We set  $B_m = \lceil \sqrt{Q_m Q_{m-1}} \rceil$  and let  $r$  be the unique integer such that  $B_r \leq N < B_{r+1}$ . We denote by  $|F|_{s,N}(x, t)$  the sum of the modulus of the summands of  $F_{s,N}(x, t)$ . Then, for

some constant  $c$  independent of  $N$ ,

$$\begin{aligned}
|F|_{s,N}(x,t) &\leq c + \sum_{\ell=1}^r \sum_{k=B_\ell}^{B_{\ell+1}-1} \left| \left( \frac{1}{k^s} - \frac{1}{(k+1)^s} \right) \sum_{j=1}^k e^{i\pi j^2 x + 2i\pi jt} \right| + \frac{1}{N^s} \left| \sum_{k=1}^N e^{i\pi k^2 x + 2i\pi kt} \right| \\
&\ll c + \sum_{\ell=1}^r \sum_{k=B_\ell}^{B_{\ell+1}-1} \frac{\frac{k}{\sqrt{Q_\ell}} + \sqrt{Q_\ell}}{k^{s+1}} + \frac{\frac{N}{\sqrt{Q_r}} + \sqrt{Q_r}}{N^s} \\
&\ll c + \sum_{\ell=1}^r \frac{1}{\sqrt{Q_\ell}} \sum_{k=B_\ell}^{B_{\ell+1}-1} \frac{1}{k^s} + \sum_{\ell=1}^r \sqrt{Q_\ell} \sum_{k=Q_\ell}^{Q_{\ell+1}-1} \frac{1}{k^{s+1}} + \frac{Q_{r+1}^{\frac{1-s}{2}}}{Q_r^{s/2}}.
\end{aligned}$$

For any  $s > \frac{1}{2}$ ,

$$\sum_{k=B_\ell}^{B_{\ell+1}-1} \frac{1}{k^{s+1}} \ll \frac{1}{B_\ell^s} \ll \frac{1}{(Q_\ell Q_{\ell-1})^{s/2}}$$

but the behavior of  $\sum_{k=B_\ell}^{B_{\ell+1}-1} \frac{1}{k^s}$  depends on whether  $s = 1$  or  $s < 1$ .

If  $s = 1$ , then

$$\sum_{k=B_\ell}^{B_{\ell+1}-1} \frac{1}{k^s} \ll \log(B_{\ell+1}) \ll \log(Q_{\ell+1})$$

so that

$$|F|_{s,N}(x,t) \ll c + \sum_{\ell=1}^r \frac{\log(Q_{\ell+1})}{\sqrt{Q_\ell}} + \sum_{\ell=1}^r \frac{1}{\sqrt{Q_\ell}} + \frac{1}{\sqrt{Q_r}}$$

and the condition

$$\sum_{\ell=0}^{\infty} \frac{\log(Q_{\ell+1})}{\sqrt{Q_\ell}} < \infty$$

ensures the absolute convergence of  $F_1(x,t)$ .

If  $\frac{1}{2} < s < 1$ , then

$$\sum_{k=B_\ell}^{B_{\ell+1}-1} \frac{1}{k^s} \ll B_{\ell+1}^{1-s} \ll (Q_{\ell+1} Q_\ell)^{\frac{1-s}{2}}.$$

Hence,

$$|F|_{s,N}(x,t) \ll c + \sum_{\ell=1}^r \frac{Q_{\ell+1}^{\frac{1-s}{2}}}{Q_\ell^{s/2}} + \sum_{\ell=1}^r \frac{Q_\ell^{\frac{1-s}{2}}}{Q_{\ell-1}^{s/2}} + \frac{Q_{r+1}^{\frac{1-s}{2}}}{Q_r^{s/2}}$$

and the condition

$$\sum_{\ell=0}^{\infty} \frac{Q_{\ell+1}^{\frac{1-s}{2}}}{Q_\ell^{s/2}} < \infty$$

ensures the absolute convergence of  $F_s(x,t)$ .

*Remark.* Our choice  $B_m = \lceil \sqrt{Q_m Q_{m-1}} \rceil$  is not arbitrary. Indeed, it is such that

$$\sum_{\ell=1}^{\infty} \frac{1}{\sqrt{Q_\ell}} \sum_{k=B_\ell}^{B_{\ell+1}-1} \frac{1}{k^s} \quad \text{and} \quad \sum_{\ell=1}^{\infty} \sqrt{Q_\ell} \sum_{k=B_\ell}^{B_{\ell+1}-1} \frac{1}{k^{s+1}}$$

both converge/diverge simultaneously when  $1/2 < s < 1$ . Its importance is lesser when  $s = 1$  where we could simply take  $B_m = Q_m$ .

In the introduction, we presented Identity (1.26) obtained by iteration of (1.6) with the operator  $T$ . It is also possible to iterate (1.6) with Gauss' operator  $G$ . Let us assume that  $x \in (0, 1)$ ,  $t \in [0, 1]$ ,  $s > \frac{1}{2}$  are such that  $F_s(x, t)$  is convergent. Then, letting  $n \rightarrow +\infty$  in (1.6), we obtain

$$\begin{aligned} F_s(x, t) &= e^{i\frac{\pi}{4}} e^{-i\pi\frac{t^2}{x}} x^{s-\frac{1}{2}} F_s\left(-\frac{1}{x}, \frac{t}{x}\right) + \Omega_s(x, t) \\ &= e^{i\frac{\pi}{4}} e^{-i\pi\frac{t^2}{x}} x^{s-\frac{1}{2}} \overline{F_s\left(G(x), \tilde{G}(x, t)\right)} + \Omega_s(x, t), \end{aligned} \quad (3.2)$$

with

$$\tilde{G}(x, t) = \left\{ \frac{1}{2} \left[ \frac{1}{x} \right] - \frac{t}{x} \right\}.$$

Skipping all the details, it can be proved that the iteration of (the finite version of) (3.2) yields the identity (1.5) stated in the introduction, which holds for all  $x$  satisfying at least the conditions (1.3)-(1.4).

#### 4. PROOF OF THEOREM 2, PART (i)

The proof is rather long and intricate. We define the following functions, which are building blocks of the function  $\Omega_s(x, t)$ :

$$U_s(x, t) = \int_{1/2-\rho\infty}^{1/2+\rho\infty} \frac{e^{i\pi z^2 x} e^{2i\pi z\{t\}}}{z^s (1 - e^{2i\pi z})} dz, \quad x > 0, t \in \mathbb{R}, s \geq 0 \quad (4.1)$$

$$V_s(x, t) = \rho x^{s-1/2} \sum_{k=1}^{\infty} e^{-i\pi(k-\{t\})^2/x} \left( \frac{1}{(k-\{t\})^s} - \frac{1}{k^s} \right), \quad x > 0, t \in \mathbb{R}, s \geq 0 \quad (4.2)$$

$$\widehat{W}_s(x, t, u) = \sum_{k=1}^{\infty} e^{-i\pi(k-\{t\})^2/x} \left( \frac{1}{(\rho x u + k - \{t\})^s} - \frac{1}{(k - \{t\})^s} \right),$$

$x > 0, t \in \mathbb{R}, s \geq 0, u \in \mathbb{R}$

and

$$W_s(x, t) = \rho x^s \int_{-\infty}^{\infty} e^{-\pi x u^2} \widehat{W}_s(x, t, u), \quad x > 0, t \in \mathbb{R}, s \geq 0. \quad (4.3)$$



Due to the identity  $\int_{-\infty}^{\infty} e^{-\pi x^2 u} du = 1/\sqrt{x}$  ( $x > 0$ ), it is easy to see that

$$V_s(x, t) + W_s(x, t) = \rho x^s \int_{-\infty}^{\infty} e^{-\pi x u^2} \sum_{k=1}^{\infty} e^{-i\pi(k-\{t\})^2/x} \left( \frac{1}{(\rho x u + k - \{t\})^s} - \frac{1}{k^s} \right) du. \quad (4.4)$$

#### 4.1. Structure of the proof.

In this section, we present all the details of the proof of the theorem except the proofs of four lemmas, which are postponed to Section 4.2. Throughout, we assume that  $x > 0$  and explain in the end how to get the case  $x < 0$ . The method is borrowed to Mordell [13] in the version presented in [1].

We introduce the parameters  $\xi = t - \lfloor (n - \frac{1}{2})x + t \rfloor$  and  $\lambda = \lfloor (n - \frac{1}{2})x + t \rfloor - \lfloor t \rfloor$ . Note that  $\lambda \geq 0$  and  $\xi + \lambda = \{t\} \in [0, 1)$ .

Let us define the function

$$g_s(z) = \frac{1}{z^s} e^{i\pi z^2 x + 2i\pi z \xi}$$

which is holomorphic as a function of  $z$  in  $\mathbb{C} \setminus (-\infty, 0]$ .

**Lemma 1.** *For all  $n \geq 1$ ,  $s \geq 0$ ,  $x > 0$  and  $t \in \mathbb{R}$ , we have*

$$F_{s, n-1}(x, t) = \int_{-1/2-\rho\infty}^{-1/2+\rho\infty} \frac{g_s(z+1)}{1-e^{2i\pi z}} dz - \int_{-1/2-\rho\infty}^{-1/2+\rho\infty} \frac{g_s(z+n)}{1-e^{2i\pi z}} dz.$$

We focus on the first integral in Lemma 1, i.e.

$$\int_{-1/2-\rho\infty}^{-1/2+\rho\infty} \frac{g_s(z+1)}{1-e^{2i\pi z}} dz$$

We have

$$\frac{g_s(z+1)}{1-e^{2i\pi z}} = \sum_{k=0}^{\lambda-1} g_s(z+1) e^{2i\pi k z} + \frac{e^{2i\pi \lambda z}}{1-e^{2i\pi z}} g_s(z+1), \quad (4.5)$$

where the assigned value of  $\lambda$  is in fact irrelevant. If  $\lambda = 0$ , the sum is empty, equal to 0 and (4.5) is a tautology; to avoid talking about empty sums, we assume from now on that  $\lambda \geq 1$  but the results also hold when  $\lambda = 0$ . Now

$$\begin{aligned} J_k &:= \int_{-1/2-\rho\infty}^{-1/2+\rho\infty} g_s(z+1) e^{2i\pi k z} dz = \int_{1/2-\rho\infty}^{1/2+\rho\infty} g_s(z) e^{2i\pi k z} dz = \int_{1/2-\rho\infty}^{1/2+\rho\infty} \frac{1}{z^s} e^{i\pi z^2 x + 2i\pi(k+\xi)z} dz \\ &= \rho e^{-i\pi(\xi+k)^2/x} \int_{-\infty}^{\infty} \frac{e^{i\pi x(\rho u + \frac{1}{2} + \frac{\xi+k}{x})^2}}{(\rho u + \frac{1}{2})^s} du = \rho^{1-s} e^{-i\pi(\xi+k)^2/x} \int_{-\infty}^{\infty} \frac{e^{-\pi x(u + \frac{\bar{\rho}}{2} + \bar{\rho} \frac{\xi+k}{x})^2}}{(u + \frac{\bar{\rho}}{2})^s} du. \end{aligned}$$

We set  $v = \frac{\bar{\rho}}{2} + \bar{\rho}\frac{\xi+k}{x}$ . We have that  $\xi + k < 0$  for each integer  $k \in \{0, \dots, \lambda - 1\}$ , so that

$$\begin{aligned} J_k &= \rho^{1-s} e^{-i\pi(\xi+k)^2/x} \int_{v-\infty}^{v+\infty} \frac{e^{-\pi x u^2}}{(u - \bar{\rho}\frac{\xi+k}{x})^s} du \\ &= \rho^{1-s} e^{-i\pi(\xi+k)^2/x} \int_{-\infty}^{\infty} \frac{e^{-\pi x u^2}}{(u - \bar{\rho}\frac{\xi+k}{x})^s} du \end{aligned} \quad (4.6)$$

where the second equality holds by Cauchy theorem because  $\bar{\rho}\frac{\xi+k}{x}$  is never in the closed horizontal strip defined by the lines  $\text{Im}(z) = 0$  and  $\text{Im}(z) = \text{Im}(v)$ . (Indeed, we have simultaneously  $\text{Im}(\bar{\rho}\frac{\xi+k}{x}) > \text{Im}(v)$  and  $\text{Im}(\bar{\rho}\frac{\xi+k}{x}) > 0$ .)

We now rewrite (4.6) as

$$\begin{aligned} J_k &= \rho e^{-i\pi(\xi+k)^2/x} \left( -\frac{x}{\xi+k} \right)^s \int_{-\infty}^{\infty} e^{-\pi x u^2} du \\ &\quad + \rho^{1-s} e^{-i\pi(\xi+k)^2/x} \int_{-\infty}^{\infty} e^{-\pi x u^2} \left( \frac{1}{(u - \bar{\rho}\frac{\xi+k}{x})^s} - \frac{1}{(-\bar{\rho}\frac{\xi+k}{x})^s} \right) du \\ &= \rho e^{-i\pi(\xi+k)^2/x} \frac{x^{s-1/2}}{(-\xi-k)^s} + \tilde{J}_k, \end{aligned}$$

where  $\tilde{J}_k$  is the integral on the second line. We observe here that

$$|\tilde{J}_k| \ll \frac{1}{|\text{Im}(\bar{\rho}\frac{\xi+k}{x})|^{s+1}} \cdot \int_{-\infty}^{\infty} |u| e^{-\pi x u^2} du \ll \frac{x^s}{|\xi+k|^{s+1}}$$

where the implicit constant is absolute.

Integrating (4.5), we thus obtain

$$\begin{aligned} &\int_{-1/2-\rho\infty}^{-1/2+\rho\infty} \frac{g_s(z+1)}{1 - e^{2i\pi z}} dz \\ &= \rho x^{s-1/2} \sum_{k=0}^{\lambda-1} \frac{e^{-i\pi(\xi+k)^2/x}}{(-\xi-k)^s} + \sum_{k=0}^{\lambda-1} \tilde{J}_k + \int_{-1/2-\rho\infty}^{-1/2+\rho\infty} \frac{g_s(z+1)e^{2i\pi\lambda z}}{1 - e^{2i\pi z}} dz. \end{aligned} \quad (4.7)$$

We will treat the three expressions in (4.7) separately.

The third term is simple because our choice of  $\lambda = \lfloor (n - \frac{1}{2})x + t \rfloor - \lfloor t \rfloor$  ensures that

$$\int_{-1/2-\rho\infty}^{-1/2+\rho\infty} \frac{g_s(z+1)e^{2i\pi\lambda z}}{1 - e^{2i\pi z}} dz = \int_{1/2-\rho\infty}^{1/2+\rho\infty} \frac{e^{i\pi z^2 x} e^{2i\pi z \{t\}}}{z^s (1 - e^{2i\pi z})} dz$$

is independent of  $n$  and is equal to the function  $U_s(x, t)$  defined in (4.1).

To study the first term in (4.7), we change the summation index  $k$  to  $\lambda - k$ :

$$\begin{aligned} \rho x^{s-1/2} \sum_{k=0}^{\lambda-1} \frac{e^{-i\pi(\xi+k)^2/x}}{(-\xi-k)^s} &= \rho x^{s-1/2} \sum_{k=1}^{\lambda} \frac{e^{-i\pi(\{t\}-k)^2/x}}{(k-\{t\})^s} \\ &= \rho x^{s-1/2} e^{-i\pi\{t\}^2/x} F_{s,\lambda} \left( -\frac{1}{x}, \frac{\{t\}}{x} \right) \\ &\quad + \rho x^{s-1/2} \sum_{k=1}^{\lambda} e^{-i\pi(k-\{t\})^2/x} \left( \frac{1}{(k-\{t\})^s} - \frac{1}{k^s} \right). \end{aligned}$$

The sum is a partial sum of the series  $V_s(x, t)$  defined in (4.2). If  $s = 0$ , then the summand is equal to 0. If  $s > 0$ , we need the following lemma.

**Lemma 2.** *For all  $N \geq 0$ ,  $s > 0$ ,  $x > 0$  and  $t \in \mathbb{R}$ , we have*

$$V_s(x, t) = \rho x^{s-1/2} \sum_{k=1}^N e^{-i\pi(k-\{t\})^2/x} \left( \frac{1}{(k-\{t\})^s} - \frac{1}{k^s} \right) + \mathcal{O} \left( \frac{x^{s-\frac{1}{2}}}{(N+1-\{t\})^s} \right)$$

where the implicit constant is absolute.

Therefore,

$$\begin{aligned} \rho x^{s-1/2} \sum_{k=0}^{\lambda-1} \frac{e^{-i\pi(\xi+k)^2/x}}{(-\xi-k)^s} &= \rho x^{s-1/2} e^{-i\pi\{t\}^2/x} F_{s,\lambda} \left( -\frac{1}{x}, \frac{\{t\}}{x} \right) + V_s(x, t) + \mathcal{O} \left( \frac{x^{s-\frac{1}{2}}}{(\lambda+1-\{t\})^s} \right) \end{aligned} \quad (4.8)$$

where the implicit constant depends on  $s$ .

To study the second term in (4.7), we do similar formal manipulations and obtain the representation

$$\sum_{k=0}^{\lambda-1} \tilde{J}_k = \rho x^s \int_{-\infty}^{\infty} e^{-\pi x u^2} \left( \sum_{k=1}^{\lambda} e^{-i\pi(k-\{t\})^2/x} \left( \frac{1}{(\rho x u + k - \{t\})^s} - \frac{1}{(k - \{t\})^s} \right) \right) du,$$

where the sum in the integrand is a partial sum of  $\widehat{W}_s(x, t, u)$ . If  $s = 0$ , then the summand is equal to 0. If  $s > 0$ , we need the following lemma.

**Lemma 3.** *For all  $N \geq 0$ ,  $s > 0$ ,  $x > 0$  and  $t \in \mathbb{R}$ , we have*

$$\widehat{W}_s(x, t, u) = \sum_{k=1}^N e^{-i\pi(k-\{t\})^2/x} \left( \frac{1}{(\rho x u + k - \{t\})^s} - \frac{1}{(k - \{t\})^s} \right) + \mathcal{O} \left( \frac{|ux|}{(N+1-\{t\})^s} \right),$$

where the implicit constant is effective and depends at most on  $s$ .

It follows that

$$\begin{aligned} \sum_{k=0}^{\lambda-1} \tilde{J}_k &= W_s(x, t) + \mathcal{O} \left( \frac{x^{s+1}}{(\lambda+1-\{t\})^s} \int_{-\infty}^{\infty} |u| e^{-\pi x u^2} du \right) \\ &= W_s(x, t) + \mathcal{O} \left( \frac{x^s}{(\lambda+1-\{t\})^s} \right) \end{aligned} \quad (4.9)$$

because

$$\int_{-\infty}^{\infty} |u| e^{-\pi x u^2} du = \frac{1}{\pi x}.$$

We now use the estimates (4.5), (4.8), (4.9) in (4.7) together with Lemma 4. This gives us

$$\begin{aligned} F_{s,n-1}(x, t) &= \rho x^{s-\frac{1}{2}} e^{-i\pi\{t\}^2/x} F_{s,\lambda}(-1/x, t) + U_s(x, t) + V_s(x, t) + W_s(x, t) \\ &\quad + \int_{-1/2-\rho\infty}^{-1/2+\rho\infty} \frac{g_s(z+n)}{e^{2i\pi z} - 1} dz + \mathcal{O} \left( \frac{x^{s-\frac{1}{2}}}{(\lambda+1-\{t\})^s} \right) + \mathcal{O} \left( \frac{x^s}{(\lambda+1-\{t\})^s} \right). \end{aligned}$$

Since  $0 < x < 1$ , the first error term absorbs the second one. We also want to replace  $F_{s,\lambda}(-1/x, \{t\}/x)$  by  $F_{s, \lfloor (n-1)x \rfloor}(-1/x, \{t\}/x)$ . This can be done at the cost of an error

$$x^{s-\frac{1}{2}} \sum_{k=1+\lfloor (n-1)x \rfloor}^{\lambda} \frac{1}{k^s} \leq \frac{2x^{s-\frac{1}{2}}}{(1+\lfloor (n-1)x \rfloor)^s}$$

because  $\lfloor (n-1)x \rfloor \leq \lambda \leq \lfloor (n-1)x \rfloor + 2$ . Hence

$$\begin{aligned} F_{s,n-1}(x, t) &= \rho x^{s-\frac{1}{2}} e^{-i\pi\{t\}^2/x} F_{s, \lfloor (n-1)x \rfloor}(-1/x, \{t\}/x) + U_s(x, t) + V_s(x, t) + W_s(x, t) \\ &\quad + \int_{-1/2-\rho\infty}^{-1/2+\rho\infty} \frac{g_s(z+n)}{e^{2i\pi z} - 1} dz + \mathcal{O} \left( \frac{x^{s-\frac{1}{2}}}{(\lambda+1-\{t\})^s} \right) + \mathcal{O} \left( \frac{x^{s-\frac{1}{2}}}{(1+\lfloor (n-1)x \rfloor)^s} \right). \end{aligned}$$

It remains to deal with the second integral in Lemma 1. For this, we have to distinguish between the case  $s = 1$  and the case  $s \neq 1$ .

**Lemma 4.** *If  $s \geq 0, s \neq 1$ , for any  $n \geq 1, x > 0, t \in \mathbb{R}$ , we have*

$$\left| \int_{-1/2-\rho\infty}^{-1/2+\rho\infty} \frac{g(z+n)}{1 - e^{2i\pi z}} dz \right| \ll \min \left( \frac{1}{n^s \sqrt{x}}, \frac{1}{x^{\frac{1-s}{2}}} \right)$$

where the implicit constant depends on  $s$ .

If  $s = 1$ , for any  $n \geq 1$ ,  $x > 0$ ,  $t \in \mathbb{R}$ , we have

$$\left| \int_{-1/2-\rho\infty}^{-1/2+\rho\infty} \frac{g(z+n)}{1-e^{2i\pi z}} dz \right| \ll \min \left( \frac{1}{n\sqrt{x}}, 1 + |\log(n\sqrt{x})| \right)$$

where the implicit constant is absolute.

In the case  $s \neq 1$ , the threshold  $n\sqrt{x} = 1$  determine which bound is the best.

We now replace  $n$  by  $n+1$  and set  $I_s(x, t) = U_s(x, t) + V_s(x, t) + W_s(x, t)$ . We get

$$\begin{aligned} F_{s,n}(x, t) &= \rho x^{s-\frac{1}{2}} e^{-i\pi\{t\}^2/x} F_{s, \lfloor nx \rfloor}(-1/x, \{t\}/x) + I_s(x, t) \\ &+ \mathcal{O} \left( \frac{x^{s-\frac{1}{2}}}{(\lfloor (n+\frac{1}{2})x + t \rfloor + 1 - t)^s} \right) + \mathcal{O} \left( \frac{x^{s-\frac{1}{2}}}{(1 + \lfloor nx \rfloor)^s} \right) \\ &+ \begin{cases} \mathcal{O} \left( \min \left( \frac{1}{(n+1)^s \sqrt{x}}, \frac{1}{x^{\frac{1-s}{2}}} \right) \right) & \text{if } s \geq 0, s \neq 1 \\ \mathcal{O} \left( \min \left( \frac{1}{(n+1)\sqrt{x}}, 1 + |\log((n+1)\sqrt{x})| \right) \right) & \text{if } s = 1. \end{cases} \end{aligned} \quad (4.10)$$

For any  $x > 0$ , any  $n \geq 0$  and any  $t \in \mathbb{R}$ ,  $\lfloor (n+\frac{1}{2})x + t \rfloor + 1 - t$  and  $\lfloor nx \rfloor + 1$  are both  $\geq \lfloor nx + t \rfloor + 1 - t$  so that we can simplify the error term in (4.10) to just

$$\mathcal{O} \left( \frac{x^{s-\frac{1}{2}}}{(\lfloor nx + t \rfloor + 1 - t)^s} \right) + \begin{cases} \mathcal{O} \left( \min \left( \frac{1}{(n+1)^s \sqrt{x}}, \frac{1}{x^{\frac{1-s}{2}}} \right) \right) & \text{if } s \geq 0, s \neq 1 \\ \mathcal{O} \left( \min \left( \frac{1}{(n+1)\sqrt{x}}, 1 + |\log((n+1)\sqrt{x})| \right) \right) & \text{if } s = 1. \end{cases}$$

To deal with the case  $x < 0$ , we simply take the complex conjugate of both sides of (4.10) and change  $x$  to  $-x$  and  $t$  to  $-t$ . We finally deduce from all this discussion that for any  $x \in (-1, 1)$ ,  $x \neq 0$ , any  $t \in \mathbb{R}$ , any  $s \geq 0$  and any  $n \geq 0$ , we have

$$\begin{aligned} F_{s,n}(x, t) &= \rho |x|^{s-\frac{1}{2}} e^{-i\pi\{\sigma(x)t\}^2/x} F_{s, \lfloor n|x| \rfloor} \left( -\frac{1}{x}, \frac{\{\sigma(x)t\}}{x} \right) + \Omega_s(x, t) \\ &+ \mathcal{O} \left( \frac{|x|^{s-\frac{1}{2}}}{(\lfloor n|x| + \sigma(x)t \rfloor + 1 - \sigma(x)t)^s} \right) \\ &+ \begin{cases} \mathcal{O} \left( \min \left( \frac{1}{(n+1)^s \sqrt{|x|}}, \frac{1}{|x|^{\frac{1-s}{2}}} \right) \right) & \text{if } s \geq 0, s \neq 1 \\ \mathcal{O} \left( \min \left( \frac{1}{(n+1)\sqrt{|x|}}, 1 + |\log((n+1)\sqrt{|x|})| \right) \right) & \text{if } s = 1. \end{cases} \end{aligned}$$

where  $\Omega_s(x, t) = I_s(x, t)$  is  $x > 0$  and  $\Omega_s(x, t) = \overline{I_s(-x, -t)}$  if  $x < 0$ . The implicit constants are effective and depend at most on  $s$ . We obtain the expression for  $I_s(x, t)$  given in the introduction by means of the expression (4.4) for  $V_s(x, t) + W_s(x, t)$ .

## 4.2. Proofs of the lemmas.

#### 4.2.1. Proof of Lemma 1.

We follow Mordell's method as it is presented in the book [1]. We define

$$f_s(z) = \frac{1}{e^{2i\pi z} - 1} \sum_{k=1}^{n-1} g_s(z+k).$$

and we integrate it over the parallelogram  $ADCB$  (positively oriented) defined by  $A = \frac{1}{2} + \rho d$ ,  $B = \frac{1}{2} - \rho d$ ,  $C = -\frac{1}{2} - \rho d$ ,  $D = -\frac{1}{2} + \rho d$  and  $d > 0$  is a parameter. Clearly,

$$\int_{ABCD} f(z) dz = 2i\pi \operatorname{Res}(f_s(z), z=0) = \sum_{k=1}^{n-1} g_s(k) = F_{s,n-1}(x, t).$$

If  $z$  is on  $CB = \{-\frac{1}{2} + \rho du, -d \leq u \leq d\}$  or  $AD = \{u + \rho d, -\frac{1}{2} \leq u \leq \frac{1}{2}\}$ , we have

$$\operatorname{Re}(i\pi(z+k)^2 + 2i\pi(z+k)t) = -\pi d^2 x \pm \sqrt{2\pi} \cdot d((u+k)x + t) = -\pi d^2 x + \mathcal{O}(d) \quad (4.11)$$

where the implicit constant does not depend on  $u$ . Moreover,  $|e^{2i\pi z}| = e^{\sigma\sqrt{2\pi}d}$  with  $\sigma = 1$  on  $CB$  and  $\sigma = -1$  on  $AD$ , so that

$$\lim_{d \rightarrow +\infty} |e^{2i\pi z} - 1| = \begin{cases} +\infty, & z \in CB \\ 1, & z \in AD. \end{cases} \quad (4.12)$$

It follows from (4.11) and (4.12) that

$$\lim_{d \rightarrow +\infty} \int_{CB \cup AD} f_s(z) dz = 0.$$

Thus

$$F_{s,n-1}(x, t) = \int_{BA} f_s(z) dz - \int_{CD} f_s(z) dz + o(1),$$

where  $o(1)$  is for  $d \rightarrow +\infty$ . We observe that  $BA = CD + 1$ , so that

$$\begin{aligned} F_{s,n-1}(x, t) &= \int_{CD} f_s(z+1) dz - \int_{CD} f_s(z) dz + o(1) \\ &= \int_{CD} (f_s(z+1) - f_s(z)) dz + o(1) \\ &= \int_{CD} \frac{g_s(z+n) - g_s(z+1)}{e^{2i\pi z} - 1} dz + o(1). \end{aligned}$$

We now let  $d \rightarrow +\infty$  to get the lemma.

#### 4.2.2. Proof of Lemma 2.

By definition of  $V_s(x, t)$ , the point is to estimate the series

$$\rho x^{s-1/2} \sum_{k=N+1}^{\infty} e^{-i\pi(k-\{t\})^2/x} \left( \frac{1}{(k-\{t\})^s} - \frac{1}{k^s} \right),$$

the modulus of which is obviously bounded by

$$x^{s-1/2} \sum_{k=N+1}^{\infty} \left| \frac{1}{(k - \{t\})^s} - \frac{1}{k^s} \right|.$$

We observe that, since  $\{t\} \in [0, 1)$  and  $N \geq 0$ ,

$$\begin{aligned} \sum_{k=N+1}^{\infty} \left| \frac{1}{(k - \{t\})^s} - \frac{1}{k^s} \right| &= \sum_{k=N+1}^{\infty} \left( \frac{1}{(k - \{t\})^s} - \frac{1}{k^s} \right) \\ &\leq \sum_{k=N+1}^{\infty} \left( \frac{1}{(k - \{t\})^s} - \frac{1}{(k + 1 - \{t\})^s} \right) \\ &= \frac{1}{(N + 1 - \{t\})^s}. \end{aligned}$$

The lemma follows.

#### 4.2.3. Proof of Lemma 3.

We want to estimate the modulus of

$$\sum_{k=N+1}^{\infty} e^{-i\pi(k-\{t\})^2/x} \left( \frac{1}{(\rho x u + k - \{t\})^s} - \frac{1}{(k - \{t\})^s} \right),$$

which is bounded above by

$$\sum_{k=1}^{\infty} \left| \frac{1}{(\rho x u + k + N - \{t\})^s} - \frac{1}{(k + N - \{t\})^s} \right|$$

This problem is similar to the one we dealt with in the proof of Lemma 2 but this is a bit more difficult here because  $\rho x u$  is not real. We set  $M = N - \{t\}$ . We observe that

$$\begin{aligned} \left| \frac{1}{(\rho x u + k + M)^s} - \frac{1}{(k + M)^s} \right| &= \left| \frac{1}{(\rho x u + k + M)^s} \cdot \left( 1 - \left( 1 + \frac{\rho x u}{k + M} \right)^s \right) \right| \\ &\ll \begin{cases} \frac{|x u|}{|k + M + \rho x u|^s (k + M)} & \text{if } |x u| \leq k + M \\ \frac{|x u|^s}{|k + M + \rho x u|^s (k + M)^s} & \text{if } |x u| \geq k + M \end{cases} \end{aligned}$$

where the implicit constant depends on  $s$  only. Hence

$$\begin{aligned} \sum_{k=1}^{\infty} \left| \frac{1}{(\rho x u + k + M)^s} - \frac{1}{(k + M)^s} \right| \\ \ll \sum_{\substack{k=1 \\ |x u| \leq k + M}}^{\infty} \frac{|x u|}{|k + M + \rho x u|^s (k + M)} + \sum_{\substack{k=1 \\ |x u| \geq k + M}}^{\infty} \frac{|x u|^s}{|k + M + \rho x u|^s (k + M)^s}. \end{aligned}$$

We denote by  $S_1$  and  $S_2$  these two series and we write  $\rho xu = \frac{\sqrt{2}}{2}(v + iv)$  for some  $v \in \mathbb{R}$ . (Note that  $|xu| = |v|$ .)

If  $v \geq 0$ , then

$$0 \leq S_1 \leq \sum_{k=1}^{\infty} \frac{v}{(k + M + v/\sqrt{2})^s (k + M)} \leq \sum_{k=1}^{\infty} \frac{v}{(k + M)^{s+1}} \ll \frac{v}{(M + 1)^s}$$

for some implicit constant that depends only on  $s$ . Moreover,

$$0 \leq S_2 \leq \sum_{1 \leq k \leq v} \frac{v^s}{(k + M + v/\sqrt{2})^s (k + M)^s} \ll \sum_{1 \leq k \leq v} \frac{v^s}{(k + M)^{2s}} \leq \frac{v^{s+1}}{(M + 1)^{2s}}$$

for some implicit constant that depends only on  $s$ . Adding the two upper bounds for  $S_1$  and  $S_2$ , we get

$$\sum_{k=1}^{\infty} \left| \frac{1}{(\rho xu + k + M)^s} - \frac{1}{(k + M)^s} \right| \ll \frac{|xu|}{(M + 1)^s}$$

which proves the lemma in this case.

Let us now consider the case  $v \leq 0$ . If  $|v| \leq k + M$ , then

$$|k + M + \rho xu| \geq (k + M - |v|/\sqrt{2}) \geq (1 - 1/\sqrt{2})(k + M)$$

so that

$$0 \leq S_1 \ll \sum_{k=1}^{\infty} \frac{|v|}{(k + M)^{s+1}} \ll \frac{|xu|}{(M + 1)^s}$$

for some implicit constants that depend only on  $s$ . If  $|v| \geq k + M$ , then  $|k + M + \rho xu|^2 \geq w^2/2 \geq (k + M)^2/2$  so that again

$$0 \leq S_2 \ll \sum_{1 \leq k \leq |v|} \frac{|v|^s}{(k + M)^{2s}} \ll \frac{|xu|^{s+1}}{(M + 1)^{2s}}$$

for some implicit constants that depend on  $s$ . We conclude exactly as above.

#### 4.2.4. Proof of Lemma 4.

Again, we follow Mordell's method in the book [1]. For any  $s \geq 0$  and  $z = -\frac{1}{2} + \rho u \in -\frac{1}{2} + \rho\mathbb{R}$ , we have

$$\left| \frac{g_s(z + n)}{e^{2i\pi z} - 1} \right| = \frac{e^{-\pi x u^2}}{|u + \bar{\rho}(n - \frac{1}{2})|^s} \cdot \frac{e^{-\sqrt{2}\pi u \theta}}{|e^{2i\pi z} - 1|}$$

with  $\theta = (n - \frac{1}{2})x + t + \xi$ . By definition of  $\xi$ , we have  $0 \leq \theta \leq 1$  and it follows that

$$\frac{e^{-\sqrt{2}\pi u \theta}}{|e^{2i\pi z} - 1|} = \mathcal{O}(1)$$

for any  $u \in \mathbb{R}$ . Moreover, for any  $u \in \mathbb{R}$  and any  $n \geq 1$ , we have

$$\frac{1}{|u + \bar{\rho}(n - \frac{1}{2})|^s} \ll \frac{1}{|u + \bar{\rho}n|^s}$$



for some effective constant that depends only on  $s$ . Hence, for any  $s \geq 0$ ,

$$\left| \int_{-1/2-\rho\infty}^{-1/2+\rho\infty} \frac{g_s(z+n)}{e^{2i\pi z}-1} dz \right| \ll \int_{-\infty}^{+\infty} \frac{e^{-\pi x u^2}}{|u+\bar{\rho}n|^s} du$$

where the implicit constant depends on  $s$ . Since  $\bar{\rho} \notin \mathbb{R}$ , we readily deduce that

$$\left| \int_{-1/2-\rho\infty}^{-1/2+\rho\infty} \frac{g_s(z+n)}{e^{2i\pi z}-1} dz \right| \leq \frac{c_1(s)}{n^s} \int_{-\infty}^{+\infty} e^{-\pi x u^2} du = \frac{c_1(s)}{n^s \sqrt{x}}.$$

for some effective constant  $c_1(s)$ .

To get the second upper bound, we need to distinguish the case  $s = 1$  and the case  $s \neq 1$ .

**Case  $s \geq 0, s \neq 1$ .** We assume for the moment that  $0 \leq n\sqrt{x} \leq 1$  and explain below how to remove this assumption. We set  $y = n\sqrt{x/2}$ ; with  $v = \sqrt{x}u$ , we get

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{e^{-\pi x u^2}}{|u+\bar{\rho}n|^s} du &\leq \sqrt{2}x^{\frac{s-1}{2}} \int_{-\infty}^{+\infty} \frac{e^{-\pi v^2}}{(|v+y|+y)^s} dv \\ &\leq \sqrt{2}x^{\frac{s-1}{2}} \int_0^{+\infty} \frac{e^{-\pi v^2}}{(v+y)^s} dv + \sqrt{2}x^{\frac{s-1}{2}} \int_0^{+\infty} \frac{e^{-\pi v^2}}{(|v-y|+y)^s} dv. \end{aligned}$$

First,

$$\begin{aligned} \int_0^{+\infty} \frac{e^{-\pi v^2}}{(v+y)^s} dv &\leq \int_0^1 \frac{dv}{(v+y)^s} + \frac{1}{(1+y)^s} \int_1^{+\infty} e^{-\pi v^2} dv \\ &\leq \frac{(1+y)^{1-s} - y^{1-s}}{1-s} + \frac{1}{(1+y)^s} \leq c_2(s) \end{aligned}$$

for some effective constant  $c_2(s)$  because  $0 \leq y \leq \sqrt{2}/2$ . Second,

$$\begin{aligned} \int_0^{+\infty} \frac{e^{-\pi v^2}}{(|v-y|+y)^s} dv &\leq \int_0^y \frac{dv}{(2y-v)^s} + \int_y^{y+1} \frac{dv}{v^s} + \frac{1}{(1+y)^s} \int_1^{+\infty} e^{-\pi v^2} dv \\ &\leq \frac{2^{1-s}-1}{1-s} y^{1-s} + \frac{(1+y)^{1-s} - y^{1-s}}{1-s} + \frac{1}{(1+y)^s} \leq c_3(s) \end{aligned}$$

for some effective constant  $c_3(s)$  because  $0 \leq y \leq \sqrt{2}/2$ . Hence,

$$\int_{-\infty}^{+\infty} \frac{e^{-\pi x u^2}}{|u+\bar{\rho}n|^s} du \leq c_4(s) x^{\frac{s-1}{2}}.$$

with  $c_4(s) = \sqrt{2}(c_2(s) + c_3(s))$ .

In summary, we have obtained so far:

$$\left| \int_{-1/2-\rho\infty}^{-1/2+\rho\infty} \frac{g_s(z+n)}{e^{2i\pi z} - 1} dz \right| \leq \begin{cases} \frac{c_1(s)}{n^s \sqrt{x}} & \text{for all } n \geq 1, x > 0 \\ c_4(s) x^{\frac{s-1}{2}} & \text{if } 0 \leq n\sqrt{x} \leq 1. \end{cases}$$

With  $c(s) = \max(c_1(s), c_4(s))$  and since  $1/(n^s \sqrt{x}) \geq x^{\frac{s-1}{2}}$  when  $0 \leq n\sqrt{x} \leq 1$ , we deduce that

$$\left| \int_{-1/2-\rho\infty}^{-1/2+\rho\infty} \frac{g_s(z+n)}{e^{2i\pi z} - 1} dz \right| \leq c(s) \min\left(\frac{1}{n^s \sqrt{x}}, \frac{1}{x^{\frac{1-s}{2}}}\right)$$

for all  $n \geq 1, x > 0$ . This completes the proof in this case.

**Case  $s = 1$ .** We set  $y = n\sqrt{x/2} \geq 0$  for simplicity. We then have (with  $v = \sqrt{x}u$ )

$$\int_{-\infty}^{+\infty} \frac{e^{-\pi x u^2}}{|u + \bar{\rho}n|} du \ll \int_0^{+\infty} \frac{e^{-\pi v^2}}{v + y} dv + \int_{-\infty}^0 \frac{e^{-\pi v^2}}{|v + y| + y} dv.$$

First,

$$\begin{aligned} 0 \leq \int_0^{+\infty} \frac{e^{-\pi v^2}}{v + y} dv &\leq \int_0^1 \frac{dv}{v + y} + \frac{1}{1 + y} \int_1^{+\infty} e^{-\pi v^2} dv \\ &\leq \log(y + 1) - \log(y) + \int_0^{\infty} e^{-\pi v^2} dv \\ &= \log(1 + n\sqrt{x/2}) - \log(n\sqrt{x/2}) + 1. \end{aligned}$$

Second,

$$\begin{aligned} 0 \leq \int_{-\infty}^0 \frac{e^{-\pi v^2}}{y + |v + y|} dv &= \int_0^{+\infty} \frac{e^{-\pi v^2}}{y + |v - y|} dv \\ &\leq \int_0^y \frac{dv}{2y - v} + \int_y^{y+1} \frac{dv}{v} + \frac{1}{y + 1} \int_{y+1}^{\infty} e^{-v} dv \\ &\leq \log(2) + \log(y + 1) - \log(y) + 1 \\ &= \log(1 + n\sqrt{x/2}) - \log(n\sqrt{x/2}) + 1 + \log(2). \end{aligned}$$

Collecting both estimates, we obtain

$$\left| \int_{-1/2-\rho\infty}^{-1/2+\rho\infty} \frac{g(z+n)}{e^{2i\pi z}-1} dz \right| \ll |\log(n\sqrt{x})| + \log(1+n\sqrt{x/2}) + 1$$

for some absolute constant. If  $n\sqrt{x} \geq 1$ , then  $\log(1+n\sqrt{x/2}) \leq \log(n\sqrt{x}) + 1$  and if  $0 < n\sqrt{x} \leq 1$ , then  $\log(1+n\sqrt{x/2}) \leq \log(1+1/\sqrt{2})$ . Consequently, there exists an absolute constant  $c$  such that

$$\left| \int_{-1/2-\rho\infty}^{-1/2+\rho\infty} \frac{g(z+n)}{e^{2i\pi z}-1} dz \right| \leq c(|\log(n\sqrt{x})| + 1)$$

for any  $n \geq 1$  and any  $x > 0$ .

## 5. PROOF OF THEOREM 2, PART (ii)

In this section, we prove that

- the function  $\Omega_s(x)$  is continuous on  $\mathbb{R} \setminus \{0\}$  for any  $s \geq 0$ .
- the function  $\Omega_s(x)$  is differentiable at any rational number  $p/q$  with  $p, q$  both odd, for any  $s \geq 0$ .
- the function  $\Omega_s(x) - \frac{\rho^{1-s}\Gamma(\frac{1-s}{2})}{2\pi^{\frac{1-s}{2}}}|x|^{\frac{s-1}{2}}$  is bounded on  $\mathbb{R}$  when  $0 \leq s < 1$ .
- the function  $\Omega_1(x) - \log(1/\sqrt{|x|})$  is bounded on  $\mathbb{R}$ .
- the function  $\Omega_s(x)$  is differentiable on  $\mathbb{R} \setminus \{0\}$  and continuous at  $x = 0$  for any  $s > 1$ .

Given the definition of  $\Omega_s(x)$  by means of the function  $I_s(x)$  (see (1.2)), it is enough to prove these facts for  $x \geq 0$ .

### 5.1. Continuity and differentiability of $\Omega_s(x)$ on $(0, +\infty)$ .

Note that  $V_s(x, 0) = 0$  and thus  $I_s(x) = U_s(x) + W_s(x)$  where

$$U_s(x) = \int_{1/2-\rho\infty}^{1/2+\rho\infty} \frac{e^{i\pi z^2 x}}{z^s(1-e^{2i\pi z})} dz$$

and

$$W_s(x) = \rho x^s \int_{-\infty}^{\infty} e^{-\pi x u^2} \left( \sum_{k=1}^{\infty} e^{-i\pi k^2/x} \left( \frac{1}{(\rho x u + k)^s} - \frac{1}{k^s} \right) \right) du.$$

It is clear that  $U_s(x)$  defines a differentiable function on  $(0, +\infty)$ . The series

$$\widehat{W}_s(x, u, 0) = \sum_{k=1}^{\infty} e^{-i\pi k^2/x} \left( \frac{1}{(\rho x u + k)^s} - \frac{1}{k^s} \right)$$

defines a continuous function of  $(u, x)$  on  $\mathbb{R} \times (0, +\infty)$ . Moreover, by Lemma 3 applied with  $N = 0$ , we have  $|\widehat{W}_s(x, u, 0)| \ll |ux|$ . This guarantees the continuity  $W_s(x)$  on  $[0, +\infty)$ , and that  $W_s(x) = \mathcal{O}(x^s)$  on  $[0, +\infty)$ . Hence,  $\Omega_s(x)$  is continuous on  $(0, +\infty)$ .

To prove that  $\Omega_s(x)$  is differentiable at any rational number  $x = p/q$  with  $p, q$  both odd, it remains to prove that this is the case of  $W_s(x)$ . Integrating by parts, we get

$$\begin{aligned} W_s(x) &= \rho x^s \int_{-\infty}^{\infty} u e^{-\pi x u^2} \left( \sum_{k=1}^{\infty} e^{-i\pi k^2/x} \frac{1}{u} \cdot \left( \frac{1}{(\rho x u + k)^s} - \frac{1}{k^s} \right) \right) du \\ &= \frac{x^{s-1}}{2\pi} \int_{-\infty}^{\infty} e^{-\pi x u^2} \left( \sum_{k=1}^{\infty} e^{-i\pi k^2/x} \frac{d}{du} \left( \frac{1}{u} \cdot \left( \frac{1}{(\rho x u + k)^s} - \frac{1}{k^s} \right) \right) \right) du, \end{aligned}$$

where differentiation under the sum is allowed by uniform convergence of  $\widehat{W}_s(x, u, 0)$ . We observe that

$$\begin{aligned} \frac{d}{du} \left( \frac{1}{u} \cdot \left( \frac{1}{(\rho x u + k)^s} - \frac{1}{k^s} \right) \right) &= \frac{-s\rho x}{u(k + \rho x u)^{s+1}} - \frac{1}{u^2(k + \rho x u)^s} + \frac{1}{u^2 k^s} \\ &= \frac{c_s(\rho x)^2}{k^{s+2}} + \mathcal{O}\left(\frac{1}{k^{s+3}}\right) \end{aligned}$$

for some constant  $c_s > 0$  independent of  $k$  and  $x$ . Therefore,

$$\begin{aligned} W_s(x) &= \frac{x^{s-1}}{2\pi} \int_{-\infty}^{\infty} e^{-\pi x u^2} \left( \sum_{k=1}^{\infty} e^{-i\pi k^2/x} \left( \frac{d}{du} \left( \frac{1}{u} \cdot \left( \frac{1}{(\rho x u + k)^s} - \frac{1}{k^s} \right) \right) - \frac{c_s(\rho x)^2}{k^{s+2}} \right) \right) du \\ &\quad + \frac{c(s)\rho^3 x^{s+1}}{2\pi} \int_{-\infty}^{\infty} e^{-\pi x u^2} du \cdot \sum_{k=1}^{\infty} \frac{e^{-i\pi k^2/x}}{k^{s+2}}. \quad (5.1) \end{aligned}$$

On the one hand, the series

$$\sum_{k=1}^{\infty} e^{-i\pi k^2/x} \left( \frac{d}{du} \left( \frac{1}{u} \cdot \left( \frac{1}{(\rho x u + k)^s} - \frac{1}{k^s} \right) \right) - \frac{c_s(\rho x)^2}{k^{s+2}} \right)$$

can be termwise differentiated with respect to  $x$  and it follows easily that the first term on the right hand side of (5.1) is differentiable for any  $x > 0$ .

On the other hand, the second term on the right hand side of (5.1) is

$$\frac{c_s \rho^3 x^{s+1/2}}{2\pi} \cdot \sum_{k=1}^{\infty} \frac{e^{-i\pi k^2/x}}{k^{s+2}}. \quad (5.2)$$

A result of Luther [12] ensures that the series  $\sum_{k=1}^{\infty} \frac{e^{i\pi k^2 x}}{k^\xi}$  is differentiable at any rational number  $x = p/q$ ,  $p, q$  odd, for any fixed  $\xi > 3/2$  (and this is no longer true when  $\xi \leq 3/2$ ). Since  $s \geq 0$ , this result is more than needed to complete the proof that  $W_s(x)$  is differentiable at any rational number  $x = p/q$ ,  $p, q$  odd.

Moreover, when  $s > 1$ , the series (5.2) is everywhere differentiable (except at 0) by uniform convergence, hence  $\Omega_s(x)$  is differentiable on  $(0, +\infty)$ .

### 5.2. Local behavior of $\Omega_s(x)$ around $x = 0$ .

It was proved above that  $W_s(x) = \mathcal{O}(x^s)$  on  $[0, +\infty)$  and  $U_s(x)$  is continuous on  $(0, +\infty)$ . We will now establish that

- $U_s(x)$  is continuous at  $x = 0$  for any  $s > 1$ .
- $U_s(x) = \frac{\rho^{1-s}\Gamma(\frac{1-s}{2})}{2\pi^{\frac{1-s}{2}}}x^{\frac{s-1}{2}} + \mathcal{O}(1)$  on  $(0, +\infty)$  for any  $0 \leq s < 1$ .
- $U_1(x) = \log(1/\sqrt{x}) + \mathcal{O}(1)$  on  $(0, +\infty)$ .

The case  $s > 1$  is easy because

$$\lim_{x \rightarrow 0^+} U_s(x) = U_s(0) = \int_{1/2-\rho\infty}^{1/2+\rho\infty} \frac{1}{z^s(1 - e^{2i\pi z})} dz$$

is finite, by Lebesgue dominated convergence theorem. This implies that  $\Omega_s(x)$  is continuous at  $x = 0$  when  $s > 1$ .

We assume from now on that  $s \in [0, 1]$ . The change of variable  $z = \frac{1}{2} + \rho u$  gives

$$U_s(x) = \rho e^{i\pi x/4} \int_{-\infty}^{\infty} \frac{e^{-\pi x u^2} e^{i\rho\pi x u}}{(\frac{1}{2} + \rho u)^s (1 + e^{2i\rho\pi u})} du.$$

If  $u \leq 0$ , then since  $e^{2i\rho\pi u} = e^{i\sqrt{2}\pi u} e^{-\sqrt{2}\pi u}$  we have

$$\left| g(u) := \frac{1}{1 + e^{2i\rho\pi u}} = \frac{e^{-i\sqrt{2}\pi u} e^{\sqrt{2}\pi u}}{1 + e^{-i\sqrt{2}\pi u} e^{\sqrt{2}\pi u}} \right| \ll e^{\pi\sqrt{2}u}$$

and thus

$$\left| \frac{e^{-\pi x u^2} e^{i\rho\pi x u}}{(\frac{1}{2} + \rho u)^s (1 + e^{2i\rho\pi u})} \right| \ll \frac{e^{\sqrt{2}\pi u}}{|\frac{1}{2} + \rho u|^s}.$$

Since the integral

$$\int_{-\infty}^0 \frac{e^{\sqrt{2}\pi u}}{|\frac{1}{2} + \rho u|^s} du$$

is convergent, by Lebesgue dominated convergence theorem, we get that

$$\lim_{x \rightarrow 0^+} \int_{1/2-\rho\infty}^{1/2} \frac{e^{i\pi z^2 x}}{z^s (e^{2i\pi z} - 1)} dz = \int_{1/2-\rho\infty}^{1/2} \frac{dz}{z^s (e^{2i\pi z} - 1)} =: \alpha_s.$$

If  $u \geq 0$ ,  $g(u) \rightarrow 1$  when  $u \rightarrow +\infty$  and we cannot invoke Lebesgue's theorem because  $1/|\frac{1}{2} + \rho u|^s$  is not integrable over  $[0, +\infty)$ . However,

$$g(u) - 1 = -\frac{e^{2i\pi\rho u}}{1 + e^{2i\pi\rho u}}$$

so that  $|g(u) - 1| \ll e^{-\sqrt{2}\pi u}$  on  $[0, +\infty)$ , and Lebesgue's theorem entails that

$$\lim_{x \rightarrow 0^+} \int_{1/2}^{1/2+\rho\infty} \frac{e^{i\pi z^2 x}}{z^s} \left( \frac{1}{e^{2i\pi z} - 1} - 1 \right) dz = \int_{1/2}^{1/2+\rho\infty} \frac{1}{z^s} \left( \frac{1}{e^{2i\pi z} - 1} - 1 \right) dz =: \beta_s.$$

It thus remains to study the simpler integral

$$P_s(x) := \int_{1/2}^{1/2+\rho\infty} \frac{e^{i\pi z^2 x}}{z^s} dz$$

because  $U_s(x) = P_s(x) + \alpha_s + \beta_s + o(1)$  as  $x \rightarrow 0^+$ . Setting  $z = \frac{1}{2} + \rho v/\sqrt{w}$ , we obtain

$$P_s(x) = \rho^{1-s} e^{i\pi x/4} x^{\frac{s-1}{2}} \int_0^{+\infty} e^{-\pi v^2} \frac{e^{i\rho\pi\sqrt{x}v}}{(v + \bar{\rho}\frac{\sqrt{x}}{2})^s} dv.$$

We see here that as  $x \rightarrow 0^+$ , the integral is “close” to the integral  $\int_0^{+\infty} e^{-\pi v^2} dv/v^s$ . The latter is convergent if  $0 \leq s < 1$  but divergent if  $s = 1$  and we now have to distinguish between both possibilities.

**Case  $0 \leq s < 1$ .** Observe that  $|\exp(i\rho\pi\sqrt{x}v) - 1| \ll \sqrt{x}v$  for  $v \in [0, +\infty)$  and any  $x \geq 0$ . (this is true when  $\sqrt{x}v$  by using the Taylor polynomial, and it goes to zero when  $\sqrt{x}v$  tends to  $+\infty$ .) Moreover,  $|v + \bar{\rho}\sqrt{x}/2| \geq |v + \sqrt{2x}|$ . Hence,

$$\left| \int_0^{+\infty} e^{-\pi v^2} \frac{e^{i\rho\pi\sqrt{x}v} - 1}{(v + \frac{\bar{\rho}\sqrt{x}}{2})^s} dv \right| \ll \int_0^{+\infty} e^{-\pi v^2} \frac{\sqrt{x}v}{(v + \sqrt{2x})^s} dv \leq \sqrt{x} \int_0^{+\infty} v^{1-s} e^{-\pi v^2} = \mathcal{O}(\sqrt{x}).$$

Therefore,

$$P_s(x) = \rho^{1-s} x^{\frac{s-1}{2}} \int_0^{+\infty} \frac{e^{-\pi v^2}}{(v + \bar{\rho}\sqrt{x}/2)^s} dv + \mathcal{O}(x^{\frac{s}{2}}). \quad (5.3)$$

We now prove that

$$\int_0^{+\infty} \frac{e^{-\pi v^2}}{(v + \bar{\rho}\sqrt{x}/2)^s} dv = \int_0^{+\infty} \frac{e^{-\pi v^2}}{v^s} dv + \mathcal{O}(x^{\frac{1-s}{2}}). \quad (5.4)$$

Indeed, with  $X = \bar{\rho}\sqrt{x}/2$ , we have

$$\int_0^{+\infty} e^{-\pi v^2} \left( \frac{1}{(v+X)^s} - \frac{1}{v^s} \right) dv \ll |X| \int_{|X|}^{\infty} \frac{e^{-v}}{v^{s+1}} dv + \int_0^{|X|} \frac{e^{-v}}{v^s} dv \ll X^{1-s},$$

where we have used that

$$\left| \frac{1}{(v+X)^s} - \frac{1}{v^s} \right| \ll \begin{cases} \frac{|X|}{v^{s+1}} & \text{if } |X| \leq v \\ \frac{1}{v^s} & \text{if } |X| \geq v. \end{cases}$$

Using (5.4) in (5.3) gives

$$P_s(x) = \frac{\rho^{1-s} \Gamma(\frac{1-s}{2})}{2\pi^{\frac{1-s}{2}}} x^{\frac{s-1}{2}} + \mathcal{O}(1)$$

and finally

$$U_s(x) = \frac{\rho^{1-s} \Gamma(\frac{1-s}{2})}{2\pi^{\frac{1-s}{2}}} x^{\frac{s-1}{2}} + \mathcal{O}(1)$$

as  $x \rightarrow 0^+$ .

**Case  $s = 1$ .** We start in a similar way:

$$\left| \int_0^{+\infty} e^{-\pi v^2} \frac{e^{i\rho\pi\sqrt{x}v} - 1}{\frac{\sqrt{x}}{2} + \rho v} dv \right| \ll \int_0^{+\infty} e^{-\pi v^2} \frac{\sqrt{x}v}{v + \sqrt{2x}} dv \leq \sqrt{x} \int_0^{+\infty} e^{-\pi v^2} dv \ll \sqrt{x}.$$

Therefore

$$P_1(x) = e^{i\pi x/4} \int_0^{+\infty} \frac{e^{-\pi v^2}}{v + \bar{\rho}\frac{\sqrt{x}}{2}} dv + \mathcal{O}(\sqrt{x}).$$

Integrating by parts, we get

$$\begin{aligned} \int_0^{+\infty} \frac{e^{-\pi v^2}}{v + \bar{\rho}\frac{\sqrt{x}}{2}} dv &= \left[ \log \left( v + \bar{\rho}\frac{\sqrt{x}}{2} \right) e^{-\pi v^2} \right]_0^{+\infty} + 2\pi \int_0^{+\infty} v \log \left( v + \bar{\rho}\frac{\sqrt{x}}{2} \right) e^{-\pi v^2} dv \\ &= \log(1/\sqrt{x}) + \mathcal{O}(1). \end{aligned}$$

Thus,  $P_1(x) = \log(1/\sqrt{x}) + \mathcal{O}(1)$  when  $x \rightarrow 0^+$  and the same estimate holds for  $U_1(x)$  as claimed in item 2.

## 6. PROOF OF THEOREM 3

For simplicity, we set  $r(x) = \exp(i\pi\sigma(x)/4)$ . For any integer  $n \geq 0$ , we define an integer  $K(\ell, n)$  as follows:  $K(-1, n) = n$  and

$$K(\ell, n) = \llbracket \cdots \llbracket \llbracket n|x| \rrbracket |T(x)| \rrbracket \cdots \rrbracket |T^\ell(x)| \rrbracket$$

for any integer  $\ell \geq 0$ . For instance,  $K(0, n) = \llbracket n|x| \rrbracket$  and  $K(1, n) = \llbracket \llbracket n|x| \rrbracket |T(x)| \rrbracket$ .

We also set  $L(n) = \min\{j \geq 0 : K(j, n) = 0\}$ . This integer is well defined. Indeed, by Proposition 1, it is obvious that

$$0 \leq K(j, n) \leq n|xT(x) \cdots T^j(x)| \leq \frac{n}{q_{j+1} - q_j}$$

and the right hand side tends to 0 as  $j \rightarrow +\infty$ . By definition, we have  $K(L(n) - 1, n) \geq 1$ . It is clear that  $\lim_n L(n) = +\infty$  because otherwise if  $L(n)$  were bounded, we would have  $\lim_n K(L(n), n) = +\infty$  which is false.

### 6.1. Proof of Theorem 3, part (i).

We write  $\Omega_s(x) = c(s)|x|^{\frac{s-1}{2}} + \Delta_s(x)$  where  $c(s) = \frac{\rho^{1-s}\Gamma(\frac{1-s}{2})}{2\pi^{\frac{1-s}{2}}}$  and the function  $\Delta_s(x)$  is bounded on  $[-1, 1]$  by Theorem 2(ii). Let us define  $E_s(n, x)$  as the error term in (1.9), which is such that

$$E_s(n, x) = \mathcal{O}\left(\min\left(\frac{1}{(n+1)^s \sqrt{|x|}}, \frac{1}{|x|^{\frac{1-s}{2}}}\right)\right)$$

For any irrational number  $x \in (-1, 1)$ , Eq. (1.9) reads

$$F_{s,n}(x) = r(x)|x|^{s-\frac{1}{2}}F_{s, \lfloor nx \rfloor}(T(x)) + c(s)|x|^{\frac{s-1}{2}} + \Delta_s(x) + E_s(n, x). \quad (6.1)$$

Since  $T(x)$  is also an irrational number in  $(-1, 1)$ , we can use (6.1) with  $x$  and  $n$  replaced by  $T(x)$  and  $\lfloor nx \rfloor$  respectively, and iterate again the result. Formally, we find that for any integer  $L \geq 0$ ,

$$\begin{aligned} F_{s,n}(x) &= c(s) \sum_{j=0}^L r(x)r(T(x)) \cdots r(T^{j-1}(x)) \frac{|xT(x) \cdots T^{j-1}(x)|^{s-\frac{1}{2}}}{|T^j(x)|^{\frac{1-s}{2}}} \\ &+ \sum_{j=0}^L r(x)r(T(x)) \cdots r(T^{j-1}(x)) |xT(x) \cdots T^{j-1}(x)|^{s-\frac{1}{2}} \Delta_s(T^j(x)) \\ &+ r(x)r(T(x)) \cdots r(T^L(x)) |xT(x) \cdots T^L(x)|^{s-\frac{1}{2}} F_{s, K(L,n)}(T^{L+1}(x)) \\ &+ \sum_{j=0}^L r(x)r(T(x)) \cdots r(T^{j-1}(x)) |xT(x) \cdots T^{j-1}(x)|^{s-\frac{1}{2}} E_s(K(j-1, n), T^j(x)). \end{aligned} \quad (6.2)$$



With  $L = L(n)$ , (6.2) becomes

$$\begin{aligned}
F_{s,n}(x) &= c(s) \sum_{j=0}^{L(n)} r(x)r(T(x)) \cdots r(T^{j-1}(x)) \frac{|xT(x) \cdots T^{j-1}(x)|^{s-\frac{1}{2}}}{|T^j(x)|^{\frac{1-s}{2}}} \\
&+ \sum_{j=0}^{L(n)} r(x)r(T(x)) \cdots r(T^{j-1}(x)) |xT(x) \cdots T^{j-1}(x)|^{s-\frac{1}{2}} \Delta_s(T^j(x)) \\
&+ \sum_{j=0}^{L(n)} r(x)r(T(x)) \cdots r(T^{j-1}(x)) |xT(x) \cdots T^{j-1}(x)|^{s-\frac{1}{2}} E_s(K(j-1, n), T^j(x)).
\end{aligned} \tag{6.3}$$

because  $F_{s,K(L(n),n)}(T^{L(n)+1}(x)) = 0$ , being an empty sum.

Under hypothesis (1.10) of Theorem 3(i), the series

$$c(s) \sum_{j=0}^{\infty} r(x)r(T(x)) \cdots r(T^{j-1}(x)) \frac{|xT(x) \cdots T^{j-1}(x)|^{s-\frac{1}{2}}}{|T^j(x)|^{\frac{1-s}{2}}}$$

converge absolutely (recall that  $|r(x)r(T(x)) \cdots r(T^{j-1}(x))| = 1$ ) and this also forces the absolute convergence of the series

$$\sum_{j=0}^{\infty} r(x)r(T(x)) \cdots r(T^{j-1}(x)) |xT(x) \cdots T^{j-1}(x)|^{s-\frac{1}{2}} \Delta_s(T^j(x))$$

because  $|T^j(x)| \leq 1$  and  $\Delta_s$  is bounded on  $[-1, 1]$ . Hence, since  $L(n) \rightarrow +\infty$  with  $n$  and by (6.2), the convergence of  $F_{s,n}(x)$  will follow from the proof that

$$\sum_{j=0}^{L(n)} r(x)r(T(x)) \cdots r(T^{j-1}(x)) |xT(x) \cdots T^{j-1}(x)|^{s-\frac{1}{2}} E_s(K(j-1, n), T^j(x))$$

tends to 0 as  $n \rightarrow +\infty$ . For this, we observe that (by Theorem 2)

$$\begin{aligned}
\lim_{n \rightarrow +\infty} E_s(K(j-1, n), T^j(x)) &= 0, \\
|E_s(K(j-1, n), T^j(x))| &\ll_s \frac{1}{|T^j(x)|^{\frac{1-s}{2}}}
\end{aligned} \tag{6.4}$$

and

$$\sum_{j=0}^{\infty} \frac{|xT(x) \cdots T^{j-1}(x)|^{s-\frac{1}{2}}}{|T^j(x)|^{\frac{1-s}{2}}} < +\infty$$

by hypothesis (1.10). Hence, Tannery's theorem can be applied and it yields that

$$\lim_{n \rightarrow +\infty} \sum_{j=0}^{L(n)} r(x)r(T(x)) \cdots r(T^{j-1}(x)) |xT(x) \cdots T^{j-1}(x)|^{s-\frac{1}{2}} E_s(K(j-1, n), T^j(x)) = 0$$

as expected. We now let  $n \rightarrow +\infty$  in (6.3) to get the identity

$$F_s(x) = \sum_{j=0}^{\infty} r(x)r(T(x)) \cdots r(T^{j-1}(x)) |xT(x) \cdots T^{j-1}(x)|^{s-\frac{1}{2}} \Omega_s(T^j(x)).$$

### 6.2. Proof of Theorem 3, part (ii).

The proof is more complicated than when  $s < 1$ . We can write a bound similar to (6.4) but the corresponding right hand side depends on  $n$  and we cannot invoke Tannery's theorem. Instead we use an ad hoc inelegant method.

For this, we first need more informations on  $K(j-1, n)$ . By multiple applications of the trivial inequality  $\lfloor \alpha \rfloor \geq \alpha - 1$ , we get

$$\begin{aligned} K(j-1, n) &\geq n|xT(x) \cdots T^{j-1}(x)| - \sum_{k=1}^j |T^k(x) \cdots T^{j-1}(x)| \\ &= n|xT(x) \cdots T^{j-1}(x)| \cdot \left( 1 - \frac{1}{n} \sum_{k=1}^j \frac{1}{|xT(x) \cdots T^{k-1}(x)|} \right). \end{aligned} \quad (6.5)$$

We define  $J = J(n)$  as the maximal integer such that  $Jq_J \leq n/4$ ; it is clear that  $J \rightarrow +\infty$  with  $n$ . We claim that

$$\sum_{k=1}^j \frac{1}{|xT(x) \cdots T^{k-1}(x)|} \leq \frac{n}{2}$$

for all  $j \in \{0, \dots, J\}$ . Indeed, the sequence  $|xT(x) \cdots T^{k-1}(x)|$  is non-increasing (because  $|T^m(x)| \leq 1$ ) and thus

$$\sum_{k=1}^j \frac{1}{|xT(x) \cdots T^{k-1}(x)|} \leq \frac{j}{|xT(x) \cdots T^{j-1}(x)|} = j|q_j + e_j x_j q_{j-1}| \leq 2jq_j \leq 2Jq_J \leq \frac{n}{2}$$

(Note that the sequence  $(jq_j)_j$  is increasing.) Therefore, (6.5) yields that

$$K(j-1, n) \geq \frac{n}{2} |xT(x) \cdots T^{j-1}(x)|$$

for any  $j \in \{0, \dots, J\}$ . Moreover, for those  $j$ , we also have  $K(j-1, n) \geq 1$ .

We write  $\Omega_1(x) = \log(1/\sqrt{|x|}) + \Delta_1(x)$  where the function  $\Delta_1(x)$  is bounded on  $[-1, 1]$  by Theorem 2(ii). Let us define  $E_1(n, x)$  as the error term in (1.9) for  $s = 1$ , i.e.

$$E_1(n, x) = \mathcal{O}\left( \min\left( \frac{1}{(n+1)\sqrt{|x|}}, |\log((n+1)\sqrt{|x|})| + 1 \right) \right)$$

For any irrational number  $x \in (-1, 1)$ , Eq. (1.9) reads

$$F_{1,n}(x) = r(x)\sqrt{|x|}F_{1, \lfloor nx \rfloor}(T(x)) - \frac{1}{2} \log|x| + \Delta_1(x) + E_1(n, x). \quad (6.6)$$

Since  $T(x)$  is also an irrational number in  $(-1, 1)$ , we can use (6.6) with  $x$  and  $n$  replaced by  $T(x)$  and  $\lfloor nx \rfloor$  respectively, and iterate again the result. Formally, we find that for any integer  $L \geq 0$ ,

$$\begin{aligned}
F_{1,n}(x) &= -\frac{1}{2} \sum_{j=0}^L r(x)r(T(x)) \cdots r(T^{j-1}(x)) \sqrt{|xT(x) \cdots T^{j-1}(x)|} \log |T^j(x)| \\
&\quad + \sum_{j=0}^L r(x)r(T(x)) \cdots r(T^{j-1}(x)) \sqrt{|xT(x) \cdots T^{j-1}(x)|} \Delta_1(T^j(x)) \\
&\quad + r(x)r(T(x)) \cdots r(T^L(x)) \sqrt{|xT(x) \cdots T^L(x)|} F_{1,K(L,n)}(T^{L+1}(x)) \\
&\quad + \sum_{j=0}^L r(x)r(T(x)) \cdots r(T^{j-1}(x)) \sqrt{|xT(x) \cdots T^{j-1}(x)|} E_1(K(j-1, n), T^j(x)).
\end{aligned} \tag{6.7}$$

From now on, the letter  $L$  will stand exclusively for  $L(n)$  (defined above), in which case (6.7) becomes

$$\begin{aligned}
F_{1,n}(x) &= -\frac{1}{2} \sum_{j=0}^L r(x)r(T(x)) \cdots r(T^{j-1}(x)) \sqrt{|xT(x) \cdots T^{j-1}(x)|} \log |T^j(x)| \\
&\quad + \sum_{j=0}^L r(x)r(T(x)) \cdots r(T^{j-1}(x)) \sqrt{|xT(x) \cdots T^{j-1}(x)|} \Delta_1(T^j(x)) \\
&\quad + \sum_{j=0}^L r(x)r(T(x)) \cdots r(T^{j-1}(x)) \sqrt{|xT(x) \cdots T^{j-1}(x)|} E_1(K(j-1, n), T^j(x)).
\end{aligned} \tag{6.8}$$

because  $F_{1,K(L,n)}(T^{L+1}(x)) = 0$ , being an empty sum.

Under hypothesis (1.13) of Theorem 3(ii), the series

$$-\frac{1}{2} \sum_{j=0}^{\infty} r(x)r(T(x)) \cdots r(T^{j-1}(x)) \sqrt{|xT(x) \cdots T^{j-1}(x)|} \log |T^j(x)|,$$

converges absolutely and hypothesis (1.12) implies the absolute convergence

$$\sum_{j=0}^{\infty} r(x)r(T(x)) \cdots r(T^{j-1}(x)) \sqrt{|xT(x) \cdots T^{j-1}(x)|},$$

which in turn forces the absolute convergence of

$$\sum_{j=0}^{\infty} r(x)r(T(x)) \cdots r(T^{j-1}(x)) \sqrt{|xT(x) \cdots T^{j-1}(x)|} \Delta_1(T^j(x))$$

by boundedness of  $\Delta_1$  on  $[-1, 1]$ . Hence, since  $L(n) \rightarrow +\infty$  with  $n$  and by (6.7), the convergence of  $F_{1,n}(x)$  will follow from the proof that

$$R_1(n, x) := \sum_{j=0}^{L(n)} \sqrt{|xT(x) \cdots T^{j-1}(x)|} E_1(K(j-1, n), T^j(x))$$

tends to 0 as  $n \rightarrow +\infty$ .

With  $H = \min(L, J)$ , we split  $R_1(n, x)$  into three parts:

$$\begin{aligned} R_1(n, x) = & \left( \sum_{j=0}^{H-1} + \sum_{j=H}^{L-1} \right) \sqrt{|xT(x) \cdots T^{j-1}(x)|} E_1(K(j-1, n), T^j(x)) \\ & + \sqrt{|xT(x) \cdots T^{L-1}(x)|} E_1(K(L-1, n), T^L(x)). \end{aligned}$$

The sum from  $H$  to  $L-1$  might be empty if  $H = L$ , which would simplify the proof below. We assume that it is not empty, i.e that  $H = J \leq L-1$ . For the two sums from  $j=0$  to  $j=L-1$ , we use the fact that (for any  $n \geq 0$ , any  $x > 0$ )

$$E_1(n, x) = \mathcal{O} \left( \frac{1}{(n+1)\sqrt{|x|}} \right).$$

First

$$\begin{aligned} & \sum_{j=0}^{H-1} \sqrt{|xT(x) \cdots T^{j-1}(x)|} E_1(K(j-1, n), T^j(x)) \\ & \ll \sum_{j=0}^{J-1} \frac{\sqrt{|xT(x) \cdots T^{j-1}(x)|}}{(K(j-1, n) + 1)\sqrt{|T^j(x)|}} \leq \frac{2}{n} \sum_{j=0}^{J-1} \frac{\sqrt{|xT(x) \cdots T^{j-1}(x)|}}{(|xT(x) \cdots T^{j-1}(x)| + 1)\sqrt{|T^j(x)|}} \\ & \leq \frac{2}{n} \sum_{j=0}^{J-1} \frac{1}{\sqrt{|xT(x) \cdots T^j(x)|}} \leq \frac{2J}{n\sqrt{|xT(x) \cdots T^{J-1}(x)|}} \\ & \leq \frac{4J\sqrt{q_J}}{n} \leq \frac{4\sqrt{Jq_J}}{n} \leq \frac{2}{\sqrt{n}} \end{aligned}$$

by definition of  $J$ .

Second, for the sum from  $j = H$  to  $j = L - 1$ , we observe that  $K(j - 1, n) \geq \frac{1}{|T^j(x)|}$  because  $1 \leq K(j, n) = \lfloor K(j - 1, n)|T^j(x)| \rfloor$  for such  $j$  by definition of  $L$ . Hence,

$$\begin{aligned}
& \sum_{j=H}^{L-1} \sqrt{|xT(x) \cdots T^{j-1}(x)|} E_1(K(j - 1, n), T^j(x)) \\
& \ll \sum_{j=H}^{L-1} \frac{\sqrt{|xT(x) \cdots T^{j-1}(x)|}}{(K(j - 1, n) + 1)\sqrt{|T^j(x)|}} \leq \sum_{j=H}^{L-1} \frac{\sqrt{|xT(x) \cdots T^{j-1}(x)|} \cdot |T^j(x)|}{\sqrt{|T^j(x)|}} \\
& \leq \sum_{j=H}^{L-1} \sqrt{|xT(x) \cdots T^j(x)|}. \tag{6.9}
\end{aligned}$$

The series with term  $\sqrt{|xT(x) \cdots T^j(x)|}$  is convergent (by hypothesis), thus the expression in (6.9) tends to 0 as  $n$  tends to infinity because  $H$  tends to  $+\infty$  with  $n$ . It remains to consider the case for  $j = L$ . Here, we use the fact that (for any  $n \geq 0$ , any  $x > 0$ )

$$E_1(n, x) \ll \left| \log((n + 1)\sqrt{|x|}) \right| + 1.$$

(In this case,  $n$  and  $x$  are replaced by  $K(L - 1, n)$  and  $T^L(x)$  respectively.) Hence,

$$\begin{aligned}
& \sqrt{|xT(x) \cdots T^{L-1}(x)|} E(K(L - 1, n), T^L(x)) \\
& \ll \sqrt{|xT(x) \cdots T^{L-1}(x)|} \cdot \left( 1 + \left| \log((K(L - 1, n) + 1)\sqrt{|T^L(x)|}) \right| \right) \tag{6.10}
\end{aligned}$$

The properties  $K(L - 1, n) \geq 1$  and  $\lfloor K(L - 1, n)|T^L(x)| \rfloor = 0$  together imply that  $|T^L(x)| \leq K(L - 1, n)|T^L(x)| < 1$ , so that

$$\begin{aligned}
\left| \log((K(L - 1, n) + 1)\sqrt{|T^L(x)|}) \right| & \leq \left| \log(K(L - 1, n)\sqrt{|T^L(x)|}) \right| + \log(2) \\
& \leq \frac{3}{2} \left| \log |T^L(x)| \right| + \log(2).
\end{aligned}$$

Therefore, (6.10) becomes

$$\begin{aligned}
& \sqrt{|xT(x) \cdots T^{L-1}(x)|} E(K(L - 1, n), T^L(x)) \\
& \ll \sqrt{|xT(x) \cdots T^{L-1}(x)|} \cdot \left( 1 + \left| \log |T^L(x)| \right| \right). \tag{6.11}
\end{aligned}$$

Hypothesis (1.12) and (1.13) in Theorem 3(ii) now ensure that the right hand side of (6.11) tends to 0 when  $L(n) \rightarrow +\infty$ . This concludes the proof that  $R_1(n, x)$  tends to 0 when  $n \rightarrow +\infty$ .

We now let  $n \rightarrow +\infty$  in (6.8) to get the identity

$$F_1(x) = \sum_{j=0}^{\infty} r(x)r(T(x)) \cdots r(T^{j-1}(x)) \sqrt{|xT(x) \cdots T^{j-1}(x)|} \Omega_1(T^j(x)).$$

## 7. PROOF OF THEOREM 5

## 7.1. Proof of Theorem 5, part (i).

We introduce a second dynamical system more adapted to our study (see the work of Schweiger [15, 16]). Let us partition the interval  $(0, 1]$  into the double-indexed sequence of intervals

$$B(+1, k) = \left( \frac{1}{2k}, \frac{1}{2k-1} \right] \quad \text{and} \quad B(-1, k) = \left( \frac{1}{2k+1}, \frac{1}{2k} \right].$$

Consider the map  $U : [0, 1] \rightarrow [0, 1]$  by  $U(0) = 0$  and if  $x \neq 0$

$$U(x) = e \cdot \left( \frac{1}{x} - 2k \right) \quad \text{when } x \in B(e, k).$$

It is trivial to check that for every irrational  $x \in [0, 1]$ ,  $|U(x)| = |T(x)|$ . So we will prove Theorem 5 using this transformation  $U$  instead of  $T$ .

A key property is that  $U$  is an ergodic transformation with a  $\sigma$ -finite invariant measure  $\mu$  with infinite mass (due to the parabolic point 1 for the mapping  $U$ ), whose density relatively to the Lebesgue measure is given by

$$d_\mu(x) = \frac{1}{x+1} + \frac{1}{1-x}.$$

Further we need to study in details the orbit of a typical point  $x$  under the action of the dynamical system  $([0, 1], U)$ . For this, we introduce the points  $x_p := \frac{p-1}{p} = 1 - \frac{1}{p}$ , for every integer  $p \geq 1$ . Observe that for every  $p \geq 2$ ,

$$U(x_p) = x_{p-1}. \tag{7.1}$$

**Lemma 5.** *Let  $p \geq 2$  be an integer, and let  $y$  be such that*

$$x_{p-1} \leq y \leq x_p.$$

*Then for every  $0 \leq m \leq p-2$ , one has*

$$\begin{aligned} x_{p-1-m} &\leq U^m(y) < x_{p-m} \\ \frac{p-m-2}{p-1} &\leq yU(y) \cdots U^m(y) \leq \frac{p-m-1}{p}. \end{aligned} \tag{7.2}$$

*Proof.* Equation (7.2) follows from (7.1) and the monotonicity of  $U$  on the interval  $[1/2, 1)$ . Further,

$$\begin{aligned} U^{P_k+p_k^1}(x) \cdots U^{P_k+p_k^1+m}(x) &\leq x_p x_{p-1} \cdots x_{p-m} \\ &\leq \frac{p-1}{p} \frac{p-2}{p-1} \cdots \frac{p-m-1}{p-m} = \frac{16}{p} = \frac{p-m-1}{p}. \end{aligned}$$

The same holds for the lower bound. □

Recall that  $\sigma(x)$  stands for the sign of  $x \in \mathbb{R}$ .

**Lemma 6.** *For every irrational  $x \in [0, 1]$ , there are two possibilities: either there exists an integer  $j_x$  such that for every  $j \geq j_x$ ,  $U^j(x) \leq x_2$ , or there exist two sequences of integers  $(p_k^1)_{k \geq 1}$  and  $(p_k^2)_{k \geq 1}$  satisfying the following: for every  $k \geq 4$ ,  $p_k^1 \geq 1$ ,  $p_k^2 \geq 1$ , and if we set  $P_{k+1} = \sum_{\ell=1}^k p_\ell^1 + p_k^2$ , then:*

- for every  $P_k \leq j \leq P_k + p_k^1 - 1$ ,  $U^j(x) \leq x_2 = 1/2$ ,
- for every  $P_k + p_k^1 \leq j \leq P_k + p_k^1 + p_k^2 - 1 = P_{k+1} - 1$ ,  $x_2 < U^{j+1}(x) < U^j(x)$ , and the  $\sigma(T^j(x))$  are all equal (i.e. the  $T^j(x)$  have all the same sign).
- $p_k^2$  is the unique integer greater than 1 such that

$$x_{p_k^2+1} \leq U^{P_k+p_k^1}(x) \leq x_{p_k^2+2}$$

*Proof.* Assume that we are not in the first case. Hence  $U^j(x) > x_2$  for infinitely many integers  $j$ .

Consider the first integer  $j$  such that  $U^j(x) > x_2$ , and call this integer  $p_1^1$ . Observe that it is possible that  $p_1^1 = 0$ , if  $x$  itself is greater than  $x_2$ . On the interval  $[x_2, 1]$ , as already stated in Lemma 5, the map  $U$  is strictly decreasing and concave, and has a derivative strictly less than one when  $x < 1$ . Consequently, as long as  $U^j(x)$  stays in the interval  $[x_2, 1)$ , the sequence  $U^j(x)$  is strictly decreasing. Moreover, using (7.2), there is a first integer  $p_1^2$  such that  $U^{p_1^1+p_1^2}(x) > x_2$  and  $U^{p_1^1+p_1^2}(x) \leq x_2$ .

Iterating this scheme allows to find the sequences  $(p_k^1)$  and  $(p_k^2)$ . The fact that the  $T^j(x)$  have all the same sign follows from the fact that  $T(\pm[2/3, 1]) \subset \pm[1/2, 1)$ .

The third item follows from the definition of  $p_k^2$  and Lemma 5.  $\square$

The rest of this section is devoted to the proof of part (i) of Theorem 5.

We denote  $M_\Omega$  a positive constant such that  $|\Omega(x)| \leq M_\Omega$  for every  $x$ . We fix an irrational number  $x \in (0, 1)$ , and for convenience we will denote  $u_j := |U^j(x)|$ . If there exists  $j_x$  such that  $u_j \leq x_{16}$  for every  $j \geq j_x$ , then the series converges. Thus, we assume that  $u_j \geq x_{16}$  for infinitely many integers  $j$ . Adapting Lemma 6, we immediately get:

**Lemma 7.** *For every irrational  $x \in [0, 1]$ , there are two possibilities: either there exists an integer  $j_x$  such that for every  $j \geq j_x$ ,  $U^j(x) \leq x_{16}$ , or there exist two sequences of integers  $(p_k^1)_{k \geq 1}$  and  $(p_k^2)_{k \geq 1}$  satisfying the following: for every  $k \geq 4$ ,  $p_k^1 \geq 1$ ,  $p_k^2 \geq 14$ , and if we set  $P_{k+1} = \sum_{\ell=1}^k p_\ell^1 + p_k^2$ , then:*

- for every  $P_k \leq j \leq P_k + p_k^1 - 1$ ,  $U^j(x) \leq x_{16}$ ,
- for every  $P_k + p_k^1 \leq j \leq P_{k+1} - 1$ ,  $x_{16} < U^{j+1}(x) < U^j(x)$ .
- $p_k^2$  is the unique integer greater than 1 such that

$$x_{p_k^2+15} \leq U^{P_k+p_k^1}(x) \leq x_{p_k^2+16}$$

The difference with Lemma 6 is that we impose  $p_k^1 \geq 14$  for every  $k$ . This follows from the fact that the sequence  $(U^m(y))$  is slowly decreasing when  $y \in [x_2, x_{16}]$ .

It is clear that the convergence of the sequence (1.21) does not depend on the first terms. Hence, without loss of generality, we assume that  $n_1 \geq 4$ . We now bound by above the

partial sums

$$\Sigma_J = \sum_{j=1}^J e^{i\frac{\pi}{4} \sum_{\ell=0}^{j-1} \sigma(T^\ell x)} |xT(x) \cdots T^{j-1}(x)|^\alpha \Omega(T^j(x)).$$

**Step 1:** We separate the cases where  $1 \leq j \leq p_1^1 - 1$  and  $n_1 \leq j \leq p_1^1 + p_1^2 - 1 = P_1 - 1$ .

- If  $j \leq p_1^1 - 1$ , then we have

$$\sum_{j=1}^{p_1^1-1} \left| e^{i\frac{\pi}{4} \sum_{\ell=0}^{j-1} \sigma(T^\ell x)} |xT(x) \cdots T^{j-1}(x)|^\alpha \Omega(T^j(x)) \right| \leq M_\Omega \frac{x_{16} - x_{16}^{p_1^1}}{1 - x_{16}} \leq M_\Omega \frac{x_{16}}{1 - x_{16}}.$$

Hence

$$|\Sigma_{p_1^1-1}| \leq M_\Omega \frac{x_{16}}{1 - x_{16}}.$$

- We now consider  $p_1^1 \leq j \leq p_1^1 + p_1^2 - 1 = P_1 - 1$ . We observe that

$$\begin{aligned} \Sigma_{P_1-1} - \Sigma_{p_1^1-1} &= (u_1 u_2 \cdots u_{p_1^1-1})^\alpha e^{i\frac{\pi}{4} \sum_{\ell=0}^{p_1^1-1} \sigma(T^\ell x)} \\ &\quad \times \sum_{j=0}^{p_1^2-1} e^{i\frac{\pi}{4} \sum_{\ell=p_1^1}^{p_1^1+j} \sigma(T^\ell x)} (u_{p_1^1} u_{p_1^1+1} \cdots u_{n_1+j})^\alpha \Omega(T^{p_1^1+j+1}(x)). \end{aligned} \quad (7.3)$$

By the second item of Lemma 6, all the  $\sigma(T^{p_1^1+j+1}(x))$  are equal. We assume, without loss of generality, that they are equal to 1. Hence, we need to take care of the sum

$$\sum_{j=0}^{p_1^2-1} e^{i\frac{\pi}{4} j} (u_{p_1^1} u_{p_1^1+1} \cdots u_{p_1^1+j})^\alpha \Omega(T^{p_1^1+j+1}(x)). \quad (7.4)$$

Now, we use that the  $T^{p_1^1+j+1}(x)$  are all close to 1. Since  $\Omega$  is differentiable at 1, we have for every  $j \in \{0, \dots, p_1^2 - 1\}$

$$\Omega(T^{p_1^1+j+1}(x)) = \Omega(1) + \Omega'(1)(T^{p_1^1+j+1}(x) - 1) + o(T^{p_1^1+j+1}(x) - 1).$$

Using this equation, we split the sum (7.4) into two parts:

- (i) The sum

$$S_1 = \sum_{j=0}^{p_1^2-1} e^{i\frac{\pi}{4} j} (u_{p_1^1} u_{p_1^1+1} \cdots u_{p_1^1+j})^\alpha \Omega(1)$$

can be rewritten (since  $e^{i\frac{\pi}{4} 8} = 1$ )

$$\Omega(1) \sum_{j=0}^{\lfloor \frac{p_1^2-1}{8} \rfloor} (u_{p_1^1} u_{p_1^1+1} \cdots u_{p_1^1+8j-1})^\alpha \left( \sum_{m=0}^7 (u_{p_1^1+8j} u_{p_1^1+8j+1} \cdots u_{p_1^1+8j+m})^\alpha e^{i\frac{\pi}{4} m} \right)$$



$$+\Omega(1) \sum_{j=8 \left\lfloor \frac{p_1^2-1}{8} \right\rfloor}^{p_1^2-1} e^{i\frac{\pi}{4}j} (u_{p_1^1} u_{p_1^1+1} \cdots u_{p_1^1+j})^\alpha.$$

The second sum contains at most 7 terms all bounded in absolute value by 1, hence it is less than  $7\Omega(1)$ .

We now consider the sum between parenthesis above. Fix  $j \in \{0, \dots, \left\lfloor \frac{p_1^2-1}{8} \right\rfloor\}$ , and let us write for every  $m = 0, \dots, 7$ ,

$$u_{p_1^1+8j} u_{p_1^1+8j+1} \cdots u_{p_1^1+8j+m} = 1 - \varepsilon_m^j.$$

Obviously from their definition, the  $\varepsilon_m^j$  are small positive quantities, and they are increasing with  $m$ . We thus have

$$\begin{aligned} \sum_{m=0}^7 (u_{p_1^1+8j} u_{p_1^1+8j+1} \cdots u_{p_1^1+8j+m})^\alpha e^{i\frac{\pi}{4}m} &= \sum_{m=0}^7 (1 - \varepsilon_m^j)^\alpha e^{i\frac{\pi}{4}m} \\ &= \sum_{m=0}^7 (1 - \alpha\varepsilon_m^j + o(\varepsilon_m^j)) e^{i\frac{\pi}{4}m} \\ &= - \sum_{m=0}^7 (\alpha\varepsilon_m^j + o(\varepsilon_m^j)) e^{i\frac{\pi}{4}m}. \end{aligned}$$

In absolute value, this sum is less than  $16\varepsilon_7^j$  (since the  $\varepsilon_m^j$  are small and increasing). In addition, since the terms  $u_{p_1^1+8j+m}$  belong to the interval  $(x_{16}, 1)$ , one also has  $8\varepsilon_7^j \leq C\varepsilon_j^0 = C(1 - u_{p_1^1+8j})$ . Finally, we get

$$|S_1| \leq C\Omega(1) \left( 1 + \sum_{j=0}^{\left\lfloor \frac{p_1^2-1}{8} \right\rfloor} (u_{p_1^1} u_{p_1^1+1} \cdots u_{p_1^1+8j-1})^\alpha (1 - u_{p_1^1+8j}) \right). \quad (7.5)$$

Using Lemma 5 (in particular that  $u_{p_1^1} \leq x_{p_2+16}$ ), we can bound by above this sum by

$$\begin{aligned} |S_1| &\leq C\Omega(1) \left( 1 + \sum_{j=0}^{\left\lfloor \frac{p_1^2-1}{8} \right\rfloor} \left( \frac{p_1^2 - 8j}{p_1^2} \right)^\alpha \frac{1}{p_1^2 - 8j} \right) \leq C\Omega(1) \left( 1 + \sum_{j=0}^{\left\lfloor \frac{p_1^2-1}{8} \right\rfloor} \frac{(p_1^2 - 8j)^{\alpha-1}}{(p_1^2)^\alpha} \right) \\ &= C\Omega(1) \end{aligned}$$

for some constant  $C$  independent of the problem.

(ii) The second sum

$$S_2 = \sum_{j=0}^{p_1^2-1} e^{i\frac{\pi}{4}j} (u_{p_1} u_{p_1+1} \cdots u_{p_1+j})^\alpha \Omega'(1) \left( T^{p_1^1+j+1}(x) - 1 \right) + o(T^{p_1^1+j+1}(x) - 1)$$

is bounded in absolute value, by using again Lemma 5 to find explicit upper bounds for the terms  $u_{p_1} u_{p_1+1} \cdots u_{p_1+j}$  and  $|T^{p_1^1+j+1}(x) - 1|$ . We find, for some constant  $C$  depending on  $\Omega$  only,

$$|S_2| \leq C \sum_{j=0}^{p_1^2-1} \left( \frac{p_1^2 - j}{p_1^2} \right)^\alpha \frac{1}{p_1^2 - j} \leq C \sum_{j=0}^{p_1^2-1} \frac{(p_1^2 - j)^{\alpha-1}}{(p_1^2)^\alpha} = C.$$

Hence, going back to (7.3), we obtain

$$|\Sigma_{P_1-1} - \Sigma_{p_1^1-1}| \leq C(u_1 u_2 \cdots u_{p_1^1-1})^\alpha$$

for some constant depending on  $\Omega$ . The same holds if the  $T^j(x)$  are all close to -1 and not to 1.

**Step 2:** We separate the cases where  $0 \leq j \leq P_1 + p_2^1 - 1$  and  $P_1 + p_2^1 \leq j \leq P_1 + p_2^1 + p_2^2 - 1 = P_2 - 1$ .

- If  $0 \leq j \leq p_2^1 - 1$ : all the terms  $u_{P_1+j}$  satisfy  $|u_{P_1+j}| \leq x_{16}$ . We deduce that

$$|\Sigma_{P_1+p_2^1-1} - \Sigma_{P_1-1}| \leq M_\Omega (u_1 u_2 \cdots u_{P_1})^\alpha \sum_{j=0}^{p_2^1-1} (x_{16})^j \leq M_\Omega \frac{(u_1 u_2 \cdots u_{P_1})^\alpha}{1 - x_{16}}.$$

- We now consider  $p_2^2 \leq j \leq P_2 - 1$ . The same procedure as above (in Step 1) yields

$$\left| \Sigma_{P_2-1} - \Sigma_{P_1+p_2^1-1} \right| \leq C(u_1 u_2 \cdots u_{P_1+p_2^1-1})^\alpha.$$

**Step k:** By an immediate recurrence, one obtains that for every  $k \geq 1$ , we have the following properties:

$$\begin{aligned} \tau_k^1 &:= \left| \Sigma_{P_k+p_{k+1}^1-1} - \Sigma_{P_k} \right| \leq M_\Omega \frac{(u_1 u_2 \cdots u_{P_k})^\alpha}{1 - x_{16}}, \\ \tau_k^2 &:= \left| \Sigma_{P_{k+1}-1} - \Sigma_{P_k+p_{k+1}^1-1} \right| \leq C(u_1 u_2 \cdots u_{P_k+p_{k+1}^1-1})^\alpha. \end{aligned}$$

We can now conclude regarding the convergence of the series (1.21). Obviously by construction  $p_k^2 \geq 14$ , and the first series  $(\sum_{k \geq 1} \tau_k^1)$  converges, since the ratio between two consecutive terms  $\frac{(u_1 u_2 \cdots u_{P_{k+1}})^\alpha}{(u_1 u_2 \cdots u_{P_k})^\alpha}$  is less than  $(x_{16})^{14\alpha} < 1$ .

For the second series  $(\sum_{k \geq 1} \tau_k^2)$ , the same argument applies (the ratio between two consecutive terms  $\frac{(u_1 u_2 \cdots u_{P_{k+1}+p_{k+2}^1-1})^\alpha}{(u_1 u_2 \cdots u_{P_k+p_{k+1}^1-1})^\alpha}$  is less than  $(x_{16})^{14\alpha} < 1$ ).

## 7.2. Proof of Theorem 5, parts (ii) and (iii).

We consider the two sums (1.24) and (1.22). Using Theorem 5(i), the only problem may come from the terms with  $|T^j(x)| \leq 1/2$ , since one can write  $\log \frac{1}{|x|} = (\log \frac{1}{|x|}) \mathbf{1}_{[0,1/2)}(x) + \Omega(x)$ , where  $\Omega$  is bounded and differentiable at 1 and  $-1$ . The same holds for  $\frac{1}{|x|^\beta}$ . Thus we will focus on the convergence of the series

$$\sum_{j=0}^{\infty} |xT(x) \cdots T^{j-1}(x)|^\alpha \log \left( \frac{1}{|T^j(x)|} \right) \mathbf{1}_{[0,1/2)}(T^j(x))$$

and

$$\sum_{j=0}^{\infty} \frac{|xT(x) \cdots T^{j-1}(x)|^\alpha}{|T^j(x)|^\beta} \mathbf{1}_{[0,1/2)}(T^j(x)).$$

We will prove the absolute convergence because the complex terms do not play any role any more in this convergence. The next lemma is key in the proof.

**Lemma 8.** *If  $|T^j(x)| \leq 1/2$ , then there exists an integer  $n$  such that*

$$|xT(x) \cdots T^{j-1}(x)| \leq \frac{1}{Q_{n-1}} \quad \text{and} \quad |T^j(x)| \geq \frac{1}{A_n}.$$

*Moreover, with two different  $j$ 's such that  $|T^j(x)| \leq 1/2$  correspond two different  $n$ 's.*

*Proof.* This follows from a fine analysis of the even and regular continued fractions. If  $x$  is an irrational number whose RCF expansion  $x = [A_1, A_2, \dots]_R$  contains only even numbers, then one knows that the ECF expansion of  $x$  is  $x = [(1, A_1), (1, A_2), \dots]_E$ , so  $T^j(x) = G^j(x)$  for every  $j \geq 1$ , and the lemma follows from (2.2) and the definition of  $G$ .

We assume that this is not the case. Let us start with two observations on the shape of the mapping  $T$ :

(R1)  $1/2 < |y = T^{m-1}(x)| < 1$ . When  $1/2 < y < 1$ , then  $T(y)$  is given by  $T(y) = 2 - \frac{1}{y}$ . Similarly, when  $-1 < y < -1/2$ ,  $T(y) = -2 - \frac{1}{y}$ . Subsequently, if  $|T^{m-1}(x)| \geq 1/2$  for some  $m$ , then, recalling the algorithm (2.6) producing the ECF from the RCF, the  $m$ -term in the ECF expansion (2.5) of  $x = [(e_1, a_1), (e_2, a_2), \dots]_E$  will be  $(-1, 2)$ .

(R2)  $y = T^{m-1}(x) \in (-1/2, 1/2)$ . If  $T(y) \in (-1/2, 0)$ , then  $\left\lfloor \frac{1}{y} \right\rfloor$  is necessarily even. In this case, the  $m$ -th term in the expansion of  $x$  is  $(1, A_n)$  for some integer  $n$ .

If  $T(y) \in (-1/2, 0)$ , then  $\left\lfloor \frac{1}{y} \right\rfloor$  is odd and  $\left\lfloor \frac{1}{T(y)} \right\rfloor$  is necessarily equal to 1. This simply follows from the form of the mapping  $T$ . Again using the algorithm (2.6), this amounts to change the RCF terms

$$A_n + \frac{1}{1 + \frac{1}{A_{n+2} + \dots}}$$

into

$$A_n + \frac{-1}{(A_{n+2} + 1) + \dots},$$

in order to get the ECF expansion of  $x$ , whose  $m$ -th term is necessarily  $(-1, A_{n+2} + 1)$  for some integer  $n$ .

We treat the case where  $0 < T^j(x) < 1/2$ , the other case is symmetric. There are two possibilities:

- If  $T^{j-1}(x) > 1/2$ : Necessarily, we are at a step  $j$  where, in the ECF expansion  $x = [(e_1, a_1), (e_2, a_2), \dots]_E$  of  $x$ , there was a sequence of  $(-1, 2)$  for some time before the index  $j$ . Then, this sequence of  $(-1, 2)$  stops at the  $(j+1)$ -th iterate, since for  $y = T^j(x)$ ,  $T(y)$  is not defined by  $T(y) = 2 - \frac{1}{y}$  any more. By our remark (R1) above, this can be translated to the ECF and RCF expansions as follows: there is an integer  $n$  such that

$$x = \frac{e_1}{a_1 + \frac{e_2}{a_2 + \frac{e_3}{\dots + \frac{-1}{2 + \frac{-1}{a_j + T^{j+1}(x)}}}}} = \frac{1}{A_1 + \frac{1}{A_2 + \frac{1}{\dots + \frac{1}{A_n + G^n(x)}}}}$$

where  $a_j = A_n + 1$ . In particular, this implies that  $\frac{p_j}{q_j} = \frac{P_n}{Q_n}$ . In this case, we know by Propositions 1 and 2 (as in Section 8) that  $q_j - q_{j-1} \geq Q_{n-1}$ . This yields

$$|xT(x) \cdots T^{j-1}(x)| \leq \frac{1}{q_{j+1} - q_j} \leq \frac{1}{Q_{n-1}}.$$

Moreover, we have  $\frac{1}{T^j(x)} - a_j \in (-1, 1)$ , with  $a_j = A_n + 1 \geq 2$ . We deduce that  $T^j(x) \geq \frac{1}{a_j - 1} = \frac{1}{A_n}$ .

- If  $-1/2 < T^{j-1}(x) < 1/2$ : then one can apply our remark (R2) above: for some integer  $n$ , one has

$$x = \frac{e_1}{a_1 + \frac{e_2}{a_2 + \frac{e_{\tilde{j}+m}}{\dots + \frac{e_{\tilde{j}+m}}{a_{\tilde{j}+m} + \frac{e_{\tilde{j}+m}}{a_{\tilde{j}+m} + T^{\tilde{j}+m+1}(x)}}}}} = \frac{1}{A_1 + \frac{1}{A_2 + \frac{1}{\dots + \frac{1}{A_n + G^n(x)}}}}$$

with either  $(e_{\tilde{j}+m}, a_{\tilde{j}+m}) = (1, A_n)$  or  $(e_{\tilde{j}+m}, a_{\tilde{j}+m}) = (-1, A_n + 1)$  for some integer  $n$ . In both case we have  $\frac{p_j}{q_j} = \frac{P_n}{Q_n}$  for some integer  $n$ , and the same estimates as above hold true.

It is obvious from the construction that the integers  $n$  corresponding to different integers  $j$  are pairwise distinct.  $\square$

Theorem 5(ii)-(iii) follows immediately. Indeed, from Lemma 8, we have

$$\sum_{j=0}^{\infty} |xT(x) \cdots T^{j-1}(x)|^\alpha \log \left( \frac{1}{T^j(x)} \right) \mathbf{1}_{[0,1/2)}(T^j(x)) \leq \sum_{n=1}^{+\infty} \frac{\log(A_{n+1})}{Q_n^\alpha} \leq \sum_{n=1}^{+\infty} \frac{\log(Q_{n+1})}{Q_n^\alpha}$$

and

$$\sum_{j=0}^{\infty} \frac{|xT(x) \cdots T^{j-1}(x)|^{\alpha}}{|T^j(x)|^{\beta}} \mathbf{1}_{[0,1/2)}(T^j(x)) \leq \sum_{n=1}^{+\infty} \frac{A_{n+1}^{\beta}}{Q_n^{\alpha}} \leq \sum_{n=1}^{+\infty} \frac{Q_{n+1}^{\beta}}{Q_n^{\alpha+\beta}},$$

where we have used that  $A_{n+1} \leq Q_{n+1}/Q_n$ .

## 8. PROOF OF THEOREM 4

### 8.1. Proof of Theorem 4, parts (i), (ii) and (iii).

Let  $\alpha > 0$ ,  $\beta \geq 0$  be two positive real numbers. Let us rewrite the general term of the sum (1.10) as

$$\frac{|xT(x) \cdots T^{j-1}(x)|^{\alpha}}{|T^j(x)|^{\beta}} = \frac{|xT(x) \cdots T^{j-1}(x)|^{\alpha+\beta}}{|xT(x) \cdots T^j(x)|^{\beta}}.$$

Using (2.7), one sees that

$$|xT(x) \cdots T^j(x)| \geq \frac{1}{2q_{j+1}} \quad \text{and} \quad |xT(x) \cdots T^{j-1}(x)| \leq \frac{1}{q_j - q_{j-1}}.$$

Hence

$$\sum_{j=1}^{\infty} \frac{|xT(x) \cdots T^{j-1}(x)|^{\alpha}}{|T^j(x)|^{\beta}} \leq 2^{\beta} \sum_{j=1}^{\infty} \frac{q_{j+1}^{\beta}}{(q_j - q_{j-1})^{\alpha+\beta}}.$$

We now use Proposition 1:

- If  $q_j = Q_n$  for some integer  $n$  and  $q_{j-1} = Q_{n-1}$ , then  $q_{j+1}$  is either equal to  $Q_{n+1}$  or to  $Q_{n+1} + Q_n$ . One also has  $q_j - q_{j-1} = Q_n - Q_{n-1} = (A_n - 1)Q_{n-1} + Q_{n-2}$ . Since in this configuration,  $A_n$  is necessarily even, we deduce that  $q_j - q_{j-1} \geq \frac{1}{2}A_n Q_{n-1} \geq \frac{1}{4}Q_n$ . Hence,

$$\frac{q_{j+1}^{\beta}}{(q_j - q_{j-1})^{\alpha+\beta}} \leq 2^{2\alpha+3\beta} \frac{Q_{n+1}^{\beta}}{Q_n^{\alpha+\beta}}.$$

- If  $q_j = Q_n$  for some integer  $n$  and  $q_{j-1} = Q_n - Q_{n-1}$ . Again,  $q_{j+1}$  is either equal to  $Q_{n+1}$  or to  $Q_{n+1} + Q_n$ . Hence,

$$\frac{q_{j+1}^{\beta}}{(q_j - q_{j-1})^{\alpha+\beta}} \leq 2^{\beta} \frac{Q_{n+1}^{\beta}}{Q_{n-1}^{\alpha+\beta}}.$$

- If  $q_j = mQ_n + Q_{n-1}$  for some integers  $n \geq 1$  and  $1 \leq m \leq A_{n+1} - 1$ , then necessarily  $q_j - q_{j-1} \geq Q_n$ , with equality when  $m \geq 2$  (i.e. when  $q_{j-1} = (m-1)Q_n + Q_{n-1}$ ). Moreover,  $q_{j+1}$  is equal to  $(m+1)Q_n + Q_{n-1} \leq 2(m+1)Q_n$ . Hence,

$$\frac{q_{j+1}^{\beta}}{(q_j - q_{j-1})^{\alpha+\beta}} \leq \frac{(2(m+1)Q_n)^{\beta}}{Q_n^{\alpha+\beta}} = 2^{\beta} \frac{(m+1)^{\beta}}{Q_n^{\alpha}}.$$

We deduce from this analysis that for some constant  $C$  depending on  $\alpha$  and  $\beta$  only,

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{|xT(x) \cdots T^{j-1}(x)|^{\alpha}}{|T^j(x)|^{\beta}} &\leq C \sum_{n=1}^{\infty} \frac{Q_{n+1}^{\beta}}{Q_n^{\alpha+\beta}} + C \sum_{n=1}^{\infty} \frac{Q_{n+2}^{\beta}}{Q_n^{\alpha+\beta}} \\ &\quad + C \sum_{n=1}^{\infty} \sum_{m=1}^{A_{n+1}-1} \frac{(m+1)^{\beta}}{Q_n^{\alpha}}. \end{aligned}$$

The first sum is clearly bounded by the second sum. For the third one, one sees that

$$\sum_{n=1}^{\infty} \sum_{m=1}^{A_{n+1}-1} \frac{(m+1)^{\beta}}{Q_n^{\alpha}} \leq C \sum_{n=1}^{+\infty} \frac{A_{n+1}^{\beta+1}}{Q_n^{\alpha}} \leq C \sum_{n=1}^{\infty} \frac{Q_{n+1}^{\beta+1}}{Q_n^{\alpha+\beta+1}},$$

where we used that  $A_{n+1} \leq \frac{Q_{n+1}}{Q_n}$ . Finally,

$$\sum_{j=1}^{\infty} \frac{|xT(x) \cdots T^{j-1}(x)|^{\alpha}}{|T^j(x)|^{\beta}} \leq C \sum_{n=1}^{\infty} \frac{Q_{n+2}^{\beta}}{Q_n^{\alpha+\beta}} + C \sum_{n=1}^{\infty} \frac{Q_{n+1}^{\beta+1}}{Q_n^{\alpha+\beta+1}}.$$

Let us call  $S_1$  and  $S_2$  the two sums in the right-hand side above. Let  $\beta_{\alpha} = \frac{\sqrt{\alpha^2 + 4} - 1}{2}$ .

•  $\beta < \beta_{\alpha}$ . We rewrite the general term of  $S_2$  as

$$\frac{Q_{n+1}^{\beta+1}}{Q_n^{\alpha+\beta+1}} = \left( \frac{Q_{n+1}}{Q_n^{1+\frac{\alpha}{\beta+1}}} \right)^{\beta+1}.$$

Then, observe that

$$\frac{Q_{n+2}^{\beta}}{Q_n^{\alpha+\beta}} = \left( \frac{Q_{n+2}}{Q_{n+1}^{1+\frac{\alpha}{\beta+1}}} \right)^{\beta} \left( \frac{Q_{n+1}}{Q_n^{1+\frac{\alpha}{\beta+1}}} \right)^{\beta(1+\frac{\alpha}{\beta+1})} Q_n^{\delta},$$

where

$$\delta = \beta \left( 1 + \frac{\alpha}{\beta+1} \right)^2 - (\alpha + \beta) = \frac{\alpha(\beta^2 + \alpha\beta - 1)}{(\beta+1)^2}.$$

Because of our choice of  $\beta$ ,  $\delta < 0$ . Hence, the convergence of the series  $S_1$  implies the convergence of  $S_2$  (because the series  $\sum_{n \geq 1} Q_n^{\delta}$  converges as soon as  $\delta < 0$  for all real numbers  $x$ ).

One can also deduce that the series  $S_1$  converges for all  $x$  whose irrationality exponent  $\mu(x)$  is smaller than  $2 + \frac{\alpha}{\beta+1}$ , which implies the convergence of the series

$$\sum_{n=1}^{\infty} \frac{Q_{n+2}}{Q_{n+1}^{1+\frac{\alpha}{\beta+1}}}.$$

•  $\beta > \beta_{\alpha}$ . In this case, the same argument yields that the convergence of the series  $S_2$  implies the convergence of  $S_1$ .

In terms of Diophantine properties,  $S_2$  converges for all real numbers whose irrationality exponent  $\mu(x)$  is smaller than  $1 + \sqrt{1 + \frac{\alpha}{\beta}}$ . Indeed, for such an  $x$ , one has  $\sum_{n \geq 1} \frac{Q_{n+1}}{Q_n^{\mu(x)-1}} < \infty$ . Writing

$$\frac{Q_{n+2}^\beta}{Q_n^{\alpha+\beta}} = \left( \frac{Q_{n+2}}{(Q_{n+1})^{\mu(x)-1}} \right)^\beta \left( \frac{Q_{n+1}}{Q_n^{\mu(x)-1}} \right)^{\beta(\mu(x)-1)} Q_n^{\beta(\mu(x)-1)^2 - \alpha - \beta},$$

we deduce that the series  $S_2$  converges, since our choice for  $\mu(x)$  implies  $\beta(\mu(x) - 1)^2 - \alpha - \beta < 0$ .

•  $\beta = \beta_\alpha$ . The convergence of the two series are close to be equivalent, but they are not, depending on the values of  $\alpha$  (and  $\beta_\alpha$ ). It is simpler to indicate that for all real numbers  $x$  with irrationality exponent less than  $1 + \sqrt{1 + \frac{\alpha}{\beta_\alpha}}$  (which coincides with  $2 + \frac{\alpha}{\beta_\alpha + 1}$ ), the two series  $S_1$  and  $S_2$  converge.

## 8.2. Proof of Theorem 4, part (iv).

The analysis is very similar to the one we performed in the last section, so we mention the main steps of the proof. We write for an irrational  $x \in (0, 1)$

$$\begin{aligned} |xT(x) \cdots T^{j-1}(x)|^\alpha \cdot \log \left( \frac{1}{T^j(x)} \right) &= |xT(x) \cdots T^{j-1}(x)|^\alpha \cdot \log (xT(x) \cdots T^{j-1}(x)) \\ &\quad - |xT(x) \cdots T^{j-1}(x)|^\alpha \cdot \log (xT(x) \cdots T^{j-1}(x)T^j(x)). \end{aligned} \quad (8.1)$$

The first term in the right hand-side is dominated in absolute value by the second term, so we focus on the convergence of this term. Equation (2.7) yields

$$|xT(x) \cdots T^{j-1}(x)|^\alpha \cdot \left| \log (xT(x) \cdots T^{j-1}(x)T^j(x)) \right| \ll \frac{\log(q_{j+1})}{(q_j - q_{j-1})^\alpha}.$$

The same distinction in three cases as in the previous section yields that

$$\begin{aligned} \sum_{j=1}^{\infty} |xT(x) \cdots T^{j-1}(x)|^\alpha \cdot \left| \log (xT(x) \cdots T^j(x)) \right| &\ll \sum_{n=1}^{\infty} \frac{\log(Q_{n+1})}{Q_n^\alpha} + \sum_{n=1}^{\infty} \frac{\log(Q_{n+2})}{Q_n^\alpha} \\ &\quad + \sum_{n=1}^{\infty} \sum_{m=1}^{A_{n+1}-1} \frac{\log(m+1)}{Q_n^\alpha}. \end{aligned}$$

The second term dominates the first one. Then we use that

$$\sum_{m=1}^{A_{n+1}-1} \log(m+1) \ll A_{n+1} \log(A_{n+1})$$

to prove that

$$\sum_{j=1}^{\infty} |xT(x) \cdots T^{j-1}(x)|^\alpha \cdot \left| \log (xT(x) \cdots T^j(x)) \right| \ll \sum_{n=2}^{\infty} \frac{\log(Q_{n+1})}{Q_{n-1}^\alpha} + \sum_{n=1}^{\infty} \frac{A_{n+1} \log(A_{n+1})}{Q_n^\alpha}.$$

Using that  $A_{n+1} \leq \frac{Q_{n+1}}{Q_n}$ , we finally get (1.18).

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