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A strategy-based proof of the existence of the value in zero-sum differential games

Pablo Maldonado and Miquel Oliu-Barton *

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Abstract

The value of a zero-sum differential games is known to exist, under Isaacs condition, as the unique viscosity solution of a Hamilton-Jacobi-Bellman equation. In this note we provide a new proof via the construction of ε-optimal strategies, which is inspired in the "extremal aiming" method from [3].

1 Introduction

Let U and V be compact subsets of some euclidean space, let $\|\cdot\|$ be the euclidean norm in $\mathbb{R}^n$, and let $f:[0,1]\times\mathbb{R}^n\times U\times V\to\mathbb{R}^n$.
Assumption 1:
1a. f is uniformly bounded, i.e. $\|f\|:=\sup_{(t,x,u,v)}\|f(t,x,u,v)\|<+\infty$,
1b. $\exists c\geq 0$ such that $\forall(u,v)\in U\times V,\ \forall s,t\in[0,1],\ \forall x,y\in\mathbb{R}^n$:

$$\|f(t,x,u,v)-f(s,y,u,v)\|\leq c\big(|t-s|+\|x-y\|\big),$$

The directional game For any $(t,x)\in[0,1]\times\mathbb{R}^n$ and any $\xi\in\mathbb{R}^n$, consider the one-shot game $\Gamma(t,x,\xi)$, with actions sets U and V and payoff function:

$$(u,v)\mapsto\langle\xi,f(t,x,u,v)\rangle.$$

Let $H^-(t,x,\xi)$ and $H^+(t,x,\xi)$ be its maxmin and minmax respectively:

$$\begin{aligned} H^-(t,x,\xi) &:= \max_{u\in U}\min_{v\in V}\langle\xi,f(t,x,u,v)\rangle,\\ H^+(t,x,\xi) &:= \min_{v\in V}\max_{u\in U}\langle\xi,f(t,x,u,v)\rangle. \end{aligned}$$

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These functions satisfy $H^- \leq H^+$. If the equality $H^+(t,x,\xi) = H^-(t,x,\xi)$ holds, the game $\Gamma(t,x,\xi)$ has a value.

Assumption 2: $\forall (t,x,\xi) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^n$, the game $\Gamma(t,x,\xi)$ has a value $H(t,x,\xi)$.

1.1 An important Lemma

Introduce the sets of controls:

$$\mathcal{U} = \{\mathbf{u} : [0,1] \to U, \text{ measurable}\}, \quad \mathcal{V} = \{\mathbf{v} : [0,1] \to V, \text{ measurable}\}.$$

Let $(\mathbf{u},\mathbf{v}) \in \mathcal{U} \times \mathcal{V}$, $t_0 \in [0,1]$, $(x_0, w_0) \in (\mathbb{R}^n)^2$ and let (u^*, v^*) be a couple of optimal actions in $\Gamma(t_0, x_0, x_0 - w_0)$. Define two continuous trajectories in $\mathbb{R}^n$, $\mathbf{x} : [t_0, 1] \to \mathbb{R}^n$ and $\mathbf{w} : [t_0, 1] \to \mathbb{R}^n$, by:

$$\begin{aligned} \mathbf{x}(t_0) = x_0, &\quad \text{and} \quad \dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t), \mathbf{u}(t), v^*), \text{ a.e.} \\ \mathbf{w}(t_0) = w_0, &\quad \text{and} \quad \dot{\mathbf{w}}(t) = f(t, \mathbf{w}(t), u^*, \mathbf{v}(t)), \text{ a.e.} \end{aligned}$$

The following lemma is inspired by Lemma 2.3.1 in [3].

Lemma 1. *Under Assumptions* 1 *and* 2*, there exists* $A, B \geq 0$ *such that* $\forall t \in [t_0, 1]$*:*

$$\|\mathbf{x}(t) - \mathbf{w}(t)\|^2 \leq (1 + (t - t_0)A)\|x_0 - w_0\|^2 + B(t - t_0)^2.$$

Proof. Notation: let $d_0 := \|x_0 - w_0\|$ and $\mathbf{d}(t) := \|\mathbf{x}(t) - \mathbf{w}(t)\|$. Then:

$$\mathbf{d}^2(t) \;=\; \|(x_0 - w_0) + \int_{t_0}^{t} f(s, \mathbf{x}(s), \mathbf{u}(s), v^*) - f(s, \mathbf{x}(s), u^*, \mathbf{v}(s)) ds\|^2. \quad (1.1)$$

The boundedness of f implies that

$$\|\int_{t_0}^{t} f(s, \mathbf{x}(s), \mathbf{u}(s), v^*) - f(s, \mathbf{w}(s), u^*, \mathbf{v}(s)) ds\|^2 \leq 4\|f\|^2 (t - t_0)^2. \quad (1.2)$$

Claim: For all $s \in [t_0, 1]$, and for all $(u, v) \in U \times V$:

$$\langle x_0 - w_0, f(s, \mathbf{x}(s), u, v^*) - f(s, \mathbf{w}(s), u^*, v) ds \rangle \leq 2C(s) d_0 + c d_0^2, \quad (1.3)$$

where $C(s) := c(1 + \|f\|)(s - t_0)$.
Let us prove this claim. Assumption 1 implies $\|\mathbf{x}(s) - x_0\| \leq (s - t_0)\|f\|$, and then:

$$\|f(s, \mathbf{x}(s), u, v^*) - f(t_0, x_0, u, v^*)\| \leq c\big((s - t_0) + \|f\|(s - t_0)\big) = C(s).$$

Then, using Cauchy-Schwartz inequality, and the optimality of v^*:

$$\begin{aligned} \langle x_0 - w_0, f(s, \mathbf{x}(s), u, v^*) \rangle &\leq \langle x_0 - w_0, f(t_0, x_0, u, v^*) \rangle + C(s) d_0, \\ &\leq H^+(t_0, x_0, x_0 - w_0) + C(s) d_0. \end{aligned}$$

Similarly, Assumption 1 implies $\|\mathbf{w}(s) - x_0\| \leq d_0 + (s - t_0)\|f\|$, and then:

$$\|f(s, \mathbf{w}(s), u^*, v) - f(t_0, x_0, u^*, v)\| \leq C(s) + c d_0.$$

Using Cauchy-Schwartz inequality, and the optimality of u^*:

$$\begin{aligned}\langle x_0 - w_0, f(s, \mathbf{x}(s), u^*, v)\rangle &\geq \langle x_0 - w_0, f(t_0, x_0, u^*, v)\rangle - (C(s) + cd_0)d_0, \\ &\geq H^-(t_0, x_0, x_0 - w_0) - C(s)d_0 - cd_0^2.\end{aligned}$$

The claim follows from Assumption 2. In particular, it holds for $(u, v) = (\mathbf{u}(s), \mathbf{v}(s))$. Note that $\int_{t_0}^t 2C(s)ds = (t - t_0)C(t)$. Thus, integrating (1.3) over $[t_0, t]$ yields:

$$\int_{t_0}^t \langle x_0 - w_0, f(s, \mathbf{x}(s), \mathbf{u}(s), v^*) - f(s, \mathbf{w}(s), u^*, \mathbf{v}(s))ds\rangle \leq (t - t_0)(C(t)d_0 + cd_0^2). \tag{1.4}$$

Go back to (1.1) using the estimates (1.2) and (1.4). We have proved:

$$\mathbf{d}^2(t) \leq d_0^2 + 4\|f\|^2(t - t_0)^2 + 2(t - t_0)C(t)d_0 + 2c(t - t_0)d_0^2.$$

Finally, use the relations $d_0 \leq 1 + d_0^2$, $C(t) \leq c(1 + \|f\|)$ and $(t - t_0)C(t) = c(1 + \|f\|)(t-t_0)^2$ to obtain the result, with $A = 3c+2\|f\|$ and $B = 4\|f\|^2+2c(1+\|f\|)$. □

1.2 Consequences

In this section, we give three direct consequences of Lemma 1. Let $d : \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}$ denote the usual distance to a set in $\mathbb{R}^n$.

1. Consider some sequence of times $\Pi = \{t_0 < t_1 < \cdots < t_N\}$ in $[0, 1]$, and let $\|\Pi\| := \max\{t_m - t_{m-1}, m = 1, \ldots, N\}$. Let $(\mathbf{u}, \mathbf{v}) \in \mathcal{U} \times \mathcal{V}$ be a fixed pair of controls. Define the trajectories $\mathbf{x}$ and $\mathbf{w}$ on $[t_0, t_N]$ inductively. Let $\mathbf{x}(t_0) = x_0$, $\mathbf{w}(t_0) = w_0$ and suppose that $\mathbf{x}(t)$ and $\mathbf{w}(t)$ are already defined on $[t_0, t_m]$. Let $(u_m^*, v_m^*) \in U \times V$ be a couple of optimal actions in $\Gamma(t_m, \mathbf{x}(t_m), \mathbf{x}(t_m) - \mathbf{w}(t_m))$. Then, on $[t_m, t_{m+1}]$, let $\mathbf{x}$ and $\mathbf{w}$ be the unique absolutely continuous solutions of:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= f(t, \mathbf{x}(t), \mathbf{u}(t), v_m^*), \\ \dot{\mathbf{w}}(t) &= f(t, \mathbf{w}(t), u_m^*, \mathbf{v}(t)).\end{aligned}$$

Corollary 1.1. *Under Assumptions* 1 *and* 2*:*

$$\|\mathbf{x}(t_N) - \mathbf{w}(t_N)\|^2 \leq e^A(\|x_0 - w_0\|^2 + B\|\Pi\|).$$

Proof. For any $0 \leq m \leq N$, let $d_m := \|\mathbf{x}(t_m) - \mathbf{w}(t_m)\|$. Lemma 1 yields:

$$d_m^2 \leq (1 + (t_m - t_{m-1})A)d_{m-1}^2 + B(t_m - t_{m-1})^2.$$

Then, by induction: $d_N^2 \leq \exp(A\sum_{m=1}^N t_m - t_{m-1})(d_0^2 + B\sum_{m=1}^N (t_m - t_{m-1})^2)$. The result follows, since $t_N - t_0 \leq 1$ and $\sum_{m=1}^N (t_m - t_{m-1})^2 \leq \|\Pi\|$. □

2. For any $(t_0, x_0) \in [0, 1] \times \mathbb{R}^n$ and $(\mathbf{u}, \mathbf{v}) \in \mathcal{U} \times \mathcal{V}$, let $\mathbf{x} = \mathbf{x}[t_0, x_0, \mathbf{u}, \mathbf{v}]$ be the unique absolutely continuous solution in $[t_0, 1]$ of:

$$\mathbf{x}(t_0) = x_0, \quad \text{and} \quad \dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t), \mathbf{u}(t), \mathbf{v}(t)), \text{ a.e.}$$

That is, $\mathbf{x}[t_0, x_0, \mathbf{u}, \mathbf{v}]$ is the trajectory induced by the initial position (t_0, x_0) and the controls $(\mathbf{u}, \mathbf{v})$. For any $u \in U$, let $\mathbf{x}[t_0, x_0, u, \mathbf{v}]$ be the trajectory induced by $(t_0, x_0, \mathbf{v})$ and the constant control $\mathbf{u} \equiv u$.

Define two properties for sets $\mathcal{W} \subset [t_0, 1] \times \mathbb{R}^n$.

- **P1:** For any $t \in [t_0, 1]$, $\mathcal{W}(t) := \{x \in \mathbb{R}^n \mid (t,x) \in \mathcal{W}\}$ is closed and nonempty.
- **P2:** For any $(t,x) \in \mathcal{W}$ and any $t_1 \in [t,1]$:

$$\sup_{u \in U} \inf_{\mathbf{v} \in \mathcal{V}} d(\mathbf{x}[t,x,u,\mathbf{v}](t_1), \mathcal{W}(t_1)) = 0,$$

  where d is the usual distance in $\mathbb{R}^n$.

Corollary 1.2. *Let $\mathcal{W} \subset [t_0,1] \times \mathbb{R}^n$ satisfy* **P1** *and* **P2**. *Under Assumptions* 1 *and* 2*, there exists $v^* \in V$ such that, $\forall t \in [t_0,1], \forall \mathbf{u} \in \mathcal{U}$:*

$$d^2(\mathbf{x}[t_0, x_0, \mathbf{u}, v^*](t), \mathcal{W}(t)) \le (1 + (t-t_0)A) d^2(x_0, \mathcal{W}(t_0)) + B(t-t_0)^2.$$

Proof. Let $w_0 \in \operatorname{argmin}_{w \in \mathcal{W}(t_0)} \|x_0 - w\|$ be some closest point (which exists by **P1**). Let (u^*, v^*) be optimal in $\Gamma(t_0, x_0, x_0 - w_0)$. By **P2**, $\forall \varepsilon > 0$, $\exists \mathbf{v}_\varepsilon$ such that $\mathbf{w}_\varepsilon(t) := \mathbf{x}[t_0, w_0, u^*, \mathbf{v}_\varepsilon](t)$ satisfies $d(\mathbf{w}_\varepsilon(t), \mathcal{W}(t)) \le \varepsilon$. The triangular equality implies $d(\mathbf{x}(t), \mathcal{W}(t)) \le \|\mathbf{x}(t) - \mathbf{w}_\varepsilon(t)\| + \varepsilon$. Taking the limit, as $\varepsilon \to 0$:

$$d^2(\mathbf{x}(t), \mathcal{W}(t)) \le \lim_{\varepsilon \to 0} \|\mathbf{x}(t) - \mathbf{w}_\varepsilon(t)\|^2,$$

where $\|\mathbf{x}(t) - \mathbf{w}_\varepsilon(t)\|^2 \le (1 + (t-t_0)A)\|x_0 - w_0\|^2 + B(t-t_0)^2$ for any $\varepsilon > 0$, by Lemma 1, and where $\|x_0 - w_0\| = d(x_0, \mathcal{W}(t_0))$ by definition. □

3. Putting Corollaries 1.1 and 1.2 together, one obtains the following result.

Corollary 1.3. *Let $\mathcal{W} \subset [t_0,1] \times \mathbb{R}^n$ satisfy* **P1** *and* **P2***, let $\Pi = \{t_0 < \dots < t_N\}$ be a sequence of times, and let $x_0 \in \mathcal{W}(t_0)$. Under Assumptions* 1 *and* 2*, there exist $v_0^*, \dots, v_{N-1}^* \in V$ such that, for $\mathbf{v} \equiv v_m^*$, on $[t_m, t_{m+1}]$, and for all $\mathbf{u} \in \mathcal{U}$:*

$$d^2(\mathbf{x}[t_0, x_0, \mathbf{u}, \mathbf{v}](t_N), \mathcal{W}(t_N)) \le e^A B \|\Pi\|.$$

2 Differential Games

For any $(t_0, x_0) \in [0,1] \times \mathbb{R}^n$, consider now the zero-sum differential with the following two-controlled dynamic

$$\mathbf{x}(t_0) = x_0, \quad \text{and} \quad \dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t), \mathbf{u}(t), \mathbf{v}(t)), \text{ a.e. on } [t_0, 1].$$

Definition 2.1. *A strategy for player* 2 *is a map $\beta : \mathcal{U} \to \mathcal{V}$ such that, for some finite partition $t_0 < t_1 < \dots < t_N = 1$ of $[t_0,1]$, $\forall \mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}$:*

$$\mathbf{u}_1 \equiv \mathbf{u}_2 \text{ a.e. on } [t_0, t_m] \implies \beta(\mathbf{u}_1) \equiv \beta(\mathbf{u}_2) \text{ a.e. on } [t_0, t_{m+1} \wedge 1].$$

These strategies are called nonanticipative strategies with delay (NAD) in [1], in contrast to the classical nonanticipative strategies. The strategies for player 1 are defined in a dual manner. Let $\mathcal{B}$ (resp. $\mathcal{A}$) the set of strategies for Player 2 (resp. 1). For any pair of strategies $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$, [1] establishes the following crucial result: there exists a unique pair $(\mathbf{u}, \mathbf{v}) \in \mathcal{U} \times \mathcal{V}$ such that $\alpha(\mathbf{v}) = \mathbf{u}$, and $\beta(\mathbf{u}) = \mathbf{v}$. Denote by $\mathbf{x}[t_0, x_0, \alpha, \beta]$ the trajectory induced by the pair $(\mathbf{u}, \mathbf{v})$.

Let $g : \mathbb{R}^n \to \mathbb{R}$ some function. The differential game with initial time t_0, initial state x_0, and terminal payoff g is denoted by $\mathcal{G}(t_0, x_0)$. Introduce the upper and lower value functions:

$$
\begin{aligned}
V^-(t_0, x_0) &:= \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} g(\mathbf{x}[t_0, x_0, \alpha, \beta](1)), \\
V^+(t_0, x_0) &:= \inf_{\beta \in \mathcal{B}} \sup_{\alpha \in \mathcal{A}} g(\mathbf{x}[t_0, x_0, \alpha, \beta](1)).
\end{aligned}
$$

The inequality $V^- \leq V^+$ holds everywhere. If $V^-(t_0, x_0) = V^+(t_0, x_0)$, the game $\mathcal{G}(t_0, x_0)$ has a value. Notice that its lower and upper Hamiltonian of are precisely the maxmin and the minmax of the directional games defined in Section 1. Consequently, Assumption 2 is precisely *Isaacs' condition*.
Assumption 3: g is c-Lipschitz continuous, i.e. $|g(x)-g(y)| \leq c\|x-y\|$, $\forall x, y \in \mathbb{R}^n$.

2.1 Existence and characterization of the value

Let $\phi : [t_0, 1] \times \mathbb{R}^n \to \mathbb{R}$ be a real function satisfying the following properties:

(i) $x \mapsto \phi(t, x)$ is lower semicontinuous, $\forall t \in [t_0, 1]$,

(ii) $\forall (t, x) \in [t_0, 1] \times \mathbb{R}^n$, $\forall t_1 \in [t, 1]$:

$$\phi(t, x) \geq \sup_{u \in U} \inf_{\mathbf{v} \in \mathcal{V}} \phi\big(t_1, \mathbf{x}[t, x, u, \mathbf{v}](t_1)\big),$$

(iii) $\phi(1, x) \geq g(x)$, $\forall x \in \mathbb{R}^n$.

For any $\ell \in \mathbb{R}$, define the *ℓ-level set of ϕ* by:

$$\mathcal{W}_\ell^\phi \quad = \quad \{(t, x) \in [t_0, 1] \times \mathbb{R}^n \mid \phi(t, x) \leq \ell\}, \tag{2.1}$$

Lemma 2. *For any $\ell \geq \phi(t_0, x_0)$, the ℓ-level set of ϕ satisfies* **P1** *and* **P2***.*

Proof. Note that $\mathcal{W}_\ell^\phi(t_0)$ is nonempty, since $x_0 \in \mathcal{W}_\ell^\phi(t_0)$. (i) implies that $\mathcal{W}_\ell^\phi(t)$ is a closed set, $\forall t \in [0, 1]$. On the other hand, by (ii) for all $(t, x) \in [t_0, 1] \times \mathbb{R}^n$, $t_1 \in [t, 1]$, $u \in U$, and $n \in \mathbb{N}^*$, there exists $\mathbf{v}_n \in \mathcal{V}$ such that:

$$\phi(t, x) \geq \phi\big(t_1, \mathbf{x}[t, x, u, \mathbf{v}_n](t_1)) - \frac{1}{n}. \tag{2.2}$$

The boundedness of f implies that $x_n := \mathbf{x}[t, x, u, \mathbf{v}_n](t_1)$ belongs to some compact set. Consider a subsequence $(x_n)_n$ such that $\lim_{n\to\infty} \phi(t_1, x_n) = \liminf_{n\to\infty} \phi(t_1, x_n)$, and such that $(x_n)_n$ converges to some $\bar{x} \in \mathbb{R}^n$. Then, taking the limit, as $n \to \infty$, in (2.2) implies, using (i) and $\ell \geq \phi(t, x)$:

$$\phi(t_1, \overline{x}) \leq \lim_{n\to\infty} \phi(t_1, x_n) \leq \phi(t, x) \leq \ell.$$

Hence $\overline{x} \in \mathcal{W}_\ell^\phi(t_1)$, and $\inf_{n \in \mathbb{N}^*} d\big(\mathbf{x}[t, x, u, \mathbf{v}_n](t_1), \mathcal{W}_\ell^\phi\big) = 0$. In particular, $\mathcal{W}_\ell^\phi(t_1)$ is nonempty, and **P1** and **P2** hold.

□

2.1.1 Extremal strategies in $\mathcal{G}(t_0, x_0)$

Let $\Pi = \{t_0 < \cdots < t_N = 1\}$ be partition of $[t_0, 1]$, let $\|\Pi\| = \max\{t_m - t_{m-1},\ m = 1, \ldots, N\}$, and let $\mathcal{W}^\phi \subset [t_0, 1] \times \mathbb{R}^n$ be the $\phi(t_0, x_0)$-level set of ϕ.

Definition 2.2. *An extremal strategy $\beta = \beta(\phi, \Pi)$ is defined inductively: suppose β is already defined on $[t_0, t_m]$ and let $x_m = \mathbf{x}[t_0, x_0, \mathbf{u}, \beta](t_m)$. Then, $\forall \mathbf{u} \in \mathcal{U}$:*

- *If $x_m \in \mathcal{W}^\phi(t_m)$, set $\beta(\mathbf{u})(s) = v$, for any $v \in V$, $\forall s \in [t_m, t_{m+1})$.*
- *If $x_m \notin \mathcal{W}^\phi(t_m)$, let $w_m \in \operatorname{argmin}_{w \in \mathcal{W}^\phi(t_m)} \|x_m - w_m\|$ be some closest point, and let v_m^* be some optimal action in the directional game $\Gamma(t_m, x_m, x_m - w_m)$. Set $\beta(\mathbf{u})(s) = v_m^*$, $\forall s \in [t_m, t_{m+1})$.*

These strategies are inspired by the *extremal aiming* method of Krasovskii and Subbotin (see Section 2.4 in [3]). Notice that β is defined up to some selection rule since V, the set of closest points and the set of minimizers may have more than one element.

Proposition 2.1. *Under Assumptions 1, 2 and 3, $\exists C \geq 0$ such that:*

$$g(\mathbf{x}[t_0, x_0, \mathbf{u}, \beta(\mathbf{u})](1)) \leq \phi(t_0, x_0) + C\sqrt{\|\Pi\|}, \quad \forall \mathbf{u} \in \mathcal{U},$$

for any extremal strategy $\beta = \beta(\phi, \Pi)$.

Proof. $\mathcal{W}^\phi$ satsifies **P1** and **P2** by Lemma 2. Applying Corollary 1.3:

$$d^2(x_N, \mathcal{W}^\phi(t_N)) \leq e^A B \|\Pi\|.$$

Now, by (iii), and since $t_N = 1$:

$$\mathcal{W}^\phi(t_N) = \{x \in \mathbb{R}^n |\ \phi(1, x) \leq \phi(t_0, x_0)\} \subset \{x \in \mathbb{R}^n |\ g(x) \leq \phi(t_0, x_0)\}.$$

Let $w_N \in \operatorname{argmin}_{w \in \mathcal{W}^\phi(1)} \|x_N - w\|$ be some closest point. By Assumption 3:

$$g(x_N) \leq g(w_N) + c\|x_N - w_N\| \leq \phi(t_0, x_0) + c d(x_N, \mathcal{W}^\phi(t_N)).$$

The result follows, recalling that $x_N = \mathbf{x}[t_0, x_0, \mathbf{u}, \beta(\mathbf{u})](1)$. Explicitly, $C = c\sqrt{e^A B}$. $\square$

Proposition 2.1 applies to any function satisfying (i), (ii) and (iii). Consequently, under Assumptions 1, 2 and 3:

$$V^+(t_0, x_0) \leq \inf\{\phi(t_0, x_0) \mid \phi : [t_0, 1] \times \mathbb{R}^n \to \mathbb{R} \text{ satisfying } (i), (ii), (iii)\}. \tag{2.3}$$

Theorem 2.3. *Under Assumptions 1, 2 and 3, the differential game $\mathcal{G}(t_0, x_0)$ has a value, characterized as:*

$$\mathbf{V}(t_0, x_0) = \min_{\substack{\phi \text{ satisfying} \\ (i),(ii),(iii)}} \phi(t_0, x_0). \tag{2.4}$$

The strategies $\beta(\mathbf{V}, \Pi)$ are asymptotically optimal for player 2, as $\|\Pi\| \to 0$.

Proof. By (2.3), it is enough to prove that V^- satisfies (i), (ii) and (iii), where (iii) is immediate. Assumption 1, and Gronwall's lemma imply that $\forall t \in [t_0, 1]$, $\forall(\mathbf{u}, \mathbf{v}) \in \mathcal{U} \times \mathcal{V}$, and $\forall x, y \in \mathbb{R}^n$:

$$\big\|\mathbf{x}[t_0, x, \mathbf{u}, \mathbf{v}](t)\big) - \mathbf{x}[t_0, y, \mathbf{u}, \mathbf{v}](t)\big)\big| \leq e^{c(t-t_0)}\|x - y\|.$$

Assumption 2 gives then, $\forall(\mathbf{u}, \mathbf{v}) \in \mathcal{U} \times \mathcal{V}$, and $\forall x, y \in \mathbb{R}^n$:

$$\big|g\big(\mathbf{x}[t_0, x, \mathbf{u}, \mathbf{v}](1)\big) - g\big(\mathbf{x}[t_0, y, \mathbf{u}, \mathbf{v}](1)\big)\big| \leq ce^{c(1-t_0)}\|x - y\|.$$

Thus, by standard arguments, $x \mapsto V^-(t, x)$ is ce^c-Lipschitz continuous $\forall t \in [t_0, 1]$ and, in particular, V^- satisfies (i). On the other hand, (ii) is a weak version of the classical dynamic programming principle (see [2], for nonanticipative strategies, and [1] for NAD strategies, defined above): $\forall(t, x) \in [t_0, 1] \times \mathbb{R}^n$, $\forall t_1 \in [t, 1]$:

$$V^-(t, x) = \sup_{\alpha \in \mathcal{A}} \inf_{\mathbf{v} \in \mathcal{V}} V^-\big(t_1, \mathbf{x}[t, x, \alpha(\mathbf{v}), \mathbf{v}](t_1)\big).$$

Finally, let $\beta(\mathbf{V}, \Pi)$ be an extremal strategy. By Corollary 2.1:

$$g\big(\mathbf{x}[t_0, x_0, \mathbf{u}, \beta(\mathbf{V}, \Pi)(\mathbf{u})](1)\big) \leq \mathbf{V}(t_0, x_0) + C\sqrt{\|\Pi\|}, \quad \forall \mathbf{u} \in \mathcal{U}.$$

Consequenly, for any $\varepsilon > 0$, $\beta(\mathbf{V}, \Pi)$ is ε-optimal for sufficiently small $\|\Pi\|$. $\square$

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