THE EXTENDED PERMUTOHEDRON ON A TRANSITIVE BINARY RELATION

LUIGI SANTOCANALE AND FRIEDRICH WEHRUNG

ABSTRACT. For a given transitive binary relation $\boldsymbol{e}$ on a set E, the transitive closures of open (i.e., co-transitive in $\boldsymbol{e}$) sets, called the *regular closed* subsets, form an ortholattice $\mathrm{Reg}(\boldsymbol{e})$, the *extended permutohedron on* $\boldsymbol{e}$. This construction, which contains the poset $\mathrm{Clop}(\boldsymbol{e})$ of all clopen sets, is a common generalization of known notions such as the generalized permutohedron on a partially ordered set on the one hand, and the bipartition lattice on a set on the other hand. We obtain a precise description of the completely join-irreducible (resp., meet-irreducible) elements of $\mathrm{Reg}(\boldsymbol{e})$ and the arrow relations between them. In particular, we prove that

— $\mathrm{Reg}(\boldsymbol{e})$ is the Dedekind-MacNeille completion of the poset $\mathrm{Clop}(\boldsymbol{e})$;
— Every open subset of $\boldsymbol{e}$ is a set-theoretical union of completely join-irreducible clopen subsets of $\boldsymbol{e}$;
— $\mathrm{Clop}(\boldsymbol{e})$ is a lattice iff every regular closed subset of $\boldsymbol{e}$ is clopen, iff $\boldsymbol{e}$ contains no "square" configuration, iff $\mathrm{Reg}(\boldsymbol{e}) = \mathrm{Clop}(\boldsymbol{e})$;
— If $\boldsymbol{e}$ is finite, then $\mathrm{Reg}(\boldsymbol{e})$ is pseudocomplemented iff it is semidistributive, iff it is a bounded homomorphic image of a free lattice, iff $\boldsymbol{e}$ is a disjoint sum of antisymmetric transitive relations and two-element full relations.

We illustrate our results by proving that, for $n \geq 3$, the congruence lattice of the lattice $\mathrm{Bip}(n)$ of all bipartitions of an n-element set is obtained by adding a new top element to a Boolean lattice with $n \cdot 2^{n-1}$ atoms. We also determine the factors of the minimal subdirect decomposition of $\mathrm{Bip}(n)$, and we prove that if $n \geq 3$, then none of them embeds into $\mathrm{Bip}(n)$ as a sublattice.

1. INTRODUCTION

The lattice $\mathsf{P}(n)$ of all permutations of an n-element chain, also known as the permutohedron, even if widely known and studied in combinatorics, is a relatively young object of study from a pure lattice theoretical perspective. Its elements, the permutations of n elements, are endowed with the weak Bruhat order; this order turns out to be a lattice.

There are many possible generalization of this order, arising from the theory of Coxeter groups (Björner [4]), from graph and order theory (Pouzet *et al.* [26], Hetyei and Krattenthaler [20]), from language theory (Flath [12], Bennett and Birkhoff [2]).

In the present paper, we shall focus on one of the most noteworthy features—at least from the lattice-theoretical viewpoint—of one of the equivalent constructions of the permutohedron, namely that it can be realized as the lattice of all *clopen*

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(i.e., both closed and open) subsets of a certain *strict ordering* relation (viewed as a set of ordered pairs), endowed with the operation of transitive closure.

It turns out that most of the theory can be done for the transitive closure operator on the pairs of a given *transitive* binary relation $\boldsymbol{e}$. While, unlike the situation for ordinary permutohedra, the poset $\mathrm{Clop}(\boldsymbol{e})$ of all clopen subsets of $\boldsymbol{e}$ may not be a lattice, it is contained in the larger lattice $\mathrm{Reg}(\boldsymbol{e})$ of all so-called *regular closed* subsets of $\boldsymbol{e}$, which we shall call the *extended permutohedron on* $\boldsymbol{e}$ (cf. Section 3). As $\mathrm{Reg}(\boldsymbol{e})$ is endowed with a natural orthocomplementation $\boldsymbol{x} \mapsto \boldsymbol{x}^{\perp}$ (cf. Definition 3.2), it becomes, in fact, an *ortholattice*. The natural question, whether $\mathrm{Clop}(\boldsymbol{e})$ is a lattice, finds a natural answer in Theorem 4.3, where we prove that this is equivalent to the preordering associated with $\boldsymbol{e}$ be *square-free*, thus extending (with completely different proofs) known results for both the case of strict orderings (Pouzet *et al.* [26]) and the case of full relations (Hetyei and Krattenthaler [20]).

However, while most earlier references deal with clopen subsets, our present paper focuses on the extended permutohedron $\mathrm{Reg}(\boldsymbol{e})$. One of our most noteworthy results is the characterization, obtained in Theorem 7.8, of all finite transitive relations $\boldsymbol{e}$ such that $\mathrm{Reg}(\boldsymbol{e})$ is *semidistributive*. It turns out that this condition is equivalent to $\mathrm{Reg}(\boldsymbol{e})$ being *pseudocomplemented*, also to $\mathrm{Reg}(\boldsymbol{e})$ being a *bounded homomorphic image of a free lattice*, and can also be expressed in terms of forbidden sub-configurations of $\boldsymbol{e}$. This result is achieved *via* a precise description, obtained in Section 5, of all completely join-irreducible elements of $\mathrm{Reg}(\boldsymbol{e})$. This is a key technical point of the present paper. This description is further extended to a description of the join-dependency relation (cf. Section 6), thus essentially completing the list of tools for proving one direction of Theorem 7.8. The other directions are achieved *via ad hoc* constructions, such as the one of Proposition 3.5.

Another noteworthy consequence of our description of completely join-irreducible elements of the lattice $\mathrm{Reg}(\boldsymbol{e})$ is the *spatiality* of that lattice: every element is a join of completely join-irreducible elements... and even more can be said (cf. Lemma 5.5 and Theorem 5.8). As a consequence, $\mathrm{Reg}(\boldsymbol{e})$ *is the Dedekind-MacNeille completion of* $\mathrm{Clop}(\boldsymbol{e})$ (cf. Corollary 5.6).

We proved in our earlier paper Santocanale and Wehrung [31] that the factors of the minimal subdirect decomposition of the permutohedron $\mathsf{P}(n)$ are exactly Reading's Cambrian lattices of type A, denoted in [31] by $\mathsf{A}_U(n)$. As a further application of our methods, we determine here the minimal subdirect decomposition of the lattice $\mathrm{Bip}(n)$ of all bipartitions (i.e., those transitive binary relations with transitive complement) of an n-element set, thus solving the "equation"

$$\frac{\text{Tamari lattice}}{\text{permutohedron}} = \frac{x}{\text{bipartition lattice}}$$

and in fact, more generally,

$$\frac{\text{Cambrian lattice of type A}}{\text{permutohedron}} = \frac{x}{\text{bipartition lattice}}\,. \tag{1.1}$$

The fractional "equation" (1.1) is a very informal notation suggesting that x stands to bipartition lattices the same way Cambrian lattices of type A stand to permutohedra. The lattices x solving the "equation" (1.1), denoted here in the form $\mathsf{S}(n,k)$ (cf. Remark 9.8), offer features quite different from those of the Cambrian lattices; in particular, their cardinality does not depend on n alone (cf. Section 11) and they are never sublattices of the corresponding bipartition lattice $\mathrm{Bip}(n)$ for

$n \geq 3$ (cf. Section 10). In fact, if $n \geq 3$, then every nonconstant lattice endomorphism of $\operatorname{Bip}(n)$ is an automorphism, and the automorphism group of $\operatorname{Bip}(n)$ is isomorphic to $\mathfrak{S}_n \times \mathfrak{S}_2$ (where $\mathfrak{S}_n$ denotes the symmetric group on n elements), see Theorem 10.1.

We also use our tools to determine the congruence lattice of every finite bipartition lattice (cf. Corollary 8.8), which, for a base set with at least three elements, turns out to be Boolean with a top element added.

2. Basic concepts and notation

We refer the reader to Grätzer [17] for basic facts, notation, and terminology about lattice theory.

We shall denote by 0 (resp., 1) the least (resp., largest) element of a partially ordered set (from now on *poset*) $(P, \leq)$, if they exist. A *lower cover* of an element $p \in P$ is an element $x \in P$ such that $x < p$ and there is no y such that $x < y < p$. If p has a unique lower cover, then we shall denote this element by p_*. *Upper covers*, and the notation p^*, are defined dually.

A nonzero element p in a lattice L is *join-irreducible* if $p = x \vee y$ implies that $p \in \{x, y\}$, for all $x, y \in L$. We say that p is *completely join-irreducible* if it has a unique lower cover p_*, and every $x < p$ satisfies $x \leq p_*$. *Completely meet-irreducible* elements are defined dually. We denote by $\operatorname{Ji} L$ (resp., $\operatorname{Mi} L$) the set of all join-irreducible (resp., meet-irreducible) elements of L.

Every completely join-irreducible element is join-irreducible, and in a finite lattice, the two concepts are equivalent. A lattice L is *spatial* if every element of L is a (possibly infinite) join of completely join-irreducible elements. Equivalently, for all $a, b \in L$, $a \nleq b$ implies that there exists a completely join-irreducible element p of L such that $p \leq a$ and $p \nleq b$.

For a completely join-irreducible element p and a completely meet-irreducible element u of L, let $p \nearrow u$ hold if $p \leq u^*$ and $p \nleq u$. Symmetrically, let $u \searrow p$ hold if $p_* \leq u$ and $p \nleq u$. The *join-dependency relation* D is defined on completely join-irreducible elements by

$$p \mathbin{D} q \underset{\text{def.}}{\iff} \big(p \neq q \text{ and } (\exists x)(p \leq q \vee x \text{ and } p \nleq q_* \vee x) \big) .$$

It is well-known (cf. Freese, Ježek, and Nation [16, Lemma 11.10]) that the join-dependency relation D on a finite lattice L can be conveniently expressed in terms of the arrow relations $\nearrow$ and $\searrow$ between $\operatorname{Ji} L$ and $\operatorname{Mi} L$.

Lemma 2.1. *Let p, q be distinct join-irreducible elements in a finite lattice L. Then $p \mathbin{D_L} q$ iff there exists $u \in \operatorname{Mi} L$ such that $p \nearrow u \searrow q$.*

We shall denote by D^n (resp., D^*) the nth relational power (resp., the reflexive and transitive closure) of the relation D, so, for example, $p \mathbin{D^2} q$ iff there exists $r \in \operatorname{Ji} L$ such that $p \mathbin{D} r$ and $r \mathbin{D} q$.

It is well-known that the congruence lattice $\operatorname{Con} L$ of a finite lattice L can be conveniently described *via* the relation D on L, as follows (cf. Freese, Ježek, and Nation [16, § II.3]). Denote by $\operatorname{con}(p)$ the least congruence of L containing (p_*, p) as an element, for each $p \in \operatorname{Ji} L$. Then $\operatorname{con}(p) \subseteq \operatorname{con}(q)$ iff $p \mathbin{D^*} q$, for all $p, q \in \operatorname{Ji} L$. Furthermore, $\operatorname{Con} L$ is a distributive lattice and its join-irreducible elements are exactly the $\operatorname{con}(p)$, for $p \in \operatorname{Ji} L$. A subset $S \subseteq \operatorname{Ji} L$ is a *D-upper subset* if $p \in S$ and

$p \mathbin{D} q$ implies that $q \in S$, for all $p, q \in \operatorname{Ji} L$. Set $S \downarrow x = \{s \in S \mid s \leq x\}$, for each $x \in L$.

Lemma 2.2. *The binary relation $\theta_S = \{(x, y) \in L \times L \mid S \downarrow x = S \downarrow y\}$ is a congruence of L, for every finite lattice L and every D-upper subset S of $\operatorname{Ji} L$, and the assignment $x \mapsto x/\theta_S$ defines an isomorphism from the $(\vee, 0)$-subsemilattice $S^\vee$ of L generated by S onto the quotient lattice L/θ_S. Furthermore, the assignment $S \mapsto \theta_S$ defines a dual isomorphism from the lattice of all D-upper subsets of $\operatorname{Ji} L$ onto $\operatorname{Con} L$. The inverse of that isomorphism is given by*

$$\theta \mapsto \{p \in \operatorname{Ji} L \mid (p, p_*) \notin \theta\}\,, \quad \textit{for each } \theta \in \operatorname{Con} L\,.$$

For each $p \in \operatorname{Ji} L$, denote by $\Psi(p)$ the largest congruence θ of L such that $p \not\equiv p_*$ $(\operatorname{mod} \theta)$. Then $\Psi(p) = \theta_{S_p}$ where we set $S_p = \{q \in \operatorname{Ji} L \mid p \mathbin{D^*} q\}$. Equivalently, $\Psi(p)$ is generated by all pairs (q, q_*) such that $p \mathbin{D^*} q$ does not hold. Say that $p \in \operatorname{Ji} L$ is *D^*-minimal* if $p \mathbin{D^*} q$ implies $q \mathbin{D^*} p$, for each $q \in \operatorname{Ji} L$. The set $\Delta(L)$ of all D^*-minimal join-irreducible elements of L defines, *via* Ψ, a subdirect product decomposition of L,

$$L \hookrightarrow \prod_{p \in \Delta(L)} \bigl(L/\Psi(p)\bigr)\,, \quad x \mapsto \bigl(x/\Psi(p) \mid p \in \Delta(L)\bigr)\,, \tag{2.1}$$

that we shall call the *minimal subdirect product decomposition of L*.

A lattice L is *join-semidistributive* if $x \vee z = y \vee z$ implies that $x \vee z = (x \wedge y) \vee z$, for all $x, y, z \in L$. *Meet-semidistributivity* is defined dually. A lattice is *semidistributive* if it is both join- and meet-semidistributive.

A lattice L is a *bounded homomorphic image of a free lattice* if there are a free lattice F and a surjective lattice homomorphism $f \colon F \twoheadrightarrow L$ such that $f^{-1}\{x\}$ has both a least and a largest element, for each $x \in L$. These lattices, introduced by McKenzie [24], form a quite important class within the theory of lattice varieties, and are often called "bounded lattices" (not to be confused with lattices with both a least and a largest element). A finite lattice is bounded iff the join-dependency relations on L and on its dual lattice are both cycle-free (cf. Freese, Ježek, and Nation [16, Corollary 2.39]). Every bounded lattice is semidistributive (cf. Freese, Ježek, and Nation [16, Theorem 2.20]), but the converse fails, even for finite lattices (cf. Freese, Ježek, and Nation [16, Figure 5.5]).

An *orthocomplementation* on a poset P with least and largest element is a map $x \mapsto x^\perp$ of P to itself such that

(O1) $x \leq y$ implies that $y^\perp \leq x^\perp$,
(O2) $x^{\perp\perp} = x$,
(O3) $x \wedge x^\perp = 0$ (in view of (O1) and (O2), this is equivalent to $x \vee x^\perp = 1$),

for all $x, y \in P$. Elements $x, y \in P$ are *orthogonal* if $x \leq y^\perp$, equivalently $y \leq x^\perp$.

An *orthocomplemented poset* is a poset with an orthocomplementation. Although an orthocomplemented poset P always has a least element 0, a largest element 1, and the elements $x \wedge x^\perp$ and $x \vee x^\perp$ exist for all $x \in P$ (with respective values 0 and 1), the poset P may not be a lattice.

Of course, any orthocomplementation of P is a dual automorphism of $(P, \leq)$. In particular, if P is a lattice, then *de Morgan's rules*

$$(x \vee y)^\perp = x^\perp \wedge y^\perp\,, \quad (x \wedge y)^\perp = x^\perp \vee y^\perp$$

hold for all $x, y \in P$. An *ortholattice* is a lattice endowed with an orthocomplementation.

A lattice L with a least element 0 is *pseudocomplemented* if $\{y \in L \mid x \wedge y = 0\}$ has a greatest element, for each $x \in P$.

We shall denote by $\operatorname{Pow} X$ the powerset of a set X, and we shall set $\operatorname{Pow}^* X = (\operatorname{Pow} X) \setminus \{\varnothing, X\}$. We shall also set $[n] = \{1, 2, \dots, n\}$, for every positive integer n.

3. Regular closed subsets of a transitive relation

Throughout the paper, by "relation" (on a possibly infinite set) we shall always mean a binary relation. For a relation $\boldsymbol{e}$ on a set E, we will often write

$$\begin{aligned} x \lhd_{\boldsymbol{e}} y &\underset{\text{def.}}{\iff} (x, y) \in \boldsymbol{e}\,, \\ x \unlhd_{\boldsymbol{e}} y &\underset{\text{def.}}{\iff} (\text{either}^{1}\ x \lhd_{\boldsymbol{e}} y \text{ or } x = y)\,, \\ x \equiv_{\boldsymbol{e}} y &\underset{\text{def.}}{\iff} (x \unlhd_{\boldsymbol{e}} y \text{ and } y \unlhd_{\boldsymbol{e}} x)\,, \end{aligned}$$

for all $x, y \in E$. We say that $\boldsymbol{e}$ is a *strict ordering* if it is irreflexive (i.e., $(x, x) \notin \boldsymbol{e}$ for every x) and transitive. In the general case, we set

$$\begin{aligned} [a, b]_{\boldsymbol{e}} &= \{x \mid a \unlhd_{\boldsymbol{e}} x \text{ and } x \unlhd_{\boldsymbol{e}} b\}\,, \\ [a, b[_{\boldsymbol{e}} &= \{x \mid a \unlhd_{\boldsymbol{e}} x \text{ and } x \lhd_{\boldsymbol{e}} b\}\,, \\ ]a, b]_{\boldsymbol{e}} &= \{x \mid a \lhd_{\boldsymbol{e}} x \text{ and } x \unlhd_{\boldsymbol{e}} b\}\,, \\ [a]_{\boldsymbol{e}} &= [a, a]_{\boldsymbol{e}}\ , \end{aligned}$$

for all $a, b \in E$. As $a \lhd_{\boldsymbol{e}} a$ may occur, a may belong to $]a, b]_{\boldsymbol{e}}$.

Denote by $\operatorname{cl}(\boldsymbol{a})$ the transitive closure of any relation $\boldsymbol{a}$. We say that $\boldsymbol{a}$ is *closed* if it is transitive. We say that $\boldsymbol{a}$ is *bipartite* if there are no x, y, z such that $(x, y) \in \boldsymbol{a}$ and $(y, z) \in \boldsymbol{a}$. It is trivial that every bipartite relation is closed.

Let $\boldsymbol{e}$ be a transitive relation on a set E. A subset $\boldsymbol{a} \subseteq \boldsymbol{e}$ is *open* (*relatively to* $\boldsymbol{e}$) if $\boldsymbol{e} \setminus \boldsymbol{a}$ is closed; equivalently,

$$\bigl(x \lhd_{\boldsymbol{e}} y \lhd_{\boldsymbol{e}} z \text{ and } (x, z) \in \boldsymbol{a}\bigr) \Rightarrow \bigl(\text{either } (x, y) \in \boldsymbol{a} \text{ or } (y, z) \in \boldsymbol{a}\bigr)\,,\ \text{for all } x, y, z \in E\,.$$

The largest open subset of $\boldsymbol{a} \subseteq \boldsymbol{e}$, called the *interior* of $\boldsymbol{a}$ and denoted by $\operatorname{int}(\boldsymbol{a})$, is exactly the set of all pairs $(x, y) \in \boldsymbol{e}$ such that for every subdivision $x = z_0 \lhd_{\boldsymbol{e}} z_1 \lhd_{\boldsymbol{e}} \cdots \lhd_{\boldsymbol{e}} z_n = y$, with $n > 0$, there exists $i < n$ such that $(z_i, z_{i+1}) \in \boldsymbol{a}$.

A subset $\boldsymbol{a} \subseteq \boldsymbol{e}$ is *clopen* if $\boldsymbol{a} = \operatorname{cl}(\boldsymbol{a}) = \operatorname{int}(\boldsymbol{a})$. We denote by $\operatorname{Clop}(\boldsymbol{e})$ the poset of all clopen subsets of $\boldsymbol{e}$. A subset $\boldsymbol{a} \subseteq \boldsymbol{e}$ is *regular closed* (resp., *regular open*) if $\boldsymbol{a} = \operatorname{cl}\operatorname{int}(\boldsymbol{a})$ (resp., $\boldsymbol{a} = \operatorname{int}\operatorname{cl}(\boldsymbol{a})$). We denote by $\operatorname{Reg}(\boldsymbol{e})$ (resp., $\operatorname{Reg}_{\mathrm{op}}(\boldsymbol{e})$) the poset of all regular closed (resp., regular open) subsets under set inclusion.

As a set $\boldsymbol{x}$ is open iff its complement $\boldsymbol{x}^{\mathsf{c}} = \boldsymbol{e} \setminus \boldsymbol{x}$ is closed (by definition), similarly a set $\boldsymbol{x}$ is closed (regular closed, regular open, clopen, respectively) iff $\boldsymbol{x}^{\mathsf{c}}$ is open (regular open, regular closed, clopen, respectively).

The terminology "open", "closed", "clopen", widely spread among lattice theorists, originates from topology, and the corresponding concepts bear some similarity with the topological ones. For example, any union of open sets is open, and any intersection of closed sets is closed. Nevertheless, an important difference between the present context and the topological one is that the union of finitely many

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[1] Here and elsewhere in the paper, "either A or B" will stand for inclusive disjunction of A and B, see for example Barwise and Etchenmendy [1, page 75].

closed sets may not be closed. This means that the closure operator cl (see, for example, Birkhoff [3, § V.1]) is not necessarily topological, that is, one may have $\mathrm{cl}(\boldsymbol{x} \cup \boldsymbol{y}) \neq \mathrm{cl}(\boldsymbol{x}) \cup \mathrm{cl}(\boldsymbol{y})$ with $\boldsymbol{x}, \boldsymbol{y} \subseteq \boldsymbol{e}$.

The following lemma gathers a few basic facts about the concepts defined above.

Lemma 3.1. *The following statements hold, for any transitive binary relation* $\boldsymbol{e}$ (*on a possibly infinite set*)*:*

(i) *The operators* $\mathrm{cl} \circ \mathrm{int}$ *and* $\mathrm{int} \circ \mathrm{cl}$ *are both idempotent.*
(ii) *A subset* $\boldsymbol{x}$ *of* $\boldsymbol{e}$ *is regular closed iff* $\boldsymbol{x} = \mathrm{cl}(\boldsymbol{u})$ *for some open set* $\boldsymbol{u}$*.*
(iii) *The poset* $\mathrm{Reg}(\boldsymbol{e})$ *is a complete lattice, with meet and join given by*

$$\bigvee_{i \in I} \boldsymbol{a}_i = \mathrm{cl}\Big(\bigcup_{i \in I} \boldsymbol{a}_i\Big)\,,$$
$$\bigwedge_{i \in I} \boldsymbol{a}_i = \mathrm{cl}\,\mathrm{int}\Big(\bigcap_{i \in I} \boldsymbol{a}_i\Big)\,,$$

for any family $(\boldsymbol{a}_i \mid i \in I)$ *of regular closed sets.*

Proof. (i). Observe that $\mathrm{int} \leq \mathrm{id} \leq \mathrm{cl}$, that is, $\mathrm{int}(\boldsymbol{x}) \subseteq \boldsymbol{x} \subseteq \mathrm{cl}(\boldsymbol{x})$ for every $\boldsymbol{x} \subseteq \boldsymbol{e}$. Since the operators cl and int are both order-preserving and idempotent, we get

$$(\mathrm{cl} \circ \mathrm{int})^2 = \mathrm{cl} \circ \mathrm{int} \circ \mathrm{cl} \circ \mathrm{int} \leq \mathrm{cl} \circ \mathrm{id} \circ \mathrm{cl} \circ \mathrm{int} = \mathrm{cl} \circ \mathrm{int}\,,$$
$$(\mathrm{cl} \circ \mathrm{int})^2 = \mathrm{cl} \circ \mathrm{int} \circ \mathrm{cl} \circ \mathrm{int} \geq \mathrm{cl} \circ \mathrm{int} \circ \mathrm{id} \circ \mathrm{int} = \mathrm{cl} \circ \mathrm{int}\,,$$

hence $(\mathrm{cl} \circ \mathrm{int})^2 = \mathrm{cl} \circ \mathrm{int}$. Likewise, $(\mathrm{int} \circ \mathrm{cl})^2 = \mathrm{int} \circ \mathrm{cl}$.

(ii). If $\boldsymbol{x} = \mathrm{cl}(\boldsymbol{u})$ with $\boldsymbol{u}$ open, then, by (i), $\boldsymbol{x} = (\mathrm{cl} \circ \mathrm{int})(\boldsymbol{u}) = (\mathrm{cl} \circ \mathrm{int})^2(\boldsymbol{u}) = (\mathrm{cl} \circ \mathrm{int})(\boldsymbol{x})$ is regular closed. Conversely, if $\boldsymbol{x}$ is regular closed, then $\boldsymbol{x} = \mathrm{cl}(\boldsymbol{u})$ for the open set $\boldsymbol{u} = \mathrm{int}(\boldsymbol{x})$.

(iii). The open set $\boldsymbol{u} = \bigcup_{i \in I} \mathrm{int}(\boldsymbol{a}_i)$ is contained in $\boldsymbol{a} = \bigcup_{i \in I} \boldsymbol{a}_i$. Furthermore, $\boldsymbol{a}_i = \mathrm{cl}(\mathrm{int}(\boldsymbol{a}_i)) \subseteq \mathrm{cl}(\boldsymbol{u})$ for each i, thus $\boldsymbol{a} \subseteq \mathrm{cl}(\boldsymbol{u})$, whence $\mathrm{cl}(\boldsymbol{a}) = \mathrm{cl}(\boldsymbol{u})$ is regular closed. It follows easily that $\mathrm{cl}(\boldsymbol{a})$ is the least upper bound of $\{\boldsymbol{a}_i \mid i \in I\}$ in $\mathrm{Reg}(\boldsymbol{e})$.

Setting $\boldsymbol{b} = \bigcap_{i \in I} \boldsymbol{a}_i$, it follows from (i) that $\mathrm{cl}\,\mathrm{int}(\boldsymbol{b})$ is regular closed. It follows easily that this set is the greatest lower bound of $\{\boldsymbol{a}_i \mid i \in I\}$ in $\mathrm{Reg}(\boldsymbol{e})$. □

The complement of a regular closed set may not be closed. Nevertheless, we shall now see that there is an obvious "complementation-like" map from the regular closed sets to the regular closed sets.

Definition 3.2. We define the *orthogonal* of $\boldsymbol{x}$ as $\boldsymbol{x}^{\perp} = \mathrm{cl}(\boldsymbol{e} \setminus \boldsymbol{x})$, for any $\boldsymbol{x} \subseteq \boldsymbol{e}$.

Lemma 3.3.

(i) $\boldsymbol{x}^{\perp}$ *is regular closed, for any closed* $\boldsymbol{x} \subseteq \boldsymbol{e}$*.*
(ii) *The assignment* ${}^{\perp} \colon \boldsymbol{x} \mapsto \boldsymbol{x}^{\perp}$ *defines an orthocomplementation of* $\mathrm{Reg}(\boldsymbol{e})$*.*

Proof. (i) follows immediately from Lemma 3.1(ii).

(ii) follows from the equations $\mathrm{int}(\boldsymbol{x}) = \boldsymbol{e} \setminus \mathrm{cl}(\boldsymbol{e} \setminus \boldsymbol{x}) = \boldsymbol{e} \setminus \boldsymbol{x}^{\perp}$, together with the idempotence of the operator $\mathrm{cl} \circ \mathrm{int}$ (cf. Lemma 3.1(i)). □

Corollary 3.4. *The lattices* $\mathrm{Reg}(\boldsymbol{e})$ *and* $\mathrm{Reg}_{\mathrm{op}}(\boldsymbol{e})$ *are pairwise isomorphic, and also self-dual, for any transitive relation* $\boldsymbol{e}$*. The isomorphisms are given by* $\mathrm{Reg}(\boldsymbol{e}) \to \mathrm{Reg}_{\mathrm{op}}(\boldsymbol{e})$*,* $\boldsymbol{x} \mapsto \mathrm{int}\,\mathrm{cl}(\boldsymbol{x})$ *and* $\mathrm{Reg}_{\mathrm{op}}(\boldsymbol{e}) \to \mathrm{Reg}(\boldsymbol{e})$*,* $\boldsymbol{x} \mapsto \mathrm{cl}\,\mathrm{int}(\boldsymbol{x})$*.*

We shall call $\mathrm{Clop}(\boldsymbol{e})$ the *permutohedron on* $\boldsymbol{e}$ and $\mathrm{Reg}(\boldsymbol{e})$ the *extended permutohedron on* $\boldsymbol{e}$. For example, if $\boldsymbol{e}$ is the strict ordering associated to a poset $(E, \leq)$, then $\mathrm{Clop}(\boldsymbol{e})$ is the poset denoted by $\mathbf{N}(E)$ in Pouzet *et al.* [26]. On the other hand, if $\boldsymbol{e} = [n] \times [n]$ for a positive integer n, then $\operatorname{Clop}(\boldsymbol{e})$ is the poset of all *bipartitions* of $[n]$ introduced in Foata and Zeilberger [13] and Han [19], see also Hetyei and Krattenthaler [20] where this poset is denoted by $\operatorname{Bip}(n)$.

While the lattice $\operatorname{Reg}(\boldsymbol{e})$ is always orthocomplemented (cf. Lemma 3.3), the following result shows that $\operatorname{Reg}(\boldsymbol{e})$ is not always pseudocomplemented.

Proposition 3.5. *Let $\boldsymbol{e}$ be a transitive relation on a (possibly infinite) set E with pairwise distinct elements $a_0, a_1, b \in E$ such that $a_0 \equiv_{\boldsymbol{e}} a_1$ and either $b \lhd_{\boldsymbol{e}} a_0$ or $a_0 \lhd_{\boldsymbol{e}} b$. Then there are clopen subsets $\boldsymbol{a}_0$, $\boldsymbol{a}_1$, $\boldsymbol{c}$ of $\boldsymbol{e}$ such that $\boldsymbol{a}_0 \wedge \boldsymbol{c} = \boldsymbol{a}_1 \wedge \boldsymbol{c} = \varnothing$ while $\varnothing \neq \boldsymbol{c} \subseteq \boldsymbol{a}_0 \vee \boldsymbol{a}_1$. In particular, the lattice $\operatorname{Reg}(\boldsymbol{e})$ is neither meet-semidistributive, nor pseudocomplemented.*

Proof. We show the proof in the case where $a_0 \lhd_{\boldsymbol{e}} b$. By applying the result to $\boldsymbol{e}^{\mathrm{op}} = \{(x, y) \mid (y, x) \in \boldsymbol{e}\}$, the result for the case $b \lhd_{\boldsymbol{e}} a_0$ will follow. We set $I = [a_0, b]_{\boldsymbol{e}} = [a_1, b]_{\boldsymbol{e}}$ and

$$\begin{aligned} \boldsymbol{a}_i &= \{a_i\} \times (I \setminus \{a_i\}) && (\text{for each } i \in \{0, 1\}), \\ \boldsymbol{c} &= \{a_0, a_1\} \times (I \setminus \{a_0, a_1\}). \end{aligned}$$

It is straightforward to verify that $\boldsymbol{a}_0$, $\boldsymbol{a}_1$, $\boldsymbol{c}$ are all clopen subsets of $\boldsymbol{e}$. Furthermore, $(a_0, b) \in \boldsymbol{c}$ thus $\boldsymbol{c} \neq \varnothing$, and $\boldsymbol{c} \subseteq \boldsymbol{a}_0 \cup \boldsymbol{a}_1 \subseteq \boldsymbol{a}_0 \vee \boldsymbol{a}_1$.

Now let (a_0, x) be an element of $\boldsymbol{a}_0 \cap \boldsymbol{c} = \{a_0\} \times (I \setminus \{a_0, a_1\})$. Observing that $a_0 \lhd_{\boldsymbol{e}} a_1 \lhd_{\boldsymbol{e}} x$ while $(a_0, a_1) \notin \boldsymbol{a}_0 \cap \boldsymbol{c}$ and $(a_1, x) \notin \boldsymbol{a}_0 \cap \boldsymbol{c}$, we obtain that $(a_0, x) \notin \operatorname{int}(\boldsymbol{a}_0 \cap \boldsymbol{c})$; whence $\boldsymbol{a}_0 \wedge \boldsymbol{c} = \varnothing$. Likewise, $\boldsymbol{a}_1 \wedge \boldsymbol{c} = \varnothing$. □

4. Lattices of clopen subsets of square-free transitive relations

Definition 4.1. A transitive relation $\boldsymbol{e}$ is *square-free* if for all $(a, b) \in \boldsymbol{e}$, any two elements of $[a, b]_{\boldsymbol{e}}$ are comparable with respect to $\trianglelefteq_{\boldsymbol{e}}$. That is,

$$(\forall a, b, x, y)\Big(\big(a \trianglelefteq_{\boldsymbol{e}} x \text{ and } a \trianglelefteq_{\boldsymbol{e}} y \text{ and } x \trianglelefteq_{\boldsymbol{e}} b \text{ and } y \trianglelefteq_{\boldsymbol{e}} b\big) \Longrightarrow (\text{either } x \trianglelefteq_{\boldsymbol{e}} y \text{ or } y \trianglelefteq_{\boldsymbol{e}} x)\Big).$$

For the particular case of the natural strict ordering $1 < 2 < \cdots < n$, the following result originates in Guilbaud and Rosenstiehl [18, § VI.A]. The case of the full relation $[n] \times [n]$ is covered by the proof of Hetyei and Krattenthaler [20, Proposition 4.2].

Lemma 4.2. *Let $\boldsymbol{e}$ be a square-free transitive relation. Then the set $\operatorname{int}(\boldsymbol{a})$ is closed, for each closed $\boldsymbol{a} \subseteq \boldsymbol{e}$. Dually, the set $\operatorname{cl}(\boldsymbol{a})$ is open, for each open $\boldsymbol{a} \subseteq \boldsymbol{e}$.*

Proof. It suffices to prove the first statement. Let $x \lhd_{\boldsymbol{e}} y \lhd_{\boldsymbol{e}} z$ with $(x, y) \in \operatorname{int}(\boldsymbol{a})$ and $(y, z) \in \operatorname{int}(\boldsymbol{a})$, we must prove that $(x, z) \in \operatorname{int}(\boldsymbol{a})$. Consider a subdivision $x = s_0 \lhd_{\boldsymbol{e}} s_1 \lhd_{\boldsymbol{e}} \cdots \lhd_{\boldsymbol{e}} s_n = z$ and suppose that

$$(s_i, s_{i+1}) \notin \boldsymbol{a} \text{ for each } i < n \tag{4.1}$$

(we say that the subdivision *fails witnessing* $(x, z) \in \operatorname{int}(\boldsymbol{a})$). Denote by l the largest integer such that $l < n$ and $s_l \trianglelefteq_{\boldsymbol{e}} y$. If $s_l = y$, then the subdivision $x = s_0 \lhd_{\boldsymbol{e}} s_1 \lhd_{\boldsymbol{e}} \cdots \lhd_{\boldsymbol{e}} s_l = y$ fails witnessing $(x, y) \in \operatorname{int}(\boldsymbol{a})$, a contradiction; so $s_l \neq y$ and $s_l \lhd_{\boldsymbol{e}} y$. From $x = s_0 \lhd_{\boldsymbol{e}} s_1 \lhd_{\boldsymbol{e}} \cdots \lhd_{\boldsymbol{e}} s_l \lhd_{\boldsymbol{e}} y$, $(x, y) \in \operatorname{int}(\boldsymbol{a})$, and (4.1) it follows that

$$(s_l, y) \in \boldsymbol{a}. \tag{4.2}$$

Since $\boldsymbol{e}$ is square-free, either $s_{l+1} \trianglelefteq_{\boldsymbol{e}} y$ or $y \vartriangleleft_{\boldsymbol{e}} s_{l+1}$. In the first case, it follows from the definition of l that $l = n-1$, thus, using (4.2) together with $(y,z) \in \boldsymbol{a}$, we get $s_{n-1} = s_l \vartriangleleft_{\boldsymbol{a}} y \vartriangleleft_{\boldsymbol{a}} z = s_n$, whence $(s_{n-1}, s_n) \in \boldsymbol{a}$, which contradicts (4.1).

Hence $y \vartriangleleft_{\boldsymbol{e}} s_{l+1}$. From $y \vartriangleleft_{\boldsymbol{e}} s_{l+1} \vartriangleleft_{\boldsymbol{e}} s_{l+2} \vartriangleleft_{\boldsymbol{e}} \cdots \vartriangleleft_{\boldsymbol{e}} s_n = z$, $(y,z) \in \operatorname{int}(\boldsymbol{a})$, and (4.1) it follows that $(y, s_{l+1}) \in \boldsymbol{a}$, thus, by (4.2), $(s_l, s_{l+1}) \in \boldsymbol{a}$, in contradiction with (4.1). □

In the particular case of *strict orderings*, most of the following result is contained (with a completely different argument) in Pouzet *et al.* [26, Lemma 12]. No finiteness assumption on $\boldsymbol{e}$ is needed.

Theorem 4.3. *The following are equivalent, for any transitive relation* $\boldsymbol{e}$*:*

(i) $\boldsymbol{e}$ *is square-free;*
(ii) $\operatorname{Clop}(\boldsymbol{e}) = \operatorname{Reg}(\boldsymbol{e})$*;*
(iii) $\operatorname{Clop}(\boldsymbol{e})$ *is a lattice;*
(iv) $\operatorname{Clop}(\boldsymbol{e})$ *has the* interpolation property*, that is, for all* $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{y}_0, \boldsymbol{y}_1 \in \operatorname{Clop}(\boldsymbol{e})$ *such that* $\boldsymbol{x}_i \subseteq \boldsymbol{y}_j$ *for all* $i, j < 2$*, there exists* $\boldsymbol{z} \in \operatorname{Clop}(\boldsymbol{e})$ *such that* $\boldsymbol{x}_i \subseteq \boldsymbol{z}$ *and* $\boldsymbol{z} \subseteq \boldsymbol{y}_i$ *for all* $i < 2$*.*

Proof. (i)⇒(ii) follows immediately from Lemma 4.2.

(ii)⇒(iii) and (iii)⇒(iv) are both trivial.

(iv)⇒(i). We prove that if $\boldsymbol{e}$ is not square-free, then $\operatorname{Clop}(\boldsymbol{e})$ does not satisfy the interpolation property. By assumption, there are $(a,b) \in \boldsymbol{e}$ and $u, v \in [a,b]_{\boldsymbol{e}}$ such that $u \ntrianglelefteq_{\boldsymbol{e}} v$ and $v \ntrianglelefteq_{\boldsymbol{e}} u$. It is easy to verify that the subsets

$$\boldsymbol{x}_0 = \{a\} \times {]a,u]}_{\boldsymbol{e}}\,, \qquad \boldsymbol{x}_1 = [u,b[_{\boldsymbol{e}} \times \{b\}\,,$$
$$\boldsymbol{y}_0 = (\{a\} \times {]a,b]}_{\boldsymbol{e}}) \cup \boldsymbol{x}_1\,, \qquad \boldsymbol{y}_1 = ([a,b[_{\boldsymbol{e}} \times \{b\}) \cup \boldsymbol{x}_0$$

are all clopen, and that $\boldsymbol{x}_i \subseteq \boldsymbol{y}_j$ for all $i, j < 2$. Suppose that there exists $\boldsymbol{z} \in \operatorname{Clop}(\boldsymbol{e})$ such that $\boldsymbol{x}_i \subseteq \boldsymbol{z} \subseteq \boldsymbol{y}_i$ for each $i < 2$. From $(a,u) \in \boldsymbol{x}_0 \subseteq \boldsymbol{z}$ and $(u,b) \in \boldsymbol{x}_1 \subseteq \boldsymbol{z}$ and the transitivity of $\boldsymbol{z}$ it follows that $(a,b) \in \boldsymbol{z}$, thus, as $a \vartriangleleft_{\boldsymbol{e}} v \vartriangleleft_{\boldsymbol{e}} b$ and $\boldsymbol{z}$ is open, either $(a,v) \in \boldsymbol{z}$ or $(v,b) \in \boldsymbol{z}$. In the first case, $(a,v) \in \boldsymbol{y}_1$, thus $v \trianglelefteq_{\boldsymbol{e}} u$, a contradiction. In the second case, $(v,b) \in \boldsymbol{y}_0$, thus $u \trianglelefteq_{\boldsymbol{e}} v$, a contradiction. □

By applying Theorem 4.3 to the full relation $[n] \times [n]$, which is trivially square-free (cf. Definition 4.1; here, $x \trianglelefteq_{\boldsymbol{e}} y$ always holds), we obtain the following result, first proved in Hetyei and Krattenthaler [20, Theorem 4.1].

Corollary 4.4 (Hetyei and Krattenthaler)**.** *The poset* $\operatorname{Bip}(n)$ *of all bipartitions of* $[n]$ *is a lattice, for every positive integer* n*.*

Example 4.5. We set $\boldsymbol{\delta}_E = \{(x,y) \in E \times E \mid x < y\}$, for any poset E, and we set $\mathsf{P}(E) = \operatorname{Clop}(\boldsymbol{\delta}_E)$ and $\mathsf{R}(E) = \operatorname{Reg}(\boldsymbol{\delta}_E)$. By Theorem 4.3 (see also Pouzet *et al.* [26, Lemma 12]), $\mathsf{P}(E)$ is a lattice iff E contains no copy of the four-element Boolean lattice $\mathsf{B}_2 = \{0, a, b, 1\}$ (represented on the left hand side diagram of Figure 4.1)—that is, by using the above terminology, $\boldsymbol{\delta}_E$ is *square-free*.

The lattice $\mathsf{R}(\mathsf{B}_2)$ has 20 elements, while its subset $\mathsf{P}(\mathsf{B}_2)$ has 18 elements. The lattice $\mathsf{R}(\mathsf{B}_2)$ is represented on the right hand side of Figure 4.1. Its join-irreducible elements, all clopen (see a general explanation in Theorem 5.8), are

$$\boldsymbol{a}_0 = \{(0,a)\}\,, \quad \boldsymbol{a}_1 = \{(a,1)\}\,, \quad \boldsymbol{b}_0 = \{(0,b)\}\,, \quad \boldsymbol{b}_1 = \{(b,1)\}\,,$$
$$\boldsymbol{c}_{00} = \{(0,a),(0,b),(0,1)\}\,, \quad \boldsymbol{c}_{01} = \{(0,a),(b,1),(0,1)\}\,,$$
$$\boldsymbol{c}_{10} = \{(a,1),(0,b),(0,1)\}\,, \quad \boldsymbol{c}_{11} = \{(a,1),(b,1),(0,1)\}\,.$$

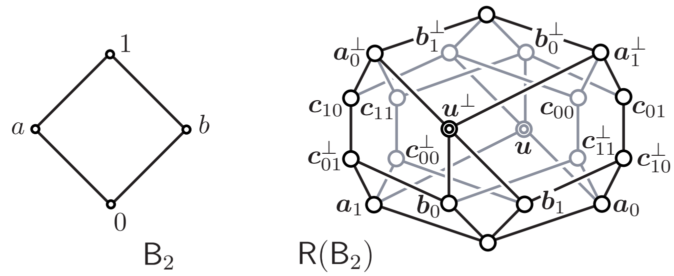

FIGURE 4.1. The lattice $\mathsf{R}(\mathsf{B}_2)$

The two elements of $\mathsf{R}(\mathsf{B}_2) \setminus \mathsf{P}(\mathsf{B}_2)$ are $\boldsymbol{u} = \{(0,a),(a,1),(0,1)\}$ together with its orthogonal, $\boldsymbol{u}^\perp = \{(0,b),(b,1),(0,1)\}$. Those elements are marked by doubled circles on the right hand side diagram of Figure 4.1.

Example 4.6. It follows from Proposition 3.5 that the lattice Bip(3) of all bipartitions of [3] is not pseudocomplemented. We can say more: Bip(3) contains a copy of the five element lattice M_3 of length two, namely $\{\varnothing, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{e}\}$, where

$$\begin{aligned}
\boldsymbol{a} &= \{(1,2),(3,1),(3,2)\}\,,\\
\boldsymbol{b} &= \{(2,1),(2,3),(3,1)\}\,,\\
\boldsymbol{c} &= \{(1,2),(1,3),(2,3)\}\,,\\
\boldsymbol{e} &= [3] \times [3]\,.
\end{aligned}$$

It is also observed in Hetyei and Krattenthaler [20, Example 7.7] that Bip(3) contains a copy of the five element nonmodular lattice N_5; hence it is not modular.

5. COMPLETELY JOIN-IRREDUCIBLE CLOPEN SETS

Throughout this section we shall fix a transitive relation $\boldsymbol{e}$ on a (possibly infinite) set E.

Definition 5.1. We denote by $\mathcal{F}(\boldsymbol{e})$ the set of all triples (a,b,U), where $(a,b) \in \boldsymbol{e}$, $U \subseteq [a,b]_{\boldsymbol{e}}$, and $a \neq b$ implies that $a \notin U$ and $b \in U$. We set $U^{\mathsf{c}} = [a,b]_{\boldsymbol{e}} \setminus U$, and

$$\langle a,b;U\rangle = \begin{cases} \{(x,y) \mid a \trianglelefteq_{\boldsymbol{e}} x \vartriangleleft_{\boldsymbol{e}} y \trianglelefteq_{\boldsymbol{e}} b\,,\ x \notin U\,,\ \text{and } y \in U\}\,, & \text{if } a \neq b\,,\\ \boldsymbol{e} \cap \big((\{a\} \cup U^{\mathsf{c}}) \times (\{a\} \cup U)\big)\,, & \text{if } a = b\,, \end{cases}$$

for each $(a,b,U) \in \mathcal{F}(\boldsymbol{e})$. Equivalently, $\langle a,b;U\rangle = \boldsymbol{e} \cap \big((\{a\} \cup U^{\mathsf{c}}) \times (\{b\} \cup U)\big)$. Observe that $\langle a,b;U\rangle$ is always a subset of $\boldsymbol{e}$.

Observe that $\langle a,b;U\rangle$ is bipartite iff $a \neq b$. If $a = b$, we shall say that $\langle a,b;U\rangle$ is a *clepsydra*.[2]

The proof of the following lemma is a straightforward exercise.

Lemma 5.2. *Let $(a,b,U),(c,d,V) \in \mathcal{F}(\boldsymbol{e})$. Then $\langle a,b;U\rangle = \langle c,d;V\rangle$ iff one of the following statements occurs:*

---

[2]Although the word "clepsydra" has Greek origins and denotes a water clock, we borrow the meaning from the Italian word "clessidra", standing for "hourglass", the latter describing the pattern of the associated transitive relation: the elements of U^{c} below; a in the middle; the elements of U above.

(i) $a \neq b$, $c \neq d$, $a \equiv_{\boldsymbol{e}} c$, $b \equiv_{\boldsymbol{e}} d$, *and* $U = V$;
(ii) $a = b = c = d$ *and* $U \setminus \{a\} = V \setminus \{a\}$.

Lemma 5.3. *The set* $\boldsymbol{p} = \langle a, b; U\rangle$ *is clopen and* $(a, b) \in \boldsymbol{p}$, *for each* $(a, b, U) \in \mathcal{F}(\boldsymbol{e})$. *Furthermore, the set* $\boldsymbol{p}_*$ *defined by*

$$\boldsymbol{p}_* = \begin{cases} \boldsymbol{p} \setminus ([a]_{\boldsymbol{e}} \times [b]_{\boldsymbol{e}})\,, & \text{if } a \neq b\,,\\ \boldsymbol{p} \setminus \{(a, a)\}\,, & \text{if } a = b \end{cases} \tag{5.1}$$

is clopen, and every open proper subset of $\boldsymbol{p}$ *is contained in* $\boldsymbol{p}_*$.

Note. The notation $\boldsymbol{p}_*$ will be validated shortly, in Corollary 5.4, by proving that $\boldsymbol{p}_*$ is, indeed, the unique lower cover of $\boldsymbol{p}$ in the lattice $\mathrm{Reg}(\boldsymbol{e})$.

Proof. In both cases (i.e., either $a \neq b$ or $a = b$) it is trivial that $(a, b) \in \boldsymbol{p}$.

Consider first the case where $a \neq b$. In that case, $\boldsymbol{p}$ is bipartite, thus closed. Let $x \vartriangleleft_{\boldsymbol{e}} y \vartriangleleft_{\boldsymbol{e}} z$ with $(x, z) \in \boldsymbol{p}$. If $y \in U$, then $(x, y) \in \boldsymbol{p}$, and if $y \notin U$, then $(y, z) \in \boldsymbol{p}$. Hence $\boldsymbol{p}$ is clopen.

As $\boldsymbol{p}_* \subseteq \boldsymbol{p}$ and $\boldsymbol{p}$ is bipartite, $\boldsymbol{p}_*$ is bipartite as well, thus $\boldsymbol{p}_*$ is closed. Let $x \vartriangleleft_{\boldsymbol{e}} y \vartriangleleft_{\boldsymbol{e}} z$ with $(x, z) \in \boldsymbol{p}_*$, and suppose by way of contradiction that $(x, y) \notin \boldsymbol{p}_*$ and $(y, z) \notin \boldsymbol{p}_*$. As $\boldsymbol{p}$ is open, either $(x, y) \in \boldsymbol{p}$ or $(y, z) \in \boldsymbol{p}$, hence either (x, y) or (y, z) belongs to $\boldsymbol{p} \cap ([a]_{\boldsymbol{e}} \times [b]_{\boldsymbol{e}})$. In the first case, $x \equiv_{\boldsymbol{e}} a$ and $x \notin U$. Furthermore, $b \equiv_{\boldsymbol{e}} y \vartriangleleft_{\boldsymbol{e}} z$, but $z \trianglelefteq_{\boldsymbol{e}} b$ (because $(x, z) \in \boldsymbol{p}_* \subseteq \boldsymbol{p}$), so $z \equiv_{\boldsymbol{e}} b$, and so we get $(x, z) \in \boldsymbol{p} \cap ([a]_{\boldsymbol{e}} \times [b]_{\boldsymbol{e}}) = \boldsymbol{p} \setminus \boldsymbol{p}_*$, a contradiction. The second case is dealt with similarly. Therefore, $\boldsymbol{p}_*$ is open.

Let $\boldsymbol{u} \subseteq \boldsymbol{p}$ be open and suppose that $\boldsymbol{u}$ is not contained in $\boldsymbol{p}_*$. This means that there exists $(a', b') \in \boldsymbol{u}$ such that $a \equiv_{\boldsymbol{e}} a'$ and $b \equiv_{\boldsymbol{e}} b'$. We must prove that $\boldsymbol{p} \subseteq \boldsymbol{u}$. Let $(x, y) \in \boldsymbol{p}$; in particular, $x \notin U$ and $y \in U$. From $x \notin U$ it follows that $(a', x) \notin \boldsymbol{p}$, thus $(a', x) \notin \boldsymbol{u}$; whence, from $a' \trianglelefteq_{\boldsymbol{e}} x \vartriangleleft_{\boldsymbol{e}} b'$, $(a', b') \in \boldsymbol{u}$, and the openness of $\boldsymbol{u}$, we get $(x, b') \in \boldsymbol{u}$. Now $y \in U$, thus $(y, b') \notin \boldsymbol{p}$, and thus $(y, b') \notin \boldsymbol{u}$, hence, as $x \vartriangleleft_{\boldsymbol{e}} y \trianglelefteq_{\boldsymbol{e}} b'$, $(x, b') \in \boldsymbol{u}$, and $\boldsymbol{u}$ is open, we get $(x, y) \in \boldsymbol{u}$, as required.

From now on suppose that $a = b$. It is trivial that $\boldsymbol{p}$ is closed (although it is no longer bipartite). The proof that $\boldsymbol{p}$ is open is similar to the one for the case where $a \neq b$.

Let $x \vartriangleleft_{\boldsymbol{e}} y \vartriangleleft_{\boldsymbol{e}} z$ with $(x, y) \in \boldsymbol{p}_*$ and $(y, z) \in \boldsymbol{p}_*$. From $\boldsymbol{p}_* \subseteq \boldsymbol{p}$ it follows that $y \in \{a\} \cup U^{\mathsf{c}}$ and $y \in \{a\} \cup U$, thus $y = a$, and thus (as $(x, a) = (x, y) \in \boldsymbol{p}_*$) $x \neq a$, and so $(x, z) \neq (a, a)$. This proves that $\boldsymbol{p}_*$ is closed.

Let $x \vartriangleleft_{\boldsymbol{e}} y \vartriangleleft_{\boldsymbol{e}} z$ with $(x, z) \in \boldsymbol{p}_*$, and suppose by way of contradiction that $(x, y) \notin \boldsymbol{p}_*$ and $(y, z) \notin \boldsymbol{p}_*$. As $\boldsymbol{p}$ is open, either $(x, y) \in \boldsymbol{p}$ or $(y, z) \in \boldsymbol{p}$, thus either $(x, y) = (a, a)$ or $(y, z) = (a, a)$. In the first case $(y, z) = (x, z) \in \boldsymbol{p}_*$, and in the second case $(x, y) = (x, z) \in \boldsymbol{p}_*$, a contradiction in both cases. This proves that $\boldsymbol{p}_*$ is open.

Finally let $\boldsymbol{u} \subseteq \boldsymbol{p}$ be open not contained in $\boldsymbol{p}_*$, so $(a, a) \in \boldsymbol{u}$. Let $(x, y) \in \boldsymbol{p}$, we must prove that $(x, y) \in \boldsymbol{u}$. If $(x, y) = (a, a)$ this is trivial. Suppose that $x = a$ and $y \in U \setminus \{a\}$. Then $a \vartriangleleft_{\boldsymbol{e}} y \vartriangleleft_{\boldsymbol{e}} a$, but $(y, a) \notin \boldsymbol{u}$ (because $(y, a) \notin \boldsymbol{p}$), $(a, a) \in \boldsymbol{u}$, and $\boldsymbol{u}$ is open, thus $(x, y) = (a, y) \in \boldsymbol{u}$, as desired. This completes the case $x = a$. The case where $y = a$ and $x \in U^{\mathsf{c}} \setminus \{a\}$ is dealt with similarly. Now suppose that $x \in U^{\mathsf{c}} \setminus \{a\}$ and $y \in U \setminus \{a\}$. As above, we prove that $(y, a) \notin \boldsymbol{u}$ and thus $(a, y) \in \boldsymbol{u}$. Now $(a, x) \notin \boldsymbol{u}$ (because $(a, x) \notin \boldsymbol{p}$), thus, as $a \vartriangleleft_{\boldsymbol{e}} x \vartriangleleft_{\boldsymbol{e}} y$, $(a, y) \in \boldsymbol{u}$, and $\boldsymbol{u}$ is open, we get $(x, y) \in \boldsymbol{u}$, as desired. □

Corollary 5.4. *Let $(a, b, U) \in \mathcal{F}(\boldsymbol{e})$. The clopen set $\boldsymbol{p} = \langle a, b; U\rangle$ is completely join-irreducible in the lattice* $\operatorname{Reg}(\boldsymbol{e})$*, and the element $\boldsymbol{p}_*$ constructed in the statement of Lemma 5.3 is the lower cover of $\boldsymbol{p}$ in that lattice.*

Proof. Let $\boldsymbol{a} \subsetneqq \boldsymbol{p}$ be a regular closed set. As $\operatorname{int}(\boldsymbol{a})$ is open and properly contained in $\boldsymbol{p}$, it follows from Lemma 5.3 that $\operatorname{int}(\boldsymbol{a}) \subseteq \boldsymbol{p}_*$. Hence, since $\boldsymbol{a}$ is regular closed and $\boldsymbol{p}_*$ is clopen, we get $\boldsymbol{a} = \operatorname{cl}\operatorname{int}(\boldsymbol{a}) \subseteq \operatorname{cl}(\boldsymbol{p}_*) = \boldsymbol{p}_*$. □

Lemma 5.5. *Every open subset $\boldsymbol{u}$ of $\boldsymbol{e}$ is the set-theoretical union of all its subsets of the form $\langle a, b; U\rangle$, where $(a, b, U) \in \mathcal{F}(\boldsymbol{e})$. In particular, every open subset of $\boldsymbol{e}$ is a union of clopen sets.*

Proof. Let $(a, b) \in \boldsymbol{u}$, we must find U such that $(a, b, U) \in \mathcal{F}(\boldsymbol{e})$ and $\langle a, b; U\rangle \subseteq \boldsymbol{u}$.

Suppose first that $a \neq b$ and set

$$U = \{x \in [a, b]_{\boldsymbol{e}} \mid x \neq a \text{ and } (a, x) \in \boldsymbol{u}\}\,.$$

It is trivial that $a \notin U$ and $b \in U$. Let $(x, y) \in \langle a, b; U\rangle$ (so $x \notin U$ and $y \in U$), we must prove that $(x, y) \in \boldsymbol{u}$. If $x = a$ then this is obvious (because $y \in U$). Now suppose that $x \neq a$. In that case, from $x \notin U$ it follows that $(a, x) \notin \boldsymbol{u}$. As $(a, y) \in \boldsymbol{u}$ (because $y \in U$), $a \lhd_{\boldsymbol{e}} x \lhd_{\boldsymbol{e}} y$, and $\boldsymbol{u}$ is open, we get $(x, y) \in \boldsymbol{u}$, as desired.

From now on suppose that $a = b$. We set

$$U = \{x \in [a]_{\boldsymbol{e}} \mid (a, x) \in \boldsymbol{u}\}\,.$$

Observe that $a \in U$, so $\{a\} \cup U = U$. Let $(x, y) \in \langle a, a; U\rangle$, we must prove that $(x, y) \in \boldsymbol{u}$. If $x = a$, then, as $y \in U$, we get $(x, y) = (a, y) \in \boldsymbol{u}$. Hence we may suppose from now on that $x \neq a$; it follows that $x \in U^{\mathsf{c}}$ (as $(x, y) \in \langle a, a; U\rangle$) and therefore $(a, x) \notin \boldsymbol{u}$. As $y \in U$, we get $(a, y) \in \boldsymbol{u}$. As $a \lhd_{\boldsymbol{e}} x \lhd_{\boldsymbol{e}} y$ and $\boldsymbol{u}$ is open, it follows again that $(x, y) \in \boldsymbol{u}$, as desired. □

Corollary 5.6. *The lattice* $\operatorname{Reg}(\boldsymbol{e})$ *is, up to isomorphism, the Dedekind-MacNeille completion of the poset* $\operatorname{Clop}(\boldsymbol{e})$*. In particular, every completely join-irreducible element of* $\operatorname{Reg}(\boldsymbol{e})$ *is clopen.*

Proof. Let $\boldsymbol{a}$ be regular closed; in particular, $\boldsymbol{a} = \operatorname{cl}(\boldsymbol{b})$, with $\boldsymbol{b}$ open. Write $\boldsymbol{b}$ as a union of clopen sets, $\boldsymbol{b} = \bigcup_{i \in I} \boldsymbol{c}_i$. Applying the closure operator cl to both sides of the equation, we obtain the equality $\boldsymbol{a} = \bigvee_{i \in I} \boldsymbol{c}_i$ in $\operatorname{Reg}(\boldsymbol{e})$.

Thus every element of $\operatorname{Reg}(\boldsymbol{e})$ is a join of elements from $\operatorname{Clop}(\boldsymbol{e})$; by duality, every element of $\operatorname{Reg}(\boldsymbol{e})$ is a meet of elements from $\operatorname{Clop}(\boldsymbol{e})$. It immediately follows, see Davey and Priestley [10, Theorem 7.41], that $\operatorname{Reg}(\boldsymbol{e})$ is the Dedekind-MacNeille completion of the poset $\operatorname{Clop}(\boldsymbol{e})$.

Suppose next that $\boldsymbol{a}$ is a completely join-irreducible element of $\operatorname{Reg}(\boldsymbol{e})$; since $\operatorname{Reg}(\boldsymbol{e})$ is join-generated by $\operatorname{Clop}(\boldsymbol{e})$, we can write $\boldsymbol{a}$ as join of clopen sets, $\boldsymbol{a} = \bigvee_{i \in I} \boldsymbol{c}_i$. As $\boldsymbol{a}$ is completely join-irreducible, it follows that $\boldsymbol{a} = \boldsymbol{c}_i$ for some $i \in I$, thus $\boldsymbol{a}$ is clopen. □

Lemma 5.5 makes it possible to extend Pouzet *et al.* [26, Lemma 11], from permutohedra on posets, to lattices of regular closed subsets of transitive relations. This result also refines the equivalence (ii)⇔(iii) of Theorem 4.3.

Corollary 5.7. *The following statements hold, for any (possibly infinite) family $(\boldsymbol{a}_i \mid i \in I)$ of clopen subsets of $\boldsymbol{e}$:*

(i) *The set* $\{\boldsymbol{a}_i \mid i \in I\}$ *has a meet in* $\operatorname{Clop}(\boldsymbol{e})$ *iff* $\operatorname{int}\bigl(\bigcap_{i\in I}\boldsymbol{a}_i\bigr)$ *is clopen, and then the two sets are equal.*
(ii) *The set* $\{\boldsymbol{a}_i \mid i \in I\}$ *has a join in* $\operatorname{Clop}(\boldsymbol{e})$ *iff* $\operatorname{cl}\bigl(\bigcup_{i\in I}\boldsymbol{a}_i\bigr)$ *is clopen, and then the two sets are equal.*

Proof. A simple application of the involution $\boldsymbol{x} \mapsto \boldsymbol{e} \setminus \boldsymbol{x}$ reduces (ii) to (i). On the way to proving (i), we define the open set $\boldsymbol{u} = \operatorname{int}\bigl(\bigcap_{i\in I}\boldsymbol{a}_i\bigr)$.

It is trivial that if $\boldsymbol{u}$ is clopen, then it is the meet of $\{\boldsymbol{a}_i \mid i \in I\}$ in $\operatorname{Clop}(\boldsymbol{e})$. Conversely, suppose that $\{\boldsymbol{a}_i \mid i \in I\}$ has a meet $\boldsymbol{a}$ in $\operatorname{Clop}(\boldsymbol{e})$. It is obvious that $\boldsymbol{a} \subseteq \boldsymbol{u}$. Let $(x, y) \in \boldsymbol{u}$. By Lemma 5.5, there exists $\boldsymbol{b} \subseteq \boldsymbol{u}$ clopen such that $(x, y) \in \boldsymbol{b}$. It follows from the definition of $\boldsymbol{a}$ that $\boldsymbol{b} \subseteq \boldsymbol{a}$, thus $(x, y) \in \boldsymbol{a}$. Therefore, $\boldsymbol{u} = \boldsymbol{a}$ is clopen. □

Notice that Corollary 5.7 can also be derived from Corollary 5.6: if the inclusion of $\operatorname{Clop}(\boldsymbol{e})$ into $\operatorname{Reg}(\boldsymbol{e})$ is, up to isomorphism, the Dedekind-MacNeille completion of $\operatorname{Clop}(\boldsymbol{e})$, then this inclusion preserves existing joins and meets. Thus, for example, suppose that $\boldsymbol{w} = \bigwedge_{i\in I}^{\operatorname{Clop}(\boldsymbol{e})}\boldsymbol{a}_i$ exists; then $\boldsymbol{w} = \bigwedge_{i\in I}^{\operatorname{Reg}(\boldsymbol{e})}\boldsymbol{a}_i$, and we get

$$\boldsymbol{w} = \bigwedge_{i\in I}^{\operatorname{Reg}(\boldsymbol{e})}\boldsymbol{a}_i = \operatorname{cl}\operatorname{int}\Bigl(\bigcap\nolimits_{i\in I}\boldsymbol{a}_i\Bigr) \supseteq \operatorname{int}\Bigl(\bigcap\nolimits_{i\in I}\boldsymbol{a}_i\Bigr)\,.$$

As $\boldsymbol{w} \subseteq \operatorname{int}\bigl(\bigcap_{i\in I}\boldsymbol{a}_i\bigr)$ follows from the openness of $\boldsymbol{w}$ together with $\boldsymbol{w} \subseteq \boldsymbol{a}_i$ for each $i \in I$, we get $\boldsymbol{w} = \operatorname{int}\bigl(\bigcap_{i\in I}\boldsymbol{a}_i\bigr)$, showing that $\operatorname{int}\bigl(\bigcap_{i\in I}\boldsymbol{a}_i\bigr)$ is closed.

Theorem 5.8. *The completely join-irreducible elements of* $\operatorname{Reg}(\boldsymbol{e})$ *are exactly the elements* $\langle a, b; U\rangle$*, where* $(a, b, U) \in \mathcal{F}(\boldsymbol{e})$*. Furthermore, the lattice* $\operatorname{Reg}(\boldsymbol{e})$ *is spatial.*

Proof. Let $\boldsymbol{a} \in \operatorname{Reg}(\boldsymbol{e})$. As $\operatorname{int}(\boldsymbol{a})$ is open, it follows from Lemma 5.5 that we can write $\operatorname{int}(\boldsymbol{a}) = \bigcup_{i\in I}\langle a_i, b_i; U_i\rangle$, for a family $((a_i, b_i, U_i) \mid i \in I)$ of elements of $\mathcal{F}(\boldsymbol{e})$. As the elements $\langle a_i, b_i; U_i\rangle$ are all clopen (thus regular closed) and $\boldsymbol{a}$ is regular closed, it follows that

$$\boldsymbol{a} = \operatorname{cl}\operatorname{int}(\boldsymbol{a}) = \bigvee_{i\in I}\langle a_i, b_i; U_i\rangle \quad \text{in } \operatorname{Reg}(\boldsymbol{e})\,.$$

In particular, if $\boldsymbol{a}$ is completely join-irreducible, then it must be one of the $\langle a_i, b_i; U_i\rangle$. Conversely, by Corollary 5.4, every element of the form $\langle a, b; U\rangle$ is completely join-irreducible in $\operatorname{Reg}(\boldsymbol{e})$. □

6. The arrow relations between clopen sets

Lemma 2.1 makes it possible to express the join-dependency relation on a finite lattice in terms of the arrow relations $\nearrow$ and $\searrow$. Throughout this section, let $\boldsymbol{e}$ be a transitive relation on a set E. We shall not necessarily assume finiteness of $\boldsymbol{e}$, except in Corollary 6.4 where we are dealing with the relation D (for Lemma 2.1 assumes finiteness). By using the dual automorphism $\boldsymbol{x} \mapsto \boldsymbol{x}^{\perp}$ (cf. Lemma 3.3), $\boldsymbol{x}^{\perp} \searrow \boldsymbol{y}$ iff $\boldsymbol{x} \nearrow \boldsymbol{y}^{\perp}$, for all $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{Reg}(\boldsymbol{e})$; hence statements involving $\searrow$ can always be expressed in terms of $\nearrow$. Furthermore, the completely meet-irreducible elements of $\operatorname{Reg}(\boldsymbol{e})$ are exactly the elements of the form $\boldsymbol{p}^{\perp}$, where $\boldsymbol{p}$ is a completely join-irreducible element of $\operatorname{Reg}(\boldsymbol{e})$. As every such $\boldsymbol{p}$ is clopen (cf. Theorem 5.8), we get $\boldsymbol{p}^{\perp} = \boldsymbol{e} \setminus \boldsymbol{p}$. Therefore, we obtain the following lemma.

Lemma 6.1. $\boldsymbol{p} \nearrow \boldsymbol{q}^{\perp}$ *iff* $\boldsymbol{p} \cap \boldsymbol{q} \neq \varnothing$ *and* $\boldsymbol{p} \cap \boldsymbol{q}_* = \varnothing$*, for all join-irreducible clopen sets* $\boldsymbol{p}$ *and* $\boldsymbol{q}$*. Furthermore, if* $\boldsymbol{q} = \langle c, d; V\rangle$*, where* $(c, d, V) \in \mathcal{F}(\boldsymbol{e})$*, then* $\boldsymbol{p} \nearrow \boldsymbol{q}^{\perp}$ *implies that* $\varnothing \neq \boldsymbol{p} \cap \boldsymbol{q} \subseteq [c]_{\boldsymbol{e}} \times [d]_{\boldsymbol{e}}$*.*

Proof. Only the second part of Lemma 6.1 requires a proof. From $\boldsymbol{p} \nearrow \boldsymbol{q}^{\perp}$ and the first part of Lemma 6.1 it follows that $\boldsymbol{p} \cap \boldsymbol{q} \neq \varnothing$ and further, using equation (5.1),

$$\boldsymbol{p} \subseteq (\boldsymbol{q}_*)^{\mathsf{c}} \subseteq \big(\boldsymbol{q} \setminus ([c]_{\boldsymbol{e}} \times [d]_{\boldsymbol{e}})\big)^{\mathsf{c}} = \boldsymbol{q}^{\mathsf{c}} \cup ([c]_{\boldsymbol{e}} \times [d]_{\boldsymbol{e}})\,,$$

so $\boldsymbol{p} \cap \boldsymbol{q} \subseteq [c]_{\boldsymbol{e}} \times [d]_{\boldsymbol{e}}$. □

From Lemma 6.2 to Lemma 6.6, we shall fix $(a, b, U), (c, d, V) \in \mathcal{F}(\boldsymbol{e})$. Further, we shall set $\boldsymbol{p} = \langle a, b; U\rangle$, $\boldsymbol{q} = \langle c, d; V\rangle$, $U^{\mathsf{c}} = [a, b]_{\boldsymbol{e}} \setminus U$, and $V^{\mathsf{c}} = [c, d]_{\boldsymbol{e}} \setminus V$.

Lemma 6.2. *$\boldsymbol{p} \nearrow \boldsymbol{q}^{\perp}$ implies that $[c, d]_{\boldsymbol{e}} \subseteq [a, b]_{\boldsymbol{e}}$.*

Proof. By Lemma 6.1 we can pick $(x, y) \in \boldsymbol{p} \cap \boldsymbol{q} \subseteq [c]_{\boldsymbol{e}} \times [d]_{\boldsymbol{e}}$, thus $x \equiv_{\boldsymbol{e}} c$ and $y \equiv_{\boldsymbol{e}} d$; furthermore, as $(x, y) \in \boldsymbol{p}$, we get $a \trianglelefteq_{\boldsymbol{e}} x \vartriangleleft_{\boldsymbol{e}} y \trianglelefteq_{\boldsymbol{e}} b$. The desired conclusion follows from the transitivity of $\trianglelefteq_{\boldsymbol{e}}$. □

Lemma 6.3. *Suppose that $c = d$. Then $\boldsymbol{p} \nearrow \boldsymbol{q}^{\perp}$ iff $a = b = c = d$, $U \cap V \subseteq \{a\}$, and $U^{\mathsf{c}} \cap V^{\mathsf{c}} \subseteq \{a\}$.*

Proof. Suppose first that $\boldsymbol{p} \nearrow \boldsymbol{q}^{\perp}$. From $\boldsymbol{q}_* = \boldsymbol{q} \setminus \{(c, c)\}$ (cf. (5.1)) it follows that

$$\boldsymbol{p} \cap \boldsymbol{q} = \{(c, c)\}\,, \tag{6.1}$$

and thus $(c, c) \in \langle a, b; U\rangle$, which rules out $a \neq b$ (cf. Definition 5.1). Hence $a = b$ and c belongs to $(\{a\} \cup U^{\mathsf{c}}) \cap (\{a\} \cup U) = \{a\}$, so $a = c$. Now let $x \in U^{\mathsf{c}} \cap V^{\mathsf{c}}$. Then (x, a) belongs to $\boldsymbol{p} \cap \boldsymbol{q}$, thus, by (6.1), $x = a$. The proof of the containment $U \cap V \subseteq \{a\}$ is similar.

Conversely, suppose that $a = b = c = d$, $U \cap V \subseteq \{a\}$, and $U^{\mathsf{c}} \cap V^{\mathsf{c}} \subseteq \{a\}$. Then any $(x, y) \in \boldsymbol{p} \cap \boldsymbol{q}$ satisfies $x \in U^{\mathsf{c}} \cap V^{\mathsf{c}}$, thus $x = a$. Likewise, $y = a$, so $\boldsymbol{p} \cap \boldsymbol{q} = \{(a, a)\}$. By Lemma 6.1, it follows that $\boldsymbol{p} \nearrow \boldsymbol{q}^{\perp}$. □

As a noteworthy consequence, we obtain that if $\boldsymbol{e}$ is finite and $\boldsymbol{q}$ is a clepsydra, then there is no $\boldsymbol{p}$ such that $\boldsymbol{p}\, D\, \boldsymbol{q}$.

Corollary 6.4. *Let $\boldsymbol{e}$ be finite. If $c = d$, then the relation $\boldsymbol{p}\, D\, \boldsymbol{q}$ does not hold.*

Proof. Suppose that $\boldsymbol{p}\, D\, \boldsymbol{q}$, so there exists a join-irreducible element $\boldsymbol{r}$ such that $\boldsymbol{p} \nearrow \boldsymbol{r}^{\perp}$ and $\boldsymbol{r} \nearrow \boldsymbol{q}^{\perp}$. By Lemma 6.3, $a = b = c = d$ and there exists $W \subseteq [a]_{\boldsymbol{e}}$ such that $\boldsymbol{r} = \langle a, a; W\rangle$ and the sets $U \cap W$, $U^{\mathsf{c}} \cap W^{\mathsf{c}}$, $V \cap W$, and $V^{\mathsf{c}} \cap W^{\mathsf{c}}$ are all contained in $\{a\}$. Since all these subsets are contained in $[a]_{\boldsymbol{e}}$, this means that each pair $(U \setminus \{a\}, W \setminus \{a\})$ and $(V \setminus \{a\}, W \setminus \{a\})$ is complementary within $[a]_{\boldsymbol{e}} \setminus \{a\}$, hence $U \setminus \{a\} = V \setminus \{a\}$, equivalently $U \cup \{a\} = V \cup \{a\}$. Therefore, $\boldsymbol{p} = \boldsymbol{q}$, contradicting the definition of the relation D. □

Lemma 6.5. *Suppose that $a = b$ and $c \neq d$. Then $\boldsymbol{p} \nearrow \boldsymbol{q}^{\perp}$ iff $a \equiv_{\boldsymbol{e}} c \equiv_{\boldsymbol{e}} d$, $(\{a\} \cup U) \cap V \neq \varnothing$, and $(\{a\} \cup U^{\mathsf{c}}) \cap V^{\mathsf{c}} \neq \varnothing$.*

Proof. Suppose first that $\boldsymbol{p} \nearrow \boldsymbol{q}^{\perp}$. It follows from Lemma 6.2 that $a \equiv_{\boldsymbol{e}} c \equiv_{\boldsymbol{e}} d$. Any element $(u, v) \in \boldsymbol{p} \cap \boldsymbol{q}$ satisfies that $u \in (\{a\} \cup U^{\mathsf{c}}) \cap V^{\mathsf{c}}$ and $v \in (\{a\} \cup U) \cap V$. Conversely, if $a \equiv_{\boldsymbol{e}} c \equiv_{\boldsymbol{e}} d$, $u \in (\{a\} \cup U^{\mathsf{c}}) \cap V^{\mathsf{c}}$, and $v \in (\{a\} \cup U) \cap V$, then $(u, v) \in \boldsymbol{p} \cap \boldsymbol{q}$. From $c \neq d$, $c \equiv_{\boldsymbol{e}} d$, and $\boldsymbol{q} \subseteq [c, d]_{\boldsymbol{e}} \times [c, d]_{\boldsymbol{e}}$ it follows that $\boldsymbol{q}_* = \boldsymbol{q} \setminus ([c]_{\boldsymbol{e}} \times [d]_{\boldsymbol{e}}) = \varnothing$; whence $\boldsymbol{p} \cap \boldsymbol{q}_* = \varnothing$. □

Lemma 6.6. *Suppose that $a \neq b$ and $c \neq d$. Then $\boldsymbol{p} \nearrow \boldsymbol{q}^{\perp}$ iff $[c, d]_{\boldsymbol{e}} \subseteq [a, b]_{\boldsymbol{e}}$ and $\varnothing \neq \boldsymbol{e} \cap \big((U^{\mathsf{c}} \cap V^{\mathsf{c}}) \times (U \cap V)\big) \subseteq [c]_{\boldsymbol{e}} \times [d]_{\boldsymbol{e}}$.*

Proof. Suppose first that $\boldsymbol{p} \nearrow \boldsymbol{q}^{\perp}$. It follows from Lemma 6.2 that $[c,d]_{\boldsymbol{e}} \subseteq [a,b]_{\boldsymbol{e}}$. Any element $(u,v) \in \boldsymbol{p} \cap \boldsymbol{q}$ belongs to $\boldsymbol{e} \cap \big((U^{\mathsf{c}} \cap V^{\mathsf{c}}) \times (U \cap V)\big)$, thus this set is nonempty. Moreover, any element (u,v) of this set belongs to $\boldsymbol{q} \setminus \boldsymbol{q}_*$, thus $u \equiv_{\boldsymbol{e}} c$ and $v \equiv_{\boldsymbol{e}} d$. Conversely, suppose that $[c,d]_{\boldsymbol{e}} \subseteq [a,b]_{\boldsymbol{e}}$ and $\varnothing \neq \boldsymbol{e} \cap \big((U^{\mathsf{c}} \cap V^{\mathsf{c}}) \times (U \cap V)\big) \subseteq [c]_{\boldsymbol{e}} \times [d]_{\boldsymbol{e}}$. Observing that $\boldsymbol{p} \cap \boldsymbol{q} = \boldsymbol{e} \cap \big((U^{\mathsf{c}} \cap V^{\mathsf{c}}) \times (U \cap V)\big)$, it follows that $\boldsymbol{p} \cap \boldsymbol{q}$ is both nonempty and contained in $[c]_{\boldsymbol{e}} \times [d]_{\boldsymbol{e}}$; the latter condition implies that $\boldsymbol{p} \cap \boldsymbol{q}_* = \varnothing$. We get therefore the relation $\boldsymbol{p} \nearrow \boldsymbol{q}^{\perp}$. □

Corollary 6.7. *Suppose that $\boldsymbol{e}$ is antisymmetric, $a \neq b$, and $c \neq d$. Then $\boldsymbol{p} \nearrow \boldsymbol{q}^{\perp}$ iff $(c,d) \in \boldsymbol{p}$ and $V = (]c,d]_{\boldsymbol{e}} \setminus U) \cup \{d\}$.*

7. Bounded lattices of regular closed sets

Let $\boldsymbol{e}$ be a transitive relation on a (possibly infinite) set E. Suppose, until the statement of Proposition 7.2, that $\boldsymbol{e}$ is *antisymmetric* (i.e., the preordering $\trianglelefteq_{\boldsymbol{e}}$ is an ordering), then some information can be added to the results of Section 6. First of all, the clepsydras (cf. Definition 5.1) are exactly the singletons $\{(a,a)\}$, where $a \vartriangleleft_{\boldsymbol{e}} a$. On the other hand, if $a \neq b$, then $\langle a,b;U\rangle$ determines both the ordered pair (a,b) and the set U. (Recall that a clepsydra is never bipartite, so it cannot have the form $\langle a,b;U\rangle$ with $a \neq b$.)

Let us focus for a while on arrow relations involving bipartite join-irreducible clopen sets. We set

$$U{\restriction}_{c,d} = (U \cap\,]c,d]_{\boldsymbol{e}}) \cup \{d\}\,, \quad \text{for all } (c,d) \in \boldsymbol{e} \text{ and all } U \subseteq E\,.$$

Observe that $(c,d,U{\restriction}_{c,d}) \in \mathcal{F}(\boldsymbol{e})$. The following lemma is a restated variant of Corollary 6.7.

Lemma 7.1. *Let $(a,b,U),(c,d,V) \in \mathcal{F}(\boldsymbol{e})$ with $a \neq b$ and $c \neq d$, and set $\widetilde{U} = (E \setminus U){\restriction}_{a,b}$. Then $\langle a,b;U\rangle \nearrow \langle c,d;V\rangle^{\perp}$ iff $[c,d]_{\boldsymbol{e}} \subseteq [a,b]_{\boldsymbol{e}}$ and $V = \widetilde{U}{\restriction}_{c,d}$.*

This yields, in the finite case, a characterization of the join-dependency relation on the join-irreducible clopen sets.

Proposition 7.2. *Suppose that E is finite, $\boldsymbol{e}$ is antisymmetric, and let (a_0,b_0,U_0), $(a_1,b_1,U_1) \in \mathcal{F}(\boldsymbol{e})$ with $a_0 \neq b_0$ and $a_1 \neq b_1$. Set $\boldsymbol{p}_i = \langle a_i,b_i;U_i\rangle$ for $i < 2$. Then $\boldsymbol{p}_0 \mathrel{D} \boldsymbol{p}_1$ in the lattice $\mathrm{Reg}(\boldsymbol{e})$ iff $[a_1,b_1]_{\boldsymbol{e}} \subsetneqq [a_0,b_0]_{\boldsymbol{e}}$ and $U_1 = U_0{\restriction}_{a_1,b_1}$.*

Proof. Suppose first that $\boldsymbol{p}_0 \mathrel{D} \boldsymbol{p}_1$. By Theorem 5.8, Lemma 2.1, and the observations at the beginning of Section 6, there exists $(c,d,V) \in \mathcal{F}(\boldsymbol{e})$ such that, setting $\boldsymbol{q} = \langle c,d;V\rangle$, the relations $\boldsymbol{p}_0 \nearrow \boldsymbol{q}^{\perp}$ and $\boldsymbol{q} \nearrow \boldsymbol{p}_1^{\perp}$ both hold. It follows from Corollary 6.4 that $c \neq d$. By Lemma 7.1, $[a_1,b_1]_{\boldsymbol{e}} \subseteq [c,d]_{\boldsymbol{e}} \subseteq [a_0,b_0]_{\boldsymbol{e}}$ and, setting $\widetilde{U}_0 = (E \setminus U_0){\restriction}_{a_0,b_0}$ and $\widetilde{V} = (E \setminus V){\restriction}_{c,d}$, $V = \widetilde{U}_0{\restriction}_{c,d}$ and $U_1 = \widetilde{V}{\restriction}_{a_1,b_1}$; clearly $\widetilde{V} = U_0{\restriction}_{c,d}$, whence $U_1 = U_0{\restriction}_{a_1,b_1}$. Since $\boldsymbol{p}_0 \neq \boldsymbol{p}_1$, it follows that $[a_1,b_1]_{\boldsymbol{e}} \subsetneqq [a_0,b_0]_{\boldsymbol{e}}$.

Conversely, suppose that $[a_1,b_1]_{\boldsymbol{e}} \subsetneqq [a_0,b_0]_{\boldsymbol{e}}$ and $U_1 = U_0{\restriction}_{a_1,b_1}$. In particular, $\boldsymbol{p}_0 \neq \boldsymbol{p}_1$. Set $U_0^{\mathsf{c}} = [a_0,b_0]_{\boldsymbol{e}} \setminus U_0$. We shall separate cases, according to whether or not a_1, b_1 belong to U_0. In each of those cases, we shall define a certain join-irreducible element $\boldsymbol{q} = \langle c,d;V\rangle$ of $\mathrm{Reg}(\boldsymbol{e})$, with $a_0 \trianglelefteq_{\boldsymbol{e}} c \vartriangleleft_{\boldsymbol{e}} d \trianglelefteq_{\boldsymbol{e}} a_1$ and $V = U_0^{\mathsf{c}}{\restriction}_{c,d}$, hence only c and d will need to be specified. Each of the desired arrow relations will be inferred with the help of Corollary 6.7.

Case 1. $a_1 \in \{a_0\} \cup U_0^{\mathsf{c}}$ and $b_1 \in \{b_0\} \cup U_0$. We set $c = a_1$ and $d = b_1$. Then $\boldsymbol{p}_0 \nearrow \boldsymbol{q}^\perp$ (because $(a_1, b_1) \in \boldsymbol{p}_0$) and $\boldsymbol{q} \nearrow \boldsymbol{p}_1^\perp$ (because $(a_1, b_1) \in \boldsymbol{q}$).

Case 2. $a_1 \in \{a_0\} \cup U_0$ and $b_1 \in \{b_0\} \cup U_0$. We set $c = a_0$ and $d = b_1$. Then $\boldsymbol{p}_0 \nearrow \boldsymbol{q}^\perp$ (because $(a_0, b_1) \in \boldsymbol{p}_0$) and $\boldsymbol{q} \nearrow \boldsymbol{p}_1^\perp$ (because $(a_1, b_1) \in \boldsymbol{q}$).

Case 3. $a_1 \in \{a_0\} \cup U_0^{\mathsf{c}}$ and $b_1 \in \{b_0\} \cup U_0^{\mathsf{c}}$. We set $c = a_1$ and $d = b_0$. Then $\boldsymbol{p}_0 \nearrow \boldsymbol{q}^\perp$ (because $(a_1, b_0) \in \boldsymbol{p}_0$) and $\boldsymbol{q} \nearrow \boldsymbol{p}_1^\perp$ (because $(a_1, b_1) \in \boldsymbol{q}$).

Case 4. $a_1 \in \{a_0\} \cup U_0$ and $b_1 \in \{b_0\} \cup U_0^{\mathsf{c}}$. We set $c = a_0$ and $d = b_0$. Then $\boldsymbol{p}_0 \nearrow \boldsymbol{q}^\perp$ (because $(a_0, b_0) \in \boldsymbol{p}_0$) and $\boldsymbol{q} \nearrow \boldsymbol{p}_1^\perp$ (because $(a_1, b_1) \in \boldsymbol{q}$).

In each of those cases, $\boldsymbol{p}_0 \nearrow \boldsymbol{q}^\perp$ and $\boldsymbol{q} \nearrow \boldsymbol{p}_1^\perp$, hence, as $\boldsymbol{p}_0 \neq \boldsymbol{p}_1$ and by Lemma 2.1, $\boldsymbol{p}_0 \, D \, \boldsymbol{p}_1$. □

By using the standard description of the congruence lattice of a finite lattice *via* the join-dependency relation (cf. Freese, Ježek, and Nation [16, § II.3]), Proposition 7.2 makes it possible to give a complete description of the congruence lattice of $\mathrm{Reg}(\boldsymbol{e})$ in case $\boldsymbol{e}$ is antisymmetric. Congruence lattices of permutohedra were originally described in Duquenne and Cherfouh [11, § 4]. An implicit description of congruences of permutohedra *via* the join-dependency relation appears in Santocanale [30]; in that paper, similar results were established for multinomial lattices.

By Lemma 6.5 together with the antisymmetry of $\boldsymbol{e}$, if $\boldsymbol{p} \nearrow \boldsymbol{q}^\perp$ and $\boldsymbol{p}$ is a clepsydra, then so is $\boldsymbol{q}$. Hence, by Corollary 6.4, $\boldsymbol{p} \, D \, \boldsymbol{q}$ implies (in the finite, antisymmetric case) that neither $\boldsymbol{p}$ nor $\boldsymbol{q}$ is a clepsydra. By Proposition 7.2, we thus obtain the following result.

Corollary 7.3. *Let $\boldsymbol{e}$ be an antisymmetric, transitive relation on a finite set. Then the join-dependency relation on the join-irreducible elements of* $\mathrm{Reg}(\boldsymbol{e})$ *is a strict ordering.*

Example 7.4. The transitivity of the relation D, holding on the join-irreducible elements of $\mathrm{Reg}(\boldsymbol{e})$ for any antisymmetric transitive relation $\boldsymbol{e}$, is quite a special property. It does not hold in all finite bounded homomorphic images of free lattices, as shows the lattice L_9 (following the notation of Jipsen and Rose [21]) represented on the left hand side of Figure 7.1. The join-irreducible elements marked there by doubled circles satisfy $p \, D \, q$ and $q \, D \, r$ but not $p \, D \, r$.

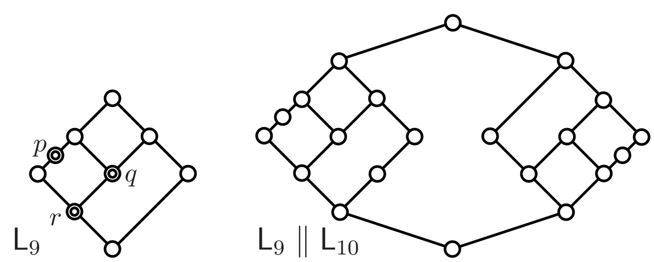

FIGURE 7.1. Bounded lattices with non-transitive join-dependency relation

The lattice L_9 is not orthocomplemented, but its parallel sum with its dual lattice L_{10}, denoted there by $\mathsf{L}_9 \parallel \mathsf{L}_{10}$, is orthocomplemented. As L_9, the parallel sum is bounded and has non-transitive relation D.

In the non-bounded case, the reflexive closure of the relation D on $\operatorname{Reg}(\boldsymbol{e})$ may not be transitive. This is witnessed by the lattice $\operatorname{Bip}(3) = \operatorname{Reg}([3] \times [3])$, see Lemmas 8.5 and 8.6.

Definition 7.5. A family $(\boldsymbol{e}_i \mid i \in I)$ of *pairwise disjoint* transitive relations is *orthogonal* if there are no distinct $i, j \in I$ and no p, q, r such that $p \neq q$, $q \neq r$, $(p, q) \in \boldsymbol{e}_i$, and $(q, r) \in \boldsymbol{e}_j$.

In particular, if $(\boldsymbol{e}_i \mid i \in I)$ is orthogonal, then $\bigcup_{i \in I} \boldsymbol{e}_i$ is itself a transitive relation.

Proposition 7.6. *The following statements hold, for any orthogonal family $(\boldsymbol{e}_i \mid i \in I)$ of transitive relations:*

(i) *A subset $\boldsymbol{x}$ of $\boldsymbol{e}$ is closed (resp., open in $\boldsymbol{e}$) iff $\boldsymbol{x} \cap \boldsymbol{e}_i$ is closed (resp., open in $\boldsymbol{e}_i$) for each $i \in I$.*
(ii) $\operatorname{Reg}(\boldsymbol{e}) \cong \prod_{i \in I} \operatorname{Reg}(\boldsymbol{e}_i)$, *via an isomorphism that carries* $\operatorname{Clop}(\boldsymbol{e})$ *onto* $\prod_{i \in I} \operatorname{Clop}(\boldsymbol{e}_i)$.

Proof. The proof of (i) is a straightforward exercise. For (ii), we define $\varphi(\boldsymbol{x}) = (\boldsymbol{x} \cap \boldsymbol{e}_i \mid i \in I)$ whenever $\boldsymbol{x} \subseteq \boldsymbol{e}$, and $\psi(\boldsymbol{x}_i \mid i \in I) = \bigcup_{i \in I} \boldsymbol{x}_i$ whenever all $\boldsymbol{x}_i \subseteq \boldsymbol{e}_i$. By using (i), it is straightforward (although somewhat tedious) to verify that φ and ψ restrict to mutually inverse isomorphisms between $\operatorname{Reg}(\boldsymbol{e})$ and $\prod_{i \in I} \operatorname{Reg}(\boldsymbol{e}_i)$, and also between $\operatorname{Clop}(\boldsymbol{e})$ and $\prod_{i \in I} \operatorname{Clop}(\boldsymbol{e}_i)$. □

Set $\Delta_A = \{(x, x) \mid x \in A\}$, for every set A. By applying Proposition 7.6 to the 2-element family $(\boldsymbol{e}, \Delta_A)$, we obtain the following result, which shows that $\operatorname{Reg}(\boldsymbol{e} \cup \Delta_A)$ is the product of $\operatorname{Reg}(\boldsymbol{e})$ by a powerset lattice.

Corollary 7.7. *Let $\boldsymbol{e}$ be a transitive relation and let A be a set with $\boldsymbol{e} \cap \Delta_A = \varnothing$. Then* $\operatorname{Reg}(\boldsymbol{e} \cup \Delta_A) \cong \operatorname{Reg}(\boldsymbol{e}) \times (\operatorname{Pow} A)$ *and* $\operatorname{Clop}(\boldsymbol{e} \cup \Delta_A) \cong \operatorname{Clop}(\boldsymbol{e}) \times (\operatorname{Pow} A)$.

Duquenne and Cherfouh [11, Theorem 3] and Le Conte de Poly-Barbut [22, Lemme 9] proved that every permutohedron is semidistributive (in the latter paper the result was extended to all *Coxeter lattices*). This result was improved by Caspard [7], who proved that every permutohedron is a bounded homomorphic image of a free lattice; and later, by Caspard, Le Conte de Poly-Barbut, and Morvan [8], who extended this result to all finite Coxeter groups. Our next result shows exactly to which transitive (not necessarily antisymmetric) relations those results can be extended.

Theorem 7.8. *The following are equivalent, for any transitive relation $\boldsymbol{e}$ on a finite set E:*

(i) *The lattice* $\operatorname{Reg}(\boldsymbol{e})$ *is a bounded homomorphic image of a free lattice.*
(ii) *The lattice* $\operatorname{Reg}(\boldsymbol{e})$ *is semidistributive.*
(iii) *The lattice* $\operatorname{Reg}(\boldsymbol{e})$ *is pseudocomplemented.*
(iv) *Every connected component of the preordering $\trianglelefteq_{\boldsymbol{e}}$ either is antisymmetric or has the form $\{a, b\}$ with $a \neq b$ while $(a, b) \in \boldsymbol{e}$ and $(b, a) \in \boldsymbol{e}$.*

Proof. (i)⇒(ii) is well-known, see for example Freese, Ježek, and Nation [16, Theorem 2.20].

(ii)⇒(iii) is trivial.

(iii)⇒(iv) follows immediately from Proposition 3.5.

(iv)⇒(i). Denote by $\{E_i \mid i < n\}$ the set of all connected components of $\trianglelefteq_{\boldsymbol{e}}$ and set $\boldsymbol{e}_i = \boldsymbol{e} \cap (E_i \times E_i)$ for each $i < n$. By Proposition 7.6, it suffices to consider the case where $\boldsymbol{e} = \boldsymbol{e}_i$ for some i, that is, $\trianglelefteq_{\boldsymbol{e}}$ is connected. It suffices to prove that the join-dependency relation on $\mathrm{Reg}(\boldsymbol{e})$ has no cycle (cf. Freese, Ježek, and Nation [16, Corollary 2.39] together with the self-duality of $\mathrm{Reg}(\boldsymbol{e})$). In the antisymmetric case, this follows from Corollary 7.3. If $E = \{a, b\}$ with $a \neq b$ while $(a, b) \in \boldsymbol{e}$ and $(b, a) \in \boldsymbol{e}$, then $\mathrm{Reg}(\boldsymbol{e})$ is isomorphic to the lattice $\mathrm{Bip}(2)$ of all bipartitions of a two-element set (cf. Section 3). The relation D on the join-irreducible elements of $\mathrm{Bip}(2)$, represented on the right hand side of Figure 7.2, has no cycle. □

The Hasse diagram of the lattice $\mathrm{Bip}(2)$ is represented on the left hand side of Figure 7.2. On the right hand side of Figure 7.2 we represent (the digraph of) the relation D on the join-irreducible elements of $\mathrm{Bip}(2)$. The join-irreducible elements of $\mathrm{Bip}(2)$ are denoted there by

$$\begin{aligned} \boldsymbol{a}_0 &= \{(1,2)\}\,, & \boldsymbol{a}_1 &= \{(1,1),(1,2)\}\,, & \boldsymbol{a}_2 &= \{(2,2),(1,2)\}\,,\\ \boldsymbol{b}_0 &= \{(2,1)\}\,, & \boldsymbol{b}_1 &= \{(1,1),(2,1)\}\,, & \boldsymbol{b}_2 &= \{(2,2),(2,1)\}\,. \end{aligned}$$

In order to allay the confusion that might arise from the ordering of the lattice and its relation D being incompatible (e.g., $p\,D\,q$ implies that $p \nleq q$), we mark the edges of the relation D with arrows, so for example $\boldsymbol{a}_0 < \boldsymbol{a}_1$, $\boldsymbol{a}_1\,D\,\boldsymbol{a}_0$, and $\neg(\boldsymbol{a}_0\,D\,\boldsymbol{a}_1)$.

Corollary 7.9. *Let $\boldsymbol{e}$ be a finite transitive relation. If $\mathrm{Reg}(\boldsymbol{e})$ is semidistributive, then the join-dependency relation defines a strict ordering on the join-irreducible elements of $\mathrm{Reg}(\boldsymbol{e})$.*

Proof. By using Proposition 7.6, together with the characterization (iv) of semidistributivity of $\mathrm{Reg}(\boldsymbol{e})$ given in Theorem 7.8, it is easy to reduce the problem to the case where $\boldsymbol{e}$ is either antisymmetric or a loop $a \vartriangleleft_{\boldsymbol{e}} b \vartriangleleft_{\boldsymbol{e}} a$ with $a \neq b$. In the first case, the conclusion follows from Corollary 7.3. In the second case, the join-dependency relation is bipartite (see the right hand side of Figure 7.2), thus transitive. □

The lattices $\mathrm{Bip}(3)$ and $\mathrm{Bip}(4)$ are represented on Figure 7.3; they have 74 and 730 elements, respectively.

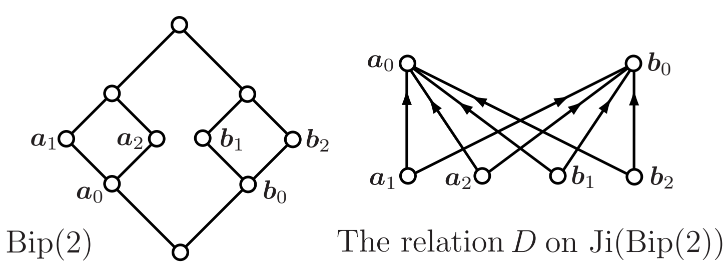

FIGURE 7.2. The lattice $\mathrm{Bip}(2)$ of all bipartitions of $\{1, 2\}$ and its relation D

Example 7.10. The following example shows that none of the implications (iv)⇒(ii) and (iv)⇒(iii) of Theorem 7.8 can be extended to the infinite case. Define $\boldsymbol{e}$ as the natural strict ordering on the ordinal $\omega + 1 = \{0, 1, 2, \dots\} \cup \{\omega\}$. As $\boldsymbol{e}$ is obviously square-free, it follows from Theorem 4.3 that $\mathrm{Clop}(\boldsymbol{e}) = \mathrm{Reg}(\boldsymbol{e})$ is a lattice, namely

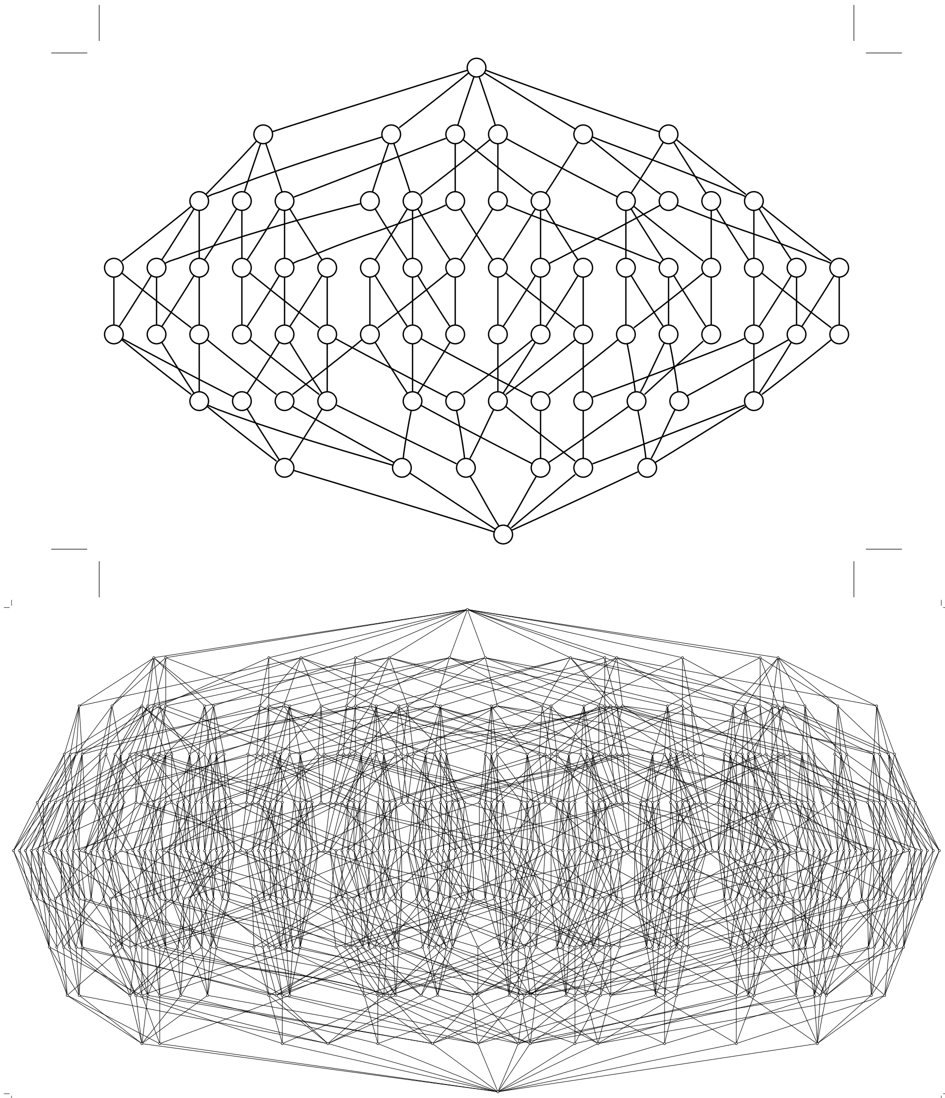

FIGURE 7.3. The lattices Bip(3) and Bip(4)

the permutohedron $\mathsf{P}(\omega+1)$ (cf. Example 4.5). It is straightforward to verify that the sets

$$\begin{aligned}
\boldsymbol{a} &= \{(2m, 2n+1) \mid m \leq n < \omega\} \cup \{(2m, \omega) \mid m < \omega\}\,,\\
\boldsymbol{b} &= \{(2m+1, 2n+2) \mid m \leq n < \omega\} \cup \{(2m+1, \omega) \mid m < \omega\}\,,\\
\boldsymbol{c} &= \{(m, \omega) \mid m < \omega\}\,.
\end{aligned}$$

are all clopen in $\boldsymbol{e}$. Furthermore,

$$\boldsymbol{a} \cap \boldsymbol{c} = \{(2m, \omega) \mid m < \omega\}$$

has empty interior, so $\boldsymbol{a} \wedge \boldsymbol{c} = \varnothing$. Likewise, $\boldsymbol{b} \wedge \boldsymbol{c} = \varnothing$. On the other hand, $\boldsymbol{c} \subseteq \boldsymbol{a} \cup \boldsymbol{b} \subseteq \boldsymbol{a} \vee \boldsymbol{b}$ and $\boldsymbol{c} \neq \varnothing$. In particular,

$$\boldsymbol{a} \wedge \boldsymbol{c} = \boldsymbol{b} \wedge \boldsymbol{c} = \varnothing \text{ and } (\boldsymbol{a} \vee \boldsymbol{b}) \wedge \boldsymbol{c} \neq \varnothing\,.$$

Therefore, the lattice $\mathsf{P}(\omega+1)$ is neither pseudocomplemented, nor semidistributive.

The result that every (finite) permutohedron $\mathsf{P}(n)$ is pseudocomplemented originates in Chameni-Nembua and Monjardet [9].

8. Join-dependency and congruences of bipartition lattices

In this paper we are using "bipartition" as a short naming for a transitive and cotransitive relation on $[n]$, that is, for an element of $\operatorname{Clop}([n] \times [n])$. These objects are more properly called "ordres bipartitionnaires" (Han [19]) or "bipartitional relations" in (Hetyei and Krattenthaler [20]). The usual representation of these objects as "ordered bipartitions", on which the naming relies, will be recalled at the beginning of Section 9.

The full relation $[n] \times [n]$ is transitive, for any positive integer n, and $\operatorname{Reg}([n] \times [n]) = \operatorname{Clop}([n] \times [n]) = \operatorname{Bip}(n)$ (cf. Section 3), the bipartition lattice of n. Due to the existence of exactly one $\equiv_{[n]\times[n]}$-class (namely the full set $[n]$), the description of the join-irreducible elements of $\operatorname{Bip}(n)$ obtained from Theorem 5.8 takes a particularly simple form.

Lemma 8.1. *The join-irreducible elements of* $\operatorname{Bip}(n)$ *consist of the sets* $\langle U \rangle = U^{\mathsf{c}} \times U$ *for* $U \in \operatorname{Pow}^*[n]$ (*those are the bipartite ones*) *together with the sets* $\langle a, U \rangle = (\{a\} \cup U^{\mathsf{c}}) \times (\{a\} \cup U)$ *for* $a \in [n]$ *and* $U \in \operatorname{Pow}[n]$ (*those are the clepsydras*).

The associated lower covers are immediately obtained from Corollary 5.4:

Lemma 8.2. *The bipartite* $\langle U \rangle$ (*with* $U \in \operatorname{Pow}^*[n]$) *are the atoms of* $\operatorname{Bip}(n)$ *while, for clepsydras* $\langle a, U \rangle$ (*with* $a \in [n]$ *and* $U \in \operatorname{Pow}[n]$), $\langle a, U \rangle_* = \langle a, U \rangle \setminus \{(a,a)\}$.

Lemma 8.3. *Let* $(a, U) \in [n] \times \operatorname{Pow}[n]$ *and let* $V \in \operatorname{Pow}^*[n]$. *Then* $\langle a, U \rangle\ D\ \langle V \rangle$ *always holds* (*within* $\operatorname{Bip}(n)$).

Proof. Suppose first that (U, V) forms a partition of $[n]$. It follows from Lemma 6.3 that $\langle a, U \rangle \nearrow \langle a, V \rangle^{\perp}$; moreover, considering that both V and V^{c} are nonempty, the relation $\langle a, V \rangle \nearrow \langle V \rangle^{\perp}$ follows from Lemma 6.5. Hence $\langle a, U \rangle\ D\ \langle V \rangle$.

Suppose from now on that (U, V) does not form a partition of $[n]$. We shall find $W \in \operatorname{Pow}[n]$ such that

$$(\{a\} \cup U^{\mathsf{c}}) \cap W^{\mathsf{c}} \neq \varnothing\,, \tag{8.1}$$

$$V^{\mathsf{c}} \cap W^{\mathsf{c}} \neq \varnothing\,, \tag{8.2}$$

$$(\{a\} \cup U) \cap W \neq \varnothing\,, \tag{8.3}$$

$$V \cap W \neq \varnothing\,. \tag{8.4}$$

By Lemmas 6.5 and 6.6, this will ensure that $\langle a, U \rangle \nearrow \langle W \rangle^{\perp}$ and $\langle W \rangle \nearrow \langle V \rangle^{\perp}$, hence that $\langle a, U \rangle\ D\ \langle V \rangle$.

If $U \cap V \neq \varnothing$, set $W = U \cap V$. Then (8.3) and (8.4) are trivial, while $V \cup W = V \neq [n]$ and $U \cup W = U \neq [n]$, so (8.1) and (8.2) are satisfied as well. If $U \cap V = \varnothing$, then, as (U, V) is not a partition of $[n]$, we get $U^{\mathsf{c}} \cap V^{\mathsf{c}} \neq \varnothing$; we set in this case $W = U \cup V$. Again, it is easy to verify that (8.1)–(8.4) are satisfied. $\square$

Definition 8.4. We say that a partition (U, U^{c}) of $[n]$ is *extremal* if either U or U^{c} is a singleton. The atom $\langle U \rangle$ will be called an *extremal atom*.

The description of the relation D on $\mathrm{Bip}(n)$ is completed by the following result.

Lemma 8.5. *Let $U, V \in \mathrm{Pow}^*[n]$. Then $\langle U \rangle \mathbin{D} \langle V \rangle$ iff (U, V) is not an extremal partition of $[n]$.*

Proof. The relation $\langle U \rangle \mathbin{D} \langle V \rangle$ holds iff there exists a join-irreducible $\boldsymbol{p} \in \mathrm{Bip}(n)$ such that $\langle U \rangle \nearrow \boldsymbol{p}^{\perp}$ and $\boldsymbol{p} \nearrow \langle V \rangle^{\perp}$. By Lemma 6.3, $\boldsymbol{p}$ cannot be a clepsydra. Hence, by Lemma 6.6, $\langle U \rangle \mathbin{D} \langle V \rangle$ iff there exists $W \in \mathrm{Pow}[n]$ such that

$$U \cap W \neq \varnothing\,, \quad V \cap W \neq \varnothing\,, \quad U^{\mathsf{c}} \cap W^{\mathsf{c}} \neq \varnothing\,, \quad V^{\mathsf{c}} \cap W^{\mathsf{c}} \neq \varnothing\,. \tag{8.5}$$

Suppose first that (U, V) is an extremal partition of $[n]$, so that either $U = \{a\}$ or $V = \{a\}$, for some $a \in [n]$. In the first case, $U \cap W \neq \varnothing$ implies that $a \in W$, hence $V \cup W = [n]$, in contradiction with (8.5). Likewise, in the second case, $a \in W$ and $U \cup W = [n]$, in contradiction with (8.5). In any case, the relation $\langle U \rangle \mathbin{D} \langle V \rangle$ does not hold. Conversely, suppose from now on that either (U, V) is not a partition of $[n]$, or it is not extremal. Suppose first that (U, V) is a partition of $[n]$ and pick $(u, v) \in U \times V$. Then the set $W = \{u, v\}$ meets both U and V. Furthermore, $U \cup W = U \cup \{v\}$ is distinct from $[n]$ as V is not a singleton. Likewise $V \cup W = V \cup \{u\} \neq [n]$. Finally, suppose that (U, V) is not a partition of $[n]$. If $U \cap V \neq \varnothing$, then $W = U \cap V$ solves our problem. If $U^{\mathsf{c}} \cap V^{\mathsf{c}} \neq \varnothing$, then $W = U \cup V$ solves our problem. In any case, (8.5) holds for our choice of W. □

Lemma 8.6. *Suppose that $n \geq 3$ and let $U, V \in \mathrm{Pow}^*[n]$. Then either $\langle U \rangle \mathbin{D} \langle V \rangle$ or $\langle U \rangle \mathbin{D^2} \langle V \rangle$.*

Proof. By Lemma 8.5, if the relation $\langle U \rangle \mathbin{D} \langle V \rangle$ fails, then there exists $a \in [n]$ such that $\{U, V\} = \{\{a\}, [n] \setminus \{a\}\}$. Pick any $b \in [n] \setminus \{a\}$ and set $W = \{a, b\}$. As $n \geq 3$, neither (U, W) nor (W, V) is a partition of $[n]$. By Lemma 8.5, it follows that $\langle U \rangle \mathbin{D} \langle W \rangle$ and $\langle W \rangle \mathbin{D} \langle V \rangle$. □

By using the end of Section 2 (in particular Lemma 2.2), the congruence lattice of $\mathrm{Bip}(n)$ can be entirely described by the relation D^* on $\mathrm{Ji}(\mathrm{Bip}(n))$. Hence it can be obtained from the following easy consequence of Corollary 6.4 together with Lemmas 8.3 and 8.6.

Corollary 8.7. *Suppose that $n \geq 3$ and let $\boldsymbol{p}, \boldsymbol{q}$ be join-irreducible elements of $\mathrm{Bip}(n)$. Then $\mathrm{con}(\boldsymbol{p}) \subseteq \mathrm{con}(\boldsymbol{q})$ iff either $\boldsymbol{q}$ is bipartite or $\boldsymbol{p}$ is a clepsydra and $\boldsymbol{p} = \boldsymbol{q}$.*

In particular, the congruences $\mathrm{con}(\boldsymbol{p})$, for $\boldsymbol{p}$ a clepsydra, are pairwise incomparable, so they are the atoms of $\mathrm{Con}\,\mathrm{Bip}(n)$. Each such congruence is thus determined by the corresponding clepsydra $\boldsymbol{p}$, and those clepsydras are in one-to-one correspondence with the associated ordered pairs $(a, U \setminus \{a\})$. Hence there are $n \cdot 2^{n-1}$ clepsydras, and we get the following result.

Corollary 8.8. *The congruence lattice of the bipartition lattice $\mathrm{Bip}(n)$ is obtained from a Boolean lattice with $n \cdot 2^{n-1}$ atoms by adding a new top element, for every integer $n \geq 3$.*

Corollary 8.8 does not extend to the case where $n = 2$: the congruence lattice of $\mathrm{Bip}(2)$ is isomorphic to the lattice of all lower subsets of the poset represented on the right hand side of Figure 7.2.

9. Minimal subdirect product decompositions of bipartition lattices

Here and in the next sections we shall use the following useful description of bipartitions introduced in Han [19] and studied further in Hetyei and Krattenthaler [20, § 2].

To every bipartition $\boldsymbol{x}$ of $[n]$, we associate the relation $\sim_{\boldsymbol{x}}$ on $[n]$ defined by

$$p \sim_{\boldsymbol{x}} q \quad \text{if} \quad \text{either } \{(p,q),(q,p)\} \subseteq \boldsymbol{x} \text{ or } \{(p,q),(q,p)\} \cap \boldsymbol{x} = \varnothing\,, \quad \text{for all } p,q \in [n]\,.$$

Then $\sim_{\boldsymbol{x}}$ is an equivalence relation, the equivalence classes of which can be enumerated as $X_1, \dots, X_m$ in a unique way such that

$$i < j \ \Leftrightarrow\ (p,q) \in \boldsymbol{x}\,, \quad \text{whenever } i \neq j \text{ in } [m]\,,\ p \in X_i\,,\ \text{and } q \in X_j\,.$$

The sets X_i, for $i \in [m]$, will be called the *blocks* of $\boldsymbol{x}$. For each $i \in [m]$, either $X_i \times X_i \subseteq \boldsymbol{x}$, in which case we set $\varepsilon_i = +1$ and say that X_i is an *underlined block*, or $(X_i \times X_i) \cap \boldsymbol{x} = \varnothing$, in which case we set $\varepsilon_i = -1$ and say that X_i is a *non-underlined block*.

The bipartition $\boldsymbol{x}$ can be recovered from the X_i and the ε_i, by

$$(p,q) \in \boldsymbol{x} \ \Leftrightarrow\ (\exists i,j \in [m])\Big(p \in X_i\,,\ q \in X_j\,, \\ \text{and } \big(\text{either } i<j \text{ or } (i=j \text{ and } \varepsilon_i = +1)\big)\Big)\,.$$

We shall call $(X_1^{\varepsilon_1}, \dots, X_m^{\varepsilon_m})$ the *ordered bipartition representation of* $\boldsymbol{x}$, and write $\boldsymbol{x} = X_1^{\varepsilon_1} \oplus \cdots \oplus X_m^{\varepsilon_m}$. We shall also, occasionally, write X_i instead of X_i^{-1} (non-underlined blocks) and $\underline{X_i}$ instead of X_i^{+1} (underlined blocks). In particular,

$$\langle U \rangle = U^{\mathsf{c}} \oplus \underline{U} \quad \text{and} \quad \langle a, V \rangle = (V^{\mathsf{c}} \setminus \{a\}) \oplus \underline{\{a\}} \oplus (V \setminus \{a\})$$

whenever $U \in \mathrm{Pow}^*[n]$, $a \in [n]$, and $V \in \mathrm{Pow}[n]$.

As in Hetyei and Krattenthaler [20, § 6], we say that a permutation ρ of $[n]$ is *compatible* with a bipartition $\boldsymbol{x}$ if

$$\big((p,q) \in \boldsymbol{x} \text{ and } (q,p) \notin \boldsymbol{x}\big) \ \Rightarrow\ \rho^{-1}(p) < \rho^{-1}(q)\,, \quad \text{for all } p,q \in [n]\,.$$

Notation 9.1. For any positive integer n, we define

- $\mathsf{G}(n)$, the set of all bipartite join-irreducible elements of $\mathrm{Bip}(n)$ (i.e., those of the form $U^{\mathsf{c}} \times U$, where $U \in \mathrm{Pow}^*[n]$);
- $\mathsf{K}(n)$, the $(\vee, 0)$-subsemilattice of $\mathrm{Bip}(n)$ generated by $\mathsf{G}(n)$;
- θ_n, the congruence of $\mathrm{Bip}(n)$ generated by all $\Psi(\boldsymbol{p})$, for $\boldsymbol{p} \in \mathsf{G}(n)$;
- $\mathsf{S}(n, \boldsymbol{p})$, the $(\vee, 0)$-subsemilattice of $\mathrm{Bip}(n)$ generated by $\mathsf{G}(n) \cup \{\boldsymbol{p}\}$, for each $\boldsymbol{p} \in \mathrm{Ji}(\mathrm{Bip}(n))$.

By the results of Section 8, the D^*-minimal join-irreducible of $\mathrm{Bip}(n)$ are exactly the clepsydras $\langle a, U \rangle$, where $a \in [n]$ and $U \subseteq [n]$ (note that the clopen set $\langle a, U \rangle$ is uniquely determined by the ordered pair $(a, U \setminus \{a\})$). This is proved in Section 8 for $n \geq 3$, but it is also trivially valid for $n \in \{1,2\}$ (cf. Figure 7.2). Hence the minimal subdirect product decomposition of $\mathrm{Bip}(n)$, given by (2.1), is the subdirect product

$$\mathrm{Bip}(n) \hookrightarrow \prod_{a \in [n]\,,\ U \subseteq [n] \setminus \{a\}} \big(\mathrm{Bip}(n)/\Psi(\langle a, U \rangle)\big)\,. \tag{9.1}$$

By Lemma 2.2, the factors of the decomposition (9.1) are exactly the lattices $\mathrm{Bip}(n)/\Psi(\langle a, U \rangle) \cong \mathsf{S}(n, \langle a, U \rangle)$. Likewise, we can also observe that $\mathrm{Bip}(n)/\theta_n \cong \mathsf{K}(n)$. We shall now identify, within $\mathrm{Bip}(n)$, the elements of $\mathsf{S}(n, \langle a, U \rangle)$.

Definition 9.2. An element $a \in [n]$ is an *isolated point* of a bipartition $\boldsymbol{x}$ if $(a,i) \in \boldsymbol{x}$ and $(i,a) \in \boldsymbol{x}$ iff $i = a$, for each $i \in [n]$. We denote by $\operatorname{isol}(\boldsymbol{x})$ the set of all isolated points of $\boldsymbol{x}$.

Lemma 9.3. *Let $\boldsymbol{x} = \bigvee_{i \in I} \boldsymbol{x}_i$ in $\operatorname{Bip}(n)$. Then any isolated point a of $\boldsymbol{x}$ is an isolated point of some $\boldsymbol{x}_i$.*

Proof. It suffices to prove that $(a,a) \in \boldsymbol{x}_i$ for some i. As $\boldsymbol{x}$ is the transitive closure of the union of the $\boldsymbol{x}_i$, there are a positive integer ℓ and $a = a_0, a_1, \dots, a_{\ell-1} \in [n]$ such that, setting $a_\ell = a$, the pair (a_k, a_{k+1}) belongs to $\bigcup_{i \in I} \boldsymbol{x}_i$ for each $k < \ell$. As $\boldsymbol{x}$ is transitive, $(a_0, a_1), (a_1, a_0) \in \boldsymbol{x}$, and $a = a_0$ is an isolated point of $\boldsymbol{x}$, we get $a_0 = a_1$, so (a,a) belongs to $\bigcup_{i \in I} \boldsymbol{x}_i$. □

Proposition 9.4. *The elements of $\mathsf{K}(n)$ are exactly the bipartitions without isolated points.*

Proof. No bipartite join-irreducible element of $\operatorname{Bip}(n)$ has any isolated point, hence, by Lemma 9.3, no element of $\mathsf{K}(n)$ has any isolated point.

Conversely, let $\boldsymbol{x} \in \operatorname{Bip}(n)$ with no isolated point and let $(i,j) \in \boldsymbol{x}$. If $i \neq j$, then, by Lemma 5.5, there exists $U \in \operatorname{Pow}^*[n]$ such that $(i,j) \in \langle U \rangle \subseteq \boldsymbol{x}$.

If $i = j$, then, as $\boldsymbol{x}$ has no isolated point, there exists $k \neq i$ such that $(i,k) \in \boldsymbol{x}$ and $(k,i) \in \boldsymbol{x}$. By the paragraph above, there are $U, V \in \operatorname{Pow}^*[n]$ such that $(i,k) \in \langle U \rangle \subseteq \boldsymbol{x}$ and $(k,i) \in \langle V \rangle \subseteq \boldsymbol{x}$. Hence $(i,i) \in \langle U \rangle \vee \langle V \rangle \subseteq \boldsymbol{x}$. Therefore, $\boldsymbol{x}$ is a join of clopen sets of the form $\langle U \rangle$. □

Proposition 9.5. *Let $a \in [n]$ and let $U \subseteq [n]$. The elements of $\mathsf{S}(n, \langle a, U \rangle)$ are exactly the bipartitions $\boldsymbol{x}$ such that*

(i) $\operatorname{isol}(\boldsymbol{x}) \subseteq \{a\}$,
(ii) *if* $\operatorname{isol}(\boldsymbol{x}) = \{a\}$, *then $U^{\mathsf{c}} \times \{a\}$ and $\{a\} \times U$ are both contained in $\boldsymbol{x}$.*

Proof. Every join-irreducible element of $\operatorname{Bip}(n)$ which is either bipartite or equal to $\langle a, U \rangle$ satisfies both (i) and (ii) above, hence, by Lemma 9.3, so do all elements of $\mathsf{S}(n, \langle a, U \rangle)$.

Conversely, let $\boldsymbol{x} \in \operatorname{Bip}(n)$ satisfy both (i) and (ii) above. If $\operatorname{isol}(\boldsymbol{x}) = \varnothing$, then, by Proposition 9.4, $\boldsymbol{x} \in \mathsf{K}(n)$, hence, *a fortiori*, $\boldsymbol{x} \in \mathsf{S}(n, \langle a, U \rangle)$.

Now suppose that $\operatorname{isol}(\boldsymbol{x}) = \{a\}$. It follows from (ii) together with the transitivity of $\boldsymbol{x}$ that $\langle a, U \rangle \subseteq \boldsymbol{x}$. For each $(i,j) \in \boldsymbol{x} \setminus \{(a,a)\}$, it follows from the argument of the proof of Proposition 9.4 that there are $U, V \in \operatorname{Pow}^*[n]$ such that $(i,j) \in \langle U \rangle \vee \langle V \rangle \subseteq \boldsymbol{x}$. Hence $\boldsymbol{x} \in \mathsf{S}(n, \langle a, U \rangle)$. □

We are indebted to the first referee for the statement of the following result, which identifies the ordered bipartition representations (cf. Han [19], Hetyei and Krattenthaler [20, § 2]) of the elements of $\mathsf{K}(n)$ and $\mathsf{S}(n, \langle a, U \rangle)$, respectively. The proof of Lemma 9.6 is a straightforward exercise.

For a positive integer n, $a \in [n]$, and $U \subseteq [n]$, a bipartition $\boldsymbol{x} \in \operatorname{Bip}(n)$ is (a,U)-*aligned* if both sets $(U^{\mathsf{c}} \setminus \{a\}) \times \{a\}$ and $\{a\} \times (U \setminus \{a\})$ are contained in $\boldsymbol{x}$ (*observe that this does not imply that $(a,a) \in \boldsymbol{x}$ as a rule*).

Lemma 9.6. *The following statements hold:*

(i) *$\boldsymbol{x} \in \mathsf{K}(n)$ iff no underlined block of $\boldsymbol{x}$ is a singleton.*
(ii) *$\boldsymbol{x} \in \mathsf{S}(n, \langle a, U \rangle)$ iff either $\boldsymbol{x} \in \mathsf{K}(n)$ or $\{a\}$ is the only underlined singleton block of $\boldsymbol{x}$ and $\boldsymbol{x}$ is (a,U)-aligned.*

Proposition 9.5 makes it possible to identify the factors of the minimal subdirect product decomposition (9.1) of the bipartition lattice $\mathrm{Bip}(n)$. Observe that a similar result is established, in Santocanale and Wehrung [31], for the *permutohedron* $\mathsf{P}(n)$. In that paper, it is proved, in particular, that the corresponding subdirect factors are exactly the *Cambrian lattices of type A* (cf. Reading [27]), denoted there by $\mathsf{A}_U(n)$, for $U \subseteq [n]$.

Hence the lattices $\mathsf{S}(n, \langle a, U\rangle)$ can be viewed as the analogues, for bipartition lattices, of the Cambrian lattices of type A (i.e., the $\mathsf{A}_U(n)$). For either $U = \varnothing$ or $U = [n]$ (the corresponding lattices are isomorphic, *via* $\boldsymbol{x} \mapsto \boldsymbol{x}^{\mathrm{op}}$), we get the bipartition analogue of the *Tamari lattice* $\mathsf{A}(n) = \mathsf{A}_{[n]}(n)$ (cf. Santocanale and Wehrung [31]), namely $\mathsf{S}(n, \langle a, \varnothing\rangle)$ (whose isomorphism class does not depend on a).

Proposition 9.7. *Every lattice $\mathsf{S}(n, \langle a, U\rangle)$ is both isomorphic and dually isomorphic to $\mathsf{S}(n, \langle a, U^{\mathsf{c}}\rangle)$. In particular, $\mathsf{S}(n, \langle a, U\rangle)$ is self-dual.*

We shall present two proofs of Proposition 9.7, the first one relying on the lattice structure of $\mathrm{Bip}(n)$, the other one on the ordered bipartition representations that we recalled at the beginning of Section 9.

First proof of Proposition 9.7. Writing $\boldsymbol{p} = \langle a, U\rangle$, the clopen set $\tilde{\boldsymbol{p}} = \langle a, U^{\mathsf{c}}\rangle$ depends only on $\boldsymbol{p}$. An isomorphism from $\mathsf{S}(n, \boldsymbol{p})$ onto $\mathsf{S}(n, \tilde{\boldsymbol{p}})$ is induced by the mapping sending a relation $\boldsymbol{x}$ to its opposite $\boldsymbol{x}^{\mathrm{op}}$.

We argue next that $\mathsf{S}(n, \boldsymbol{p})$ is dually isomorphic to $\mathsf{S}(n, \tilde{\boldsymbol{p}})$. To this goal, denote by $\mathsf{M}(\boldsymbol{p})$ the $(\wedge, 1)$-subsemilattice of $\mathrm{Bip}(n)$ generated by the set

$$\{\boldsymbol{u} \in \mathrm{Mi}\,\mathrm{Bip}(n) \mid (\boldsymbol{u}, \boldsymbol{u}^*) \notin \Psi(\boldsymbol{p})\}\,.$$

It follows from Lemma 2.2, applied to the dual lattice of $\mathrm{Bip}(n)$, that $\mathrm{Bip}(n)^{\mathrm{op}}/\Psi(\boldsymbol{p})$ is isomorphic to $\mathsf{M}(\boldsymbol{p})$ endowed with the dual ordering of L; hence,

$$\mathsf{M}(\boldsymbol{p}) \cong \mathsf{S}(n, \boldsymbol{p}) \cong \mathrm{Bip}(n)/\Psi(\boldsymbol{p})\,.$$

On the other hand, $\mathsf{M}(\boldsymbol{p})$ is dually isomorphic, *via* the operation $\boldsymbol{x} \mapsto \boldsymbol{x}^{\perp}$ of complementation on $\mathrm{Bip}(n)$, to the $(\vee, 0)$-subsemilattice $\mathsf{S}'(\boldsymbol{p})$ of $\mathrm{Bip}(n)$ generated by the subset

$$\mathsf{G}'(\boldsymbol{p}) = \{\boldsymbol{r} \in \mathrm{Ji}(\mathrm{Bip}(n)) \mid (\boldsymbol{r}^{\perp}, (\boldsymbol{r}_*)^{\perp}) \notin \Psi(\boldsymbol{p})\}\,.$$

Now for each $\boldsymbol{r} \in \mathrm{Ji}(\mathrm{Bip}(n))$,

$$\begin{aligned}(\boldsymbol{r}^{\perp}, (\boldsymbol{r}_*)^{\perp}) \notin \Psi(\boldsymbol{p}) &\Leftrightarrow (\mathsf{G}(n) \cup \{\boldsymbol{p}\}) \downarrow \boldsymbol{r}^{\perp} \neq (\mathsf{G}(n) \cup \{\boldsymbol{p}\}) \downarrow (\boldsymbol{r}_*)^{\perp} \qquad \text{(by Lemma 2.2)}\\ &\Leftrightarrow (\exists \boldsymbol{q} \in \mathsf{G}(n) \cup \{\boldsymbol{p}\})(\boldsymbol{q} \leq (\boldsymbol{r}_*)^{\perp} \text{ and } \boldsymbol{q} \nleq \boldsymbol{r}^{\perp})\\ &\Leftrightarrow (\exists \boldsymbol{q} \in \mathsf{G}(n) \cup \{\boldsymbol{p}\})(\boldsymbol{q} \nearrow \boldsymbol{r}^{\perp})\,.\end{aligned}$$

If $\boldsymbol{r}$ is bipartite, then, by Lemma 6.6, there is always $\boldsymbol{q} \in \mathsf{G}(n)$ such that $\boldsymbol{q} \nearrow \boldsymbol{r}^{\perp}$ (for example $\boldsymbol{q} = \boldsymbol{r}$). Now suppose that $\boldsymbol{r} = \langle b, W\rangle$ is a clepsydra. By Lemma 6.3, $\boldsymbol{q} \nearrow \boldsymbol{r}^{\perp}$ can occur only in case $\boldsymbol{q} = \langle b, W^{\mathsf{c}}\rangle$; furthermore, this element belongs to $\mathsf{G}(n) \cup \{\boldsymbol{p}\}$ iff $a = b$ and $U \setminus \{a\} = W^{\mathsf{c}} \setminus \{a\}$ (cf. Lemma 5.2). Therefore,

$$\mathsf{G}'(\boldsymbol{p}) = \mathsf{G}(n) \cup \{\tilde{\boldsymbol{p}}\}\,,$$

and therefore $\mathsf{S}'(\boldsymbol{p}) = \mathsf{S}(n, \tilde{\boldsymbol{p}})$ is dually isomorphic to $\mathsf{S}(n, \boldsymbol{p})$.

Second proof of Proposition 9.7. We are indebted to the first referee of our paper for pointing to us the dual automorphism τ defined below. Observe that this proof yields the immediate observation that τ is *involutive*.

First, writing bipartition representations in reverse order, and keeping the underlinings, defines the automorphism

$$\kappa\colon \operatorname{Bip}(n) \to \operatorname{Bip}(n)\,, \quad \boldsymbol{x} \mapsto \boldsymbol{x}^{\mathrm{op}}\,. \tag{9.2}$$

It follows immediately from Lemma 9.6 that this automorphism maps $\mathsf{S}(n, \langle a, U\rangle)$ onto $\mathsf{S}(n, \langle a, U^{\mathrm{c}}\rangle)$.

We shall now define an involutive dual automorphism τ of $\mathsf{S}(n, \langle a, U\rangle)$. For every $\boldsymbol{x} \in \mathsf{S}(n, \langle a, U\rangle)$ with ordered bipartition representation $(X_1^{\varepsilon_1}, \dots, X_p^{\varepsilon_p})$, we denote by $\tau(\boldsymbol{x}) = \tau_{a,U}(\boldsymbol{x})$ the ordered bipartition with ordered bipartition representation $(X_1^{\varepsilon'_1}, \dots, X_p^{\varepsilon'_p})$, where for every $i \in [p]$, we set

(i) $\varepsilon'_i = -\varepsilon_i$ if either $\operatorname{card} X_i \geq 2$ or $X_i = \{a\}$ and $\boldsymbol{x}$ is (a, U)-aligned.
(ii) $\varepsilon'_i = \varepsilon_i$ if $X_i = \{b\}$ where either $b \neq a$ or $\boldsymbol{x}$ is not (a, U)-aligned. (Necessarily, $\varepsilon_i = -1$; see Lemma 9.6.)

It is obvious that τ is an involutive bijection of $\mathsf{S}(n, \langle a, U\rangle)$ onto itself. Hence we only need to prove that τ is order-reversing. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathsf{S}(n, \langle a, U\rangle)$, with respective ordered bipartition representations $(X_1^{\xi_1}, \dots, X_p^{\xi_p})$ and $(Y_1^{\eta_1}, \dots, Y_q^{\eta_q})$, such that $\boldsymbol{x} \subseteq \boldsymbol{y}$. We shall prove that $\tau(\boldsymbol{y}) \subseteq \tau(\boldsymbol{x})$. By Hetyei and Krattenthaler [20, Proposition 6.4], there exists a permutation ρ of $[n]$ which is compatible with both $\boldsymbol{x}$ and $\boldsymbol{y}$, thus also with both $\tau(\boldsymbol{x})$ and $\tau(\boldsymbol{y})$; and furthermore,

$$\text{Every underlined } X_i \text{ is contained in some underlined } Y_j\,, \tag{9.3}$$

$$\text{Every non-underlined } Y_j \text{ is contained in some non-underlined } X_i \tag{9.4}$$

(recall that the *underlined blocks* of $\boldsymbol{x}$ are the X_i such that $\xi_i = 1$). We must verify (ii) and (iii) of [20, Proposition 6.4] for the pair $(\tau(\boldsymbol{y}), \tau(\boldsymbol{x}))$.

First statement. Let $j \in [q]$ such that $\eta'_j = 1$, we prove that there exists $i \in [p]$ such that $\xi'_i = 1$ and $Y_j \subseteq X_i$. If Y_j is not a singleton, then $\eta_j = -1$, thus, by (9.4), there exists $i \in [p]$ such that $\xi_i = -1$ and $Y_j \subseteq X_i$. In particular, X_i is not a singleton, thus $\xi'_i = 1$ and we are done. Now suppose that $Y_j = \{b\}$. From $\eta'_j = 1$ it follows that $b = a$, $\eta_j = -1$, and $\boldsymbol{y}$ is (a, U)-aligned. By (9.4), there exists $i \in [p]$ such that $a \in X_i$ and $\xi_i = -1$. If X_i is not a singleton, then $\xi'_i = 1$ and we are done. The remaining case is where $X_i = \{a\}$. Since ρ is compatible with both $\boldsymbol{x}$ and $\boldsymbol{y}$, $X_i = Y_j = \{a\}$, and $\boldsymbol{y}$ is (a, U)-aligned, $\boldsymbol{x}$ is also (a, U)-aligned, thus $\xi'_i = -\xi_i = 1$ and we are done.

Second statement. Let $i \in [p]$ such that $\xi'_i = -1$, we prove that there exists $j \in [q]$ such that $\eta'_j = -1$ and $X_i \subseteq Y_j$. Suppose first that X_i is not a singleton. Hence $\xi_i = 1$, thus, by (9.3), there exists $j \in [q]$ such that $X_i \subseteq Y_j$ and $\eta_j = 1$. Since Y_j cannot be a singleton, $\eta'_j = -1$ and we are done. Suppose from now on that $X_i = \{b\}$. There exists a unique $j \in [q]$ such that $b \in Y_j$, and we must prove that $\eta'_j = -1$. Suppose, to the contrary, that $\eta'_j = 1$. Necessarily, $\eta_j = -1$. By (9.4), there exists $\overline{\imath} \in [p]$ such that $\xi_{\overline{\imath}} = -1$ and $Y_j \subseteq X_{\overline{\imath}}$. From $\{b\} = X_i \subseteq Y_j \subseteq X_{\overline{\imath}}$ it follows that $\overline{\imath} = i$ and $X_i = Y_j = \{b\}$. Since $\eta'_j = 1$, it follows that $b = a$ and $\boldsymbol{y}$ is (a, U)-aligned. Since $X_i = Y_j = \{a\}$ and ρ is compatible with both $\boldsymbol{x}$ and $\boldsymbol{y}$, it follows that $\boldsymbol{x}$ is also (a, U)-aligned, thus $-1 = \xi_{\overline{\imath}} = \xi_i = -\xi'_i = 1$, a contradiction. □

Remark 9.8. Set $\mathsf{S}(n, k) = \mathsf{S}(n, \langle 1, \{2, 3, \dots, k+1\}\rangle)$, for all integers n and k with $0 \leq k < n$. Observe that $\mathsf{S}(n, \langle a, U\rangle) \cong \mathsf{S}(n, \operatorname{card} U)$, for each $a \in [n]$ and each

$U \subseteq [n] \setminus \{a\}$. Furthermore, it follows from roposition 9.7 that $\mathsf{S}(n, \langle a, U\rangle) \cong \mathsf{S}(n, \langle a, [n] \setminus (U \cup \{a\})\rangle)$, hence $\mathsf{S}(n, k) \cong \mathsf{S}(n, n-1-k)$, and hence the factors of the minimal subdirect product decomposition (9.1) of $\mathrm{Bip}(n)$ are exactly the lattices $\mathsf{S}(n, k)$ where $n > 0$ and $0 \leq 2k < n$.

The lattices $\mathsf{S}(3, 0)$ and $\mathsf{S}(3, 1)$ are represented on the left hand side and the right hand side of Figure 9.1, respectively.

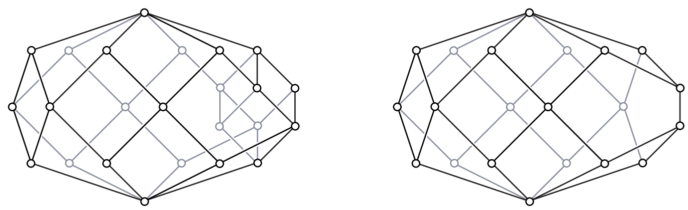

FIGURE 9.1. The bipartition-Cambrian lattices $\mathsf{S}(3, 0)$ and $\mathsf{S}(3, 1)$

10. The lattice endomorphisms of $\mathrm{Bip}(n)$

For a Coxeter element c in a finite Coxeter group, Reading's projection map $\pi_{\downarrow}^{c}$ (cf. Reading [29, page 419]) exhibits the Cambrian lattice associated to c as a lattice retract of the corresponding Coxeter lattice (weak order), see Reading [29, Theorem 1.2]. In particular, the lattices $\mathsf{A}(n)$ and $\mathsf{A}_U(n)$ are lattice-theoretical retracts of $\mathsf{P}(n)$ (this originates in Björner and Wachs [5] and is stated formally in Santocanale and Wehrung [31]).

In this section, we shall show that the behavior of the "bipartition-Cambrian lattices" $\mathsf{S}(n, k)$ (cf. Remark 9.8) is quite opposite to the one of Cambrian lattices of type A. In particular, for $n \geq 3$, $\mathsf{S}(n, k)$ is never a lattice retract of $\mathrm{Bip}(n)$ (Corollary 10.16). This result will be achieved by proving that $\mathrm{Bip}(n)$ has no other lattice endomorphism than the "obvious" ones (Theorem 10.1).

Throughout this section we shall fix an integer $n \geq 3$. Further, we shall denote by $\mathbf{1}$ the unit element of $\mathrm{Bip}(n)$, that is, $\mathbf{1} = [n] \times [n]$.

Denote by $\mathfrak{S}_n$ the permutation group of $[n]$. For every $\sigma \in \mathfrak{S}_n$, the assignment

$$f_\sigma \colon \boldsymbol{a} \mapsto \{(\sigma(x), \sigma(y)) \mid (x, y) \in \boldsymbol{a}\} \tag{10.1}$$

defines an automorphism of the lattice $\mathrm{Bip}(n)$, which commutes with the transposition automorphism κ defined in (9.2).

Define the *standard automorphisms* of $\mathrm{Bip}(n)$ as those of the form either f_σ or $\kappa \circ f_\sigma$. The standard automorphisms of $\mathrm{Bip}(n)$ form a subgroup of the automorphism group of $\mathrm{Bip}(n)$, isomorphic to $\mathfrak{S}_n \times \mathfrak{S}_2$. It will follow from Theorem 10.1 that every automorphism of $\mathrm{Bip}(n)$, for $n \geq 3$, is standard. Theorem 10.1 does not extend to $n = 2$: there are 570 nonconstant lattice endomorphisms of $\mathrm{Bip}(2)$, 514 lattice endomorphisms preserving the bounds, while the group of automorphisms of $\mathrm{Bip}(2)$ is the 8-element dihedral group (cf. Figure 7.2).

Theorem 10.1. *Let n be an integer, $n \geq 3$. Then every nonconstant lattice endomorphism of* $\mathrm{Bip}(n)$ *is a standard automorphism.*

The proof of Theorem 10.1 will be organized as follows. We will first prove, in Lemma 10.3, that the zero element in $\mathrm{Bip}(n)$ has more upper covers than any other element of $\mathrm{Bip}(n)$. This, together with the observation that the atoms of $\mathrm{Bip}(n)$ all generate the same congruence (Corollary 8.7) and join to $\mathbf{1}$ (Lemma 10.4) will show (Lemma 10.10) that any nonconstant lattice endomorphism f of $\mathrm{Bip}(n)$ must preserve the zero, and, dually, the unit. This will yield, for every $U \in \mathrm{Pow}^*[n]$, a unique atom $\langle g(U)\rangle$ below $f(\langle U\rangle)$ (Lemma 10.12), with g a permutation of $\mathrm{Pow}^*[n]$ (Lemma 10.11) preserving complementary pairs (Lemma 10.13). A preliminary study of join-covering relations among join-irreducible elements of $\mathrm{Bip}(n)$ (Lemma 10.8), will enable us to prove that g preserves unordered extremal partitions (Lemma 10.15), which will yield a standard automorphism $\overline{g} \leq f$ (i.e., $\overline{g}(\boldsymbol{x}) \subseteq f(\boldsymbol{x})$ for every $\boldsymbol{x} \in \mathrm{Bip}(n)$). By applying that result to the dual endomorphism $\boldsymbol{x} \mapsto f(\boldsymbol{x}^{\mathsf{c}})^{\mathsf{c}}$, we will obtain a standard automorphism $\overline{g}'$ with $f \leq \overline{g}'$, and then our conclusion will be a consequence of the following well-known result.

Lemma 10.2 (folklore)**.** *Let f and g be automorphisms of a finite poset P, and let $x \in P$. If $f(x) \leq g(x)$, then $f(x) = g(x)$. In particular, distinct automorphisms are incomparable with respect to the pointwise ordering.*

Proof. If $f(x) < g(x)$, then $x < f^{-1}(g(x))$, hence, setting $h = f^{-1}g$, we get an infinite ascending sequence $x < h(x) < h^2(x) < \cdots < h^n(x) < \cdots$ in P, a contradiction. □

Lemma 10.3. *If $n \geq 3$, then the degree (i.e., the number of upper covers plus the number of lower covers) of every $\boldsymbol{x} \in \mathrm{Bip}(n)$ is at most $2^n - 2$. In particular, if $\boldsymbol{x} \neq \varnothing$, then $\boldsymbol{x}$ has less than $2^n - 2$ upper covers.*

Note that Lemma 10.3 does not not extend to the case $n = 2$, as it can easily be seen from the Hasse diagram of $\mathrm{Bip}(2)$ in Figure 7.2.

Proof of Lemma 10.3. Let $\boldsymbol{x} = X_1^{\varepsilon_1} \oplus \cdots \oplus X_m^{\varepsilon_m}$ (cf. Section 9) and set $n_i = \operatorname{card} X_i$, for each $i \in [m]$. If $\boldsymbol{x} = \varnothing$, then the upper covers of $\boldsymbol{x}$ are the $\langle U\rangle$, for $U \in \mathrm{Pow}^*[n]$ (cf. Lemma 8.2) and we are done. The case where $\boldsymbol{x} = \mathbf{1}$ follows from applying the dual automorphism $\boldsymbol{x} \mapsto \boldsymbol{x}^{\mathsf{c}}$.

Suppose from now on that $\boldsymbol{x} \neq \varnothing$ and $\boldsymbol{x} \neq \mathbf{1}$; that is, $m \geq 2$. By Hetyei and Krattenthaler [20, Theorem 5.1], the upper and lower covers of $\boldsymbol{x}$ can be evaluated according to the following rules:

1. Merge two adjacent blocks $X_i^{\varepsilon_i}, X_{i+1}^{\varepsilon_{i+1}}$ with $\varepsilon_i = \varepsilon_{i+1}$ into a new block $(X_i \cup X_{i+1})^{\varepsilon_i}$.
2. Split a block $X_i^{\varepsilon_i}$ into two new adjacent blocks $Y^{\varepsilon_i}, Z^{\varepsilon_i}$.
3. Exchange the polarity ε_i of a singleton block $X_i^{\varepsilon_i}$.

Let N_i, for $i \in \{1, 2, 3\}$, be the number of upper and lower covers of $\boldsymbol{x}$ obtained according to rules (1), (2), and (3), respectively. Let N be the degree of $\boldsymbol{x}$. It is immediate that $N_1 \leq m - 1$ and

$$N_2 + N_3 \leq \sum_{n_i \geq 2} (2^{n_i} - 2) + \operatorname{card}\{i \mid n_i = 1\} \leq \sum_{i=1}^{m} (2^{n_i} - 1) = \Bigl(\sum_{i=1}^{m} 2^{n_i}\Bigr) - m\,,$$

whence

$$N = N_1 + N_2 + N_3 \leq \Bigl(\sum_{i=1}^{m} 2^{n_i}\Bigr) - 1\,. \tag{10.2}$$

We claim that $\sum_{i=1}^{m} 2^{n_i} \leq 2^n - 1$. Pick $b_i \in X_i$, for $i \in [m]$. Since $m \geq 2$ and $n \geq 3$, the sets $\mathcal{X} = \bigcup_{i=1}^{m} \bigl(\mathrm{Pow}(X_i) \setminus \{\varnothing\}\bigr)$ and $\mathcal{B} = \bigl\{\{b_i, b_j\} \mid i \neq j\bigr\}$ are both contained in $\operatorname{Pow}^*[n]$. Since $m \geq 2$ and $\mathcal{B} \cap \mathcal{X} = \varnothing$, it follows that

$$\sum_{i=1}^{m} 2^{n_i} - 1 = \sum_{i=1}^{m} \Big(2^{n_i} - 1\Big) + m - 1 \leq \sum_{i=1}^{m} \Big(2^{n_i} - 1\Big) + \binom{m}{2} = \operatorname{card}(\mathcal{X} \cup \mathcal{B}) \leq 2^n - 2\,,$$

which proves our claim. The desired conclusion follows from (10.2). □

Lemma 10.4. *The following statements hold.*

(i) $J \times I \in \operatorname{Bip}(n)$ *iff* $[n] = I \cup J$, *for all* $I, J \subseteq [n]$.

(ii) $\langle U \rangle \vee \langle V \rangle = \mathbf{1}$ *iff* $V = U^{\mathsf{c}}$, *for all* $U, V \in \operatorname{Pow}^*[n]$.

Proof. (i) is obvious.

(ii) It follows from (i) that $\boldsymbol{a} = (U^{\mathsf{c}} \cup V^{\mathsf{c}}) \times (U \cup V)$ belongs to $\operatorname{Bip}(n)$. Since $\langle U \rangle \vee \langle V \rangle$ is contained in $\boldsymbol{a}$, it follows that $\langle U \rangle \vee \langle V \rangle = \mathbf{1}$ implies that $\boldsymbol{a} = \mathbf{1}$, that is, $V = U^{\mathsf{c}}$.

Assume, conversely, that $V = U^{\mathsf{c}}$, and pick $(u, v) \in U \times U^{\mathsf{c}}$. Let $x, y \in [n]$, we prove that $(x, y) \in \langle U \rangle \vee \langle U^{\mathsf{c}} \rangle$. Since this holds whenever $(x, y) \in \langle U \rangle \cup \langle U^{\mathsf{c}} \rangle$, it remains to check the case where $\{x, y\} \subseteq U$ and the case where $\{x, y\} \subseteq U^{\mathsf{c}}$. In the first case, the desired conclusion follows from $(x, v) \in \langle U^{\mathsf{c}} \rangle$ and $(v, y) \in \langle U \rangle$. In the second case, it follows from $(x, u) \in \langle U \rangle$ and $(u, y) \in \langle U^{\mathsf{c}} \rangle$. □

From now on we shall denote by $\boldsymbol{P}$ the set of all atoms of $\operatorname{Bip}(n)$ (that is, the set of all $\langle U \rangle$, for $U \in \operatorname{Pow}^*[n]$) and by $\boldsymbol{Q}$ the set of all nonzero elements of $\operatorname{Bip}(n)$ containing exactly one atom. The clepsydras $\tilde{\boldsymbol{a}}_x$ and $\tilde{\boldsymbol{b}}_x$ defined by

$$\tilde{\boldsymbol{a}}_x = [n] \times \{x\} = \{x\}^{\mathsf{c}} \oplus \underline{\{x\}} = \langle x, \varnothing \rangle \text{ and } \tilde{\boldsymbol{b}}_x = \{x\} \times [n] = \underline{\{x\}} \oplus \{x\}^{\mathsf{c}} = \langle x, [n] \rangle\,,$$

for $x \in [n]$, all belong to $\boldsymbol{Q}$, with respective lower covers

$$\boldsymbol{a}_x = \{x\}^{\mathsf{c}} \times \{x\} = \{x\}^{\mathsf{c}} \oplus \{x\} = \langle \{x\} \rangle \quad \text{and} \quad \boldsymbol{b}_x = \{x\} \times \{x\}^{\mathsf{c}} = \{x\} \oplus \{x\}^{\mathsf{c}} = \langle \{x\}^{\mathsf{c}} \rangle\,.$$

Lemma 10.5. $\boldsymbol{Q} = \boldsymbol{P} \cup \{\tilde{\boldsymbol{a}}_x \mid x \in [n]\} \cup \{\tilde{\boldsymbol{b}}_x \mid x \in [n]\}$.

Proof. Let $\boldsymbol{p} = \langle x, U \rangle$, with $U \subseteq \{x\}^{\mathsf{c}}$, be a clepsydra. The distinct bipartitions $\boldsymbol{u} = U^{\mathsf{c}} \times U$ and $\boldsymbol{v} = (U^{\mathsf{c}} \setminus \{x\}) \times (U \cup \{x\})$ are properly contained in $\boldsymbol{p}$, thus, if $\boldsymbol{p} \in \boldsymbol{Q}$, then either $\boldsymbol{u}$ or $\boldsymbol{v}$ is not an atom, that is, either $\boldsymbol{u} = \varnothing$ or $\boldsymbol{v} = \varnothing$, which easily implies that either $\boldsymbol{p} = \tilde{\boldsymbol{a}}_x$ or $\boldsymbol{p} = \tilde{\boldsymbol{b}}_x$.

Now let $\boldsymbol{c} \in \boldsymbol{Q} \setminus \boldsymbol{P}$. By Lemmas 8.1 and 8.2, $\boldsymbol{c}$ is the join of the atoms and of the clepsydras below it. Since $\boldsymbol{c} \in \boldsymbol{Q} \setminus \boldsymbol{P}$, there is at least a clepsydra below $\boldsymbol{c}$. Every such clepsydra belongs to $\boldsymbol{Q}$, thus it has the form either $\tilde{\boldsymbol{a}}_x$ or $\tilde{\boldsymbol{b}}_x$. Since the lower covers of those clepsydras, that is, all the $\boldsymbol{a}_x$ and $\boldsymbol{b}_x$, are pairwise distinct, it follows that there is exactly one clepsydra below $\boldsymbol{c}$, which, by symmetry, may be assumed to be $\tilde{\boldsymbol{a}}_x$. Since $\boldsymbol{c} \in \boldsymbol{Q}$, the only atom below $\boldsymbol{c}$ is $\boldsymbol{a}_x$. Therefore, $\boldsymbol{c} = \tilde{\boldsymbol{a}}_x \vee \boldsymbol{a}_x = \tilde{\boldsymbol{a}}_x$. □

Lemma 10.6. *Let* $I \subseteq [n]$. *Then* $\bigvee_{i \in I} \boldsymbol{a}_i \subseteq \bigvee_{i \in I} \tilde{\boldsymbol{a}}_i = [n] \times I$ *and* $\bigvee_{i \in I} \boldsymbol{b}_i \subseteq \bigvee_{i \in I} \tilde{\boldsymbol{b}}_i = I \times [n]$. *Furthermore, if* $\operatorname{card} I \geq 2$, *then the equality holds everywhere.*

Proof. A simple application of the dual automorphism $\boldsymbol{x} \mapsto \boldsymbol{x}^{\mathrm{op}}$ reduces the statement about the $\boldsymbol{b}_i$ to the one about the $\boldsymbol{a}_i$. Then the only nontrivial statement to establish is that if $\operatorname{card} I \geq 2$, then $[n] \times I$ is contained in $\boldsymbol{a} = \bigvee_{i \in I} \boldsymbol{a}_i$. We must prove that any $(x, i) \in [n] \times I$ belongs to $\boldsymbol{a}$. If $x \neq i$, then $(x, i) \in \boldsymbol{a}_i \subseteq \boldsymbol{a}$. Suppose now that $x = i$. Since $\operatorname{card} I \geq 2$, there exists $j \in I \setminus \{i\}$. From $(i, j) \in \boldsymbol{a}_j$ and $(j, i) \in \boldsymbol{a}_i$ it follows again that $(i, i) \in \boldsymbol{a}$. □

Lemma 10.7. *The set* $\{\boldsymbol{a}_i \mid i \in [n]\} \cup \{\boldsymbol{b}_i \mid i \in [n]\}$ *generates* $\mathrm{Bip}(n)$ *as a sublattice.*

Proof. It follows immediately from Lemma 10.6 that $\tilde{\boldsymbol{a}}_i = (\boldsymbol{a}_i \vee \boldsymbol{a}_j) \cap (\boldsymbol{a}_i \vee \boldsymbol{a}_k)$, thus, *a fortiori*, $\tilde{\boldsymbol{a}}_i = (\boldsymbol{a}_i \vee \boldsymbol{a}_j) \wedge (\boldsymbol{a}_i \vee \boldsymbol{a}_k)$ whenever i, j, k are distinct. A similar formula holds for $\tilde{\boldsymbol{b}}_i$. Finally, if $\operatorname{card} I, \operatorname{card} J \geq 2$ we get $J \times I = \bigvee_{i \in I} \boldsymbol{a}_i \cap \bigvee_{j \in J} \boldsymbol{b}_j$; if $I \cup J = [n]$, then $J \times I \in \mathrm{Bip}(n)$, thus, *a fortiori*, $J \times I = \bigvee_{i \in I} \boldsymbol{a}_i \wedge \bigvee_{j \in J} \boldsymbol{b}_j$.

Since all join-irreducible elements of $\mathrm{Bip}(n)$ have one of the forms $\boldsymbol{a}_i$, $\boldsymbol{b}_i$, $\tilde{\boldsymbol{a}}_i$, $\tilde{\boldsymbol{b}}_i$, or $J \times I$ above (with $\operatorname{card} I, \operatorname{card} J \geq 2$), and since the join-irreducible elements generate $\mathrm{Bip}(n)$ as a sublattice, the conclusion follows. $\square$

Lemma 10.8. *Let* $k \in [n]$ *and let* I *and* J *be disjoint subsets of* $[n]$. *Then* $\boldsymbol{a}_k \subseteq \bigvee_{i \in I} \boldsymbol{a}_i \vee \bigvee_{j \in J} \boldsymbol{b}_j$ *iff* $\boldsymbol{a}_k \subseteq \bigvee_{i \in I} \tilde{\boldsymbol{a}}_i \vee \bigvee_{j \in J} \tilde{\boldsymbol{b}}_j$, *iff either* $k \in I$ *or* $\{k\}^{\mathsf{c}} \subseteq J$.

Proof. If $\{k\}^{\mathsf{c}} \subseteq J$, then, since $n \geq 3$ and by Lemma 10.6,

$$\boldsymbol{a}_k \subseteq \{k\}^{\mathsf{c}} \times [n] \subseteq \bigvee_{j \in J} \boldsymbol{b}_j \subseteq \bigvee_{i \in I} \boldsymbol{a}_i \vee \bigvee_{j \in J} \boldsymbol{b}_j\,.$$

Further, it is trivial that $k \in I$ implies $\boldsymbol{a}_k \subseteq \bigvee_{i \in I} \boldsymbol{a}_i \vee \bigvee_{j \in J} \boldsymbol{b}_j$.

Conversely, suppose that $\boldsymbol{a}_k \subseteq \bigvee_{i \in I} \tilde{\boldsymbol{a}}_i \vee \bigvee_{j \in J} \tilde{\boldsymbol{b}}_j$. By Lemma 10.6, it follows that $\{k\}^{\mathsf{c}} \times \{k\} = \boldsymbol{a}_k$ is contained in $([n] \times I) \vee (J \times [n]) = ([n] \times I) \cup (J \times [n])$ (*we use here the disjointness of* I *and* J). It follows easily that either $k \in I$ or $\{k\}^{\mathsf{c}} \subseteq J$. $\square$

Remark 10.9. Lemma 10.8 shows that $\{\boldsymbol{b}_j \mid j \neq i\}$ is the unique nontrivial minimal join-cover of $\boldsymbol{a}_i$ made of extremal atoms (cf. Definition 8.4). However, by Lemma 8.5, $\boldsymbol{a}_i$ has many other minimal join-covers.

From now on we shall fix a nonconstant lattice endomorphism f of $\mathrm{Bip}(n)$. We shall prove that f is a standard automorphism. The main part of the argument consists of proving that there exists a standard automorphism $\overline{g}$ such that $\overline{g} \leq f$.

Lemma 10.10. $f^{-1}\{\varnothing\} = \varnothing$ *and* $f^{-1}\{\mathbf{1}\} = \{\mathbf{1}\}$.

Proof. We claim that $f(\boldsymbol{p}) \neq f(\varnothing)$, for every atom $\boldsymbol{p}$ of $\mathrm{Bip}(n)$. Suppose, to the contrary, that $f(\boldsymbol{p}) = f(\varnothing)$. This means that $\mathrm{con}(\boldsymbol{p})$ is contained in the kernel of f, hence, by Corollary 8.7, $\mathrm{con}(\boldsymbol{q})$ is also contained in the kernel of f, that is, $f(\boldsymbol{q}) = f(\varnothing)$, for every atom $\boldsymbol{q}$. By Lemma 10.4(ii), it follows that $f(\mathbf{1}) = f(\varnothing)$, a contradiction since f is nonconstant.

Now suppose that $f(\varnothing) \neq \varnothing$. It follows from Lemma 10.3 that $f(\varnothing)$ has less than $2^n - 2$ upper covers in $\mathrm{Bip}(n)$. For every atom $\boldsymbol{p}$ of $\mathrm{Bip}(n)$, it follows from our claim that there exists an upper cover $\boldsymbol{a}_{\boldsymbol{p}}$ of $f(\varnothing)$ such that $\boldsymbol{a}_{\boldsymbol{p}} \subseteq f(\boldsymbol{p})$. Since $\mathrm{Bip}(n)$ has $2^n - 2$ atoms, there are distinct atoms $\boldsymbol{p}$ and $\boldsymbol{q}$ of $\mathrm{Bip}(n)$ such that $\boldsymbol{a}_{\boldsymbol{p}} = \boldsymbol{a}_{\boldsymbol{q}}$; denote by $\boldsymbol{a}$ the common value. It follows that $\boldsymbol{a} \subseteq f(\boldsymbol{p}) \wedge f(\boldsymbol{q}) = f(\boldsymbol{p} \wedge \boldsymbol{q}) = f(\varnothing)$, a contradiction. This proves that $f(\varnothing) = \varnothing$. If $f(\boldsymbol{a}) = \varnothing$ for some $\boldsymbol{a} \neq \varnothing$, then $f(\boldsymbol{p}) = \varnothing$ for some atom $\boldsymbol{p}$, in contradiction with the claim. Hence $f^{-1}\{\varnothing\} = \{\varnothing\}$.

By applying the above result to the endomorphism $\boldsymbol{x} \mapsto f(\boldsymbol{x}^{\mathsf{c}})^{\mathsf{c}}$, we obtain that $f^{-1}\{\mathbf{1}\} = \{\mathbf{1}\}$. $\square$

It follows from Lemma 10.10 that for every $U \in \mathrm{Pow}^*[n]$, there exists $g(U) \in \mathrm{Pow}^*[n]$ such that $\langle g(U) \rangle \subseteq f(\langle U \rangle)$.

Lemma 10.11. *The map* g *is a permutation of* $\mathrm{Pow}^*[n]$.

Proof. It suffices to prove that g is one-to-one. Suppose that $g(U_0) = g(U_1) = V$, for $V \in \mathrm{Pow}^*[n]$ and distinct $U_0, U_1 \in \mathrm{Pow}^*[n]$. By using Lemma 10.10, we get

$$\langle V\rangle \subseteq f(\langle U_0\rangle) \wedge f(\langle U_1\rangle) = f(\langle U_0\rangle \wedge \langle U_1\rangle) = f(\varnothing) = \varnothing\,,$$

a contradiction. □

Our next lemma shows that the elements $f(\boldsymbol{p})$, for an atom $\boldsymbol{p}$, all belong to $\boldsymbol{Q}$.

Lemma 10.12. *Let $U \in \mathrm{Pow}^*[n]$. Then $g(U)$ is the unique $V \in \mathrm{Pow}^*[n]$ such that $\langle V\rangle \subseteq f(\langle U\rangle)$. In particular, $f(\langle U\rangle)$ belongs to $\boldsymbol{Q}$.*

Proof. Let $V \in \mathrm{Pow}^*[n]$ such that $\langle V\rangle \subseteq f(\langle U\rangle)$ and suppose that $V \neq g(U)$. By Lemma 10.11, there exists $U' \in \mathrm{Pow}^*[n]$ such that $V = g(U')$. Necessarily, $U' \neq U$, hence

$$\langle V\rangle \subseteq f(\langle U\rangle) \wedge f(\langle U'\rangle) = f(\langle U\rangle \wedge \langle U'\rangle) = f(\varnothing) = \varnothing\,,$$

a contradiction. □

Lemma 10.13. *$g(U^{\mathsf{c}}) = g(U)^{\mathsf{c}}$, for every $U \in \mathrm{Pow}^*[n]$.*

Proof. Set $V = g(U)$. By Lemma 10.11, $V^{\mathsf{c}} = g(U')$ for some $U' \in \mathrm{Pow}^*[n]$. From Lemma 10.4 it follows that $\mathbf{1} = \langle V\rangle \vee \langle V^{\mathsf{c}}\rangle$, thus $\mathbf{1} = f(\langle U\rangle) \vee f(\langle U'\rangle) = f(\langle U\rangle \vee \langle U'\rangle)$, and thus, by Lemma 10.10, $\mathbf{1} = \langle U\rangle \vee \langle U'\rangle$. By Lemma 10.4 again, it follows that $U' = U^{\mathsf{c}}$. □

Lemma 10.14. *Let m be a positive integer, let $\boldsymbol{x} \in \mathrm{Bip}(n)$, let $U \in \mathrm{Pow}^*[n]$, and let $U_1, \dots, U_m \in \mathrm{Pow}^*[n]$. Then $\langle U\rangle \subseteq \bigvee_{i=1}^{m}\langle U_i\rangle \vee \boldsymbol{x}$ implies that $\langle g(U)\rangle \subseteq \bigvee_{i=1}^{m}\langle g(U_i)\rangle \vee f(\boldsymbol{x})$.*

Proof. We may assume that $U \neq U_i$ for each $i \in [m]$. By Lemmas 10.5 and 10.12, for every i, the element $\boldsymbol{p}_i = f(\langle U_i\rangle)$ is either equal to $\langle g(U_i)\rangle$ or to a clepsydra with lower cover $\langle g(U_i)\rangle$. Since f is a join-homomorphism and by the definition of g, we get $\langle g(U)\rangle \subseteq f(\langle U\rangle) \subseteq \bigvee_{i=1}^{m} \boldsymbol{p}_i \vee f(\boldsymbol{x})$. Let I be a minimal subset of $[m]$ such that

$$\langle g(U)\rangle \subseteq \bigvee\nolimits_{i\in I} \boldsymbol{p}_i \vee \bigvee\nolimits_{i\in[m]\setminus I}\langle g(U_i)\rangle \vee f(\boldsymbol{x})\,,$$

and suppose that I is nonempty. Pick $j \in I$. By the minimality assumption on I, $\boldsymbol{p}_j$ is a clepsydra with lower cover $(\boldsymbol{p}_j)_* = \langle g(U_j)\rangle$ and we obtain

$$\langle g(U)\rangle \nsubseteq \bigvee\nolimits_{i\in I\setminus\{j\}} \boldsymbol{p}_i \vee \bigvee\nolimits_{i\in[m]\setminus I}\langle g(U_i)\rangle \vee (\boldsymbol{p}_j)_* \vee f(\boldsymbol{x})\,,$$

hence $\langle g(U)\rangle \mathrel{D} \boldsymbol{p}_j$, which is impossible since $\boldsymbol{p}_j$ is a clepsydra and by Lemma 6.4. Hence $I = \varnothing$, which proves the desired containment. □

Lemma 10.15. *Let $U \in \mathrm{Pow}^*[n]$. If either $g(U)$ or $g(U)^{\mathsf{c}}$ is a singleton, then either U or U^{c} is a singleton.*

Proof. Suppose otherwise. It follows from Lemma 8.5 that $\langle U\rangle \mathrel{D} \langle U\rangle^{\mathsf{c}}$, so there exists $\boldsymbol{x} \in \mathrm{Bip}(n)$ such that $\langle U\rangle \subseteq \langle U^{\mathsf{c}}\rangle \vee \boldsymbol{x}$ and $\langle U\rangle \nsubseteq \boldsymbol{x}$. A direct application of Lemma 10.14, for $m = 1$, yields that

$$\langle g(U)\rangle \leq \langle g(U^{\mathsf{c}})\rangle \vee f(\boldsymbol{x})\,. \tag{10.3}$$

Since $\langle U\rangle \nsubseteq \boldsymbol{x}$ and $\langle U\rangle$ is an atom, $f(\langle U\rangle) \wedge f(\boldsymbol{x}) = f(\langle U\rangle \wedge \boldsymbol{x}) = f(\varnothing) = \varnothing$, thus, *a fortiori*, $\langle g(U)\rangle \wedge f(\boldsymbol{x}) = \varnothing$, and thus $\langle g(U)\rangle \nsubseteq f(\boldsymbol{x})$. By (10.3) and since $\langle g(U^{\mathsf{c}})\rangle$ is an atom, it follows that $\langle g(U)\rangle \mathrel{D} \langle g(U^{\mathsf{c}})\rangle$.

Now by Lemma 10.13, $g(U^{\mathsf{c}}) = g(U)^{\mathsf{c}}$. By Lemma 8.5, it follows that neither $g(U)$ nor $g(U)^{\mathsf{c}}$ is a singleton, which contradicts our assumption. □

End of the proof of Theorem 10.1. Lemma 10.15 shows that the necessarily bijective correspondence sending $\{U, U^{\mathsf{c}}\}$ to $\{g(U), g(U)^{\mathsf{c}}\}$ preserves extremal (unordered) partitions, see Definition 8.4. As $n \geq 3$, an extremal unordered partition is uniquely identified by its singleton; thus, there exists a permutation $\gamma \in \mathfrak{S}_n$ such that $\{g(\{i\}), g(\{i\}^{\mathsf{c}})\} = \{\{\gamma(i)\}, \{\gamma(i)\}^{\mathsf{c}}\}$ for all $i \in [m]$; so $g(\{i\})$ is either equal to $\{\gamma(i)\}$ or to $\{\gamma(i)\}^{\mathsf{c}}$. Say that g is *positive at* i, in the first case, and *negative at* i, in the second case.

We want to prove that there exists a standard automorphism $\overline{g} \leq f$. By possibly precomposing f with κ, we reduce the problem to the case where g is positive at 1. From $\langle\{1\}\rangle \subseteq \bigvee_{j \neq 1} \langle\{j\}^{\mathsf{c}}\rangle$ together with Lemma 10.14 we get

$$\boldsymbol{a}_{\gamma(1)} = \langle\{\gamma(1)\}\rangle = \langle g(\{1\})\rangle \subseteq \bigvee_{j \neq 1} \langle g(\{j\}^{\mathsf{c}})\rangle \,,$$

with each $\langle g(\{j\}^{\mathsf{c}})\rangle \in \{\boldsymbol{a}_{\gamma(j)}, \boldsymbol{b}_{\gamma(j)}\}$. By Lemma 10.8, it follows that $\langle g(\{j\}^{\mathsf{c}})\rangle = \boldsymbol{b}_{\gamma(j)}$ for all $j \neq 1$, thus, by Lemma 10.13, $\langle g(\{j\})\rangle = \boldsymbol{a}_{\gamma(j)}$, and thus g is positive at every element of $[n]$. Hence, for every $i \in [n]$,

$$f_\gamma(\boldsymbol{a}_i) = \boldsymbol{a}_{\gamma(i)} = \langle g(\{i\})\rangle \subseteq f(\langle\{i\}\rangle) = f(\boldsymbol{a}_i) \,,$$

and, similarly (using Lemma 10.13), $f_\gamma(\boldsymbol{b}_i) \subseteq f(\boldsymbol{b}_i)$. Since $\{\boldsymbol{x} \mid f_\gamma(\boldsymbol{x}) \subseteq f(\boldsymbol{x})\}$ is a sublattice of $\mathrm{Bip}(n)$, it follows from Lemma 10.7 that $f_\gamma \leq f$.

By applying this result to the endomorphism $\boldsymbol{x} \mapsto f(\boldsymbol{x}^{\mathsf{c}})^{\mathsf{c}}$, it follows that there exists $\gamma' \in \mathfrak{S}_n$ such that $f \leq f_{\gamma'}$. Since $f_\gamma \leq f \leq f_{\gamma'}$ and by Lemma 10.2, it follows that $f = f_\gamma = f_{\gamma'}$. □

Corollary 10.16. *Let $n \geq 3$. Then no nontrivial lattice quotient of* $\mathrm{Bip}(n)$ *can be embedded into* $\mathrm{Bip}(n)$ *as a sublattice. In particular, neither* $\mathsf{K}(n)$ *nor* $\mathsf{S}(n,k)$*, where $0 \leq 2k < n$, can be embedded into* $\mathrm{Bip}(n)$ *as a sublattice.*

Proof. Let L be a lattice and let $f \colon \mathrm{Bip}(n) \twoheadrightarrow L$ be a surjective lattice homomorphism. If $g \colon L \hookrightarrow \mathrm{Bip}(n)$ is a lattice embedding, then $g \circ f$ is a lattice endomorphism of $\mathrm{Bip}(n)$, thus, by Theorem 10.1, either $g \circ f$ is constant (in which case f is constant) or $g \circ f$ is an isomorphism (in which case f is an isomorphism). □

Remark 10.17. For any sets E and F and any surjective map $f \colon F \twoheadrightarrow E$, we can define a lattice embedding $f^{[-1]} \colon \mathrm{Bip}(E) \hookrightarrow \mathrm{Bip}(F)$, preserving the bounds, by setting

$$f^{[-1]}(\boldsymbol{a}) = \{(x,y) \in F \times F \mid (f(x), f(y)) \in \boldsymbol{a}\} \,, \qquad \text{for all } \boldsymbol{a} \in \mathrm{Bip}(E) \,.$$

For an infinite set E, there is a surjective, non-injective map $f \colon E \twoheadrightarrow E$, and then $f^{[-1]}$ is not surjective (for if $x \neq y$ and $f(x) = f(y)$, then $\{x\}^{\mathsf{c}} \times \{x\}$ does not belong to the range of $f^{[-1]}$). This shows that *Theorem* 10.1 *does not extend to the infinite case, even for lattice embeddings preserving the bounds*.

On the other hand, an easy modification of the proof of Theorem 10.1, with joins and meets no longer necessarily finite, makes it possible to prove that *Every automorphism of the poset* $\mathrm{Bip}(E)$ *is standard* (with the obvious extension of the definition of "standard" to arbitrary sets), *for any* (*possibly infinite*) *set E with at least three elements*. The proof of this result is, actually, noticeably easier than the one of Theorem 10.1, as atoms are automatically sent to atoms, so the proof yields directly $f_\gamma = f$ (instead of only $f_\gamma \le f$).

11. Cardinalities of the bipartition-Cambrian lattices

It is known since Reading [27, Theorem 1.3] that all Cambrian lattices of type A, associated with the permutohedron of index n (i.e., the lattices $\mathsf{A}_U(n)$), have cardinality $\frac{1}{n+1}\binom{2n}{n}$, which is independent of U (i.e., of the orientation). This result actually extends to all finite Coxeter groups, as it can be seen by combining Reading [28, Theorem 9.1] with Reading [29, Theorem 1.1]: the former gives the enumeration of all sortable elements, while the latter says that the sortable elements are exactly the minimal elements of the Cambrian congruence classes. Hohlweg and Lange give in [14] a geometric interpretation of the undirected covering graphs of Cambrian lattices of type either A or B as oriented vertex-edge graphs of the corresponding associahedra; see Section 1 of that paper.

The following computations emphasize that the situation is quite different for the subdirectly irreducible factors $\mathsf{S}(n,k)$ of $\operatorname{Bip}(n)$, whose size depends on k.

We shall set $M(n) = \operatorname{card}\operatorname{Bip}(n)$, $K(n) = \operatorname{card}\mathsf{K}(n)$, and $S(n,k) = \operatorname{card}\mathsf{S}(n,k)$ (cf. Remark 9.8), for all possible values of n and k. It is established in Wagner [33] that the $M(n)$ are characterized by the induction formula

$$M(0) = 1\,;\quad M(n) = 2\cdot\sum_{k=1}^{n}\binom{n}{k}M(n-k)\quad \text{if } n>0\,. \tag{11.1}$$

The first entries of the sequence of numbers $M(n)$ are

$$M(1) = 2;\ M(2) = 10;\ M(3) = 74\,;\ M(4) = 730\,;\ M(5) = 9{,}002\,;\ M(6) = 133{,}210\,.$$

The sequence of numbers $M(n)$ is `A004123` of Sloane's Encyclopedia of Integer Sequences [25].

We are indebted to the first referee for suggesting us the following formulas (11.2) and (11.3) for calculating $K(n)$ and $S(n,k)$, for all possible values of n and k. We shall use the characterizations of the elements of $\mathsf{K}(n)$ and $\mathsf{S}(n,k)$ given by Lemma 9.6. Let $\boldsymbol{x} \in \operatorname{Bip}(n)$ with ordered bipartition representation $(X_1^{\varepsilon_1},\dots,X_p^{\varepsilon_p})$. We first count all the possibilities where $\boldsymbol{x} \in \mathsf{K}(n)$. Set $l = \operatorname{card} X_p$. If $2 \le l \le n$, then both underlinings of X_p are possible, and there are $K(n-l)$ possiblities for $(X_1^{\varepsilon_1},\dots,X_{p-1}^{\varepsilon_{p-1}})$, which gives $2\binom{n}{l}K(n-l)$ possibilities. If $l=1$, then only the underlining $\varepsilon_p = -1$ is possible, which gives $n\cdot K(n-1)$ possibilities. Hence the $K(n)$ are given by the induction formula

$$K(0) = 1;\quad K(n) = n\cdot K(n-1) + 2\sum_{l=2}^{n}\binom{n}{l}K(n-l) \text{ if } n>0\,. \tag{11.2}$$

For small values of n, these numbers are the following:

$$K(1) = 1\,;\ K(2) = 4\,;\ K(3) = 20\,;\ K(4) = 138\,;\ K(5) = 1{,}182\,;\ K(6) = 12{,}166\,.$$

Finally, we can compute the cardinalities of the lattices $\mathsf{S}(n,k)$ as follows:

$$S(n,k) = K(n) + K(n-1-k)\cdot K(k)\,. \tag{11.3}$$

Indeed, recalling that $\mathsf{S}(n,k) = \mathsf{S}(n,\langle 1,\{2,3,\dots,k+1\}\rangle$, if a bipartition $\boldsymbol{x} \in \mathsf{S}(n,k)$ has an underlined singleton block, then this block is $\{1\}$, and furthermore, both $\{1\}\times[2,k+1]$ and $[k+2,n]\times\{1\}$ are contained in $\boldsymbol{x}$; thus, besides all the bipartitions with no underlined singleton block (i.e., the elements of $\mathsf{K}(n)$), we must count the bipartitions whose ordered bipartition representation has the form $(\pi_1, \{1\}^{+1}, \pi_2)$, where π_1 represents a bipartition on $[k+2, n]$, π_2 represents a bipartition on $[2, k+1]$, and neither π_1 nor π_2 have any underlined singleton block.

For small values of n, the orders of the lattices $\mathsf{S}(n, k)$ are the following:

$$S(1,0) = 2\,. \quad S(2,0) = 5\,.$$
$$S(3,0) = 24\,; \quad S(3,1) = 21\,.$$
$$S(4,0) = 158\,; \quad S(4,1) = 142\,.$$
$$S(5,0) = 1{,}320\,; \quad S(5,1) = 1{,}202\,; \quad S(5,2) = 1{,}198\,.$$
$$S(6,0) = 13{,}348\,; \quad S(6,1) = 12{,}304\,; \quad S(6,2) = 12{,}246\,.$$

12. Open problems

Our first problem asks for a converse to Theorem 7.8.

Problem 1. Can every finite ortholattice, which is also a bounded homomorphic image of a free lattice, be embedded into $\mathrm{Reg}(\boldsymbol{e})$, for some finite strict ordering $\boldsymbol{e}$?

A variant of Problem 1, for arbitrary finite ortholattices, is the following.

Problem 2. Can every finite ortholattice be embedded into $\mathrm{Bip}(n)$, for some positive integer n?

On the opposite side of Problems 1 and 2, it is natural to state the following problems.

Problem 3. Is there a nontrivial lattice (resp., ortholattice) identity that holds in $\mathrm{Reg}(\boldsymbol{e})$ for every finite strict ordering $\boldsymbol{e}$?

Problem 4. Is there a nontrivial lattice (resp., ortholattice) identity that holds in $\mathrm{Bip}(n)$ for every positive integer n?

Bruns observes in [6, §(4.2)] that the variety of all ortholattices is generated by its finite members (actually, the argument presented there shows that "variety" can even be replaced by "quasivariety"). This shows, for example, that Problems 2 and 4 cannot simultaneously have a positive answer.

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Luigi Santocanale, LIF, UMR 7279, Aix-Marseille Université, Parc Scientifique et Technologique de Luminy, 163, avenue de Luminy, Case 901, F-13288 Marseille cedex 9, France

E-mail address: `luigi.santocanale@lif.univ-mrs.fr`
URL: `http://www.lif.univ-mrs.fr/~lsantoca/`

Friedrich Wehrung, LMNO, CNRS UMR 6139, Département de Mathématiques, Université de Caen, 14032 Caen Cedex, France

E-mail address: `friedrich.wehrung01@unicaen.fr`
URL: `http://www.math.unicaen.fr/~wehrung`