Notes on the analytic solution for the Dahl's friction model

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***Abstract*— This document provides the analytical solution of the friction model for different changes in the sign of the vehicle speed.**

I. Analytic solution of the Dahl's friction model

The load torque τ_a is described by the Dahl's model and is given by the following differential equation:

$$\frac{\mathrm{d}F}{\mathrm{d}\theta} = \sigma_0 \left(1 - \frac{F}{F_c}\operatorname{sgn}\left(\dot{\theta}\right)\right) \tag{1}$$

Let make the hypothesis to have a positive speed signal during the time interval $t \in [0, t_1]$. At the time $t = 0$, we have the corresponding value for the angle $\theta(0) = \theta_0$. This hypothesis implies that we have $\operatorname{sgn}\left(\dot{\theta}\right) = +1$ in Eq. (1) for $\theta \in [\theta_0, \theta]$. We make also the hypothesis to start from null initial conditions, this implies that we need to solve the following differential equation:

$$\frac{\mathrm{d}F}{\mathrm{d}\theta} = \sigma_0 \left(1 - \frac{F}{F_c}\right) \tag{2}$$

with the following initial condition:

$$F(\theta_0) = 0 \tag{3}$$

We divide both terms of Eq. (2) by $\left(1 - \frac{F}{F_c}\right)$ and F_c, in order to obtain a suited primitive:

$$\frac{-\frac{1}{F_c}\left(\frac{\mathrm{d}F}{\mathrm{d}\theta}\right)}{\left(1 - \frac{F}{F_c}\right)} = -\frac{\sigma_0}{F_c} \tag{4}$$

To obtain the analytic solution of Eq. (4) for positive speeds, we integrate both terms during the interval $(\theta_0, \theta(t))$:

$$\int_{\theta_0}^{\theta(t)} \frac{-\frac{1}{F_c}\frac{\mathrm{d}F}{\mathrm{d}\theta}}{\left(1 - \frac{F}{F_c}\right)} \mathrm{d}\theta = -\frac{\sigma_0}{F_c}\int_{\theta_0}^{\theta(t)} \mathrm{d}\theta \tag{5}$$

The result of this integral is:

$$\ln\left|1 - \frac{F(\theta(t))}{F_c}\right| - \ln\left|1 - \frac{F(\theta_0)}{F_c}\right| = -\frac{\sigma_0}{F_c}(\theta(t) - \theta_0) \tag{6}$$

By substituting the initial condition (3), it is easily to obtain the solution for positive speeds:

$$F(\theta(t)) = F_c\left[1 - e^{-\frac{\sigma_0}{F_c}(\theta(t) - \theta_0)}\right] \tag{7}$$

This work was partially funded by the project VolHand ANR-09-VTT-14.

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At the time instant $t = t_1$, the speed changes in sign and becomes negative ($\operatorname{sgn}\left(\dot{\theta}\right) = -1$ in Eq. (1)) during the time interval $t \in [t_1, t_2]$. To guarantee the continuity of the function, we require that the initial condition corresponds to the last values assumed from $F(\theta(t))$ during the interval $t \in [0, t_1]$, i.e. $F(\theta(t_1^-)) = F(\theta(t_1^+))$. Under these hypothesis, the new Cauchy problem to solve is:

$$\frac{\mathrm{d}F}{\mathrm{d}\theta} = \sigma_0\left(1 + \frac{F}{F_c}\right) \tag{8}$$

$$F(\theta(t_1^-)) = F(\theta(t_1^+)) \tag{9}$$

To solve this problem, we proceed as for Eqs. (4) and (5), then we integrate between $\theta(t_1)$ and $\theta(t)$, as follows:

$$\int_{\theta(t_1)}^{\theta(t)} \frac{\frac{1}{F_c}\frac{\mathrm{d}F}{\mathrm{d}\theta}}{\left(1 + \frac{F}{F_c}\right)} \mathrm{d}\theta = \int_{\theta(t_1)}^{\theta(t)} \frac{\sigma_0}{F_c}\mathrm{d}\theta \tag{10}$$

The solution of this integral is:

$$\ln\left|1 + \frac{F(\theta(t))}{F_c}\right| - k_1 = \frac{\sigma_0}{F_c}(\theta(t) - \theta(t_1)) \tag{11}$$

Where k_1 is a constant, depending from the initial conditions. By inverting this equation, we obtain:

$$F(\theta(t)) = F_c\left[-1 + e^{\frac{\sigma_0}{F_c}(\theta(t) - \theta(t_1))} \cdot e^{k_1}\right] \tag{12}$$

To determine the constant k_1, we introduce the initial condition $F(\theta(t_1^-)) = F(\theta(t_1^+))$:

$$F(\theta(t_1^-)) = F_c\left[1 - e^{-\frac{\sigma_0}{F_c}\theta(t_1)}\right] \tag{13}$$

$$F(\theta(t_1^+)) = F_c\left[-1 + e^{\frac{\sigma_0}{F_c}(\theta(t_1) - \theta(t_1))} \cdot e^{k_1}\right] \tag{14}$$

It follows:

$$F_c\left[1 - e^{-\frac{\sigma_0}{F_c}\theta(t_1)}\right] = F_c\left[-1 + e^{k_1}\right] \tag{15}$$

By solving for k_1, we obtain:

$$k_1 = \ln\left[2 - e^{-\frac{\sigma_0}{F_c}\theta(t_1)}\right] \tag{16}$$

So the solution for negative speeds is:

$$F(\theta(t)) = F_c\left[-1 + e^{\frac{\sigma_0}{F_c}(\theta(t) - \theta(t_1))} \cdot e^{\ln\left[2 - e^{-\frac{\sigma_0}{F_c}\theta(t_1)}\right]}\right] \tag{17}$$

By rearranging all terms, this expression can be simplified as follows:

$$F(\theta(t)) = F_c\left[-1 + 2e^{\frac{\sigma_0}{F_c}(\theta(t) - \theta(t_1))} - e^{\frac{\sigma_0}{F_c}(\theta(t) - 2\theta(t_1))}\right] \tag{18}$$

If we make the hypothesis that the speed changes in sign and becomes positive during the time interval $t \in [t_2, t_f]$, we can find the analytic solution $F(\theta)$ for this last interval, by repeating the same steps as for the previous cases. In this case, the boundary condition is $F(\theta(t_2^-)) = F(\theta(t_2^+))$, so we need to solve:

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}\theta} &= \sigma_0\left(1-\frac{F}{F_c}\right) && (19)\\ F(\theta(t_2^-)) &= F(\theta(t_2^+)) && (20) \end{aligned}$$

We proceed as in the previous cases and we integrate the primitive between $[\theta_2, \theta(t)]$:

$$\int_{\theta_2}^{\theta(t)} \frac{-\frac{1}{F_c}\frac{\mathrm{d}F}{\mathrm{d}\theta}}{\left(1-\frac{F}{F_c}\right)}\mathrm{d}\theta = -\frac{\sigma_0}{F_c}\int_{\theta_2}^{\theta(t)}\mathrm{d}\theta \tag{21}$$

We obtain the following analytic solution:

$$F(\theta(t)) = F_c\left[1-e^{-\frac{\sigma_0}{F_c}(\theta(t)-\theta(t_2))}\cdot e^{k_2}\right] \tag{22}$$

We can determine the constant k_2 by introducing the boundary condition $F(\theta(t_2^-)) = F(\theta(t_2^+))$:

$$F(\theta(t_2^-)) = F_c\left[-1+2e^{\frac{\sigma_0}{F_c}(\theta(t_2)-\theta(t_1))} - e^{\frac{\sigma_0}{F_c}(\theta(t_2)-2\theta(t_1))}\right] \tag{23}$$

$$F(\theta(t_2^+)) = F_c\left[1-e^{-\frac{\sigma_0}{F_c}(\theta(t_2)-\theta(t_2))}\cdot e^{k_2}\right] \tag{24}$$

It follows:

$$F_c\left[1-e^{k_2}\right] = F_c\left[-1+2e^{\frac{\sigma_0}{F_c}(\theta(t_2)-\theta(t_1))} - e^{\frac{\sigma_0}{F_c}(\theta(t_2)-2\theta(t_1))}\right] \tag{25}$$

From Eq. (25) we get the following expression for the constant k_2:

$$k_2 = \ln\left[2-2e^{-\frac{\sigma_0}{F_c}(\theta(t_2)-\theta(t_1))} + e^{\frac{\sigma_0}{F_c}(\theta(t_2)-2\theta(t_1))}\right] \tag{26}$$

If we substitute Eq. (26) in Eq. (22) and we simplify, we get the solution for the friction torque when the speed is positive and the initial condition is different from zero:

$$\begin{aligned} F(\theta(t)) = F_c\Big[&1-2e^{-\frac{\sigma_0}{F_c}(\theta(t)-\theta(t_2))}+\\ &2e^{-\frac{\sigma_0}{F_c}(\theta(t)+\theta(t_1)-2\theta(t_2))} - e^{-\frac{\sigma_0}{F_c}(\theta(t)+2\theta(t_1)-2\theta(t_2))}\Big] \end{aligned} \tag{27}$$

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