Three Brouwer fixed point theorems for homeomorphisms of the plane

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Abstract

We prove three theorems giving fixed points for orientation preserving homeomorphisms of the plane following forgotten results of Brouwer.

1 Introduction

In 1910, Brouwer [Bro10b, Bro11] proved the following three fixed point theorems (the first one is well known as the Cartwright-Littlewood theorem, see the historical remark below). In all three cases we consider an orientation preserving homeomorphism h of $\mathbf{R}^2$. A continuum is a non empty connected compact set.

Theorem 1.1. *Let K be a non separating continuum in $\mathbf{R}^2$ such that $h(K) = K$. Then h admits a fixed point in K.*

Let us recall that a set X is compactly connected if given any two points in X, there exists a subcontinuum of X which contains the two given points.

Theorem 1.2. *Let F be a closed, compactly connected, non separating subset of $\mathbf{R}^2$ without interior such that $h(F) = F$. Then h admits a fixed point in F. More precisely, we will prove that (if F is non compact) if h has no fixed point in F, given any neighborhood V of F, there is a simple closed curve with h-index $+1$ inside V.*

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We now consider again a non degenerated and non separating continuum K in $\mathbf{R}^2$ such that $h(K) = K$. We further suppose that the circle of prime ends of $\mathbf{R}^2 \setminus K$ splits into two non degenerated arcs a_1 and a_2 with the same endpoints such that $\cup_{p \in a_i} \mathrm{I}(p) = K, i = 1, 2$, where $\mathrm{I}(p)$ is the impression of the prime end p (and therefore $\mathrm{int} K = \emptyset$). Equivalently, the end points of all accessible arcs in a_i are dense in K, $i = 1, 2$ (see section 4).

Theorem 1.3. *Suppose that the orientation preserving homeomorphism $\hat{h}$ of the circle of prime ends naturally induced by h preserves a_1 and a_2 (that is fixes the common end points of a_1 and a_2). Then h admits two fixed points in K.*

Historical remark 1.4. In their Annals paper of 1951, Cartwright and Littlewood [CL51] proved Theorem 1.1 using the theory of prime ends. This was probably the first application of that theory to dynamical systems (though as we will see with Theorems 1.2 and 1.3, Brouwer can be considered as a precursor on that matter too) and the result as well as the ideas of its proof have been of great importance and have generated a large body of literature. Nevertheless, four years later, in the same Annals, Reifenberg [Rei55] explained a short elementary proof due to Brouwer of the same result. Strangely enough, that paper of Reifenberg went unnoticed and several papers have been written giving various proofs of Theorem 1.1 [Ham54, Bro77, Med87, BG92, MN11, BFM$^+$10] but without recovering the original ideas.

What we offer here is another presentation of Brouwer clever but elementary ideas with two other fixed point results he obtained in that vein and which are seemingly new knowledge even today. I wish to thank Alexis Marin for his comments on this paper.

2 Index along a curve

Given an continuous path $\alpha : [a, b] \to \mathbf{R}^2$ and a continuous map $f : \mathrm{Im}\alpha \to \mathbf{R}^2$ without fixed point, we can define the index of f along α, denoted $i(f, \alpha)$, as follows. Write $f(\alpha(t)) - \alpha(t) = |f(\alpha(t)) - \alpha(t)| e^{2i\pi\theta(t)}$ for some continuous map $\theta : [a, b] \to \mathbf{R}$.

Definition 2.1. Set $i(f, \alpha) = \theta(b) - \theta(a)$. The following properties follow immediately from those of the covering map $t \mapsto e^{2i\pi t}$ from $\mathbb{R}$ to $\mathbf{S}^1$.

1. $i(f,\alpha)$ does not depend on the choice of the lift θ of the map $\dfrac{f\circ\alpha-\alpha}{|f\circ\alpha-\alpha|}$ from $[a,b]$ to $\mathbf{S}^1$.

2. α and $\alpha\circ\phi$, where ϕ is a map from $[a,b]$ to itself fixing a and b give rise to the same index. More generally, if $\alpha,\beta : [a,b]\to\mathbb{R}^2$ are two paths homotopic rel $\{a,b\}$ such that f has no fixed point on the image of the homotopy, then $i(f,\alpha)=i(f,\beta)$.

3. If α is a closed curve (i.e. $\alpha(a)=\alpha(b)$), then $i(f,\alpha)$ is an integer.

4. If α^{-1} is defined by $\alpha^{-1}(t)=\alpha(b-t+a), a\le t\le b$, then $i(f,\alpha^{-1})=-i(f,\alpha)$

5. If g is an orientation preserving homeomorphism of $\mathbb{R}^2$ and α a closed curve, then $i(gfg^{-1},g(\alpha))=i(f,\alpha)$ (for a proof of this fact, one can use that g is isotopic to id).

   (Notice that the index along a non closed curve is not invariant by conjugation.)

One often thinks equivalently of the vector field without zero ξ on $\mathrm{Im}\alpha$ defined by $\xi(x)=f(\alpha(t))-\alpha(t)$ if $x=\alpha(t)$ and one defines $i(\xi,\alpha)$ as $i(f,\alpha)$.

Lemma 2.2. *$i(f,\alpha)=0$ if α is a simple closed curve and f extend without fixed point to* $\mathrm{int}\alpha$.

As usual, we will denote by $\mathrm{int}\alpha$ the bounded component of $\mathbb{R}^2\setminus\mathrm{Im}\alpha$ when α is a simple closed curve, but if α is an arc (i.e. an injective path) from $[a,b]$ to $\mathbb{R}^2$, $\mathrm{int}\alpha$ will denote $\alpha(]a,b[)$. This should cause no confusion.

Proof. One can suppose that α is given as a map from the unit circle $\mathbf{S}^1=\{e^{2i\pi t}|0\le t\le 1\}$ and, using Schoenflies Theorem, that it extends to a map ϕ from the open unit disc $\mathbf{D}^2$ to $\mathrm{Int}\alpha$. Then the map $F:(t,u)\to \dfrac{f(\phi(ue^{2i\pi t}))-\phi(ue^{2i\pi t})}{|f(\phi(ue^{2i\pi t}))-\phi(ue^{2i\pi t})|}$, $0\le u\le 1$, is a well defined homotopy which lifts to a homotopy θ_u with θ_0 a constant map; now $\theta_u(1)-\theta_u(0)$ is an integer (since $F(0,u)=F(1,u)$) depending continuously on u which is 0 if $u=0$ and so it is 0 also when $u=1$. $\square$

Lemma 2.3. *Suppose that α is an arc and that we are given two maps f and g without fixed point on* $\mathrm{Im}\alpha$ *such that $f(\alpha(a))=g(\alpha(a))$, $f(\alpha(b))=g(\alpha(b))$. Then*

1. *$i(f,\alpha) - i(g,\alpha) = 0$ if the images of f and g lie inside $\mathbb{R}^2 \setminus (L \bigcup \mathrm{Im}\alpha)$ where L is a proper half-line from one endpoint of α towards infinity such that $L \bigcap \mathrm{Im}\alpha$ is reduced to that endpoint.*

2. *$i(f,\alpha) - i(g,\alpha) = 1$ if the images of f and g make up a Jordan curve $\mathcal{C}$, $\mathrm{Im}\alpha \subset \mathrm{int}\mathcal{C}$ and the orientation of $\mathcal{C}$ induced by that of f (from $f(\alpha(a))$ to $f(\alpha(b))$) is positive.*

Proof. 1) Since $\mathbb{R}^2 \setminus (L \bigcup \mathrm{Im}\alpha)$ is simply connected, there exists a homotopy F between f and g relative to the endpoints inside $\mathbb{R}^2 \setminus (L \bigcup \mathrm{Im}\alpha)$. The homotopy $(t,u) \mapsto \dfrac{F(t,u) - \alpha(t)}{|F(t,u) - \alpha(t)|}$ lifts to a homotopy θ_u which gives the result since $\theta_u(b)$ and $\theta_u(a)$ lift constantly $\dfrac{f(\alpha(b)) - \alpha(b)}{|f(\alpha(b)) - \alpha(b)|}$ and $\dfrac{f(\alpha(a)) - \alpha(a)}{|f(\alpha(a)) - \alpha(a)|}$ as u varies.

2) Let $\tilde{f}$ denote the parametrization of $\mathcal{C}$ given by the path composition of f and g^{-1} (where $g^{-1}(\alpha(t)) = g(\alpha(b - t + a))$). Applying Schoenflies theorem, we can suppose that $\mathcal{C}$ is the circle $\mathbf{S}^1$. The homotopy $F(u,t) = \tilde{f}(\alpha(t)) - u\alpha(t)$, for $0 \leq u \leq 1$ gives the conclusion given our orientation hypothesis.

□

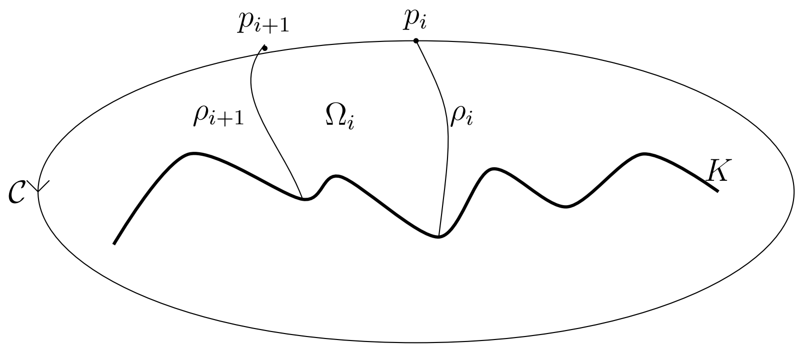
Lemma 2.4. *If $\mathcal{C}$ is a simple closed curve and f a map of $\mathcal{C}$ into $\mathbb{R}^2$ without fixed point such that $f(\mathcal{C}) \subset \mathrm{int}\mathcal{C} \bigcup \mathcal{C}$, then $i(f,\mathcal{C}) = 1$.*

Proof. Once again we apply Schoenflies theorem to reduce the proof to the case $\mathcal{C} = \mathbf{S}^1$ and consider the homotopy $F(u,t) = uf(e^{2i\pi t}) - e^{2i\pi t}$. □

The next Lemma is the key to the ingenious index computation of Brouwer giving the proof of Theorem 1.1. A variant of this computation was rediscovered (but unpublished !) in the eighties by Bell, see [Aki99, BFM$^+$10].

We deal with the following situation: h is an orientation preserving homeomorphism of $\mathbf{R}^2$, $\mathcal{C}$ is a simple closed curve in $\mathbf{R}^2$ positively oriented (int$\mathcal{C}$ lies on the left of $\mathcal{C}$) and K is a h-invariant continuum inside $\mathcal{C}$.

We suppose there exist successive points $p_0, p_1, \ldots p_n \in \mathcal{C}$ with $p_0 = p_n$ and disjoint (except perhaps for their endpoints in K) irreducible arcs $\rho_0, \rho_1, \ldots, \rho_n$ from p_i to K, $0 \leq i \leq n$, $\rho_0 = \rho_n$, such that $h(\rho_i p_i p_{i+1} \rho_{i+1}) \bigcap \rho_i p_i p_{i+1} \rho_{i+1} = \emptyset$. Let Ω_i be the bounded region determined by $K \bigcup \rho_i \bigcup p_i p_{i+1} \bigcup \rho_{i+1}$.



Let ξ be the vector field $\xi(x) = h(x) - x$ which is supposed to be without zero on $\mathcal{C}$. We have:

Lemma 2.5. *There exist a non zero vector field ξ' on $\mathcal{C}$ with endpoints in int$\mathcal{C} \bigcup \mathcal{C}$ such that $i(\xi, \mathcal{C}) - i(\xi', \mathcal{C}) = k \in \mathbf{N}$. And so $i(\xi, \mathcal{C}) = 1 + k$. That integer k is given by the number of i such that $\Omega_i \subset h(\Omega_i)$.*

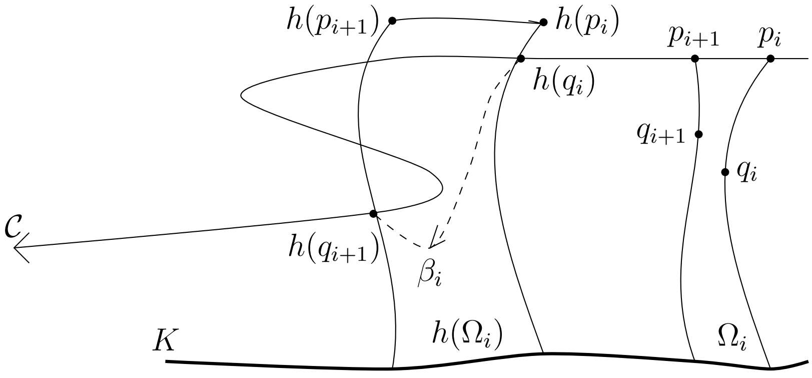
Proof. The second sentence of the Lemma follows from the fact that $i(\xi', \mathcal{C}) = 1$ since the endpoint of $\xi' \in \text{int}\mathcal{C} \bigcup \mathcal{C}$ (see Lemma 2.3).

On ρ_i described from p_i towards K define q_i as p_i if $h^{-1}(\mathcal{C}) \bigcap \rho_i = \emptyset$ or the last point of $h^{-1}(\mathcal{C}) \bigcap \rho_i$ if this set is not empty. Let ϕ_i be an orientation preserving homeomorphism from p_ip_{i+1} to $q_ip_ip_{i+1}q_{i+1}$ and let $\tilde{\xi}$ be the vector field along $\mathcal{C}$ defined by $\tilde{\xi}(x) = h(\phi_i(x)) - x$ for $x \in p_ip_{i+1}$ (this is non zero since $h(\rho_ip_ip_{i+1}\rho_{i+1}) \bigcap p_ip_{i+1} = \emptyset$).

One has obviously $i(\xi, \mathcal{C}) = \sum_{i=0}^{n-1} i(\xi, p_ip_{i+1})$ and also $i(\xi, \mathcal{C}) = \sum_{i=0}^{n-1} i(\tilde{\xi}, p_ip_{i+1})$. Indeed, the path composition of the maps ϕ_i and ϕ_{i+1} from $p_ip_{i+1}p_{i+2}$ to $q_ip_ip_{i+1}q_{i+1}p_{i+1}p_{i+2}q_{i+2}$ is homotopic rel endpoints to a map onto the arc $q_ip_ip_{i+1}p_{i+2}q_{i+2}$. Combining all the ϕ_i we get that $\sum_{i=0}^{n-1} i(\tilde{\xi}, p_ip_{i+1}) = i(\hat{\xi}, \mathcal{C})$ where $\hat{\xi} = h(\phi(x)) - x$ and ϕ is an orientation preserving homeomorphism of $\mathcal{C}$ fixing p_0 and therefore homotopic to the identity rel p_0, so that $i(\hat{\xi}, \mathcal{C}) = i(\xi, \mathcal{C})$.

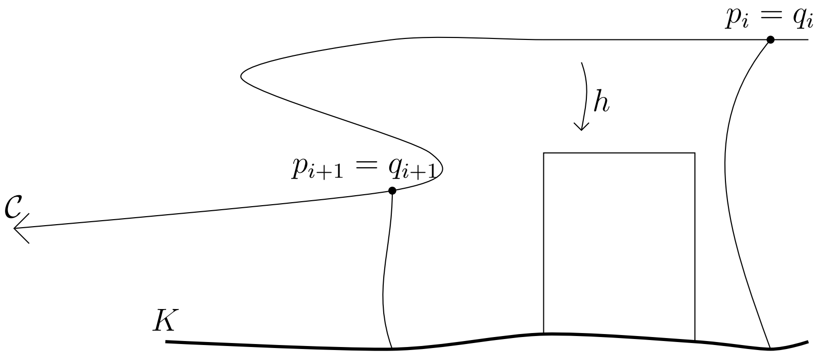
We now distinguish three cases.

1) $h(\Omega_i) \bigcap \Omega_i = \emptyset$.



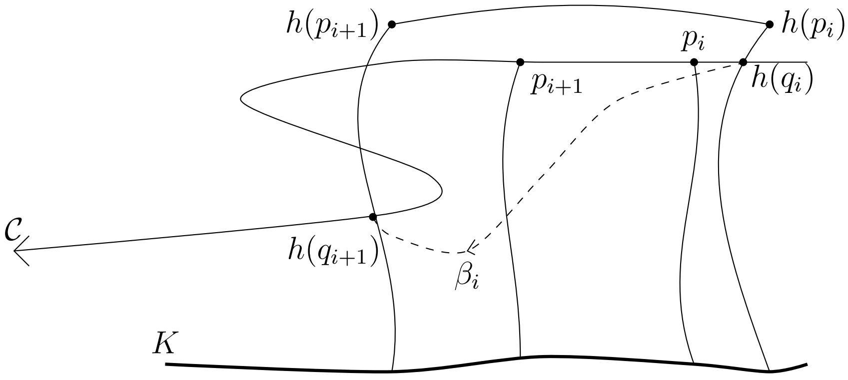
Choose any arc β_i from $h(q_i)$ to $h(q_{i+1})$ inside $\mathcal{C}$ except for its endpoints. Let λ and μ be parametrisations by $[0,1]$ of the arcs p_ip_{i+1} and β_i respectively and define ξ' on p_ip_{i+1} by $\xi'(\lambda(t)) = \mu(t) - \lambda(t)$. One has $i(\tilde{\xi}, p_ip_{i+1}) = i(\xi', p_ip_{i+1})$ by Lemma 2.3(1) since the arc p_ip_{i+1} lies outside the Jordan curve made of β_i and $h(q_ip_ip_{i+1}q_{i+1})$.

2) $h(\Omega_i) \subset \Omega_i$.



In that case $q_i = p_i$ and $q_{i+1} = p_{i+1}$ and we let $\xi' = \tilde{\xi}$ along p_ip_{i+1}. Obviously, $i(\tilde{\xi}, p_ip_{i+1}) = i(\xi', p_ip_{i+1})$

3) $\Omega_i \subset h(\Omega_i)$.



We define ξ' as in case 1). We have $i(\tilde{\xi}, p_i p_{i+1}) - i(\xi', p_i p_{i+1}) = 1$ since the arc $p_i p_{i+1}$ is contained inside the Jordan curve made of β_i and $h(q_i p_i p_{i+1} q_{i+1})$ and h is orientation preserving so that $h(q_i p_i p_{i+1} q_{i+1})$ is oriented from $h(q_i)$ to $h(q_{i+1})$, see Lemma 2.3.

Since $h(\rho_i p_i p_{i+1} \rho_{i+1}) \bigcap \rho_i p_i p_{i+1} \rho_{i+1} = \emptyset$ these three cases exhaust all possibilities and this concludes the proof of the Lemma. □

3 Proof of Theorem 1.1

The next Lemma is very classical [Sch04] and is a key step to show that $\mathbb{R}^2 \setminus K$ is homeomorphic to $\mathbb{R}^2 \setminus \{0\}$ if and only if K is a non separating continuum in $\mathbb{R}^2$ (which is not really needed or used here but explain the Brouwer terminology "circular continum": $\mathbb{R}^2 \setminus K$ looks like a circular region).

We will follow a presentation of Sieklucki [Sie68].

Lemma 3.1. *Let $K \subset \mathbb{R}^2$ a non empty non separating continuum. Then there exists a sequence B_n,$n \geq 0$, of topological closed discs such that*

1. $K = \bigcap B_n$
2. $B_{n+1} \subset B_n$
3. *For every $b \in \mathrm{Fr} B_n$ there exists a rectilinear segment $\rho = \rho(b)$ from b to some point $x(b) \in \mathrm{Fr} K$ such that $\rho(b) \setminus \{b, x(b)\} \subset \mathrm{Int} B_n \setminus K$, $\mathrm{diam}(\rho(b)) < \sqrt{2}.2^{-n}$ and $\rho(b) \bigcap \rho(b')$ is empty or reduced to $x(b) = x(b')$ $(b \neq b')$.*

Proof. For $n \geq 1$, we consider the tiling C_n of the plane by the closed squares centered at $(\frac{k}{2^n}, \frac{l}{2^n})$ of side length $\frac{1}{2^n}$ $(k, l \in \mathbb{Z})$. Let Q_n be the union of all squares of C_n which meet K and B_n be the union of Q_n and all bounded components of $\mathbb{R}^2 \setminus Q_n$. Then B_n is a topological disc and $B_{n+1} \subset B_n$ (since $\mathrm{Fr} Q_{n+1} \subset Q_n$).

To show that $K = \bigcap_{n>0} B_n$, let $x \in \mathbb{R}^2 \setminus K$ and use that K is non separating to find a half line l from x to ∞ such that $l \bigcap K = \emptyset$. Then $\mathrm{d}(l, K) > 0$ and if $\frac{\sqrt{2}}{2^n} < \mathrm{d}(l, K)$ then $x \notin B_n$ (otherwise there exists $y \in l \bigcap \mathrm{Fr} B_n$ and $\mathrm{d}(y, K) < \frac{\sqrt{2}}{2^n} < \mathrm{d}(l, K)$ which is absurd).

As for 3), given $b \in \mathrm{Fr} B_n$, then b belongs to a side of some square Q of C_n. Let $x(b)$ be some nearest point of K in Q and $\rho(b)$ be the rectilinear segment from b to $x(b)$. Given $b, b' \in \mathrm{Fr}B_n$, if $\rho(b)$ and $\rho(b')$ do not belong to the same square, clearly $\rho(b) \bigcap \rho(b')$ is empty or reduced to $x(b) = x(b')$. If $\rho(b)$ and $\rho(b')$ belong to the same square Q, we conclude with the following elementary Lemma. □

Lemma 3.2. *Let Q is a square and K a closed subset in Q. For $b \in \mathrm{Fr}Q$, let $x(b)$ be a nearest point of K in Q and $\rho(b)$ be the rectilinear segment from b to $x(b)$. Then $\rho(b) \bigcap \rho(b')$ is empty or reduced to $x(b) = x(b')$ if $b \neq b'$.*

Proof. Suppose there exists a point c in $\rho(b) \bigcap \rho(b')$. The inequalities

$$|b - c| + |c - x(b)| = |b - x(b)| \leq |b - x(b')| \leq |b - c| + |c - x(b')|$$

give $|c - x(b)| \leq |c - x(b')|$ and by symmetry $|c - x(b)| = |c - x(b')|$. Therefore $|b - x(b')| \geq |c - b| + |c - x(b)| = |c - b| + |c - x(b')|$ so that $|b - x(b')| = |b - c| + |c - x(b')|$ and, by symmetry, $|b' - x(b)| = |b' - c| + |c - x(b)|$. This implies, if $b \neq b'$, that $c = x(b) = x(b')$.

□

We can now complete the proof of Theorem 1.1. We have an orientation preserving homeomorphism h of $\mathbf{R}^2$ and a non separating continuum K in $\mathbf{R}^2$ such that $h(K) = K$. We have to prove that h admits a fixed point in K.

Proof. Let us suppose that h has no fixed point in K. Then we can find a neighborhood V of K and an $\epsilon > 0$ such that $\mathrm{dist}(h(x), x) > 3\epsilon$ on V. According to Lemma 3.1, one can find a Jordan curve $\mathcal{C}$ contained in V and the ϵ-neighborhood of K such that $K \subset \mathrm{int}\mathcal{C}$, successive points $p_0, p_1, \ldots, p_n = p_0$ on $\mathcal{C}$ and disjoint arcs (except perhaps for their extremities on K) $\rho_0, \ldots, \rho_{n-1}$ from p_i to K such that $\mathrm{diam}p_i p_{i+1}$ and $\mathrm{diam}\rho_i$ are less than ϵ and consequentely $h(\rho_i p_i p_{i+1} \rho_{i+1}) \bigcap \rho_i p_i p_{i+1} \rho_{i+1} = \emptyset$. We are then in position to apply Lemma 2.5 which implies that $i(\xi, \mathcal{C}) > 0$ in contradiction to Lemma 2.2.

□

Remark 3.3. Another fixed point theorem of Brouwer [Bro12] is as follows.

Theorem 3.4. *Let h be an orientation preserving homeomorphism of $\mathbb{R}^2$ and let K be a non empty compact subset of $\mathbb{R}^2$ such that $h(K) = K$ then h admits a fixed point (in $\mathbb{R}^2$).*

This result follows quickly (using a perturbation argument for example) from the fact that an orientation preserving homeomorphism of $\mathbb{R}^2$ without fixed point has no periodic point which can also be proved by an index computation [Bro12, Gui94].

Notice that the "short" proof of Theorem 1.1 by Hamilton [Ham54] and the "short short" proof by Brown [Bro77] are indeed merely reduction of Theorem 1.1 to the above result. In fact, if h had no fixed point inside the non separating continuum K, these authors construct (by bare hands for Hamilton, by a simple covering argument for Brown) another extension h' to $\mathbb{R}^2$ of the restriction of h to K which is orientation preserving and without any fixed point: this is a contradiction to Theorem 3.4.

4 Prime ends

We give only a brief sketch of the theory of prime ends in a pre-Caratheodory style, based on the notion of accessible arc and the cyclic order that can be given to equivalence classes of such arcs and as used by Brouwer. See [Mil06] or [Pom92] for modern and more complete expositions.

We consider a non empty continuum $K \subset \mathbf{S}^2 = \mathbb{R}^2 \bigcup \{\infty\}$ such that $U = \mathbf{S}^2 \setminus K$ is non empty and connected.

Definition 4.1. A point $x \in \mathrm{Fr}U = \mathrm{Fr}K$ is said *accessible* (from U) if there is an arc $\gamma : [a, b] \to \overline{U}$ such that $\gamma([a, b)) \subset U$ and $\gamma(b) = x$. The point x is the endpoint of γ and γ is an *access arc* to x.

We define an equivalence relation on the set of access arcs from x_0 to $\mathrm{Fr}U$ by $\gamma \sim \gamma'$ if γ and γ' have the same endpoint $x \in \mathrm{Fr}U$ and γ is isotopic to γ' in $U \bigcup \{x\}$ rel $\{x\}$. Notice that the end point of the class $p = [\gamma]$ is well defined.

Some basic facts are:

-The set of accessible points is dense in $\mathrm{Fr}U = \mathrm{Fr}K$.

-Given a finite number of distinct equivalence classes of access arcs $p_1, p_2, \ldots, p_n$, one can find disjoint access arcs $\gamma_1, \ldots, \gamma_n$ where $\gamma_i \in p_i$.

-Using a circle surrounding K and meeting $\gamma_1, \ldots, \gamma_n$ (see Lemma 3.1) we can transfer a cyclic order on this circle to the set $\{p_1, p_2, \ldots, p_n\}$ and thus define a cyclic order on the set of equivalence classes of access arcs (given coherent choices of orientation for the circles surrounding K; we will assume that K is to the left of each such circle). We can therefore talk of the closed interval $[p, p']$ given two equivalence classes p and p'.

-Given two distinct equivalence classes p and p', there exists a third one p'' such that $p < p'' < p'$.

-Given an equivalence class p, there exist sequences of equivalence classes $(p_n)_{n\geq 0}$ and $(p'_n)_{n\geq 0}$ such that $\bigcap_{n\geq 0}[p_n, p'_n] = \{p\}$.

Definition 4.2. We now consider sequences of decreasing intervals $[p_n, p'_n]$ such that $\bigcap_{n\geq 0}[p_n, p'_n]$ is empty or reduced to one point, where p_n and p'_n are sequences of equivalence classes of access arcs. Two such sequences $[p_n, p'_n]$ and $[q_n, q'_n]$ are considered equivalent if for each n there exist r such that $[p_n, p'_n] \supset [q_r, q'_r]$ and s such that $[q_n, q'_n] \supset [p_s, p'_s]$. Equivalence classes of such sequences of intervals define the *prime ends*.

Given the last fact above, equivalence classes of access arcs are naturally seen as prime ends.

The cyclic order on the equivalence classes of access arcs can be extended to the set of all prime ends and a classical result of the theory of ordered sets gives a cyclic order preserving bijection of the set of prime ends to the circle. If we give the order topology to the set of prime ends such a bijection becomes a homeomophism. Also, any homeomorphism h of $\overline{U}$ extend to the set of equivalence classes of access arcs (by $h([\gamma]) = [h(\gamma)]$) and so to the circle of prime ends.

Definition 4.3. A prime end p being defined by a sequence $[p_n, p'_n]$, we define its *impression* as the set of all points of $\mathrm{Fr}\overline{U}$ which are limits of a sequence of end points of access arcs β_k such that $[\beta_k] \in [p_{n_k}, p'_{n_k}]$ for some increasing subsequence n_k of the integers.

This impression, denoted $\mathrm{I}(p)$, does not depend on the choice of the sequences p_n and p'_n and can be shown to be a subcontinuum of $\mathrm{Fr}U \subset \mathbf{S}^2$. The union over all prime ends of these impressions form a covering of $\mathrm{Fr}U$ but notice that different prime ends may have the same impression and it is even possible that some impressions are equal to $\mathrm{Fr}U$ (see the indecomposable continuum of Brouwer [Bro10a, Rut35, Rog93]).

Definition 4.4. A *cut* c of U is an arc $c : [a, b] \to \overline{U} \setminus \{x_0\}$ such that $c(a), c(b) \in \mathrm{Fr}U$ and $c(a, b) \subset U$.

As is well known, a cut separate U into exactly two regions and we will call the region not containing ∞ the bounded region determined by c (and $\mathrm{Fr}U$).

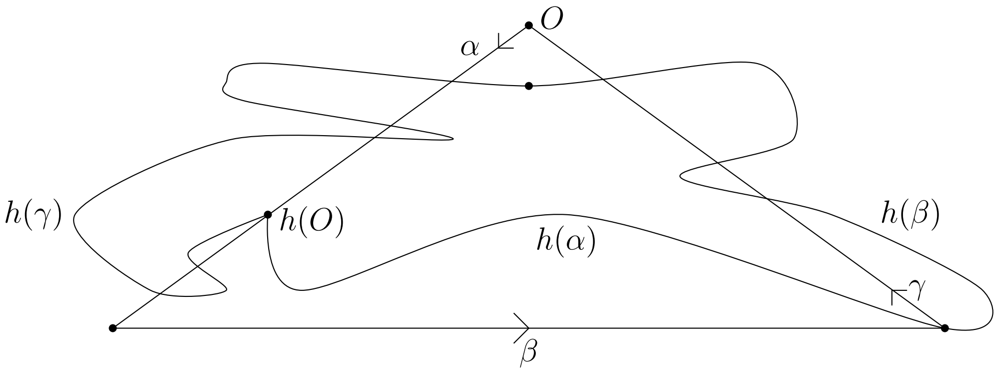
5 Proof of Theorem 1.2

We begin with some preliminary lemmas.

Lemma 5.1. *Let a simple closed curve $\mathcal{C}$ be composed of three consecutive arcs α, β, $\gamma : [0,1] \to \mathbb{R}^2$ with disjoint interiors and $h : \mathcal{C} \to \mathbb{R}^2$ a map without fixed point. Suppose that*

1. $h(\alpha) \subset \mathrm{int}\mathcal{C} \bigcup \mathcal{C}$ *and that* $h(\alpha(0)) \in \mathrm{int}\alpha$.
2. $h(\beta(0)) = \beta(1)$ *and* $h(\beta) \bigcap \beta = \beta(1)$.
3. $\beta \setminus \beta(1)$ *lies in the unbounded region of* $\mathbb{R}^2 \setminus h(\mathcal{C})$.
4. $h(\gamma) \bigcap \gamma = \emptyset$.

Then $i(h, \mathcal{C}) = 1$.



Proof. Let $\star$ denote the path composition and O be $\alpha(0)$. By hypothesis, $\beta \setminus \beta(1)$ lies in the unbounded complementary region of the closed curves $h(\mathcal{C})$ and $h(\alpha) \star \gamma \star Oh(O)$ (where $Oh(O)$ is a subarc of α). Inside $\mathbb{R}^2 \setminus (\beta \setminus \beta(1))$, we have $h(\beta)$ homotopic rel endpoints to $h(\alpha)^{-1} \star h(\gamma)^{-1}$ and $h(\alpha)^{-1}$ homotopic rel endpoints to $\gamma \star Oh(O)$. Therefore, according to Lemma 2.3, we can replace the field $\xi = h(x) - x$ on $\mathcal{C}$ by a field $\xi' = h'(x) - x$, (where $h' : \mathcal{C} \to \mathbb{R}^2$ is without fixed point and equal to h on γ), with the same index as ξ and whose endpoint describes $\gamma \star Oh(O) \star h(\gamma)^{-1} \star h(\gamma)$ as its origin describes $\beta \star \gamma$. The natural homotopy of $h(\gamma)^{-1} \star h(\gamma)$ (supported by $\mathrm{Im}(h \circ \gamma)$) to the constant map on $h(O)$ does not meet fixed points of h' since $\beta \star \gamma$ does not meet $h'(\gamma) = h(\gamma)$. Finally we get a new field on $\mathcal{C}$ with the same index as ξ whose endpoint describes $h(\alpha) \star \gamma \star Oh(O)$, that is a curve inside $\mathrm{int}\mathcal{C} \bigcup \mathcal{C}$ and we conclude with Lemma 2.4. $\square$

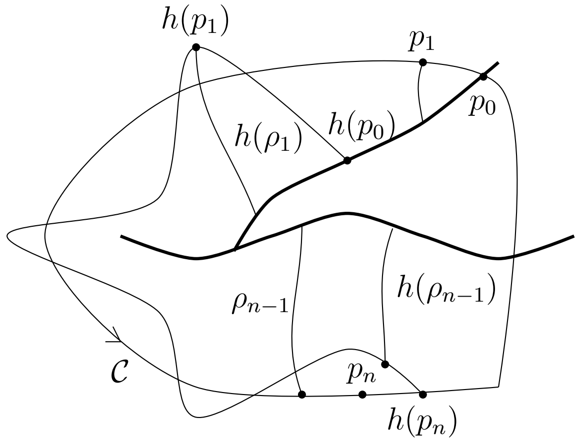
We will need a variation on Lemma 2.5.

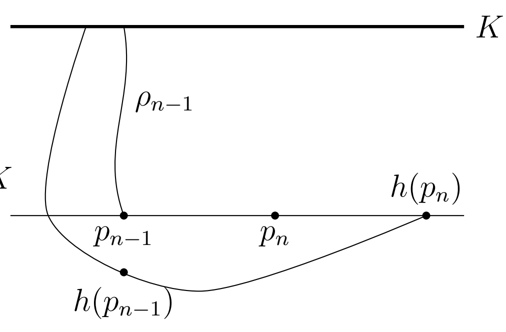
We consider the following situation: $\mathcal{C}$ is a simple closed curve positively oriented, K a closed connected set such that $K \bigcap \mathrm{int}\mathcal{C} \neq \emptyset$ and $h(K) = K$, and ξ is a vector field without zero along $\mathcal{C}$. Suppose there is an arc $\alpha \subset \mathcal{C}$ with $\mathrm{int}\alpha \bigcap K = \emptyset$ and successive points $p_0, p_1, \cdots, p_n \in \alpha$ (where p_0 is the origin of α and p_n the endpoint of α) such that $h(p_0), h(p_n) \in \mathrm{int}\mathcal{C} \bigcup \mathcal{C}$ and arcs $\rho_1, \rho_2, \cdots, \rho_{n-1}$ where ρ_i joins p_i to K irreductibly which are disjoint except perhaps for their endpoint in K. Let Ω_i be the bounded region determined by $\rho_i p_i p_{i+1} \rho_{i+1}$ and K $(1 \le i \le n-1)$.

Lemma 5.2. *If the preceding data satisfy*

1. $\rho_i \subset \mathrm{int}\mathcal{C} \bigcup \{p_i\}$.
2. $h(\rho_i p_i p_{i+1} \rho_{i+1}) \bigcap \rho_i p_i p_{i+1} \rho_{i+1} = \emptyset$, $1 \le i \le n-2$.
3. $h(\rho_i) \bigcap \mathrm{int}\mathcal{C} \neq \emptyset$.
4. *Either p_0 and $h(p_0)$ belong to $\mathcal{C}$ and $h(p_0p_1) \bigcup h(\rho_1)$ does not separate p_0p_1 from infinity in $\mathbb{R}^2 \setminus \mathrm{int}\mathcal{C}$ or p_0 (and $h(p_0)$) belong to K and the bounded region Ω_0 determined by $p_0p_1 \bigcup \rho_1$ and K satisfies $\Omega_0 \bigcap h(\Omega_0) = \emptyset$. And similarly for p_n.*

Then there exists another vector field without zero ξ' along $\mathcal{C}$ equal to ξ outside α such that the endpoints of ξ' belong to $\mathrm{int}\mathcal{C} \bigcup \mathcal{C}$ along α and which satisfies $i(\xi, \alpha) - i(\xi', \alpha) = k$ where $k \ge 0$ is given by the number of i such that $\Omega_i \subset h(\Omega_i)$ $(1 \le i \le n-1)$.





this is forbidden by hypothesis 4.

Proof. The proof is very similar to the proof of Lemma 2.5. We define again, on ρ_i described from p_i towards K, q_i as p_i if $h^{-1}(\mathcal{C}) \bigcap \rho_i = \emptyset$ or the last point of $h^{-1}(\mathcal{C}) \bigcap \rho_i$ if this set is not empty. Let $q_0 = p_0$ and $q_n = p_n$ and proceed now exactly as in the proof of Lemma 2.5, the arcs β_i such that $\beta_i \subset \mathrm{int}\mathcal{C} \bigcup \mathcal{C}$ existing trivially using 3), $\mathrm{int}\mathcal{C} \bigcup \mathcal{C}$ being arc connected. Given hypothesis 4., the contributions of ξ and ξ' to their index are equal on p_0p_1 and $p_{n-1}p_n$, whence the last formula.

$\square$

We will also need the following slight extension of Lemma 3.1.

We consider a non empty non separating continuum in the plane and a finite number of pairwise disjoints arcs $\gamma_i : [-1, 0] \to \mathbb{R}^2$, $1 \leq i \leq k$, such that $\gamma_i(0) \in K$ and $\gamma_i([-1, 0))$ lies in $\mathbb{R}^2 \setminus K$.

Lemma 5.3. *There exists a sequence B_n, $n \geq 0$, of topological closed discs such that*

1. $K = \bigcap B_n$
2. $B_{n+1} \subset B_n$
3. *For every $b \in \mathrm{Fr}B_n \setminus \bigcup \mathrm{Im}\gamma_i$ there exists an arc $\rho = \rho(b)$ from b to some point $x(b) \in \mathrm{Fr}K$ such that $\rho(b) \setminus \{b, x(b)\} \subset \mathrm{Int}B_n \setminus (K \bigcup (\bigcup_i \mathrm{Im}\gamma_i))$, and $\rho(b) \bigcap \rho(b')$ is at most a point in K if $b \neq b'$.*
4. *$\mathrm{diam}\rho(b) \to 0$ uniformly in $b \in \mathrm{Fr}B_n$ as $n \to +\infty$.*

Proof. Schoenflies theorem gives us a homeomorphism ϕ of $\mathbb{R}^2$ such that each $\phi^{-1}(\gamma_i)$ is a vertical segment with abscissa an integer. We now apply the proof of Lemma 3.1 to the continuum $\phi^{-1}(K)$, considering the tiling C_n of the plane by the closed squares of center $(\frac{k}{2^n}, \frac{l}{2^n})$ and side length $\frac{1}{2^n}$, $k, l \in \mathbb{Z}$. Since the γ_i are contained in the 1-skeleton of C_n, we get the desired result for the continuum $\phi^{-1}(K)$ and the $\phi^{-1}(\gamma_i)$. We conclude using the uniform continuity of ϕ on any big ball containing the whole sequence of discs associated to $\phi^{-1}(K)$. $\square$

Lemma 5.4. *A closed compactly connected subset of $\mathbb{R}^2$ can be written as an increasing union of subcontinua.*

Proof. If $F \subset \mathbb{R}^2$ is closed and compactly connected, choose $x_0 \in F$ and let K_n be the connected component of x_0 in $F \bigcap B(O, n)$. We are left to show that $F \subset \bigcup_{n>0} K_n$: but if $x \in F$, there exist a continuum C such that $x_0, x \in C$ and therefore $x \in K_n$ as soon as $C \subset B(O, n)$. $\square$

Lemma 5.5. *Let F be a closed, compactly connected, non compact, non separating subset of $\mathbb{R}^2$, with int$F = \emptyset$, then any neighborhood of F contains a neighborhood of F homeomorphic to $\mathbb{R}^2$ bounded by a proper line. Consequently, $\mathbb{R}^2 \setminus F$ is homeomorphic to $\mathbb{R}^2$.*

Proof. Given any neighbohood W of F, write F as a union of compact, connected, non separating sets : $F = \bigcup_{n>0} K_n$, $K_n \subset \mathrm{int}K_{n+1}$ and use Lemma 3.1 to find a ball B_n such that $K_n \subset \mathrm{int}B_n \subset W$. We choose B_n as a subset a tiling of the plane by squares of side length decreasing with n. Then, the family $(\mathrm{Fr}B_n)_{n>0}$ is locally finite and if V is the union of $\bigcup_{n>0} B_n$ with all the components of $\mathbb{R}^2 \setminus \bigcup_{n>0} B_n$ which lie inside W, then FrV is a non compact connected (since F is non separating) one-manifold properly embedded in $\mathbb{R}^2$, that is a proper line. This implies that $\mathbb{R}^2 \setminus F$ is homeomorphic to an increasing sequence of half-planes and therefore homeomorphic to $\mathbb{R}^2$. □

To prove Theorem 1.2, according to Theorem 1.1, we can suppose F non compact and in all the rest of this section, we consider F a closed, compactly connected subset of $\mathbb{R}^2$ without interior such that $\mathbb{R}^2 \setminus F$ is homeomorphic to $\mathbb{R}^2$ and a homeomorphism h of $\mathbb{R}^2$ preserving F: $h(F) = F$.

It follows from Lemma 5.5 that ∞ is an accessible point of $F \bigcup\{\infty\}$ from $\mathbf{S}^2 \setminus F \bigcup\{\infty\}$, and that if we let γ_∞ be an access arc to ∞, then $h([\gamma_\infty]) = [\gamma_\infty]$. Therefore the set of prime ends of $\mathbf{S}^2 \setminus F \bigcup\{\infty\}$ minus $[\gamma_\infty]$, which we call the prime ends of $\mathbb{R}^2 \setminus F$, is linearly orderable, in fact homeomorphic to a line, and invariant under the homeomorphim induced by h. Given a cyclic order on the prime ends of $\mathbf{S}^2 \setminus F \bigcup\{\infty\}$, the prime ends of $\mathbb{R}^2 \setminus F$ are given by equivalence classes of sequences $([p_n, p'_n])_{n\geq 0}$ with $p_n < p'_n < [\gamma_\infty]$ or $[\gamma_\infty] < p_n < p'_n$ where p_n and p'_n can be represented by access arcs inside $\mathbb{R}^2 \setminus F$ except for their endpoint in F.

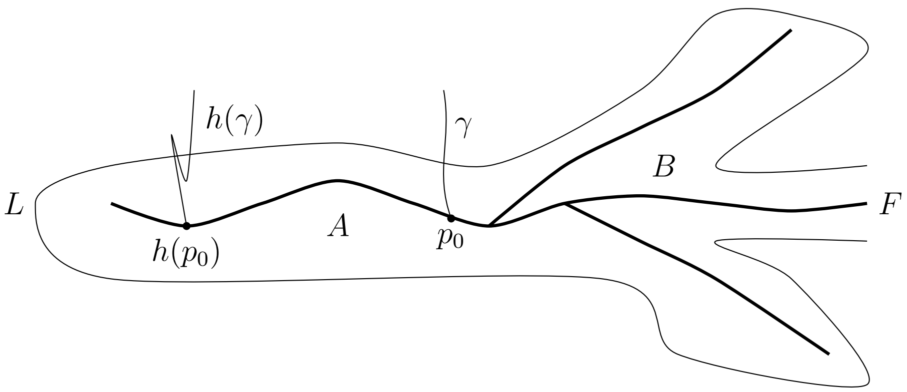
To prove Theorem 1.2, we will suppose, aiming to a contradiction, that h has no fixed point in F and therefore no fixed point on a neighborhood V of F.

Lemma 5.6. *On the line of prime ends h has no fixed point and therefore, for any interval $[a, b]$, neither $[h(a), h(b)]$ or $[h^{-1}(a), h^{-1}(b)]$ is contained in $[a, b]$.*

Proof. Suppose q, defined by $[[\gamma_n], [\gamma'_n]]$, is a fixed prime end and let c be a cut obtained by joining irreductibely γ_0 to γ'_0 by an arc inside $\mathbb{R}^2 \setminus F$. The endpoints of c are both in some compact connected subset D of F since

F is compactly connected and therefore, the region cut out by c in $\mathbb{R}^2 \setminus F$ and containing the endpoints of access arcs β such that $[\beta] \in [[\gamma_0], [\gamma_0']]$ is bounded. This implies that the impression associated to $q = h(q)$ is a non separating compact connected set invariant under h. Theorem 1.1 would then give a fixed point of h in this impression and therefore in F: a contradiction. $\square$

Let γ an access arc to some point $p_0 \in F$ short enough so that $h^{-1}(\gamma)$, γ, $h(\gamma)$, $h^2(\gamma)$ are all disjoint (except perhaps in their endpoints) and let L be a proper line in $V \setminus F$, boundary of a neighborhood of F, close enough to F so that L meets $h^{-1}(\gamma)$, γ, $h(\gamma)$, $h^2(\gamma)$. There exists a subarc pp_0 of γ joining irreductibely L to F and the arc pp_0 separates the region R between L and F into two sub-regions A and B which are unbounded. Lemma 5.6 says that $h(\gamma) \bigcap R$ and $h^{-1}(\gamma) \bigcap R$ are not in the same region and we call A the one containing $h(\gamma) \bigcap R$ and B the one containing $h^{-1}(\gamma) \bigcap R$. By definition γ is on the frontiers of A and B. Notice that A contains $h^k(\gamma) \bigcap R, k \geq 0$ and B contains $h^{-k}(\gamma) \bigcap R, k \geq 1$. Also, all regions cut out in R by $h^k(\gamma)$ and $h^{k+1}(\gamma)$, as k varies in $\mathbb{Z}$, are disjoint (assuming that $h^k(\gamma)$ and $h^{k+1}(\gamma)$ meet L).

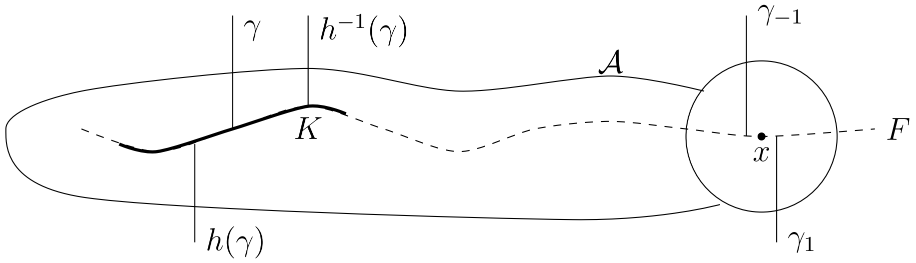


Lemma 5.7. $\mathrm{Fr}A \bigcap \mathrm{Fr}B \bigcap F$ *is unbounded.*

Proof. Let F_A (resp. F_B) be the set of points of F which admit a neighborhood contained in $A \bigcup F$ (resp. $B \bigcup F$). The sets $A \bigcup F_A$ and $B \bigcup F_B$ are disjoint (since $\mathrm{int}F = \emptyset$) and open, therefore their complement in $R \bigcup F \bigcup \setminus(\gamma \setminus \{p\})$ (which complement is the set of points of F for which every neighborhood meets A and B, that is $\mathrm{Fr}A \bigcap \mathrm{Fr}B \bigcap F$) separates $R \bigcup F \setminus (\gamma \setminus \{p\})$ and $R \bigcup F \setminus (\gamma \setminus \{p\})$ can be written as the disjoint union
$(A \bigcup F_A) \coprod (B \bigcup F_B) \coprod (\mathrm{Fr}A \bigcap \mathrm{Fr}B \bigcap F)$.

On the other hand, if $\mathrm{Fr}A \bigcap \mathrm{Fr}B \bigcap F$ was compact in $\mathbb{R}^2$ or equivalently in $R \bigcup F$ (which is homeomorphic to $\mathbb{R}^2$), thinking of L as a straight line and of γ as a segment orthogonal to L (as it is legitimate by Schoenflies theorem), one can find a large rectangle in $R \bigcup F$ with a side parallel to L , containing $\mathrm{Fr}A \bigcap \mathrm{Fr}B \bigcap F$ and whose boundary cuts γ transversaly in a single point. The boundary of this rectangle joins a point of A near γ to a point of B near γ in contradiction to the above decomposition of $R \bigcup F \setminus (\gamma \setminus \{p\})$. $\square$

Given the fact that F is compactly connected, there exist a connected compact K in F containing p_0, $h(p_0)$ and $h^{-1}(p_0)$. The preceding Lemma gives $x \in \mathrm{Fr}A \bigcap \mathrm{Fr}B \bigcap F$ outside of K and we let U be an euclidean disc inside $R \bigcup F$ containing x such that $h(U) \bigcap U = \emptyset$. We can assume that U does not meet K, γ, $h(\gamma)$ and $h^{-1}(\gamma)$.



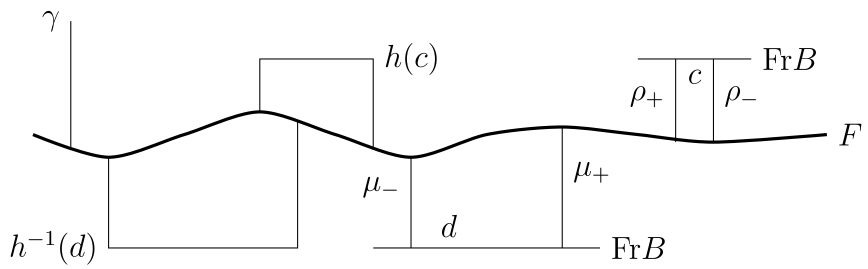
Choose any arc joining the boundary of U to itself in the region R between L and F so that, united to an arc of ∂U it gives a simple closed curve $\mathcal{A}$ containing K and x (to get such an arc, we can consider the boundary of a neighborhood of a continuum in F containing $K \bigcup \{x\}$). Choose then $\epsilon > 0$ so that $\mathrm{dist}(h(u), v) > \epsilon$ for all u, v inside the curve $\mathcal{A}$ with $\mathrm{dist}(u, v) < 3\epsilon$. In particular, if $\mathrm{diam}X < 3\epsilon$, then $h(X) \bigcap X = \emptyset$. We also ask that $\epsilon < \mathrm{dist}(\gamma \bigcap R, h(\gamma) \bigcap R)$. Choose also δ, $\epsilon > \delta > 0$, such that $\mathrm{dist}(u, v) < \delta$ implies $\mathrm{dist}(h(u), h(v)) < \epsilon$ and $\mathrm{dist}(h^{-1}(u), h^{-1}(v)) < \epsilon$ for $u, v \in \mathrm{int}\mathcal{A}$.

Let p_{-1} a point of $U \bigcap F$ accessible from B, p_1 a point of $U \bigcap F$ accessible from A with corresponding access arcs γ_{-1} and γ_1 such that $\mathrm{dist}(p_{-1}, p_1) < \epsilon$. We order the line of prime ends so that $[\gamma_{-1}] < [\gamma] < [\gamma_1]$. By Lemma 5.6, exchanging h and h^{-1} if necessary, we can (and we will) suppose that $h(p) > p > h^{-1}(p)$ for every prime end p. We can assume that γ_{-1}, $h(\gamma_{-1})$, γ, $h^{-1}(\gamma_1)$, γ_1, $h(\gamma_1)$ are all disjoint and meet L (choosing, if necessary, a new

L closer to F). By construction γ separates $h(\gamma_{-1})$ and $h^{-1}(\gamma_1)$ (inside R) and we have the order $[\gamma_{-1}] < [h(\gamma_{-1})] < [\gamma] < [h^{-1}(\gamma_1)] < [\gamma_1]$.

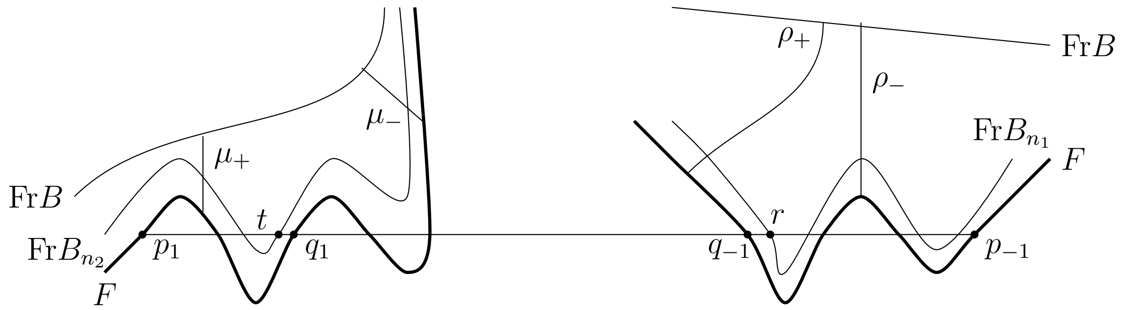
5.1 A simple closed curve

We consider the rectilinear segment p_1p_{-1} and u_{-1} the infimum on the line of prime ends of classes of access arcs between $[\gamma]$ and $[\gamma_{-1}]$ which do not cut p_1p_{-1} and u_1 the supremum of classes of access arcs between $[\gamma]$ and $[\gamma_1]$ which do not cut p_1p_{-1}. Choose a disc B in the ϵ-neighborhood of F as given by Lemma 5.3 applied to a subcontinuum M of F containing $F \bigcap \overline{\text{int}\mathcal{A}}$, p_1 and p_{-1}, and to the arcs γ_1, γ_{-1}. We choose B close enough to M so that $\text{Fr}B$ meets γ_1 and γ_{-1}. Let c be a cut inside $\mathcal{A}$ made of three arcs : two end parts ρ_-, ρ_+ of access arcs such that $[\rho_-] < u_{-1} < [\rho_+]$, $\rho_- \bigcap \gamma_{-1} = \emptyset = \rho_+ \bigcap \gamma_{-1}$ and an arc of $\text{Fr}B$. We can suppose that c and $h(c)$ do not meet γ and that $\text{diam}c < 3\epsilon$ so that $c \bigcap h(c) = \emptyset$. Similarly, we define a cut d inside $\mathcal{A}$ made of (end parts) of access arcs μ_- and μ_+, $[\mu_-] < u_1 < [\mu_+]$ and an arc of $\text{Fr}B$ such that $d \bigcap \gamma = \emptyset = h^{-1}(d) \bigcap \gamma$ and $d \bigcap h^{-1}(d) = \emptyset$.

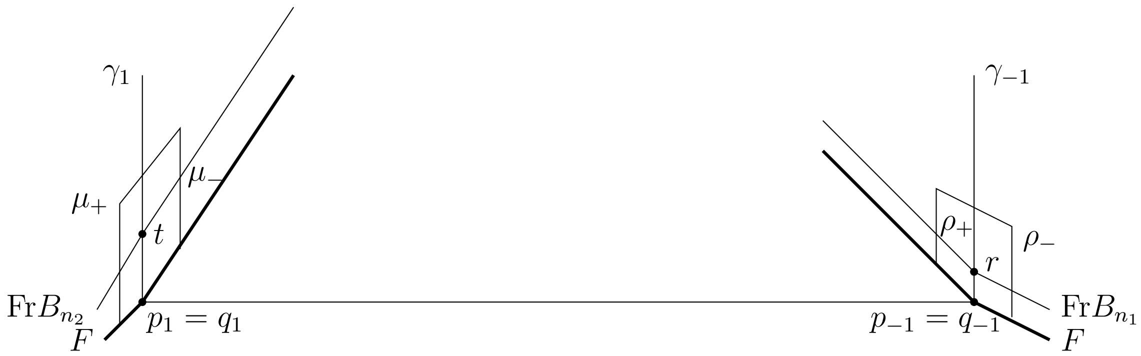


Now, choose a sequence of discs $(B_n)_{n\geq 1}$ inside B as given by Lemma 5.3 relative to M and the arcs ρ_+, $h(\rho_-)$, $h(\rho_+)$, $h^{-1}(\mu_-)$.

If $u_1 > [\gamma_1]$, that is, if between $[\rho_-]$ and $[\rho_+]$ on the line of prime ends there exist prime ends of the form $[\delta]$ such that δ cuts the segment p_1p_{-1} for every representative δ, then we choose n_1 large enough so that $\text{Fr}B_{n_1}$ cuts p_1p_{-1} inside the bounded region determined by c and M and we call r the last point of intersection on p_1p_{-1} of p_1p_{-1} with $\text{Fr}B_{n_1}$.



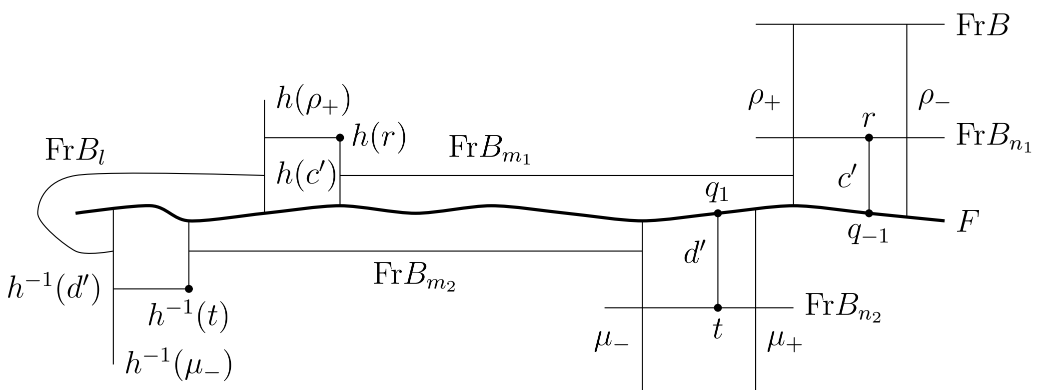
In the opposite case (that is $u_{-1} = [\gamma_{-1}]$) we let r be the last point on γ_{-1} (from x_0 to F) of $\gamma_{-1} \bigcap \mathrm{Fr}B_{n_1}$ where n_1 is choosen large enough so that the diameter of the subarc rp_{-1} of γ_{-1} is less than ϵ.



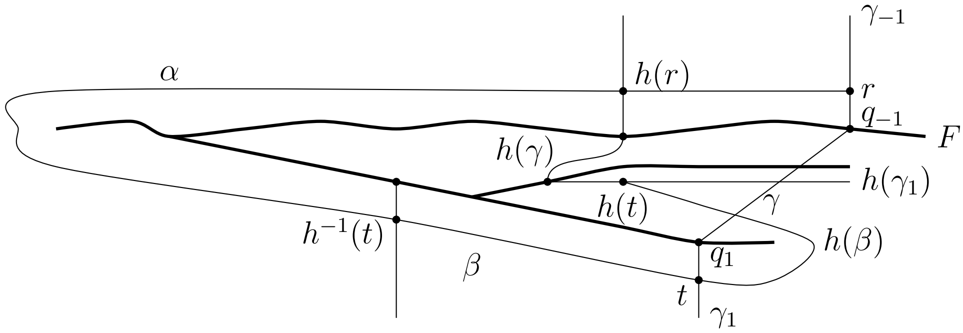
Similarly, if between $[\mu_-]$ and $[\mu_+]$ on the line of prime ends there exist prime ends of the form $[\delta]$ such that δ cuts the segment p_1p_{-1} for every representative δ, then we choose n_2 large enough so that $\mathrm{Fr}B_{n_2}$ cuts p_1p_{-1} inside the bounded region determined by d and M and we call t the first point of intersection on p_1p_{-1} of p_1p_{-1} with $\mathrm{Fr}B_{n_2}$.

In the opposite case (that is $u_1 = [\gamma_1]$) we let t be the last point on γ_1 (from x_0 to F) of $\gamma_1 \bigcap \mathrm{Fr}B_{n_2}$ where n_2 is choosen large enough so that the diameter of the subarc tp_1 of γ_1 is less than ϵ.

On the arc rt (which is a subarc of $\gamma_{-1}p_{-1}p_1\gamma_1$), let q_{-1} and q_1 be the first and last point on M. We now consider the cut c' made of the arc $q_{-1}r$, the subarc of $\mathrm{Fr}B_{n_1}$ from r to ρ_+ and the subarc of ρ_+ from this last point to F, and we join irreductibly the part of ρ_+ in c' to $h(q_{-1}r)$ by an arc of $\mathrm{Fr}B_{m_1}$ for some $m_1 > n_1$ large enough so that $\mathrm{Fr}B_{m_1}$ cuts the part of ρ_+ in c' and $h(q_{-1}r)$.



Similarly, we now consider the cut d' made of the arc q_1t, the subarc of FrB_{n_2} from t to μ_- and the subarc of μ_- from this last point to F, and we join irreductibly the part of μ_- in d' to $h^{-1}(q_1t)$ by an arc of FrB_{m_2} for some $m_2 > n_2$ large enough so that FrB_{m_2} cuts the part of μ_- in c' and $h^{-1}(q_1t)$.



Finally, we join irreductibly $h(\rho_+)$ and $h^{-1}(\mu_-)$ by a subarc of FrB_l, $l > m_1, m_2$, and we consider the simple closed curve from r to $h(r)$ to $h^{-1}(t)$, to t to r composed of subarcs of the cuts c', $h(c')$, $h^{-1}(d')$, d', of subarcs of FrB_{m_1}, FrB_l, FrB_{m_2} and of the arc tr.

In every case, we have constructed a simple closed curve $\mathcal{C}$ in the ϵ-neighborhood of F composed of three consecutive arcs α (from r to $h^{-1}(t)$), β (from $h^{-1}(t)$ to t), γ (from t to r) with disjoints interiors such that, if O denotes the origin of α, $h(O) \in \alpha$, the endpoint of β is the origin of $h(\beta)$ and $h(\beta) \bigcap \beta$ is reduced to the endpoint of β, $h(\beta) \bigcap \alpha = \emptyset$ and $h(\gamma) \bigcap \gamma = \emptyset$. Furthermore, according to Lemma 5.3, we can find a sequence of points $r_0 = r, r_1, r_2, \ldots, r_n = h^{-1}(t)$ on α and for each $i, 1 \leq i \leq n-1$ an arc ρ_i inside $\mathcal{C}$, in the region determined by γ_{-1}, $h^{-1}(\gamma_1)$ and F, irreducible from r_i to F such that $\rho_i \bigcap \rho_{i+1}$ is at most a point in F. We can make these choices (choosing n_1, m_1, l large enough) so that diam$\rho_i r_i r_{i+1} \rho_{i+1}$ is less than 3ϵ so that $\rho_i r_i r_{i+1} \rho_{i+1}$ is disjoint from its image under h.

5.2 An index computation

Our aim is now to compute the index of the non vanishing vector field $\zeta(u) = h(u) - u$ along $\mathcal{C}$: if it is non zero, we will have reached the desired final contradiction.

We observe that an arc ρ from $\alpha \subset \mathcal{C}$ towards F in $\overline{\mathrm{int}\mathcal{C}}$ preceding $h^{-1}(\gamma_1)$ (on the line of prime ends) verifies that $h(\rho)$, from $h(\alpha)$ to F, precedes γ_1 and so must meet $\mathrm{int}\mathcal{C}$. Therefore, hypothesis 3) of Lemma 5.2 is satisfied. Lemma 5.6 justifies hypothesis 4) and also says that in fact $h(\rho_i r_i r_{i+1} \rho_{i+1})$ lies outside the bounded region Ω_i cut out from $\mathbf{R}^2 \setminus F$ by $\rho_i r_i r_{i+1} \rho_{i+1}$. Therefore, we can find a new non vanishing vector field which points inward the curve $\mathcal{C}$ or on $\mathcal{C}$ as its origin describes the arc α and which has the same index as ζ on $\mathcal{C}$.

Since $h(\gamma) \bigcap \gamma = \emptyset$, the distance between $h(\gamma)$ and the endpoint of β (which is also the origin of γ) is positive and we can find an isotopy supported in a small neighborhood, disjoint of γ, of a closed subarc of $\beta \setminus \mathrm{endpoint}(\beta)$ which moves $h(\gamma)$ outside of β. This gives a new non zero vector field, which we write as $f(x) - x$, on $\mathcal{C}$ with the same index as ζ. Since the origin of β can be joined to ∞ using $h^{-1}(\gamma_1)$, we see that β lies in the unbounded region of $\mathbb{R}^2 \setminus f(\mathcal{C})$ except for its endpoint and we can apply Lemma 5.1 to conclude that our original vector field has index 1, which concludes the proof by contradiction of Theorem 1.2.

Example 5.8. Simple examples show the necessity of the hypothesis. For $\mathrm{int}F = \emptyset$, consider a translation and an invariant half-plane, for $\mathbb{R}^2 \setminus F$ connected, consider a translation and an invariant line. For F compactly connected, consider the translation τ given by $\tau(x, y) = (x + 2, y)$ and for F the set $\bigcup_{n \in \mathbb{Z}} \tau^n(G)$ where G is the union of the half-lines $\{(0, y), y \geq 0\}$, $\{(1, y), y \geq 0\}$, $\{(2, y), y \geq 0\}$ and of all the segments from $(1, n)$ to $(\frac{1}{n}, 0)$ and to $(2 - \frac{1}{n}, 0)$, $n \geq 2$.

Remark 5.9. In [Gui11], Theorem 1.2 is proved under the further assumption that h is fixed point free on $\mathbb{R}^2 \setminus F$ using some Brouwer theory related to the plane translation theorem [Bro12] (compare to Remark 3.2). One can reduce the present Theorem 1.2 to the one in [Gui11] using a covering argument as in [Bro77].

6 Proof of Theorem 1.3

We will need some more elementary index computations.

Lemma 6.1. *Let $\mathcal{C}$ be a simple closed curve positively oriented (int$\mathcal{C}$ is on the left) in the plane and $\alpha \subset \mathcal{C}$ an arc from a to b. Let also $f, g : \alpha \to \mathbb{R}^2$ be maps without fixed point such that $f(\alpha) \subset$ int$\mathcal{C} \bigcup \mathcal{C}$ and $g(\alpha) \subset$ ext$\mathcal{C} \bigcup \mathcal{C}$.*

1. *If $f(a) = g(a)$ and $f(b) = g(b)$ lie in intα, then $i(f, \alpha) - i(g, \alpha) = -1$. Idem if a and b lie inside the arc of $\mathcal{C}$ from $f(a) = g(a)$ to $f(b) = g(b)$.*

2. *If $f(a) = g(a)$ lies in intα and $fb) = g(b)$ lies after b outside of α on $\mathcal{C}$, then $i(f, \alpha) - i(g, \alpha) = 0$. Idem if $f(b) = g(b)$ lies in intα and $f(a) = g(a)$ lies before a outside of α.*

Proof. As for 1), using Schoenflies theorem, we can think of the arc ab as a vertical segment with $\mathcal{C}$ on the left of the line supporting this segment. In the first case, the vector $\dfrac{f(\alpha(t)) - \alpha(t)}{|f(\alpha(t)) - \alpha(t)|}$ goes from $(0, 1)$ to $(0, -1)$ without ever pointing to the right so that $i(f, \alpha) = -\dfrac{1}{2}$ and similarly $i(g, \alpha) = \dfrac{1}{2}$. The second case and point 2) are treated in the same way. $\square$

We consider now a non degenerated non separating compact connected set $K \subset \mathbf{R}^2$ and an orientation preserving homeomorphism $h : \mathbf{R}^2 \to \mathbf{R}^2$ preserving K: $h(K) = K$. According to Theorem 1.1, h admits a fixed point in K.

The simple case $K = [-1, 1] \times \{0\} \subset \mathbf{R}^2$ and h a π-rotation around $(0, 0)$, shows that some extra hypothesis in Theorem 1.1 is needed in order to get two fixed points in K. What follows is a formal version of the idea of preserving the sides of $[0, 1] \times \{0\}$.

We suppose further that the circle of prime ends of $\mathbf{R}^2 \setminus K$ splits into two (non degenerated) arcs a_1 and a_2 with the same endpoints such that $\bigcup_{p \in a_i} \mathrm{I}(p) = K$, $i = 1, 2$, where $\mathrm{I}(p)$ is the impression of the prime end p (and therefore int$K = \emptyset$).

Theorem 1.3 then states that if the orientation preserving homeomorphism of the circle of prime ends induced by h preserves a_1 and a_2 (that is fixes the common endpoints of a_1 and a_2), then h admits two fixed points in K

For the proof we will argue by contradiction and suppose that h has only one fixed point $p_0 \in K$.

Lemma 6.2. *There is no accessible periodic point of period $k > 1$ in K.*

Proof. Suppose there exist an accessible periodic point p in K of period $k > 1$ and let γ be an access arc for p. We can suppose that $[\gamma] \in a_1$ and, since the orientation preserving homeomorphim of the circle of prime ends induced by h has no periodic point (only fixed points), we can suppose that $[\gamma]$, $[h(\gamma])$, ..., $[h^k(\gamma)]$ are represented by disjoint arcs γ_0, γ_1, ..., γ_k except that $\gamma_0 \bigcap \gamma_k = \{p\}$. We can find Jordan curve J by joining irreductibely γ and γ_k inside $\mathbb{R}^2 \setminus K$, such that int$J$ contains $h(p)$, ..., $h^{k-1}(p)$, so that int$J \bigcap K \neq \emptyset$ but no point of int$J \bigcap K$ is endpoint of an access arc δ with $[\delta] \in a_2$: a contradiction. $\square$

Lemma 6.3. *There exist access arcs γ with $[\gamma] \in a_1$ and δ with $[\delta] \in a_2$ with endpoints p and q and an arc α from p to q such that $\gamma \bigcup \alpha \bigcup \delta$ is a arc in $\mathbb{R}^2$, $h(\alpha) \bigcap \alpha = \emptyset = h^2(\alpha) \bigcap \alpha$ and $h(\gamma) \bigcap \alpha = \emptyset = h^{-1}(\gamma) \bigcap \alpha$. (Possibly $p = q$ and α is reduced to a point).*

Proof. Choose some access arc γ with $[\gamma] \in a_1$ and with endpoint $p \neq p_0$ in K. Using Schoenflies theorem one can think of γ as a straight segment. Let then B be an euclidean disc such that $p \in$ int$B$, $p_0 \notin B$, $B \bigcap h(\gamma) = \emptyset = B \bigcap h^{-1}(\gamma)$, $B \bigcap h^{-1}(B) = \emptyset = B \bigcap h(B)$ and $B \bigcap h^2(B) = \emptyset$. Let $\tilde{\delta}$ be an access arc with $[\tilde{\delta}] \in a_2$ and with endpoint $\tilde{q} \in B \bigcap K$. Inside B the segment $\tilde{\alpha}$ from p to $\tilde{q}$ satisfies $\tilde{\alpha} \bigcap \gamma = \{p\}$. We can suppose $\tilde{\delta}$ short enough so that $\tilde{\delta} \subset B$ and therefore $\tilde{\delta} \bigcap h(\tilde{\alpha}) = \emptyset$. We now follow $\tilde{\delta}$ from its origin to $\tilde{q}$ until we meet $\tilde{\alpha}$. We then follow $\tilde{\alpha}$ towards p until we reach an accessible point q on $K \bigcap \tilde{\alpha}$; the path followed is an access arc δ for q and we define α as the part of $\tilde{\alpha}$ between p and q. Surely $[\delta] \in a_2$ for otherwise, between $[\delta]$ and $[\tilde{\delta}]$ there would be an endpoint of a_1 which is a fixed prime end in contradiction to $B \bigcap h(B) = \emptyset$. $\square$

For the rest of this section we suppose, without loss of generality, that $g = h$. We will also assume that γ and δ are short enough so that the eight arcs $h^{-1}(\gamma)$, γ, $h(\gamma)$, $h^2(\gamma)$ and $h^{-1}(\delta)$, δ, $h(\delta)$, $h^2(\delta)$ are all disjoints.

Let $\mathcal{A}$ be a simple closed curve such that $K^+ = K \bigcup \alpha \bigcup h(\alpha) \bigcup h^2(\alpha) \subset$ int$\mathcal{A}$ close enough to K^+ so that $\mathcal{A}$ cuts $h^{-1}(\gamma)$, γ, $h(\gamma)$, $h^2(\gamma)$ and $h^{-1}(\delta)$, δ, $h(\delta)$, $h^2(\delta)$. That curve is split by an irreductible subarc of $\gamma \bigcup \alpha \bigcup \delta$ from $\mathcal{A}$ to itself, containing α, into two arcs $\mathcal{A}_1$ and $\mathcal{A}_2$ with the same endpoints. These arcs, joined with the preceding irreducible subarc of $\gamma \bigcup \alpha \bigcup \delta$, give rise to two simple closed curves $\tilde{\mathcal{A}}_1$ and $\tilde{\mathcal{A}}_2$ with disjoint interiors such that one of them, say $\tilde{\mathcal{A}}_1$, does not contain the fixed point p_0. Denote by $\tilde{\mathcal{D}}_1$ the closure of the interior of $\tilde{\mathcal{A}}_1$ and let $L = K \bigcap (\tilde{\mathcal{D}}_1 \bigcup h(\tilde{\mathcal{D}}_1) \bigcup h^2(\tilde{\mathcal{D}}_1))$. Notice that $p_0 \notin L$.

We orient $\mathcal{A}$ by going from γ to δ on $\tilde{\mathcal{A}}_1$ without meeting α (equivalently, if α is non degenerated, we orient α from q to p).

Given our hypothesis that there is only p_0 as fixed point in K, we can now find a neighborhood U of $L \bigcup \alpha \bigcup h(\alpha) \bigcup h^2(\alpha)$ such that $p_0 \notin U$ and $\epsilon > 0$ such that $\mathrm{dist}(h(x), x) > 3\epsilon$ and $\mathrm{dist}(h^{-1}(x), x) > 3\epsilon$ on U. Furthermore, we ask that $2\epsilon < \mathrm{dist}(\gamma \bigcap U, h(\gamma) \bigcap U), \mathrm{dist}(\delta \bigcap U, h(\delta) \bigcap U)$, $2\epsilon < \mathrm{dist}(\alpha, h(\alpha))$, $\mathrm{dist}(h(\alpha), h^2(\alpha))$ and $\mathrm{dist}(h^2(\alpha), h^3(\alpha))$. Finally, let $\epsilon > 3\epsilon' > 0$ be such that if $\mathrm{dist}(x, y) < 3\epsilon'$ then $\mathrm{dist}(h(x), h(y)) < \epsilon$ and $\mathrm{dist}(h^{-1}(x), h^{-1}(y)) < \epsilon$.

Our aim is now to find a closed curve $\mathcal{C}$ such that $L \subset \mathrm{int}\mathcal{C} \subset U$ and to compute the index of the vector field $\zeta(x) = h(x) - x$ on $\mathcal{C}$ (or the one of $\zeta'(x) = h^{-1}(x) - x$). If it is non zero, we will have reached a contradiction proving the theorem.

Let $\hat{L}$ be $L^+ = L \bigcup \alpha \bigcup h(\alpha) \bigcup h^2(\alpha)$ plus all the bounded components of $\mathbb{R}^2 \setminus L^+$. We now apply Lemma 5.3 to $\hat{L}$ and the arcs $h^{-1}(\gamma)$, γ , $h(\gamma)$, $h^2(\gamma)$ and $h^{-1}(\delta)$, δ , $h(\delta)$, $h^2(\delta)$ to get an arc η from $h(\gamma)$ to $h(\delta)$ in $U \setminus \hat{L}$ which is ϵ'-close to $\hat{L}$. Adding subarcs of $h(\gamma)$ and $h(\delta)$ to $\eta \bigcup h(\alpha)$ we get an oriented simple closed curve $\mathcal{C}$. By construction the fixed point p_0 does not belong to $\mathrm{int}\mathcal{C}$.

The arc η comes equiped with a sequence of successive points, $r_0 \in h(\gamma), r_1, \ldots, r_n \in h(\delta)$ such that $\mathrm{diam} r_i r_{i+1} < \epsilon'$ for each $i, 0 \leq i \leq n-1$, and for each $i, 1 \leq i \leq n-1$, an arc ρ_i inside $\mathcal{C}$, disjoint from all the arcs $h^{-1}(\gamma)$, γ , $h(\gamma)$, $h^2(\gamma)$ and $h^{-1}(\delta)$, δ , $h(\delta)$, $h^2(\delta)$, irreducible from r_i to $\hat{L}$, such that $\mathrm{diam}\rho_i < \epsilon'$ and therefore so that each one of the cuts $\rho_i r_i r_{i+1} \rho_{i+1}$ of $\mathbf{R}^2 \setminus K$ is disjoint from its image under h or h^{-1}.

We forget the ρ_i with endpoint on α, $h(\alpha)$ or $h^2(\alpha)$ (recall that these three arcs are disjoint). By choice of ϵ', we still have cuts disjoint from their images under $h$or h^{-1}. Indeed, if c is a cut of $\mathbf{R}^2 \setminus K$ subarc of α, $h(\alpha)$ or $h^2(\alpha)$, $\rho_{k+1}, \ldots, \rho_{l-1}$ the ρ_i with endpoint on c and d is the cut $\rho_k r_k r_l \rho_l$ obtained by forgetting $\rho_{k+1}, \cdots, \rho_{l-1}$, then every point of d has distance less than $3\epsilon'$ to c, therefore every point of $h(d)$ has distance less than ϵ to $h(c)$ and $h(d) \bigcap d = \emptyset$ since $\mathrm{dist}(c, h(c)) > 2\epsilon$.

We will distinguish four cases according to the order of the pairs of prime ends $([\gamma], h([\gamma])) \in a_1$ and $([\delta], h([\delta])) \in a_2$ on the circle of prime ends.

6.1 First case

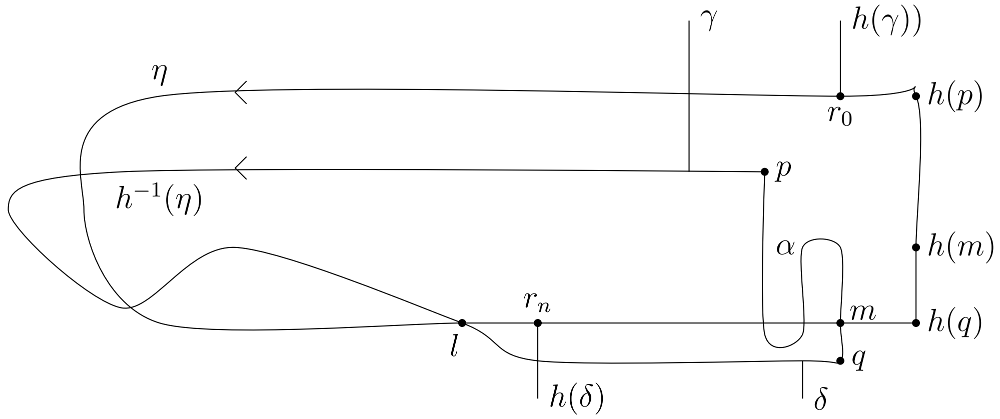
$h([\gamma])$ precedes $[\gamma]$ and $h([\delta])$ precedes $[\delta]$.

First remark that $\gamma \bigcup \alpha \bigcup \delta$ separates $\mathrm{int}\mathcal{A}$ into two regions and by hypothesis in this case the parts of $h(\gamma)$ and $h(\delta)$ close to $\mathcal{A}$ do not belong to the same region. Therefore $h(\delta)$ has to meet α before ending in $h(q)$ and p

and q are separated by $h(\delta)\bigcup h(\alpha)\bigcup h(\gamma)$. Therefore $p\in \mathrm{int}\mathcal{C}$ and $q\in \mathrm{ext}\mathcal{C}$ (even $\delta\subset \mathrm{ext}\mathcal{C}$).

Let l be the last point of intersection of $\eta\bigcup r_n h(q)$ and $h^{-1}(\eta)$ (on $r_0h(q)\subset\mathcal{C}$ oriented from r_0 to $h(q)$), and m be the first point of intersection of α and $h(\delta)$ on α (oriented from q to p). Notice that l precedes m on $\mathcal{C}$ and that the arc lm on $h^{-1}(\mathcal{C})$ lies outside $\mathcal{C}$. Also if $l\in h(\delta)$, then, since $h(l)\in\eta$, $h(l)$ precedes l on $\eta\bigcup r_n h(q)$.

Notice that the intersections $\alpha\bigcap\eta$, $h^{-1}(\eta)\bigcap h(\alpha)$ and $h(\alpha)\bigcap\alpha$ are empty.



We will compute the index of h^{-1} along $\mathcal{C}$ as the sum of three contributions: the index of h^{-1} along the subarc $h(p)h(l)$ on $\mathcal{C}$, then along the subarc $h(l)h(m)$ and finally along the subarc $h(m)h(p)\subset h(\alpha)$ which we denote by i_1, i_2 and i_3 respectively.

We will distinguish two subcases according to the position of $h(l)$ which lies before or after l on $\eta\bigcup r_n h(q)\subset\mathcal{C}$.

6.1.1 Subcase 1

$h(l)$ lies after l on $\eta\bigcup r_n h(q)\subset\mathcal{C}$ (then, since $h(l)\in\eta$, l and $h(l)\in\eta$):

Let k be the index such that $h(l)$ lies between r_k and r_{k+1}. Using Lemma 5.2 on K and the arc $h(p)h(l)\subset\mathcal{C}$ subdivided by the points $h(p), r_1,\ldots,r_k, h(l)$ with the arcs $\rho_1,\ldots,\rho_k$ we get $i_1=j_1+n, n\geq 0$ where j_1 is the index of a vector field whose origin describes $h(p)h(l)$ while its extremity describes an arc from p to l inside $\mathcal{C}$. Indeed, hypothesis 3) of this Lemma is verified, for if an arc ρ goes from $h(p)h(l)$ towards K then $h^{-1}(\rho)$, which is issued from $h^{-1}(\eta)$ before l must step into $\mathcal{C}$ since K does not meet the components of $\mathrm{int}h^{-1}(\mathcal{C})\bigcap\mathrm{ext}\mathcal{C}$ except perhaps the one which contains lm in its frontier. Hypothesis 4) too is verified at $h(p)$ since ρ_1 lies in the region determined by γ, $h(\gamma)$ and $\hat{L}$ by choice of ϵ and therefore the region Ω_0 determined by

$h(p)r_0r_0r_1\rho_1$ and $\hat{L}$ is disjoint from its image by h^{-1}. It is verified also at $h(l) \in \eta$ since by choice of ϵ and ϵ', l lies before r_k on η, $h^{-1}(r_k)$ precedes l on $h^{-1}(\eta)$ and $h^{-1}(\rho_k)$ lie inside $h^{-1}(\mathcal{C})$.

Lemma 6.1 imply that $i_2 = j_2$ where j_2 is the index of a vector field whose origin describes $h(l)h(m)$ while its extremity describes an arc from l to m inside $\mathcal{C}$ and Lemma 2.4 that $i_3 = j_3$ where j_3 is the index of a vector field whose origin describes $h(m)h(p)$ while its extremity describes an arc from m to p inside $\mathcal{C}$.

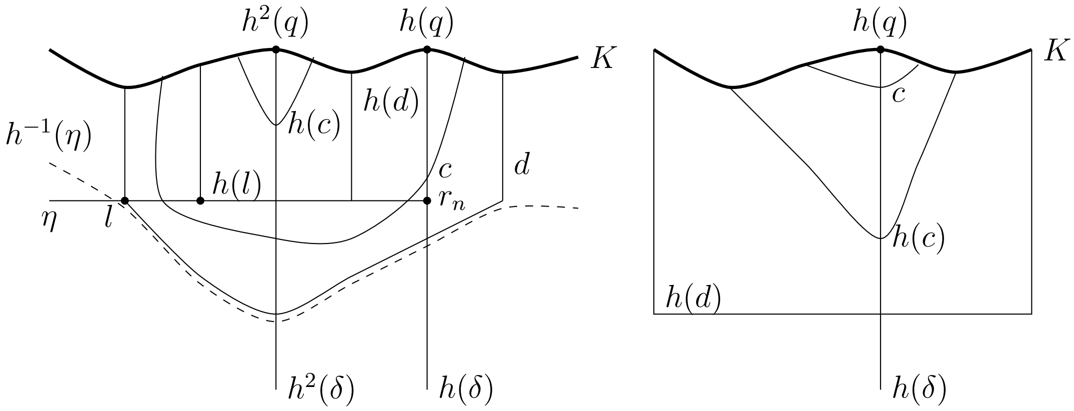
The sum $j_1 + j_2 + j_3$ is equal to 1 since it computes the index of a vector field whose origin describes $\mathcal{C}$ while its end point stays inside $\mathcal{C}$. Therefore we get that the index of h^{-1} along $\mathcal{C}$, which is $i_1+i_2+i_3$, is equal to $1+n \geq 1$ in contradiction to the hypothesis that there is no fixed point for h^{-1} inside $\mathcal{C}$.

6.1.2 Subcase 2

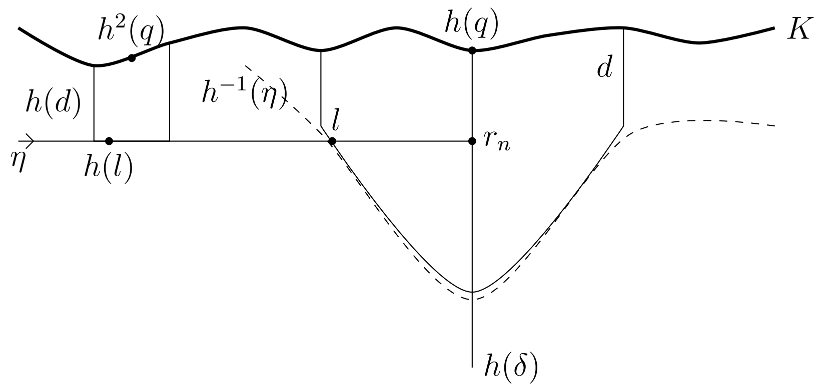
$h(l)$ lies before l on $\eta \bigcup r_n h(q)$:

For the computation of i_1, we can repeat everything said in subcase 1, except for the verification of hypothesis 4 of Lemma 5.2 at $h(l)$. But now we want to get $i_1 = j_1 + n$ for some $n \geq 1$ and we need a more detailed study of the curves $\mathcal{C}$ and $h^{-1}(\mathcal{C})$ near $h(l)$ and l.

Notice first that since a cut c subarc of α separates $h(q)$ from ∞, the cut $h(c)$ separates $h^2(q)$ from ∞ and there is a *special cut* (that is one of the form $\rho_i r_i r_{i+1} \rho_{i+1}$) which contains $h(c)$ in the bounded region it determines with K and which separates $h^2(q)$ from ∞. If we call $h(d)$ this special cut, then the cut d separates $h(q)$ from ∞ and contains c in its associated bounded region. Since $d \bigcap h(d) = \emptyset$ there are apriori three possibilities for the relative position of d and $h(d)$. But d cannot be contained in the bounded region associated to $h(d)$ since $h(\delta) \bigcap h(\alpha) = \{h(q)\}$ and $h(d)$ cannot be contained in the bounded region determined by d since in that case we would have l before $h(l)$ on $\eta \bigcup r_n h(q)$.



Therefore the bounded regions associated to d and $h(d)$ are disjoint and l is the last point on d (starting from the endpoint of d before $h(\delta)$ on the circle of prime ends) of $\eta \bigcup r_n h(q)$: between l and m, $\eta \bigcup h(\alpha)$ and $h^{-1}(\eta) \bigcup \alpha$ are disjoint.



Therefore, between $h(\gamma)$ and $h^2(\delta)$, there exists a special cut $\tilde{c}$ which contains an end point of a_1 (invariant subarc of the circle of prime ends) that is a fixed point on the circle of prime ends. Let $\tilde{\Omega}$ be the bounded region determined by $\tilde{c}$ and K. If $\tilde{\Omega} \subset h^{-1}(\tilde{\Omega})$ we get $n \geq 1$ (see Lemma 5.2). If $h^{-1}(\tilde{\Omega}) \subset \tilde{\Omega}$, then that endpoint of a_1 is a repulsor for the map $\hat{h}$ induced by h on the circle of prime ends and the position of δ and $h(\delta)$ (or γ and $h(\gamma)$) on that circle imply that there is another fixed point between $[h(\gamma)]$ and $[h^2(\delta)]$ which is an attractor for $\hat{h}$. This gives a special cut $\hat{c}$ with $\hat{\Omega} \subset h^{-1}(\hat{\Omega})$ so that in any case, $n \geq 1$.

Also $i_2 = j_2 - 1$ by Lemma 6.1, $i_3 = j_3$ by Lemma 2.4 and $j_1 + j_2 + j_3 = 1$. We conclude again that $i_1 + i_2 + i_3$ is equal to $1 + n - 1 \geq 1$ to get the same contradiction.

Remark 6.4. In the situation of subcase 1, $h(d)$ is in the bounded region determined by d and one gets also $n \geq 1$ in that subcase (see the picture), but this information was not necessary there.

6.2 Second case

$h([\gamma])$ follows $[\gamma]$ and $[h(\delta)]$ precedes $[\delta]$.

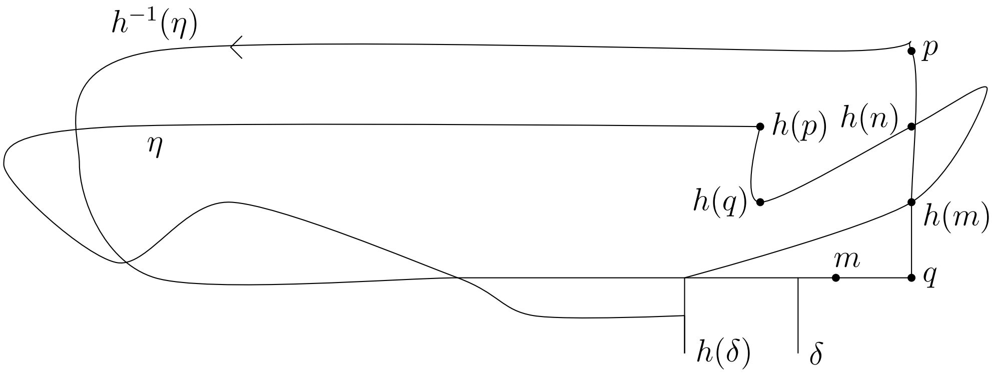
We will compute the index of h along the curve $h^{-1}(\mathcal{C})$.

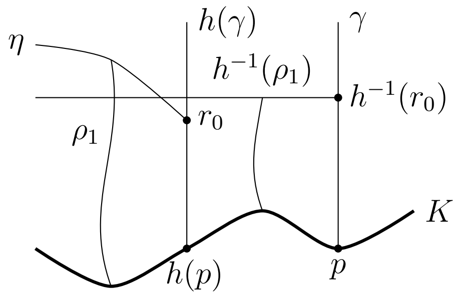
Since $h(\gamma)\bigcap\gamma\bigcup\alpha\bigcup\delta = \emptyset$, one has $h(p) \in \mathrm{int}h^{-1}(\mathcal{C})$. Also $h(q) \in \mathrm{int}h^{-1}(\mathcal{C})$, otherwise, since $h(\alpha)\bigcap(h^{-1}(\eta)\bigcup\alpha\bigcup\gamma) = \emptyset$, we would have that $h(\alpha)$ cuts δ and so $h(\delta)$ would cut δ.

Again, we will distinguish two subcases.

6.2.1 Subcase 1

$h(\delta)$ cuts α. And therefore $h(\alpha)\bigcap\delta = \emptyset$ and $h(\alpha) \subset \mathrm{int}h^{-1}(\mathcal{C})$.





Let $h(m)$ (resp. $h(n)$) denote the first (resp. last) intersection point of $h(\delta)$ and α on α oriented from q to p.

We compute our index as the sum $i_1 + i_2$ where i_1 is the index of h along the subarc pm of $h^{-1}(\mathcal{C})$ and i_2 the index of h along the subarc mp of $h^{-1}(\mathcal{C})$.

The arc pm comes equipped with the points $h^{-1}(r_i)$ and the arcs $h^{-1}(\rho_i)$ which gives a sequence of special cuts on pm disjoint from their images under h. To apply Lemma 5.2, we first verify its third hypothesis. If an arc ρ

goes from $pm \subset h^{-1}(\mathcal{C})$ to K, then $h(\rho)$ from $h(p)h(m)$ towards K must go inside $h^{-1}(\mathcal{C})$ since K does not meet the components of $\mathrm{ext}h^{-1}(\mathcal{C}) \bigcap \mathrm{int}\mathcal{C}$. As for the fourth hypothesis, note that since $h^{-1}(\rho_1) \bigcap h(\gamma) = \emptyset$, the bounded region determined by $ph^{-1}(r_0)h^{-1}(r_1)h^{-1}(\rho_1)$ is contained in the the region between γ and $h(\gamma)$ and does not meet its image under h. At the other end of the arc pm, we have $h^{-1}(\rho_{n-1}) \bigcap h(\delta) = \emptyset$ and so $h^{-1}(\rho_{n-1})h^{-1}(r_{n-1})h^{-1}(r_n)m$ is contained in the region bounded by δ and $h(\delta)$, whence hypothesis 4). Lemma 5.2, which can now be applied gives then $i_1 = j_1 + n$, $n \geq 0$, where j_1 be the index along pm of a vector field whose origin describes pm while its endpoint describes a path inside $h^{-1}(\mathcal{C})$ from $h(p)$ to $h(m)$.

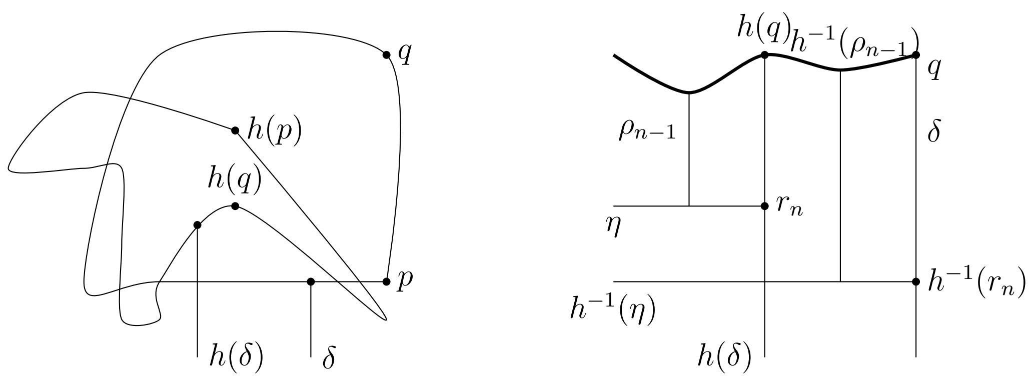
Let j_2 be the index along mp of a vector field along mp whose origin describes mp while its endpoint follows the curve obtained by replacing in $h(m)h(p) \subset \mathcal{C}$ the subarc $h(m)h(n) \subset \mathcal{C}$ by the subarc of α with the same endpoints.

One has $i_2 = j_2$ since the subarcs from $h(m)$ to $h(n)$ on $\mathcal{C}$ and α are homotopic rel their endpoints in $\mathbb{R}^2 \setminus \delta$ (and $mn \subset \delta$)

Since $j_1 + j_2 = 1$ (Lemma 2.4), we get $i_1 + i_2 = 1 + n \geq 1$, a contradiction.

6.2.2 Subcase 2

$h(\delta)$ does not cut α



Let i_1 be the index of h along $pq \subset h^{-1}(\mathcal{C})$. The arc pq is again equipped with the points $h^{-1}(r_i)$ and the arcs $h^{-1}(\rho_i)$ which gives a sequence of special cuts on pm disjoint from their images under h. Hypothesis 3. and 4. of Lemma 5.2 are verified as in subcase 1 above except for hypothesis 4. at q where we use that $h^{-1}(\rho_{n-1}) \bigcap h(\alpha) = \emptyset$ to show that the region determined by $h^{-1}(\rho_{n-1})h^{-1}(r_n)q$ is disjoint from its image. Therefore, there exists a vector field whose origin describes pq while its endpoint describes first an arc inside $h^{-1}(\mathcal{C})$ from $h(p)$ to $h(q)$ and whose index j_1 satisfies $i_1 = j_1 + n, n \geq 0$.

Since $h(\alpha)$ is homotopic rel endpoints in $\mathbb{R}^2 \setminus \gamma \bigcup \alpha$ to an arc inside $h^{-1}(\mathcal{C})$, the index i_2 of h along qp is equal to the index j_2 of a vector field whose origin describes qp while its endpoint describes an arc inside $h^{-1}(\mathcal{C})$.

Since, by Lemma 2.4 $j_1 + j_2 = 1$, we have again a contradiction.

6.3 Third case

$h([\gamma])$ follows $[\gamma]$ and $h([\delta])$ follows $[\delta]$.

This case reduces to the first one by exchanging h and h^{-1}.

6.4 Fourth case

$h([\gamma])$ precedes $[\gamma]$ and $h([\delta])$ follows $[\delta]$.

This case reduces to the second one by exchanging h and h^{-1}.

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