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On the Hougaard subordinated Gaussian Lévy processes

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Abstract

The Hougaard subordinated multivariate Gaussian Lévy processes are characterized. Necessary and sufficient conditions of their selfdecomposability are given and related Ornstein-Uhlenbeck type processes are described.

Keywords: Gaussian Lévy process, Hougaard process, Ornstein-Uhlenbeck type process, self-decomposability, subordination, Thorin class, Thorin measure, Tweedie distribution

1. INTRODUCTION

We start with the families of Tweedie or power-variance distributions $\{\text{Tw}_p(\mu, \lambda), \mu > 0, \lambda > 0\}$, $p \in \mathbb{R}^1 \setminus [0, 1)$, as exponential dispersion models, characterized by the following properties: for each $\mu > 0$, $\lambda > 0$ and given p

$$\int_{\mathbb{R}^1} x \text{Tw}_p(\mu, \lambda)(dx) = \mu,$$

$$\int_{\mathbb{R}^1} (x - \mu)^2 \text{Tw}_p(\mu, \lambda)(dx) = \lambda^{-1} \mu^p;$$

$\text{Tw}_0(\mu, \lambda) := \mathcal{N}(\mu, \lambda^{-1})$, $\mu \in \mathbb{R}^1$, $\lambda > 0$, are Gaussian distributions.

It is well known (see, e.g., Jørgensen (1997), Vinogradov (2008)) that all these distributions are infinitely divisible and, moreover, for $p \geq 1$ are supported by $\mathbb{R}_+ := [0, \infty)$.

A Lévy process $H_{p,\mu,\lambda} = \{H_{p,\mu,\lambda}(t), t \geq 0\}$ is called the Hougaard process if the probability law $\mathcal{L}(H_{p,\mu,\lambda}(1)) = \text{Tw}_p(\mu, \lambda)$. Thus, if $p \geq 1$, the Hougaard

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processes are subordinators. Basing on Grigelionis (2007, 2008), in this paper we characterize the Hougard subordinated multivariate Gaussian Lévy processes. Necessary and sufficient conditions of their selfdecomposability are given and related Ornstein-Uhlenbeck type processes are described.

2. The Hougard subordinated multivariate Gaussian Lévy processes

Let $G = \{G(t), t \geq 0\}$ be a Gaussian Lévy process in $\mathbb{R}^d$ with the mean vector $a = \mathbb{E} G(1)$ and the covariance matrix $A = \mathbb{E} [(G(1) - a)^\top (G(1) - a)]$, assuming that $|A| := \det A > 0$, where $\top$ means the transposition.

Let

$$\begin{aligned} GH_p(t) &= G(H_{p,\mu,\lambda}(t)), \\ \mu_p^t &= \mathcal{L}(GH_p(t)), \quad t \geq 0, \quad p \geq 1, \end{aligned}$$

assuming that processes G and $H_{p,\mu,\lambda}$ are independent,

$$G_{a,A}(B) = \int_B g_{a,A}(x) dx, \quad B \in \mathcal{B}(\mathbb{R}^d),$$

where

$$g_{a,A}(x) = \frac{1}{\sqrt{|A|}(2\pi)^{d/2}} \exp \left\{ -\frac{1}{2} ((x - a)A^{-1}, (x - a)) \right\}, \quad x \in \mathbb{R}^d. \quad (1)$$

Theorem 1. (i) GH_1 is a Lévy process with the triplet of Lévy characteristics $(a_1, 0, \nu_1)$ and

$$\mu_1^t(B) = \sum_{k=0}^{\infty} G_{\lambda^{-1}ak, \lambda^{-1}Ak}(B) e^{-\mu\lambda t} \frac{(\mu\lambda t)^k}{k!}, \quad t \geq 0, \quad (2)$$

where

$$a_1 = \mu\lambda \int_{|x| \leq 1} x G_{\lambda^{-1}a, \lambda^{-1}A}(dx), \quad (3)$$

and

$$\nu_1(B) = \mu\lambda G_{\lambda^{-1}a, \lambda^{-1}A}(B), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}); \quad (4)$$

(ii) for $p > 1$, GH_p is a Lévy process with the triplet $(a_p, 0, \nu_p)$ of Lévy characteristics and

$$\mu_p^t(B) = \int_0^\infty G_{sa, sA}(B) \text{Tw}_p(\mu t, \lambda t^{p-1})(ds), \quad t \geq 0, \quad (5)$$

where

$$a_p = \int_{|x| \leq 1} x g_p(x) dx, \quad \nu_p(B) = \int_B g_p(x) dx, \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \quad (6)$$

$$g_p(x) = \frac{2c_{p,\lambda} \exp\{(xA^{-1}, a)\}}{\sqrt{|A|}(2\pi)^{d/2}} \left(\frac{\frac{2\lambda\mu^{1-p}}{p-1} + (aA^{-1}, a)}{(xA^{-1}, x)} \right)^{\frac{p-2}{2(p-1)} + d/4} \cdot K_{\frac{p-2}{p-1} + \frac{d}{2}} \left(\left[\left(\frac{2\lambda\mu^{1-p}}{p-1} + (aA^{-1}, a) \right) (xA^{-1}, x) \right]^{\frac{1}{2}} \right), \quad (7)$$

$$c_{p,\lambda} = \frac{\lambda^{\frac{1}{p-1}}}{\Gamma\left(\frac{p}{p-1}\right) (p-1)^{\frac{p}{p-1}}} \quad (8)$$

and

$$K_\nu(z) = \frac{1}{2} \left(\frac{z}{2} \right)^\nu \int_0^\infty t^{-\nu-1} \exp \left\{ -t - \frac{z^2}{4t} \right\} dt, \quad z, \nu \in \mathbb{R}^1, \quad (9)$$

is the modified Bessel function of the third kind.

Proof. Let $p \geq 1$. It is known (see Jorgensen and Martínez (1996)) that $H_{p,\mu,\lambda}$ is a Lévy subordinator with zero drift. Let ρ_p be its Lévy measure. From Theorem 30.1 of Sato (1999) it follows that GH_p is a Lévy process with

$$\mu_p^t(B) = \int_0^\infty G_{sa, sA}(B) \mathcal{L}(H_{p,\mu,\lambda}(t))(ds)$$

and with the triplet of Lévy characteristics $(a_p, 0, \nu_p)$, where

$$a_p = \int_0^\infty \int_{|x| \leq 1} x g_{sa, sA}(x) dx \rho_p(ds)$$

and the Lévy measure ν_p is given by

$$\nu_p(B) = \int_0^\infty G_{sa, sA}(B) \rho_p(ds), \quad B \in \mathcal{B}(R^d \setminus \{0\}).$$

If $p = 1$, then $\rho_1(ds) = \mu \lambda \varepsilon_{\lambda^{-1}}$, where ε_Δ is the Dirac measure, implying that $T_{W_1}(\mu, \lambda)$ is the scaled Poisson distribution, i.e.,

$$Tw_1(\mu, \lambda)(\{\lambda^{-1}k\}) = e^{-\mu\lambda} \frac{(\mu\lambda)^k}{k!}, \quad k = 0, 1, \dots,$$

and showing formulas (3) and (4). Since the Hougaard process has the following scaling property

$$\mathcal{L}(H_{p, \mu, \lambda}(t)) = Tw_p(\mu t, \lambda t^{p-1}),$$

we also obtain (2) and (5).

If $p > 1$, then

$$\rho_p(dx) = c_{p, \lambda} x^{\frac{3-2p}{p-1}} \exp \left\{ -\frac{\lambda \mu^{1-p}}{p-1} x \right\} dx. \quad (10)$$

Now (6) and (7) are consequences of the following formula: for all $\alpha, \beta \geq 0$, $\nu \in R^1$ and $x \in R^d$

$$\begin{aligned} & \int_0^\infty g_{sa, sA}(x) s^\nu e^{-as-\beta/s} ds \\ &= \frac{2 \exp\{(xA^{-1}, \alpha)\}}{\sqrt{|A|}(2\pi)^{d/2}} \left(\frac{2\alpha + (aA^{-1}, \alpha)}{2\beta + (xA^{-1}, x)} \right)^{\frac{-\nu}{2} + \frac{d}{4} - \frac{1}{2}} \\ & \cdot K_{\nu-d/2+1} \left([(2\alpha + (aA^{-1}a))(2\beta + (xA^{-1}, x))]^{1/2} \right). \end{aligned} \quad (11)$$

Indeed, from (9) we derive that for all $\gamma, \delta > 0$

$$\int_0^\infty t^{-\nu-1} e^{-\gamma t - \delta/t} dt = 2 \left(\frac{\gamma}{\delta} \right)^{\frac{\nu}{2}} K_\nu(2\sqrt{\gamma\delta}), \quad (12)$$

having in mind that $K_\nu(z) = K_{-\nu}(z)$ (see, e.g., Watson (1958)). Now (11) follows from (1) and (12) after elementary calculations. Formulas (5), (6) easily follow from (11) and Theorem 30.1 of Sato (1999). $\square$

Remark 2. If $p = \frac{3}{2}$ (the compound Poisson-exponential distribution case), we have (see Vinogradov (2007))

$$\begin{aligned} \text{Tw}_{3/2}(\mu t, \lambda\sqrt{t})(C) &= e^{-2\lambda\sqrt{\mu}t} \varepsilon_0(C) + \\ &+ \int_C \frac{2\lambda\sqrt{t}}{\sqrt{s}} e^{-2\frac{\lambda}{\sqrt{\mu}}(s+\mu t)} I_1(4\lambda\sqrt{st}) ds, \quad C \in \mathcal{B}(\mathbb{R}_+), \end{aligned} \quad (13)$$

where $I_\nu(z)$ is the modified Bessel function of the first kind, i.e.,

$$I_\nu(z) = \left(\frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2} \right)^{2k}}{k! \Gamma(\nu + k + 1)}, \quad \nu \geq -1, \quad (14)$$

and $\Gamma(z)$ is the Euler's gamma function.

From (10), (13) and (14) it follows that

$$\begin{aligned} \mu_{3/2}^t(B) &= e^{-2\lambda\sqrt{\mu}t} \varepsilon_0(B) + \frac{4\lambda^2 t e^{2\lambda\sqrt{\mu}t}}{\sqrt{|A|} (2\pi)^{d/2}} \left(\frac{4\lambda}{\sqrt{\mu}} + (aA^{-1}, a) \right)^{d/4-1/2} \\ &\cdot \sum_{k=0}^{\infty} \int_B \frac{(4\lambda^2 t (xA^{-1}, x)^{1/2})^k}{k!(k+1)!} \\ &\cdot K_{k-d/2+1} \left(\left[\left(\frac{4\lambda}{\sqrt{\mu}} + (aA^{-1}, a) \right) (xA^{-1}, x) \right]^{1/2} \right) dx, \quad t \geq 0, \quad B \in \mathcal{B}(\mathbb{R}^d). \end{aligned}$$

Remark 3. If $p = 2$ (the gamma distribution case), we have

$$\text{Tw}_2(\mu t, \lambda t)(C) = \int_C \frac{\left(\frac{\lambda}{\mu} \right)^{\lambda t} s^{\lambda t-1}}{\Gamma(\lambda t)} e^{-\frac{\lambda}{\mu} s} ds, \quad C \in \mathcal{B}(\mathbb{R}_+),$$

implying that

$$\begin{aligned} \mu_2^t(B) &= \frac{2 \left(\frac{\lambda}{\mu}\right)^{\lambda t}}{\sqrt{|A|}(2\pi)^{d/2} \Gamma(\lambda t)} \int_B \exp\{(xA^{-1}, a)\} \left(\frac{\frac{2\lambda}{\mu} + (aA^{-1}, a)}{(xA^{-1}, x)}\right)^{-\lambda t/2 + d/4} \\ &\quad \cdot K_{\lambda t - d/2} \left(\left[\left(\frac{2\lambda}{\mu} + (aA^{-1}, a) \right) (xA^{-1}, x) \right]^{1/2} \right) dx, \quad t \geq 0, \quad B \in \mathcal{B}(\mathbb{R}^d). \end{aligned}$$

Remark 4. If $p = 3$ (the inverse Gaussian distribution case), we have

$$\text{Tw}_3(\mu t, \lambda t^2)(C) = \int_C \sqrt{\frac{\lambda}{2\pi}} t e^{\frac{\lambda}{\mu} t} s^{-\frac{3}{2}} e^{-\frac{\lambda s}{2\mu^2} - \frac{\lambda t^2}{2s}} ds, \quad t > 0, \quad C \in \mathcal{B}(\mathbb{R}^d),$$

implying that

$$\begin{aligned} \mu_3^t(B) &= \frac{\sqrt{\lambda} t e^{\frac{\lambda}{\mu} t}}{\sqrt{|A|}(2\pi)^{\frac{d+1}{2}}} \int_B \left(\frac{\frac{\lambda}{\mu^2} + (aA^{-1}, a)}{\lambda t^2 + (xA^{-1}, x)} \right)^{\frac{d+1}{4}} \\ &\quad \cdot K_{\frac{d+1}{2}} \left(\left[\left(\frac{\lambda}{\mu^2} + (aA^{-1}, a) \right) (\lambda t^2 + (xA^{-1}, x)) \right]^{1/2} \right) dx, \\ &\quad t > 0, B \in \mathcal{B}(\mathbb{R}^d). \end{aligned}$$

3. Selfdecomposability of subordinated processes

A Lévy process $X = \{X(t), t \geq 0\}$ in $\mathbb{R}^d$ with characteristic function $\hat{\mu}(z) := \mathbb{E} \exp\{i(z, X(1))\}$, $z \in \mathbb{R}^d$, is said to be selfdecomposable or of class $L(\mathbb{R}^d)$ if, for any $b > 1$, there is a characteristic function $\hat{\mu}_b$ such that $\hat{\mu}(z) = \hat{\mu}(b^{-1}z)\hat{\mu}_b(z)$, $z \in \mathbb{R}^d$. If the Lévy measure ν of X equals

$$\nu(B) = \int_B g(x) dx, \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),$$

then, according to Proposition 1 in Grigelionis (2007), X is selfdecomposable if and only if for a.e. $\xi \in S^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ with respect to the surface Lebesgue measure on S^{d-1} the function $k(\xi, r) := r^d g(r\xi)$, $r > 0$, is decreasing.

Theorem 5. *The Hougard subordinated Gaussian Lévy process GH_p is self-decomposable if and only if $p \geq 2$, $d = 1$ or $p \geq 2$, $d \geq 2$ and $a = 0$.*

Proof. Let $p = 1$. From (4) we find that

$$\nu_1(B) = \int_B \mu \lambda g_{\lambda^{-1}a, \lambda^{-1}A}(x) dx, \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).$$

Because, obviously, the function

$$k_1(\xi, r) = \mu \lambda r^d g_{\lambda^{-1}a, \lambda^{-1}A}(r\xi), \quad r > 0,$$

is not decreasing for each $\xi \in S^{d-1}$, the process GH_1 is not selfdecomposable.

Now let $p > 1$. From (10) it follows that for each $\mu > 0$, $\lambda > 0$, $H_{p,\mu,\lambda}$ is subordinator of the Thorin class $T^{(\frac{1}{p-1})}(\mathbb{R}_+)$ (see Grigelionis (2008)) with the Thorin measure

$$Q_{\frac{1}{p-1}}(dv) = c_{p,\lambda} \exp \left\{ -\frac{\lambda \mu^{1-p}}{p-1} v \right\} dv. \quad (15)$$

If $1 < p < 2$, from (15) and Proposition 4 (v) of Grigelionis (2008), we find that the processes GH_p are not selfdecomposable.

If $p \geq 2$, then $T^{(\frac{1}{p-1})}(\mathbb{R}_+) \subseteq T^{(1)}(\mathbb{R}_+)$ and the assertion of the theorem follows from Proposition 4 (i) of Grigelionis (2008). $\square$

Finally, we describe the background driving Lévy processes

$$Z_p^{(c)} = \{Z_p^{(c)}(t), t \geq 0\}, \quad c > 0,$$

such that the Ornstein-Uhlenbeck type processes

$$X_p(t) = e^{-ct} X_p(0) + \int_0^t e^{-c(t-s)} dZ_p^{(c)}(s), \quad t \geq 0, \quad c > 0, \quad (16)$$

are strictly stationary, provided $\mathcal{L}(X_p(0)) = \mathcal{L}(GH_p(1))$, GH_p is selfdecomposable and $Z_p^{(c)}$ is independent of $X_p(0)$.

Theorem 6. If $p \geq 2$, $d = 1$ or $p \geq 2$, $d \geq 2$, $a = 0$ and the background driving Lévy process $Z_p^{(c)}$ has the triplet $(a_p^{(c)}, 0, \nu_p^{(c)})$ of Lévy characteristics, where

$$a_p^{(c)} = \int_{|x| \leq 1} x g_p^{(c)}(x) dx, \quad \nu_p^{(c)}(B) = \int_B g_p^{(c)}(x) dx, \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}),$$

and

$$g_p^{(c)}(x) = -\frac{d}{dr}(cr^d g_p(r\xi))|_{r\xi=x},$$

then X_p in (16) is strictly stationary.

Proof. The statement of Theorem 6 follows directly from Proposition 1 in Grigelionis (2007), and Theorem 17.5 in Sato (1999). $\square$

Remark 7. If $p \geq 2$, $d \geq 2$ and $a = 0$, from (7) we have

$$r^d g_p(r\xi) = \frac{2c_{p,\lambda}}{\sqrt{|A|}(2\pi)^{d/2}} \left(\frac{2\lambda\mu^{1-p}}{r^2(\xi A^{-1}, \xi)} \right)^{\frac{p-2}{2(p-1)} + \frac{d}{4}} \cdot K_{\frac{p-2}{p-1} + \frac{d}{2}} \left(\left[\frac{2r^2\lambda\mu^{1-p}}{p-1}(\xi A^{-1}, \xi) \right]^{1/2} \right),$$

and, using the equality (see, e.g., Watson (1958))

$$K'_\nu(z) = -\left(K_{\nu-1}(z) + \frac{\nu}{2} K_\nu(z) \right),$$

we find that

$$g_p^{(c)}(x) = C_p(xA^{-1}, x)^{\frac{2p-3}{p-1} + \frac{d}{4}} K_{\frac{1}{p-1} - \frac{d}{2}} \left(\left[\frac{2\lambda\mu^{1-p}}{p-1}(xA^{-1}, x) \right]^{1/2} \right),$$

where

$$C_p = \frac{2cc_{p,\lambda}(2\lambda\mu^{1-p})^{\frac{2p-3}{2(p-1)} + \frac{d}{4}}}{\sqrt{|A|}(2\pi)^{d/2}\sqrt{p-1}}.$$

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