Towards a proof of Pillai and Fermat-Catalan conjectures

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Abstract

(MSC=11D04) We begin with Fermat-Catalan equation $Z^c = X^a + Y^b > 2Y^b$ and see if it has solutions for $q + 1 = 2p$ or $q + 2 = 2p$ or $q = 2p$.

(Keywords : Diophantine equations, Fermat-Catalan equation ; Approach)

An attempt of resolution of Fermat-Catalan and Pillai equations

Let Fermat-Catalan equation

$$Y^p = X^q + Z^a > 2Z^a$$

The truth is Fermat-Catalan consists in three equations : the one higher and $Y^p = Z^a - X^q$ and $Y^p = X^q - Z^a$, we will treat only the first as the others have the same solutions by the same reasoning. Then let $Z^a = 1_a 1_a$. And Pillai one

$$Y^p = X^q + a = X^q + 1_a 1_a$$

Thus

$$Y^p = X^q + 1_a 1_a$$

We suppose firstly that

$$c = \frac{X^p - 1_a}{Y^{\frac{p}{2}}} > 0, \quad c' = \frac{7_a - X^p}{Y^{\frac{p}{2}}} < 0$$

Hence

$$Y^{\frac{p}{2}} = \frac{6_a}{c + c'}$$

$$X^p = cY^{\frac{p}{2}} + 1_a = \frac{6_a c}{c + c'} + 1_a = \frac{7_a c + 1_a c'}{c + c'}$$

And

$$X^q = Y^p - 1_a 1_a = \frac{36_a 1_a - 1_a 1_a (c + c')^2}{(c + c')^2}$$

As $X^p > 7_a$ then $c' < 0$ and $c > c'$ but $Y^{\frac{p}{2}} \geq 3$ then $0 < c + c' \leq 2_a$ Two cases $c^2 > 1$ et $0 < c^2 \leq 1$

$c^2 > 1$ We have

$$X^q - X^{2p} = \frac{36_a 1_a - 1_a 1_a (c+c')^2 - (7_a c + 1_a c')^2}{(c+c')^2}$$

$$< \frac{36_a 1_a - 36_a 1_a c^2}{(c+c')^2} < 0$$

Or

$$q + 1 \leq 2p$$

But

$$c' Y^{\frac{p}{2}} = 7_a - X^p < -Y^{\frac{p}{2}}$$

Then $c' < -1$. But, let u, v verifying

$$\frac{7uc + vc'}{c + c'} = 4$$

Or

$$(7u - 4)c = (4 - v)c' > 0$$

(The reasoning is the same if it is negative). As $c > 0$ and $c' < 0$ it means $7u - 4 > 0$ and $4 - v < 0$. Then

$$(7u - 4)(X^p - 1_a) = (4 - v)(7_a - X^p)$$

Or

$$(7u - v)X^p = 24_a + 7_a(u - v)$$

Hence

$$X^p = \frac{7_a(u - v) + 24_a}{7u - v}$$

But

$$c + c' = \frac{4 - v}{7u - 4}c' + c' = \frac{7u - v}{7u - 4}c'$$

Then

$$Y^{\frac{p}{2}} = \frac{6_a}{c + c'} = \frac{6_a(7u - 4)}{(7u - v)c'}$$

And

$$\frac{X^p}{Y^{\frac{p}{2}}} = \frac{(7_a(u - v) + 24_a)c'}{6_a(7u - 4)} = \frac{(7(u - v) + 24)c'}{6(7u - 4)}$$

But as $X^p > 28_a$ (else X, p can be calculated and Pillai equation has a finite number of solutions !) Thus

$$(9c' + 7c)Y^{\frac{p}{2}} = 9(7_a - X^p) + 7(X^p - 1_a) = 56_a - 2X^p < 0 \Rightarrow 9c' + 7c < 0$$

And

$$(4 - v)(9c' + 7c) > 0$$

Thus

$$9(7u - 4)c = 9(4 - v)c' > 7(v - 4)c$$

It means

$$7(7u-4) > 7v - 14u - 20 > 7(4-v)c' + 2(4-7u)$$

But if $X^p < X^{\frac{q+1}{2}} + 7 < X^{\frac{q+2}{2}}$ then $X^{q+1} \leq X^{2p} < X^{q+2}$ else

$$X^q(X-49) > X^q > 49_a 1_a \Rightarrow (X^p - 7_a)^2 - 49_a 1_a \geq X^{q+1} - 49_a 1_a > 49X^q$$

$$\Rightarrow (X^p - 7_a)^2 \geq 49(X^q + 1_a 1_a) = 49Y^p \Rightarrow X^p - 7_a \geq 7Y^{\frac{p}{2}}$$

And

$$X^p - 7_a > 7Y^{\frac{p}{2}} > Y^{\frac{p}{2}}(\frac{7(v-4) + 2(4-7u)}{7u-4})$$

$$> Y^{\frac{p}{2}}(\frac{7(4-v)c' + 2(4-7u)}{7u-4})$$

Then

$$-7_a + X^p + 2Y^{\frac{p}{2}} > (\frac{7(4-v)c'}{7u-4})Y^{\frac{p}{2}}$$

Or

$$\frac{-7_a + X^p + 2Y^{\frac{p}{2}}}{6c'}$$

$$= (\frac{2}{6c'} - \frac{1}{6})Y^{\frac{p}{2}} < \frac{7(4-v)}{6(7u-4)}Y^{\frac{p}{2}}$$

Or

$$(\frac{2}{6c'} - \frac{1}{6})c' > \frac{7(4-v)c'}{6(7u-4)}$$

And

$$\frac{c'}{6} + \frac{7(4-v)c'}{6(7u-4)}$$

$$= \frac{(7(u-v)+24)c'}{6(7u-4)} = \frac{X^p}{Y^{\frac{p}{2}}} < \frac{2}{6} < 1$$

Thus

$$X^{q+1} \leq X^{2p} < Y^p = X^q + 1_a 1_a < X^{q+1}$$

It is impossible ! Thus $q+1 = 2p$ or $q+2 = 2p$

$0 < c^2 \leq 1$

$$2X^q - X^{2p} = \frac{72_a 1_a - 2_a 1_a(c+c')^2 - (7_a c + 1_a c')^2}{(c+c')^2}$$

$$> 1_a 1_a \frac{64 - 49c^2}{(c+c')^2} > 0$$

Or

$$X^q \geq X^{2p-1}$$

Or

$$q + 1 \geq 2p$$

We have
$$c'Y^{\frac{p}{2}} = 7_a - X^p > -Y^{\frac{p}{2}}$$
Thus $c' > -1$. But, let u, v verifying
$$\frac{7uc + vc'}{c + c'} = 4$$
Or
$$(7u - 4)c = (4 - v)c' > 0$$
As $c > 0$ and $c' < 0$ it means $7u - 4 > 0$ and $4 - v < 0$. Then $7u - 4 > 0$ and $4 - v < 0$ and
$$(7u - 4)(X^p - 1_a) = (4 - v)(7_a - X^p)$$
Or
$$(7u - v)X^p = 24_a + 7_a(u - v)$$
Hence
$$X^p = 1_a\frac{7(u - v) + 24}{7u - v}$$
But
$$c + c' = \frac{4 - v}{7u - 4}c' + c' = \frac{7u - v}{7u - 4}c'$$
Then
$$Y^{\frac{p}{2}} = \frac{6_a}{c + c'} = \frac{6_a(7u - 4)}{(7u - v)c'}$$
And
$$\frac{X^p}{Y^{\frac{p}{2}}} = \frac{(7(u - v) + 24)c'}{6(7u - 4)}$$
But
$$(3c' + 7c)Y^{\frac{p}{2}} = 3(7_a - X^p) + 7(X^p - 1_a) = 4X^p + 14_a > 0 \Rightarrow 3c' + 7c > 0$$
And
$$(4 - v)(3c' + 7c) < 0$$
Or
$$3(7u - 4)c = 3(4 - v)c' < 7(v - 4)c$$
It means
$$7u - 4 < 7v - 14u - 20 < 7(4 - v)c' + 2(4 - 7u)$$
But
$$X^q + 1_a1_a = Y^p \geq X^{2p} + 1_a1_a > (X^p - 7_a)^2$$
Thus
$$X^p - 7_a < Y^{\frac{p}{2}} < Y^{\frac{p}{2}}(\frac{7(v - 4) + 2(4 - 7u)}{7u - 4})$$
$$< Y^{\frac{p}{2}}(\frac{7(4 - v)c' + 2(4 - 7u)}{7u - 4})$$
Then
$$-7_a + X^p + 2Y^{\frac{p}{2}} < (\frac{7(4 - v)c'}{7u - 4})Y^{\frac{p}{2}}$$

Or

$$\frac{-7_a + X^p + 2Y^{\frac{p}{2}}}{6c'} = (\frac{2}{6c'} - \frac{1}{6})Y^{\frac{p}{2}}$$
$$> \frac{7(4-v)}{6(7u-4)}Y^{\frac{p}{2}}$$

Or

$$(\frac{2}{6c'} - \frac{1}{6})c' < \frac{7(4-v)c'}{6(7u-4)}$$

And

$$\frac{c'}{6} + \frac{7(4-v)c'}{6(7u-4)}$$
$$= \frac{(7(u-v)+24)c'}{6(7u-4)} = \frac{X^p}{Y^{\frac{p}{2}}} > \frac{2}{6}$$

Thus

$$3X^{2p+1} > 36X^{2p} > 4Y^p > 3X^q$$

It means that finally $q = 2p$ or $q + 1 = 2p$ or $q + 2 = 2p$. We suppose now that

$$c = \frac{X^p - 1_a}{Y^{\frac{p}{2}}} < 0, \quad c' = \frac{7_a - X^p}{Y^{\frac{p}{2}}} > 0$$

Hence

$$Y^{\frac{p}{2}} = \frac{6_a}{c + c'} > 3$$

Thus

$$0 < c + c' < 2_a$$

And

$$X^p = cY^{\frac{p}{2}} + 1_a = \frac{6_a c}{c + c'} + 1_a = \frac{7_a c + 1_a c'}{c + c'}$$

And

$$X^q = Y^p - 1_a 1_a = \frac{36_a 1_a - 1_a 1_a (c + c')^2}{(c + c')^2}$$

As $X^p < 1_a < 7_a$ then $c < 0$ and $c' > -c > c$ but $Y^{\frac{p}{2}} \geq 3$ then $0 < c + c' \leq 2_a$ Two cases $c'^2 > 1$ et $0 < c'^2 \leq 1$

$c'^2 > 1$ We have

$$c^2 Y^p = (X^p - 1_a)^2 > (X^p - 7_a)^2 = c'^2 Y^p > Y^p = X^q + 1_a 1_a > (X^{\frac{q}{2}} - 1_a)^2$$

Or

$$q + 1 \leq 2p$$

But if $7_a < X^p + Y^{\frac{p}{2}}$ then $7_a < 3X^p < 21_a$ and X, p can be calculated and Pillai equation has a finite number of solutions, else

$$c'Y^{\frac{p}{2}} = 7_a - X^p > Y^{\frac{p}{2}}$$

Then $-c' < -1$. But, let u, v verifying

$$\frac{7uc + vc'}{c + c'} = 4$$

Or

$$(7u - 4)c = (4 - v)c' > 0$$

As $c < 0$ and $c' > 0$ it means $7u - 4 < 0$ and $4 - v > 0$. Then

$$(7u - 4)(X^p - 1_a) = (4 - v)(7_a - X^p)$$

Or

$$(7u - v)X^p = 24_a + 7_a(u - v)$$

Hence

$$X^p = \frac{7_a(u - v) + 24_a}{7u - v}$$

But

$$c + c' = \frac{4 - v}{7u - 4}c' + c' = \frac{7u - v}{7u - 4}c'$$

Then

$$Y^{\frac{p}{2}} = \frac{6_a}{c + c'} = \frac{6_a(7u - 4)}{(7u - v)c'}$$

And

$$\frac{X^p}{Y^{\frac{p}{2}}} = \frac{(7_a(u - v) + 24_a)c'}{6_a(7u - 4)} = \frac{(7(u - v) + 24)c'}{6(7u - 4)}$$

But

$$(9c' + 7c)Y^{\frac{p}{2}} = 9(7_a - X^p) + 7(X^p - 1_a) = 56_a - 2X^p > 0 \Rightarrow 9c' + 7c > 0$$

And

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But if $X^p < X^{\frac{q+1}{2}} + 7 < X^{\frac{q+2}{2}}$ then $X^{q+1} \leq X^{2p} < X^{q+2}$ else

$$X^q(X - 49) > X^q > 49_a 1_a \Rightarrow (X^p - 7_a)^2 - 49_a 1_a \geq X^{q+1} - 49_a 1_a > 49X^q$$

$$\Rightarrow (X^p - 7_a)^2 \geq 49(X^q + 1_a 1_a) = 49Y^p \Rightarrow X^p - 7_a \geq 7Y^{\frac{p}{2}}$$

And

$$X^p - 7_a > 7Y^{\frac{p}{2}} > Y^{\frac{p}{2}}(\frac{7(v - 4) + 2(4 - 7u)}{7u - 4})$$

$$> Y^{\frac{p}{2}}(\frac{7(4 - v)c' + 2(4 - 7u)}{7u - 4})$$

Then

$$-7_a + X^p + 2Y^{\frac{p}{2}} > (\frac{7(4 - v)c'}{7u - 4})Y^{\frac{p}{2}}$$

Or

$$\frac{-7_a + X^p + 2Y^{\frac{p}{2}}}{6c'}$$

$$= (\frac{2}{6c'} - \frac{1}{6})Y^{\frac{p}{2}} < \frac{7(4-v)}{6(7u-4)}Y^{\frac{p}{2}}$$

Or

$$(\frac{2}{6c'} - \frac{1}{6})c' > \frac{7(4-v)c'}{6(7u-4)}$$

And

$$\frac{c'}{6} + \frac{7(4-v)c'}{6(7u-4)}$$

$$= \frac{(7(u-v)+24)c'}{6(7u-4)} = \frac{X^p}{Y^{\frac{p}{2}}} < \frac{2}{6} < 1$$

Thus

$$X^{q+1} \leq X^{2p} < Y^p = X^q + 1_a 1_a < X^{q+1}$$

It is impossible ! Thus $q+1 = 2p$ or $q+2 = 2p$

$0 < c'^2 \leq 1$

$$(X^p - 7_a)^2 < Y^p = X^q + 1_a 1_a$$

Or

$$X^q \geq X^{2p-1}$$

Or

$$q + 1 \geq 2p$$

We have

$$c'Y^{\frac{p}{2}} = 7_a - X^p > -Y^{\frac{p}{2}}$$

Thus $c' > -1$. But, let u, v verifying

$$\frac{7uc + vc'}{c + c'} = 4$$

Or

$$(7u - 4)c = (4 - v)c' > 0$$

As $c < 0$ and $c' > 0$ it means $7u - 4 < 0$ and $4 - v > 0$. And

$$(7u - 4)(X^p - 1_a) = (4 - v)(7_a - X^p)$$

Or

$$(7u - v)X^p = 24_a + 7_a(u - v)$$

Hence

$$X^p = 1_a \frac{7(u-v)+24}{7u-v}$$

But

$$c + c' = \frac{4-v}{7u-4}c' + c' = \frac{7u-v}{7u-4}c'$$

Then

$$Y^{\frac{p}{2}} = \frac{6_a}{c+c'} = \frac{6_a(7u-4)}{(7u-v)c'}$$

And

$$\frac{X^p}{Y^{\frac{p}{2}}} = \frac{(7(u-v)+24)c'}{6(7u-4)}$$

But

$$(3c'+7c)Y^{\frac{p}{2}} = 3(7_a - X^p) + 7(X^p - 1_a) = 4X^p + 14_a > 0 \Rightarrow 3c' + 7c > 0$$

And

$$(4-v)(3c'+7c) > 0$$

Or

$$3(7u-4)c = 3(4-v)c' > 7(v-4)c$$

It means $3(7u-4) < 7(v-4)$ and

$$7u - 4 < 7v - 14u - 20 < 7(4-v)c' + 2(4-7u)$$

But

$$X^q + 1_a 1_a = Y^p \geq X^{2p} + 1_a 1_a > (X^p - 7_a)^2$$

Thus

$$X^p - 7_a < Y^{\frac{p}{2}} < Y^{\frac{p}{2}}(\frac{7(v-4)+2(4-7u)}{7u-4})$$
$$< Y^{\frac{p}{2}}(\frac{7(4-v)c'+2(4-7u)}{7u-4})$$

Then

$$-7_a + X^p + 2Y^{\frac{p}{2}} < (\frac{7(4-v)c'}{7u-4})Y^{\frac{p}{2}}$$

Or

$$\frac{-7_a + X^p + 2Y^{\frac{p}{2}}}{6c'} = (\frac{2}{6c'} - \frac{1}{6})Y^{\frac{p}{2}}$$
$$> \frac{7(4-v)}{6(7u-4)}Y^{\frac{p}{2}}$$

Or

$$(\frac{2}{6c'} - \frac{1}{6})c' < \frac{7(4-v)c'}{6(7u-4)}$$

And

$$\frac{c'}{6} + \frac{7(4-v)c'}{6(7u-4)}$$
$$= \frac{(7(u-v)+24)c'}{6(7u-4)} = \frac{X^p}{Y^{\frac{p}{2}}} > \frac{2}{6}$$

Thus

$$3X^{2p+1} > 36X^{2p} > 4Y^p > 3X^q$$

It means that finally $q = 2p$ or $q+1 = 2p$ or $q+2 = 2p$. We suppose finally

$$c = \frac{X^p - 1_a}{Y^{\frac{p}{2}}} > 0, \quad c' = \frac{7_a - X^p}{Y^{\frac{p}{2}}} > 0$$

Hence

$$Y^{\frac{p}{2}} = \frac{6_a}{c+c'}$$

$$X^p = cY^{\frac{p}{2}} + 1_a = \frac{6_a c}{c+c'} + 1_a = \frac{7_a c + 1_a c'}{c+c'}$$

And

$$X^q = Y^p - 1_a 1_a = \frac{36_a 1_a - 1_a 1_a (c+c')^2}{(c+c')^2}$$

As $1_a < X^p < 7_a$ then X, p can be calculated and Pillai equation has a finite number of solutions !

main result and relation with Fermat and Catalan equations It means that finally $q = 2p$ or $q+1 = 2p$ or $q+2 = 2p$. We see that for Catalan equation (as q and $2p$ do not have the same parity) $q+1 = 2p$ (for this equation, we have $c > 0$ and $c' < 0$) and for Fermat equation there are only two solutions $q+2 = 2p = 2n = n+2 = 4$ or $q+1 = 2p = n+1 = 2n = 2$. We will prove now that there is only one value of p for the solution, it is $p = 2$. If the exponent of X is even, we must do as it follows

$$Y^p = X^{2p-2u} + X + 1_a 1_a - X = (X+1)X(X^{2p-2u-2} - X^{2p-2u-3} + \ldots - X + 1) + 1_a 1_a - X = uX(X+1) + 1_a 1_a - X$$

We have also

$$Y^p = X^{2p-2u} - X + 1_a 1_a + X = (X-1)X(X^{2p-2u-2} + X^{2p-2u-3} + \ldots + X + 1) + 1_a 1_a + X = vX(X-1) + 1_a 1_a + X$$

Else if the exponent of X is odd and it will be the case that we will treat :

$$Y^p = X^{2p-1} + 1 + 1_a 1_a - 1 = (X+1)(X^{2p-2} - X^{2p-3} + \ldots - X + 1) + 1_a 1_a - 1 = u(X+1) + 1_a 1_a - 1$$

We have also

$$Y^p = X^{2p-2u} - 1 + 1_a 1_a + 1 = (X-1)(X^{2p-2} + X^{2p-3} + \ldots + X + 1) + 1_a 1_a + 1 = v(X-1) + 1_a 1_a + 1$$

Hence in all cases

$$2 = u(X+1) - v(X-1) = X + 1 - (X-1)$$

And

$$2Y^p - 2_a 1_a = uX(X+1) + vX(X-1)$$

Thus

$$(u-1)(X+1) = (v-1)(X-1)$$

Thus

$$X = \frac{u+v-2}{v-u}$$

And

$$2Y^p - 2_a 1_a = (u+v)X + u - v = \frac{(u+v)(u+v-2)}{v-u} + u - v$$

$$= \frac{u^2 + v^2 + 2uv - 2u - 2v - u^2 - v^2 + 2uv}{v-u} = \frac{2u(v-1) + 2v(u-1)}{v-u}$$

Hence
$$Y^p = \frac{u(v-1)+v(u-1)+1_a1_a(v-u)}{v-u} = X^{2p-1}+1_a1_a$$
If we suppose p even
$$Y^p - Y = v'Y(Y-1) = X^{2p-1}+1_a1_a - Y$$
$$Y^p + Y = u'Y(Y+1) = X^{2p-1}+1_a1_a + Y$$
$$2Y = u'Y(Y+1) - v'Y(Y-1) = Y(Y+1) - Y(Y-1)$$
And
$$2X^{2p'-1} + 2_a1_a = u'Y(Y+1) + v'Y(Y-1)$$
Or
$$(u'-1)(Y+1) = (v'-1)(Y-1)$$
Thus
$$Y = \frac{u'+v'-2}{v'-u'}$$
And
$$2X^{2p-1} + 2_a1_a = Y((u'+v')Y + (u'-v')) = \frac{u'+v'-2}{v'-u'}(\frac{2u'(v'-1)+2v'(u'-1)}{v'-u'})$$
Or
$$Y^p = X^{2p-1}+1_a1_a = \frac{u'+v'-2}{v'-u'}(\frac{u'(v'-1)+v'(u'-1)}{v'-u'})$$
We have for equations with four unknowns
$$Y^p = (\frac{u'+v'-2}{v'-u'})^p = X^{2p-1}+1_a1_a = (\frac{u+v-2}{v-u})^{2p-1}+1_a1_a$$
$$= \frac{u'+v'-2}{v'-u'}(\frac{u'(v'-1)+v'(u'-1)}{v'-u'}) = \frac{u(v-1)+v(u-1)+1_a1_a(v-u)}{v-u}$$
The solutions of those equations are $u' = v' = 2$. Let us prove it : we have also
$$Y^p = Y^p - Y + Y = Y(Y-1)v' + Y = X^{2p-1}+1_a1_a = u(X+1)+1_a1_a-1 \quad (1)$$
$$Y^p = Y^p - Y + Y = Y(Y-1)v' + Y = X^{2p-1}-1+1_a1_a+1 = v(X-1)+1_a1_a+1 \quad (2)$$
$$Y^p = Y^p + Y - Y = Y(Y+1)u' - Y = X^{2p-1}+1+1_a1_a-1 = u(X+1)+1_a1_a-1 \quad (3)$$
$$Y^p = Y^p + Y - Y = Y(Y+1)u' - Y = X^{2p-1}-1+1_a1_a+1 = v(X-1)+1_a1_a+1 \quad (4)$$
Hence, with
$$X = \frac{u+v-2}{v-u}, \quad Y = \frac{u'+v'-2}{v'-u'}$$
We have
$$(1) \quad v'\frac{u'+v'-2}{v'-u'}\frac{2u'-2}{v'-u'} + \frac{u'+v'-2}{v'-u'} = u\frac{2v-2}{v-u} + 1_a1_a - 1$$
$$(2) \quad v'\frac{u'+v'-2}{v'-u'}\frac{2u'-2}{v'-u'} + \frac{u'+v'-2}{v'-u'} = v\frac{2u-2}{v-u} + 1_a1_a + 1$$

$$(3)\quad u'\frac{u'+v'-2}{v'-u'}\frac{2v'-2}{v'-u'}-\frac{u'+v'-2}{v'-u'}=u\frac{2v-2}{v-u}+1_a1_a-1$$

$$(4)\quad u'\frac{u'+v'-2}{v'-u'}\frac{2v'-2}{v'-u'}-\frac{u'+v'-2}{v'-u'}=v\frac{2u-2}{v-u}+1_a1_a+1$$

Or

$$(1)\quad (v-u)(u'+v'-2)(2v'(u'-1)+v'-u')=(2u(v-1)+(1_a1_a-1)(v-u))(v'-u')^2$$

$$(2)\quad (v-u)(u'+v'-2)(2v'(u'-1)+v'-u')=2v(u-1)(v'-u')^2+(1_a1_a+1)(v-u)(v'-u')^2$$

$$(3)\quad (v-u)(u'+v'-2)(2u'(v'-1)-(v'-u'))=(2u(v-1)+(1_a1_a-1)(v-u))(v'-u')^2$$

$$(4)\quad (v-u)(u'+v'-2)(2u'(v'-1)-(v'-u'))=2v(u-1)(v'-u')^2+(1_a1_a+1)(v-u)(v'-u')^2$$

But

$$(1),(3)\quad (v-u)(u'+v'-2)(2u'v'-u'-v')=(2uv-2u)(v'-u')^2+(1_a1_a-1)(v-u)(v'-u')^2$$

$$(2),(4)\quad (v-u)(u'+v'-2)(2u'v'-u'-v')=(2vu-2v)(v'-u')^2+(1_a1_a+1)(v-u)(v'-u')^2$$

Hence if we suppose $u'-v'\neq 0$

$$(v-u)((u'-1)(Y-1)+(v'-1)(Y-1))(u'(v'-1)(Y-1))+v'(u'-1)(Y-1))$$

$$=(v-u)((u'-1)(Y-1)+(u'-1)(Y+1))(u'(u'-1)(Y+1)+v'(u'-1)(Y-1))$$

$$=2(v-u)(u'-1)^2Y(u'(Y+1)+v'(Y-1))$$

$$=4(v-u)(u'-1)^2Y(X^{2p-1}+1)$$

$$=(2vu-2u)(v'-u')^2(Y-1)(Y-1)+(1_a1_a-1)(v-u)(v'-u')^2(Y-1)(Y+1)$$

$$=(2vu-2u+(1_a1_a-1)(v-u))(v'-1-(u'-1))(Y-1)(v'-1-(u'-1))(Y-1)$$

$$=(2vu-2u+(1_a1_a-1)(v-u))((u'-1)(Y+1)-(u'-1)(Y-1))((u'-1)(Y+1)-(u'-1)(Y-1))$$

$$=(8u(v-1)+4(1_a1_a-1)(v-u))(u'-1)^2$$

$$(u'-1)^2(2u(v-1)+(1_a1_a-1)(v-u)-(v-u)Y(X^{2p-1}+1_a1_a))=0$$

If the expression in the right of the parenthesis is equel to zero

$$(v-1)(2u+1_a1_a-1-Y(X^{2p-1}+1_a1_a))=-(u-1)Y(X^{2p-1}+1_a1_a-1_a1_a+1)$$

$$=\frac{(u-1)(X+1)}{X-1}(2u+1_a1_a-1-Y(X^{2p-1}+1_a1_a))$$

Thus

$$(u-1)((X+1)(2u+1_a1_a-1-Y(X^{2p-1}+1_a1_a))+(X-1)Y(X^{2p-1}+1+1_a1_a-1_a1_a))=0$$

$$=(u-1)(2Y^p+(1_a1_a-1)(X-1)-2Y(X^{2p-1}+1_a1_a)+(1-1_a1_a)(X-1))=0$$

$$=(u-1)Y^p(2-2Y)=0$$

Consequently

$$u-1=0\Rightarrow (u-1)(X+1)=(v-1)(X-1)=0\Rightarrow v=1$$

Impossible (because Y^p is not equal to $X+1$) ! Thus

$$v' = u'$$

$$(1),(3) \quad (v-u)(u'+v'-2) = \frac{2u(v-1)(v'-u')^2}{2v'u'-v'-u'}$$

$$(2),(4) \quad (v-u)(u'+v'-2) = \frac{(2uv+2v-4u)(v'-u')^2}{u'v'-u'-v'} = \frac{(2uv-2u)(v'-u')^2}{u'v'-u'-v'}$$

If $v-u=0$ then

$$(v-u)X = u+v-2 = 2u-2 = 2v-2 = 0$$

And it is impossible, because Y^p is different of $X+1$. Thus $v'-u'=0$, it means

$$(v'-u')Y = u'+v'-2 = 2u'-2 = 2v'-2 = 0$$

Thus

$$v'Y(Y-1)+Y = u(X+1) = Y^2 = Y^p$$

And $p=2$. And if p is odd

$$Y^p - 1 = v'(Y-1)$$

$$Y^p + 1 = u'(Y+1)$$

$$2 = u'(Y+1) - v'(Y-1) = (Y+1)-(Y-1)$$

$$2Y^p = 2X^{2p-1} + 2_a 1_a = (u'+v')Y + u' - v'$$

And

$$(u'-1)(Y+1) = (v'-1)(Y-1)$$

Thus

$$Y = \frac{u'+v'-2}{v'-u'}$$

$$X^{2p-1} + 1_a 1_a = Y^p = \frac{u'(v'-1)+v'(u'-1)}{v'-u'}$$

We have four equations with four unknowns

$$Y^p = (\frac{u'+v'-2}{v'-u'})^p = X^{2p-1} + 1_a 1_a = (\frac{u+v-2}{v-u})^{2p'-1} + 1_a 1_a$$

$$= \frac{u'(v'-1)+v'(u'-1)}{v'-u'} = \frac{u(v-1)+v(u-1)+1_a 1_a(v-u)}{v-u}$$

The solutions are $u'=v'=1$. Let us prove it : we have also

$$Y^p = Y^p - 1 + 1 = (Y-1)v' + 1 = X^{2p-1} + 1_a 1_a = u(X+1) + 1_a 1_a - 1 \quad (1)$$

$$Y^p = Y^p - 1 + 1 = (Y-1)v' + 1 = X^{2p-1} - 1 + 1_a 1_a + 1 = v(X-1) + 1_a 1_a + 1 \quad (2)$$

$$Y^p = Y^p + 1 - 1 = (Y+1)u' - 1 = X^{2p-1} + 1 + 1_a 1_a - 1 = u(X+1) + 1_a 1_a - 1 \quad (3)$$

$$Y^p = Y^p + 1 - 1 = (Y+1)u' - 1 = X^{2p-1} - 1 + 1_a 1_a + 1 + 2 = v(X-1) + 1_a 1_a + 1 \quad (4)$$

Hence, with

$$X = \frac{u+v-2}{v-u}, \quad Y = \frac{u'+v'-2}{v'-u'}$$

We have

$$\begin{aligned}
&(1) \quad v'\frac{2u'-2}{v'-u'}+1 = u\frac{2v-2}{v-u}+1_a1_a-1\\
&(2) \quad v'\frac{2u'-2}{v'-u'}+1 = v\frac{2u-2}{v-u}+1_a1_a+1\\
&(3) \quad u'\frac{2v'-2}{v'-u'}-1 = u\frac{2v-2}{v-u}+1_a1_a-1\\
&(4) \quad u'\frac{2v'-2}{v'-u'}-1 = v\frac{2u-2}{v-u}+1_a1_a+1
\end{aligned}$$

Or

$$(1) \quad (v-u)(2v'(u'-1)+v'-u') = 2u(v-1)(v'-u')+(1_a1_a-1)(v-u)(v'-u')$$

$$(2) \quad (v-u)(2v'(u'-1)+v'-u') = 2v(u-1)(v'-u')+(1_a1_a+1)(v-u)(v'-u')$$

$$(3) \quad (v-u)(2u'(v'-2)-(v'-u')) = 2u(v-1)(v'-u')+(1_a1_a-1)(v-u)(v'-u')$$

$$(4) \quad (v-u)(2u'(v'-1)-(v'-u')) = 2v(u-1)(v'-u')+(1_a1_a+1)(v-u)(v'-u')$$

But

$$(1),(3) \quad (v-u)(2u'v'-u'-v') = 2u(v-1)(v'-u')+(1_a1_a-1)(v-u)(v'-u')$$

$$(2),(4) \quad (v-u)(2u'v'-u'-v') = (2vu-2v)(v'-u')+(1_a1_a+1)(v-u)(v'-u')$$

Hence if we suppose $u'-v' \neq 0$

$$\begin{aligned}
&(v-u)(u'(v'-1)(Y-1)+v'(u'-1)(Y-1))\\
&= (v-u)(u'(u'-1)(Y+1)+v'(u'-1)(Y-1))\\
&= (v-u)(u'-1)(2X^{2p-1}+2_a1_a-2)\\
&= (2vu-2u+(1_a1_a-1))(v'-u')(Y-1) = (2u(v-1)+(1_a1_a-1)(v-u))(v'-1-(u'-1))(Y-1)\\
&= (2u(v-1)+(1_a1_a-1)(v-u))((u'-1)(Y+1)-(u'-1)(Y-1))\\
&= (4u(v-1)+2(1_a1_a-1)(v-u))(u'-1)
\end{aligned}$$

Thus

$$(u'-1)(2u(v-1)+(1_a1_a-1)(v-u)-(v-u)(X^{2p-1}+1_a1_a)) = 0$$

If

$$\begin{aligned}
(v-1)(X^{2p'-1}+1-2u) &= (u-1)(X^{2p'-1}+1)\\
&= \frac{(u-1)(X+1)}{X-1}(X^{2p'-1}+1-2u)
\end{aligned}$$

And

$$\begin{aligned}
&(u-1)((X+1)(X^{2p'-1}+1-2u-(X-1)(X^{2p'-1}+1)) = 0\\
&= (u-1)(2X^{2p'-1}+3-2u) = 0
\end{aligned}$$

Consequently

$$u - 1 = 0 \Rightarrow (u-1)(X+1) = (v-1)(X-1) = 0 \Rightarrow v = 1$$

Impossible (because Y^p is not equal to $X+1$) ! Thus

$$v' = u'$$

And

$$u' + v' = 2 = 2u' = 2v'$$

Thus

$$v'(Y-1) + 1 = u(X+1) = Y = Y^p$$

It is impossible ! Thus $p = 2$. We deduce that Fermat-Catalan equation implies six equations-solutions which are

$$(I) \quad Y^2 = X^3 + Z^a$$

$$(II) \quad Y^2 = X^4 + Z^a$$

$$(III) \quad Y^2 + X^3 = Z^a$$

$$(IV) \quad Y^2 + X^4 = Z^a$$

$$(V) \quad Y^2 + Z^a = X^3$$

$$(VI) \quad Y^2 + Z^a = X^4$$

Known solutions We have

$$1 + 2^3 = 3^2$$

$$7^2 + 3^5 = 3^4$$

$$13^2 + 7^3 = 2^9$$

$$71^2 = 17^3 + 2^7$$

$$3^5 + 11^4 = 122^2$$

$$1549304^2 + 33^8 = 15613^3$$

$$2213459^2 + 1413^3 = 65^7$$

$$15312283^2 = 113^7 - 9262^3$$

$$21062938^2 = 76271^3 + 17^7$$

$$30042907^2 = 96222^3 + 43^8$$

Conclusion

Fermat-Catalan equation $Z^c = X^a + Y^b$ does not have solutions for both $a > 2$, $b > 2$ and $c > 2$. In fact, at least one of the exponents must be equal to 2. We have shown a way to discuss the solutions.