Fast Weak-Kam Integrators for separable Hamiltonian systems

Anne Bouillard, Erwan Faou, Maxime Zavidovique

FAST WEAK-KAM INTEGRATORS

by

Anne Bouillard, Erwan Faou & Maxime Zavidovique

Abstract. — We consider a numerical scheme for Hamilton-Jacobi equations based on a direct discretization of the Lax-Oleinik semi-group. We prove that this method is convergent with respect to the time and space stepsizes provided the solution is Lipschitz, and give an error estimate. Moreover, we prove that the numerical scheme is a *geometric integrator* satisfying a discrete weak-KAM theorem which allows to control its long time behavior. Taking advantage of a fast algorithm for computing min-plus convolutions based on the decomposition of the function into concave and convex parts, we show that the numerical scheme can be implemented in a very efficient way.

1. Introduction

We consider Hamilton-Jacobi equations of the form

$$\partial_t u + H(t,x,\nabla u) = 0, \quad u(0,x) = u_0(x), \tag{1}$$

where $H(t,x,v)$ is a Hamiltonian function

$$H : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \tag{2}$$

and where u_0 is a given global Lipschitz function.

We will mainly consider the case where H is separable, in the sense that we can write $H(t,x,p) = K(p) + V(t,x)$, for some convex function K and some

2000 ***Mathematics Subject Classification.*** — 35F21, 65M12, 06F05 .
Key words and phrases. — Hamilton-Jacobi equations, Weak-KAM theorem, Geometric integration, Min-plus convolution.

smooth and bounded function V. The typical cases of study we have in mind are the so called mechanical Hamiltonians, of the form

$$H(t,x,p) = \frac{1}{2}|p+P|^2 + V(t,x) \tag{3}$$

where $P \in \mathbb{R}^n$ is a given vector, $|v|^2 = v_1^2 + \cdots + v_n^2$ for $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$, and where $V(t,x)$ is a suitably smooth and bounded function.

Since the pioneering works of Crandall and Lions [CL84] and Souganidis [Sou85], the study of numerical schemes for the Hamilton-Jacobi equation (1) has known many recent progresses, see for instance [LS95, LT01, Abg96, JX98, BJ05] and the references therein, and more specifically [OS88, OS91, JP00] for the popular (weighted) essentially nonoscillatory (ENO and WENO) methods which are now widely commonly used in many application fields.

Following a different approach, more in the spirit of [FF02, Ror06] (or [AGL08] in an optimal control setting), the main aim of this paper is to show how a direct discretization of the Lax-Oleinik representation of the viscosity solution of (1) allows to define a new fast algorithm for computing $u(t,x)$ possessing strong geometrical properties allowing to control its long time behavior and obtain error estimates when the solution is Lipschitz.

Let us recall, see [Lio82, Fat05], that under some assumptions on H (smoothness, uniform superlinearity and strict convexity over the fibers, see Section 2 below), we can write

$$u(t,x) = \inf_{\gamma(t)=x} u_0(\gamma(0)) + \int_0^t L(s,\gamma(s),\dot\gamma(s))\mathrm{d}s, \tag{4}$$

where the infimum is taken over over all absolutely continuous curves $\gamma : [0,t] \to \mathbb{R}^n$ such that $\gamma(t) = x$, and where $L(t,x,v)$ is the Lagrangian associated with H. The idea of this paper is to discretize directly (4) on a space time grid, by replacing the set of curves γ by the set of piecewise linear curves across the space grid points.

We first prove that such an approximation is convergent with respect to the size of the space and time stepsizes, and under an anti-CFL condition (namely that the ratio between the space and time stepsize should be small). We give an error estimate under the assumption that u_0 is Lipschitz.

Moreover, this numerical integrator turns out to be a *geometric integrator* (see for instance [HLW06, LR04]) in the sense that is respects the long time behavior of the exact solution $u(t,x)$. Let us recall that in the case of periodic Hamiltonians (both in time and space variables), the weak-KAM theorem (see [Fat05, CISM00]) shows the existence of a constant $\overline{H}$ such that

$$\frac{1}{t}u(t,x) \to \overline{H} \quad \text{when} \quad t \to +\infty.$$

Here, using a discrete weak-KAM theorem, see [**BB07, Zav12**], we prove that the numerical scheme possesses the same long time property, with a constant that is close to the exact constant $\overline{H}$.

Finally, we show that in the separable case mainly considered in this paper (see (5) below), the discrete version of (4) is a min-plus convolution that can be approximated using a fast algorithm with $\mathcal{O}(N)$ operations in many situations if N is the number of grid points. This algorithm uses the decomposition of u into concave and convex parts. Moreover, it easily extends to any space dimension n using a splitting strategy, when the kinetic part of the Hamiltonian is separable - see Remark 2.8 - which includes the case (3).

We then conclude by numerical simulations in dimension 1 to illustrate the good behavior of our algorithm, as well as its very low cost in general situations.

The paper is divided into three parts: in a first part (Section 2) we give a convergence result over a finite time interval of the form $[0, T]$ where T is fixed. In a second part (Section 3), we consider the case where the Hamiltonian is periodic in time t and x. In this case, we can derive explicitly the dependence in T in the error estimates, and prove a weak-KAM theorem for the numerical scheme which gives informations concerning the long time behavior of the scheme. In the third part (Section 4), we describe the implementation of the method based on a fast algorithm to compute min-plus convolutions. We conclude this part by showing numerical simulations.

Acknowledgement. — This work owes a lot to Vincent Calvez, who put the authors in touch and took part to preliminary discussions. It is a great pleasure to thank him a lot. We also would like to thank Vinh Nguyen for careful reading through previous versions of the paper. Finally, the last author would like to thank Antonio Siconolfi for bringing him to this subject.

2. Description of the scheme and convergence results

2.1. Hypotheses. — We consider a Hamiltonian $H(t, x, p)$ of the form

$$H(t, x, p) = K(p) + V(t, x) \tag{5}$$

defined on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$. With this Hamiltonian we can associate by Legendre transform the Lagrangian

$$L(t, x, v) = \sup_{p \in \mathbb{R}^n} \Big(p \cdot v - H(t, x, p) \Big),$$

and we calculate that in our case,

$$L(t, x, v) = K^*(v) - V(t, x),$$

where $K^*(v)$ is the Legendre transform of K. For instance in the special case (3) we have

$$L(t,x,v) = \frac{1}{2}|v|^2 - P\cdot v - V(t,x).$$

We make the following assumptions on K and V:

(i) The function $K^* \in \mathcal{C}^2(\mathbb{R}^n)$ is uniformly strictly convex in the sense that there exists a constant $c > 0$ such that for all $Y \in \mathbb{R}^n$, and for all $v \in \mathbb{R}^n$,

$$\frac{\partial^2 K^*}{\partial v^2}(v)(Y,Y) \geqslant c|Y|^2. \tag{6}$$

(ii) The function $V(t,x) \in \mathcal{C}^2(\mathbb{R}\times\mathbb{R}^n)$ is such that there exists a constant B such that for $j+q \leqslant 2$, and all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$,

$$|\partial_t^j \partial_x^q V(t,x)| \leqslant B, \tag{7}$$

where $|\cdot|$ denote the norm of differential operators acting on $\mathbb{R}\times\mathbb{R}^n$.

Note that the bound (7) is straightforward under the supplementary assumption that $V(t,x,v)$ is *periodic* in (t,x), the case studied in the next section.

Remark 2.1. — The previous hypotheses imply that the Hamiltonian H and the Lagrangian L are $\mathcal{C}^2$, convex and superlinear in respectively p and v:

$$\forall\, k > 0, \quad \forall\, t > 0, \quad \exists\, A(k) < \infty, \quad L(t,x,v) \geqslant k|v| - A(k). \tag{8}$$

Under these assumptions, the viscosity solution of (1) can be represented by the formula: for all $t, \delta > 0$,

$$\forall x \in \mathbb{R}^n, \quad u(t+\delta,x) := T_t^\delta u(x) = \inf_{\gamma(t+\delta)=x} u(t,\gamma(t)) + \int_t^{t+\delta} L(s,\gamma(s),\dot\gamma(s))\mathrm{d}s, \tag{9}$$

where the infimum is taken on all absolutely continuous curves $\gamma : (t,t+\delta) \to \mathbb{R}^n$ verifying $\gamma(t+\delta) = x$, see [**Lio82, Fat05**]. Moreover, the infimum is achieved on a curve $\gamma_{t,x}^\delta(s)$ that is $\mathcal{C}^2$ and satisfies the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial L}{\partial v}(s,\gamma(s),\dot\gamma(s)) = \frac{\partial L}{\partial x}(s,\gamma(s),\dot\gamma(s)). \tag{10}$$

The notation T_t^δ defines the Lax-Oleinik semi-group. In particular, we have $T_{t+\delta}^\sigma \circ T_t^\delta = T_t^{\delta+\sigma}$ for non negative δ and σ. With these assumptions, we have the following Proposition.

Proposition 2.2. — *For all $T > 0$, and for all $R > 0$, there exists $M(R,T)$ such that for all $x, y \in \mathbb{R}^n$ satisfying $|x-y| \leqslant R$ and for all $t \in \mathbb{R}$, then every* *solution of the Euler-Lagrange equation* (10) *minimizing the action*

$$\int_t^{t+T} L\big(s,\gamma(s),\dot\gamma(s)\big)\mathrm{d}s, \tag{11}$$

with fixed endpoints $\gamma(t)=x$ *and* $\gamma(t+T)=y$, *satisfies* $|\dot\gamma(s)|\leqslant M(R,T)$ *for all* $s\in[t,t+T]$.

Proof. — The Euler-Lagrange equation is written

$$\frac{\partial^2 K^*}{\partial v^2}\big(\dot\gamma(s)\big)\big(\ddot\gamma(s)\big) = -\frac{\partial V}{\partial x}\big(s,\gamma(s)\big).$$

Using the uniform strict convexity of K^* and the fact that $\partial_x V$ is uniformly bounded, there exists a constant C depending only on T and K, such that

$$\forall\, s\in[t,t+T]\quad |\ddot\gamma(s)|^2\leqslant C. \tag{12}$$

This implies that for all $s\in[t,t+T]$,

$$|\dot\gamma(s)-\dot\gamma(t)|\ \leqslant \int_t^{t+T}|\ddot\gamma(s)|\,\mathrm{d}s\leqslant T\sqrt{C}. \tag{13}$$

Now as γ minimizes the action between t and $t+T$, comparing with the trivial curve $t\mapsto x+t(y-x)/T$ from x to y, we get

$$\begin{aligned}\int_t^{t+T} L\big(s,\gamma(s),\dot\gamma(s)\big)\mathrm{d}s\ &\leqslant\ \int_t^{t+T} K^*\Big(\frac{y-x}{T}\Big)-V\Big(s,x+\frac{t}{T}(y-x)\Big)\mathrm{d}s\\ &\leqslant\ TD(R/T)+TB,\end{aligned}$$

where $D(M)=\sup_{|v|\leqslant M}|K^*(v)|$, and B is given by (7). By superlinearity, we deduce that

$$\int_t^{t+T}|\dot\gamma(s)|\,\mathrm{d}s\leqslant TD(R/T)+TA(1)+TB,$$

where $A(1)$ is given by (8). Using (13), we thus obtain

$$T|\dot\gamma(t)|\leqslant\int_t^{t+T}|\dot\gamma(s)|\mathrm{d}s+T^2\sqrt{C}\leqslant TD(R/T)+TA(1)+TB+T^2\sqrt{C}.$$

This shows that $|\dot\gamma(t)|$ is bounded, and hence using again (13) that $|\dot\gamma(s)|$ is bounded for all s, with a constant depending only on T, R and the constants appearing in (6) and (7). □

Remark 2.3. — The previous lemma is one of the main keys in the proof of the convergence of our schemes. Here, it is established thanks to the particular form of H, but it can be noted that it remains valid under other technical assumptions (for example if H is autonomous and Tonelli as established in [**Fat05**, **FM07**], or if it is Tonelli and periodic both in the space and in the time variable as proven in [**Mat91**, **CISM00**, **Itu96**]). Actually, in these cases, it can be established that $M(R,T)$ only depends on the ratio R/T.

We will come back on these matters in Section 3 and give a proof of this result in the Appendix. Therefore, this section and the next would still be valid for general Hamiltonians chosen in these classes, however the convolution techniques of section 4 would fail.

Finally, a clear consequence of Equation (12) is that the Euler-Lagrange flow of L is complete.

2.2. An approximate semi-group. — For a given $\varepsilon > 0$ we define the ε-grid $G_\varepsilon = \varepsilon\mathbb{Z}^n$ endowed with the metric induced by the euclidian metric on $\mathbb{R}^n$. For a given continuous function u, we define $u|_{G_\varepsilon} : G_\varepsilon \to \mathbb{R}$ its restriction to the grid G_ε. Given $t, \tau > 0$, let us define $c^\tau_{t,\varepsilon} : G^2_\varepsilon \mapsto \mathbb{R}$ as follows :

$$\forall (x,y) \in G^2_\varepsilon, \quad c^\tau_{t,\varepsilon}(x,y) = \int_t^{t+\tau} L\left(s, x + (s-t)\frac{y-x}{\tau}, \frac{y-x}{\tau}\right) \mathrm{d}s. \tag{14}$$

Let us introduce the following discrete Lax-Oleinik semi-group: if $u : G_\varepsilon \to \mathbb{R}$ is any function, we set

$$\forall x \in G_\varepsilon, \quad T^\tau_{t,\varepsilon} u(x) = \inf_{y \in G_\varepsilon} u(y) + c^\tau_{t,\varepsilon}(y,x).$$

A good setting to apply this semi-group is the one of functions u with linear growth, which means that the quantity $\big(u(y) - u(x)\big)/\big(1 + \|y - x\|\big)$ is uniformly bounded. Note that for such a function u, for a given x, the hypotheses made on L ensure the existence of a minimizing $y \in G_\varepsilon$ attaining the previous infimum. Indeed, the cost $c^\tau_{t,\varepsilon}$ inherits from L a superlinearity property which implies that the function $y \mapsto u(y) + c^\tau_{t,\varepsilon}(y,x)$ goes to ∞ at ∞. Moreover, the set of functions with at most linear growth is invariant by our semi-group. For more details, we refer the reader to the appendix of [**Zav12**]. Note that Lipschitz functions, which we will only consider in the following, have linear growth.

For a given integer N, we define

$$T^{N\tau}_{t,\varepsilon} u = T^\tau_{t_{N-1},\varepsilon} \circ \cdots \circ T^\tau_{t_1,\varepsilon} \circ T^\tau_{t_0,\varepsilon} u \tag{15}$$

the composition of N times the discrete semi-group $T^\tau_{t,\varepsilon}$, where for all $i = 1, \dots, N-1$, $t_i = t + i\tau$.

Remark 2.4. — The following monotonicity property holds true: if $u(x) \leqslant v(x)$ for all $x \in \mathbb{R}^n$, we easily observe that $T^{N\tau}_{t,\varepsilon} u \leqslant T^{N\tau}_{t,\varepsilon} v$.

Proposition 2.5. — *Let $u : \mathbb{R}^n \mapsto \mathbb{R}$ and $N \geqslant 1$ an integer, $t \in \mathbb{R}$ and $\tau > 0$. Then*

$$(T^{N\tau}_t u)|_{G_\varepsilon} \leqslant T^{N\tau}_{t,\varepsilon}(u|_{G_\varepsilon}). \tag{16}$$

Proof. — Let $x \in G_\varepsilon$. With this point, we can associate $N+1$ points x_i, $i = 0, \dots, N$ such that $x_N = x$, and such that for all $i = 0, \dots, N-1$,

$$T^{\tau}_{t_i,\varepsilon} \circ T^{i\tau}_{t,\varepsilon} u(x_{i+1}) = T^{i\tau}_{t,\varepsilon} u(x_i) + c^{\tau}_{t_i,\varepsilon}(x_i, x_{i+1}).$$

With this notation, we can verify that

$$\begin{aligned} T^{N\tau}_{t,\varepsilon}(u|_{G_\varepsilon})(x) &= u(x_0) + \sum_{i=0}^{N-1} c^{\tau}_{t_i,\varepsilon}(x_i, x_{i+1}) \\ &= u(x_0) + \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} L\left(s, x_i + (s - t_i)\frac{x_{i+1} - x_i}{\tau}, \frac{x_{i+1} - x_i}{\tau}\right) \mathrm{d}s \\ &= u(x_0) + \int_0^{N\tau} L\big(s, \gamma_{t,\varepsilon,\tau}(s), \dot\gamma_{t,\varepsilon,\tau}(s)\big) \mathrm{d}s, \end{aligned}$$

where $\gamma_{t,\varepsilon,\tau}(s)$ is the continuous piecewise linear curve defined by

$$\gamma_{t,\varepsilon,\tau}(s) = x_i + (s - t_i)\frac{x_{i+1} - x_i}{\tau}, \quad \text{for} \quad s \in [t_i, t_{i+1}]. \tag{17}$$

The result then easily follows by definition of $T^{N\tau}_t u$. □

As we will see now, the reverse inequality (16) is true up to a small error term coming from the time and space discretization. This is stated in the following convergence result:

Theorem 2.6. — *Let $T > 0$, ε_0, τ_0 and $h_0 > 0$ and $u : \mathbb{R}^n \to \mathbb{R}$ a bounded Lipschitz function. There exists a constant M such that for all $\varepsilon > 0$ and $\tau > 0$ such that $\varepsilon < \varepsilon_0$, $\tau < \tau_0$ and*

$$\frac{\varepsilon}{\tau} < h_0, \tag{18}$$

for all N satisfying $N\tau \leqslant T$,

$$\Big|(T^{N\tau}_t u)|_{G_\varepsilon} - T^{N\tau}_{t,\varepsilon}(u|_{G_\varepsilon})\Big|_\infty \leqslant M\big(\frac{\varepsilon}{\tau} + \tau\big). \tag{19}$$

Proof. — Let us denote by $\gamma_t : [t, t+T] \to \mathbb{R}^n$ a minimizer of (9). Recall that the curve $\gamma_t(s)$ is C^2. Let us set $y := \gamma_t(t)$ and $x := \gamma_t(t+T)$. We have

$$T^T_t u(x) = u(y) + \int_t^{t+T} L\big(s, \gamma_t(s), \dot\gamma_t(s)\big) \mathrm{d}s.$$

By superlinearity (8), this implies that

$$\int_t^{t+T} |\dot\gamma_t(s)| \mathrm{d}s \leqslant TA(1) + |T^T_t u(x) - u(y)|.$$

Comparing with the trivial curve $\gamma \equiv x$ in the definition of the Lax-Oleinik semi-group, we have that

$$T_t^T u(x) \leqslant |u|_\infty + T\big(B + K^*(0)\big) \tag{20}$$

where B is the constant in (7). Moreover, since L is bounded below ($L(t,x,v) \geqslant b$ for some constant b, for all (t,x,v)), clearly, the action of any curve defined for a time T is greater than Tb which implies immediately that

$$T_t^T u(x) \geqslant -|u|_\infty + Tb. \tag{21}$$

Hence, there exists a constant B_1 depending only on L and $|u|_\infty$ such that

$$|x-y| = |\gamma_t(t+T) - \gamma_t(t)| \leqslant \int_t^{t+T} |\dot\gamma_t(s)|\mathrm{d}s \leqslant B_1(1+T). \tag{22}$$

Remarking that γ_t is also a minimizer of the action (11) under the constraint $\gamma(t) = y$ and $\gamma(t+T) = x$, we can apply Proposition 2.2 which shows that there exists a constant $M_1 = M\big(B_1(1+T), T\big)$ depending only on T, L and $|u|_\infty$ such that

$$\forall\, s \in [t, t+T], \quad |\dot\gamma_t(s)| \leqslant M_1. \tag{23}$$

Assume now that N is an integer such that $N\tau \leqslant T$. For all $i = 0,\dots,N$ we define

$$x_i = \varepsilon \left\lfloor \frac{1}{\varepsilon}\gamma_t(t_i) \right\rfloor$$

where for $i = 0,\dots,N$, $t_i = t + i\tau$ and where the function $\lfloor\cdot\rfloor$ is the floor function, coordinate by coordinate. With these points, we associate the continuous piecewise linear path $\gamma_{t,\varepsilon,\tau}$ defined as in formula (17). Notice however that the points x_i are no longer the same. By definition of the points x_i, we have

$$\forall\, i \in [0,N], \quad |x_i - \gamma_t(t_i)| = |\gamma_{t,\varepsilon,\tau}(t_i) - \gamma_t(t_i)| \leqslant \varepsilon\sqrt{n}. \tag{24}$$

Now, using the bound (23), we have for all $i = 0,\dots,N-1$,

$$|x_{i+1} - x_i| \leqslant 2\varepsilon\sqrt{n} + \int_{t_i}^{t_{i+1}} |\dot\gamma_t(s)|\mathrm{d}s \leqslant 2\varepsilon\sqrt{n} + \tau M_1.$$

But this inequality implies that for all $i = 0,\dots,N-1$,

$$\forall\, s \in [t_i, t_{i+1}], \quad |\gamma_{t,\varepsilon,\tau}(s) - x_i| \leqslant 2\varepsilon\sqrt{n} + \tau M_1,$$

while $|\gamma_t(s) - \gamma_t(t_i)| \leqslant \tau M_1$ upon using (23). Hence we get

$$\forall\, s \in [t_i, t_{i+1}], \quad |\gamma_{t,\varepsilon,\tau}(s) - \gamma_t(s)| \leqslant 3\varepsilon\sqrt{n} + 2\tau M_1. \tag{25}$$

Moreover, we have for $s, \sigma \in [t_i, t_{i+1}]$,

$$|\dot\gamma_t(\sigma) - \dot\gamma_t(s)| \leqslant \tau C$$

upon using (12). Hence for $s \in [t_i, t_{i+1}]$, we have

$$|\gamma_t(t_{i+1}) - \gamma_t(t_i) - \tau\dot\gamma_t(s)| \leqslant \int_{t_i}^{t_{i+1}} |\dot\gamma_t(\sigma) - \dot\gamma_t(s)| \mathrm{d}\sigma \leqslant \tau^2 C$$

and hence for all $s \in [t_i, t_{i+1}]$

$$\left| \frac{\gamma_t(t_{i+1}) - \gamma_t(t_i)}{\tau} - \dot\gamma_t(s) \right| \leqslant \tau C.$$

Using (24), we obtain easily that for all $i = 0, \dots, N-1$,

$$\forall\, s \in [t_i, t_{i+1}], \quad |\dot\gamma_{t,\varepsilon,\tau}(s) - \dot\gamma_t(s)| \leqslant \tau C + \frac{2\varepsilon}{\tau}\sqrt{n}. \tag{26}$$

Note that using (18) and (23), the previous equation implies that for all $i = 0, \dots, N-1$,

$$\forall\, s \in [t_i, t_{i+1}], \quad |\dot\gamma_{t,\varepsilon,\tau}(s)| \leqslant M_2 \tag{27}$$

for some constant $M_2 = \tau_0 C + h_0\sqrt{n} + M_1$ independent of ε and τ.

Now by definition of $\gamma_{t,\varepsilon,\tau}$, we have

$$\begin{aligned}
&\Big| \int_t^{t+N\tau} L(s, \gamma_t(s), \dot\gamma_t(s))\mathrm{d}s - \sum_{i=0}^{N-1} c^{\tau}_{t_i,\varepsilon}(x_i, x_{i+1}) \Big| \\
&\quad = \Big| \int_0^{N\tau} L(s, \gamma_t(s), \dot\gamma_t(s))\mathrm{d}s - \int_0^{N\tau} L(s, \gamma_{t,\varepsilon,\tau}(s), \dot\gamma_{t,\varepsilon,\tau}(s))\mathrm{d}s \Big| \\
&\qquad \leqslant \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \Big| L(s, \gamma_t(s), \dot\gamma_t(s)) - L(s, \gamma_{t,\varepsilon,\tau}(s), \dot\gamma_{t,\varepsilon,\tau}(s)) \Big| \mathrm{d}s.
\end{aligned} \tag{28}$$

Using (7), the fact that K^* is C^2 and the bounds (23) and (27), there exists a constant M_3, depending on L, M_1 and M_2, such that the previous error term is bounded by

$$M_3 \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \Big(|\gamma_t(s) - \gamma_{t,\varepsilon,\tau}(s)| + |\dot\gamma_t(s) - \dot\gamma_{t,\varepsilon,\tau}(s)| \Big) \mathrm{d}s.$$

Using (25) and (26), this shows that there exists a constant M_4 independent on ε and τ, such that

$$\left| \int_t^{t+N\tau} L(s, \gamma_t(s), \dot\gamma_t(s))\mathrm{d}s - \sum_{i=0}^{N-1} c^{\tau}_{t_i,\varepsilon}(x_i, x_{i+1}) \right| \leqslant M_4\big(\frac{\varepsilon}{\tau} + \tau\big), \tag{29}$$

where we used the fact that $\varepsilon \leqslant \tau_0\varepsilon/\tau$.

Now by definition of $T_{t,\varepsilon}^{N\tau}$, we have using (24) and the Lipschitz nature of u,

$$\begin{aligned}
T_{t,\varepsilon}^{N\tau}(u|_{G_\varepsilon})(x) &\leqslant u(x_0) + \sum_{i=0}^{N-1} c_{t_i,\varepsilon}^{\tau}(x_i, x_{i+1}) \\
&\leqslant (T_t^{N\tau}u)|_{G_\varepsilon}(x) + \big|u(x_0) - u\big(\gamma_t(0)\big)\big| + M_4\big(\frac{\varepsilon}{\tau} + \tau\big) \\
&\leqslant (T_t^{N\tau}u)|_{G_\varepsilon}(x) + M\big(\frac{\varepsilon}{\tau} + \tau\big) \qquad (30)
\end{aligned}$$

for some constant M independent on ε and τ. This inequality and (16) show the final estimate (19). $\square$

2.3. Fully discrete semi-group. — In the previous Section, we have seen that the Lax-Oleinik semi-group can be approximated using the cost (14) defined on the grid. To compute this cost, a quadrature rule in time has to be used. In this subsection, we prove how the Euler approximation of this integral yields a convergent scheme which still satisfies a weak-KAM theorem similar to Proposition 3.1 under suitable periodicity assumptions.

For a given $\varepsilon > 0$ and $\tau > 0$, we define the following cost function:

$$\forall (x, y) \in G_\varepsilon^2, \quad \kappa_{t,\varepsilon}^{\tau}(y, x) = \tau L\left(t, x, \frac{x - y}{\tau}\right). \qquad (31)$$

and the associated fully discrete Lax-Oleinik semi-group

$$\forall x \in G_\varepsilon, \quad \mathcal{T}_{t,\varepsilon}^{\tau} u(x) = \inf_{y \in G_\varepsilon} u(y) + \kappa_{t,\varepsilon}^{\tau}(y, x),$$

if $u : G_\varepsilon \to \mathbb{R}$ is a function. Using the explicit expression of L, we can rewrite this fully-discrete semi-group as

$$\forall x \in G_\varepsilon, \quad \mathcal{T}_{t,\varepsilon}^{\tau} u(x) = \inf_{y \in G_\varepsilon} \left(u(y) + \tau K^*\left(\frac{x - y}{\tau}\right)\right) - \tau V(t, x). \qquad (32)$$

involving the (min,plus)-convolution of u and K^*.

Remark 2.7. — We can interpret this scheme as a discretization of the splitting scheme (see for instance [**JKR01**]) with time step τ based on the decomposition

$$\partial_t u(t, x) + K(\nabla u(t, x)) = 0, \quad \text{and} \quad \partial_t u(t, x) + V(t, x) = 0,$$

where the first part is integrated using the method described in the previous section.

Remark 2.8. — In dimension $n \geqslant 1$, if we assume that for $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$, $K(p) = K_1(p_1) + \cdots + K_n(p_n)$ with convex Hamiltonian functions K_i^*, $i = 1, \ldots, n$, satisfying all the hypotheses ***(i)***, ***(ii)*** on $\mathbb{R}$, then we immediately see that for a given function $u(x) = u(x_1, \ldots, x_n)$, with $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we have

$$\begin{aligned}
&\inf_{y \in G_\varepsilon} u(y) + \tau K^*(\frac{x-y}{\tau}) = \\
&\qquad \inf_{y_n \in G_\varepsilon^n} \Bigg[\tau K_n^*(\frac{x_n - y_n}{\tau}) + \Bigg[\inf_{y_{n-1} \in G_\varepsilon^{n-1}} \tau K_{n-1}^*(\frac{x_{n-1} - y_{n-1}}{\tau}) + \cdots \\
&\qquad\qquad + \Bigg[\inf_{y_1 \in G_\varepsilon^1} \tau K_n^*(\frac{x_1 - y_1}{\tau}) + u(y_1, \ldots, y_n) \Bigg] \cdots \Bigg] \Bigg] = \\
&\qquad\qquad\qquad\qquad \mathcal{T}_{t,\varepsilon}^{\tau,1} \circ \cdots \circ \mathcal{T}_{t,\varepsilon}^{\tau,n} u(x),
\end{aligned}$$

where we have decomposed $G_\varepsilon = G_\varepsilon^1 \times \cdots \times G_\varepsilon^n$ and where

$$\forall i \in [1, n], \quad \mathcal{T}_{t,\varepsilon}^{\tau,i} u(x) = \inf_{y_i \in G_\varepsilon^i} \tau K_i^*(\frac{x_i - y_i}{\tau}) + u(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots x_n).$$

This formula is essentially due to the fact that the Hamiltonians K_i commute, i.e. satisfy $\{K_i, K_j\} = 0$ for $(i,j) \in \{1, \ldots, n\}^2$ which ensures that the flows of $\partial_t u = K_i(\nabla u)$, $i = 1, \ldots, n$ commute. In this case, this allows to reduce the computation of the minimum over the n dimensional grid G_ε to n minimization problems over the 1 dimensional grids G_ε^i.

For a given integer N, we define - compare (15)

$$\mathcal{T}_{t,\varepsilon}^{N\tau} u = \mathcal{T}_{t_{N-1},\varepsilon}^{\tau} \circ \cdots \circ \mathcal{T}_{t_1,\varepsilon}^{\tau} \circ \mathcal{T}_{t_0,\varepsilon}^{\tau} u$$

where $t_i = t + i\tau$. Note that with these notations, an estimate of the form (16) is no longer valid. However, we have the following convergence result:

Theorem 2.9. — *Let $T > 0$, ε_0, τ_0 and $h_0 > 0$ and $u : \mathbb{R}^n \to \mathbb{R}$ a bounded Lipschitz function. There exists a constant M such that for all $t > 0$, and all $\varepsilon > 0$ and $\tau > 0$ such that $\varepsilon < \varepsilon_0$, $\tau < \tau_0$ and the bound $\varepsilon/\tau < h_0$ are satisfied, then for all N verifying $N\tau \leqslant T$,*

$$\Big| (T_t^{N\tau} u)|_{G_\varepsilon} - \mathcal{T}_{t,\varepsilon}^{N\tau}(u|_{G_\varepsilon}) \Big|_\infty \leqslant M\big(\frac{\varepsilon}{\tau} + \tau\big).$$

Proof. — The proof is exactly the same as the proof of Theorem 2.6 until Equation (28) that has to be replaced by the following estimate:

$$\begin{aligned}\Big|\int_t^{t+N\tau} L\big(s,\gamma_t(s),\dot\gamma_t(s)\big)\mathrm{d}s - \sum_{i=0}^{N-1}\kappa^\tau_{t,\varepsilon}(x_i,x_{i+1})\Big| \\ \leqslant \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\Big|L\big(s,\gamma_t(s),\dot\gamma_t(s)\big) - L\big(s,\gamma_{t,\varepsilon,\tau}(s),\dot\gamma_{t,\varepsilon,\tau}(s)\big)\Big|\mathrm{d}s \\ + \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}\Big|L\big(s,\gamma_{t,\varepsilon,\tau}(s),\dot\gamma_{t,\varepsilon,\tau}(s)\big) - L\big(t_i,\gamma_{t,\varepsilon,\tau}(t_i),\dot\gamma_{t,\varepsilon,\tau}(t_i)\big)\Big|\mathrm{d}s.\end{aligned}$$

In the right-hand side of this inequality, the first term is bounded by $M_4\big(\frac{\varepsilon}{\tau}+\tau\big)$ see (29). To bound the second term, we observe first that for $s\in[t_i,t_{i+1}]$, the derivative $\dot\gamma_{t,\varepsilon,\tau}(s) = (x_{i+1}-x_i)/\tau$ does not depend on s. Hence using (7) and (27) the function

$$[t_i,t_{i+1}] \ni s \mapsto L\big(s,\gamma_{t,\varepsilon,\tau}(s),\dot\gamma_{t,\varepsilon,\tau}(s)\big)$$

is $\mathcal{C}^1$ with uniformly bounded derivative. Thus we obtain that there exists a constant M_5 such that

$$\Big|L\big(s,\gamma_{t,\varepsilon,\tau}(s),\dot\gamma_{t,\varepsilon,\tau}(s)\big) - L\big(t_i,\gamma_{t,\varepsilon,\tau}(t_i),\dot\gamma_{t,\varepsilon,\tau}(t_i)\big)\Big| \leqslant M_5(s-t_i).$$

This proves that $\big($compare (29)$\big)$

$$\Big|\int_t^{t+N\tau} L\big(s,\gamma_t(s),\dot\gamma_t(s)\big)\mathrm{d}s - \sum_{i=0}^{N-1}\kappa^\tau_{t_i,\varepsilon}(x_i,x_{i+1})\Big| \leqslant M_6\big(\frac{\varepsilon}{\tau}+\tau\big), \tag{33}$$

for some constant M_6 independent of ε and τ. As in (30), we obtain that

$$\mathcal{T}^{N\tau}_{t,\varepsilon}(u|_{G_\varepsilon})(x) \leqslant (T^{N\tau}_t u)|_{G_\varepsilon}(x) + M\big(\frac{\varepsilon}{\tau}+\tau\big),$$

for some constant M independent on ε and τ.

To prove the reverse inequality, let us fix $x\in G_\varepsilon$. We consider a sequence y_i, $i=0,\dots,N$ with $y_N = x$ and

$$\mathcal{T}^{N\tau}_{t,\varepsilon}(u|_{G_\varepsilon})(x) = u(y_0) + \sum_{i=0}^{N-1}\kappa^\tau_{t_i,\varepsilon}(y_i,y_{i+1}), \tag{34}$$

and we define the curve

$$\eta_{t,\varepsilon,\tau}(s) = y_i + (s-t_i)\frac{y_{i+1}-y_i}{\tau}, \quad \text{for} \quad s\in[t_i,t_{i+1}].$$

Note that in a similar manner to what we did to prove the inequalities 20 and 21, using the fact that u is bounded, and comparing with the trivial sequence made of a constant point (with (7)) on the one hand, and the fact that L, hence the $\kappa^{\tau}_{t_i,\varepsilon}$ are bounded below on the second hand show that there exists a constant D_1 such that

$$|\mathcal{T}^{N\tau}_{t,\varepsilon}(u|_{G_\varepsilon})|_\infty \leqslant D_1.$$

By superlinearity of L (and of the $\kappa^{\tau}_{t_i,\varepsilon}$) and using again the fact that u is bounded, we thus see, as in 22 that there exists a constant D_2 such that for all $i = 0, \dots, N-1$,

$$\left|\frac{y_{i+1}-y_i}{\tau}\right| \leqslant D_2,$$

which in turn implies that

$$\forall\, s \in [t_i, t_{i+1}], \quad |\eta_{t,\varepsilon,\tau}(s) - \eta_{t,\varepsilon,\tau}(t_i)| \leqslant \tau D_2.$$

As the derivative of $\eta_{t,\varepsilon,\tau}(s)$ with respect to s is uniformly bounded by D_2 and constant on the time intervals $[t_i, t_{i+1}]$, and as L is $\mathcal{C}^1$ with uniformly bounded derivative on $\mathbb{R}\times\mathbb{R}^n \times B(0, D_2)$, we obtain

$$\left|\sum_{i=0}^{N-1} \kappa^{\tau}_{t_i,\varepsilon}(y_{i+1}, y_i) - \int_0^{N\tau} L\big(s, \eta_{t,\varepsilon,\tau}(s), \dot\eta_{t,\varepsilon,\tau}(s)\big)\mathrm{d}s\right| \leqslant \tau D_3 \tag{35}$$

for some constant D_3. Using the definition of the exact semi-group, we thus have

$$\begin{aligned}(T^{N\tau}_t u)|_{G_\varepsilon}(x) &\leqslant u\big(\eta_{t,\varepsilon,\tau}(t_N)\big) + \int_0^{N\tau} L\big(s, \eta_{t,\varepsilon,\tau}(s), \dot\eta_{t,\varepsilon,\tau}(s)\big)\mathrm{d}s \\ &\leqslant \mathcal{T}^{N\tau}_{t,\varepsilon}(u|_{G_\varepsilon})(x) + \tau D_3\end{aligned}$$

upon using (34) and (35). This proves the result. □

3. Long time behavior in the periodic case

We will now make the supplementary assumption that the potential function $V(t,x)$ is periodic, namely

(iii) The function V is $\mathbb{Z}\times\mathbb{Z}^n$-periodic, in the sense that

$$\forall (t,x) \in \mathbb{R}\times\mathbb{R}^n, \quad \forall (m, M) \in \mathbb{Z}\times\mathbb{Z}^n, \quad V(t,x) = V(t+m, x+M).$$

Moreover, $V \in \mathcal{C}^2(\mathbb{R}\times\mathbb{R}^n)$.

Note that under this assumption, the estimate (7) is automatically satisfied.

3.1. Weak-KAM theorem and a priori compactness. — In this periodic case, the weak-KAM theorem allows to study the long time behavior of the solution of (1) defined by the Lax-Oleinik semi-group:

Proposition 3.1. — *Assume that the hypothesis **(i)** and **(iii)** are satisfied. Then there exists a unique constant $\overline{H}$ such that there exists a $\mathbb{Z}\times\mathbb{Z}^n$-periodic continuous function $u^*:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ verifying for all $t>0$,*

$$T_t^1 u^*(t,\cdot)=\overline{H}+u^*(t,\cdot).$$

Moreover, for any uniformly bounded $u:\mathbb{R}^n\to\mathbb{R}$, there exists a constant C_u such that we have

$$\forall\, t>0,\quad |T_0^t u - t\overline{H}|_\infty \leqslant C_u.$$

Proof. — The existence of $\overline{H}$ and of u^* is exactly the content of the weak-KAM theorem (see [**Fat97, Fat05**] for the autonomous case, and [**CISM00**] for the time periodic case). The second assertion is a consequence of the fact that

$$|T_0^t u - T_0^t v|_\infty \leqslant |u-v|_\infty$$

for all continuous bounded functions u and v on $\mathbb{R}^n$, where $|\cdot|_\infty$ denotes the L^∞ norm on $\mathbb{R}^n$. □

The goal of this section is to prove that a similar result holds for the numerical scheme described in the previous section, under the assumptions ***(i)*** and ***(iii)***.

In order to study the long time behavior of the method in this case, we first give an a priori compactness result which refines the estimates given in Proposition 2.2. The following proposition is mainly due to Mather (see [**Mat91**] for the case of time periodic Lagrangians, or [**Itu96**, Lemma 7 and Corollary 8] for space periodic Lagrangians).

Proposition 3.2. — *Assume that H satisfies the hypotheses **(i)** and **(iii)**. For all $\Gamma>0$, there exists a constant Γ' such that for any minimizer of the Lagrangian action $\gamma:[a,b]\to\mathbb{R}^n$ with $b-a\geqslant 1$ and $|\gamma(b)-\gamma(a)|/(b-a)\leqslant\Gamma$ then we have*

$$\forall t\in[a,b],\quad |\dot\gamma(t)|\leqslant\Gamma'.$$

In other words, the constant $M(R,T)$ of Proposition 2.2 can be chosen to be an increasing function of R/T.

For the sake of completeness, we will give in appendix a proof of this proposition. Note that most of the proof - essentially taken from [**Mat91**] - does not require the Hamiltonian to be periodic in space. In [**Itu96**] a similar result is proven which requires the Lagrangian to be periodic in space, but not any more in time.

3.2. Convergence estimates in the periodic case. — Using the previous proposition, we can compute explicitly the time dependence in the error estimates of Theorems 2.6 and 2.9 in the periodic case.

Theorem 3.3. — *Let $T_0 > 1$, ε_0, τ_0 and $h_0 > 0$ and $u : \mathbb{R}^n \to \mathbb{R}$ a bounded Lipschitz function. There exists a constant M such that for all $\varepsilon > 0$ and $\tau > 0$ such that $\varepsilon < \varepsilon_0$, $\tau < \tau_0$ and*

$$\frac{\varepsilon}{\tau} < h_0, \tag{36}$$

for all N satisfying $N\tau \geqslant T_0$,

$$\left|(T_t^{N\tau} u)|_{G_\varepsilon} - T_{t,\varepsilon}^{N\tau}(u|_{G_\varepsilon})\right|_\infty \leqslant MN\tau\big(\frac{\varepsilon}{\tau} + \tau\big),$$

and in similar way,

$$\left|(T_t^{N\tau} u)|_{G_\varepsilon} - \mathcal{T}_{t,\varepsilon}^{N\tau}(u|_{G_\varepsilon})\right|_\infty \leqslant MN\tau\big(\frac{\varepsilon}{\tau} + \tau\big). \tag{37}$$

Proof. — In the proof of Theorem 2.6, equation (22) then gives using Proposition 3.2 (with $T \geqslant T_0 > 1$) that the constant M_1 defined in (23) does not depend on $T = N\tau$ and depend in fact only on T_0. It then follows that M_2 and M_3 also are independent of $T = N\tau$, while M_4 is proportional to the time of integration, that is $N\tau$. In a similar way, in Theorem 2.9, the additional error made is proportional to the time of integration, hence giving the second part of Theorem 3.3. □

3.3. Discrete weak-KAM theorem and effective Hamiltonian. — Recall that the function $u(t,x)$ is defined on $\mathbb{T}^1 \times \mathbb{T}^n = \big(\mathbb{R}/\mathbb{Z}\big) \times \big(\mathbb{R}^n/\mathbb{Z}^n\big)$. For convenience, we will only treat the cases of rational time and space discretizations: We set

$$\Lambda = \Big\{\Big(\frac{1}{k}, \frac{1}{\ell}\Big) \mid (k,\ell) \in \mathbb{N}^* \times \mathbb{N}^*\Big\},$$

and in the sequel, we will only consider stepsizes $(\varepsilon,\tau) \in \Lambda$. We then will denote by p the canonical projection from $\mathbb{R}^n$ to $\mathbb{T}^n$, and by $\widetilde{G}_\varepsilon = G_\varepsilon/\mathbb{Z}^n$ the quotiented grid, where G_ε is the grid above, defined on $\mathbb{R}^n$.

Finally, we define two new cost functions: For $(\varepsilon,\tau) \in \Lambda$ and $t > 0$,

$$\forall\, (\tilde{x},\tilde{y}) \in (\widetilde{G}_\varepsilon)^2, \quad \tilde{c}_{t,\varepsilon}^{\tau}(\tilde{x},\tilde{y}) = \inf_{\substack{p(x)=\tilde{x}\\ p(y)=\tilde{y}}} c_{t,\varepsilon}^{\tau}(x,y),$$

where $c_{t,\varepsilon}^{\tau}(x,y)$ is defined in (14), and similarly

$$\forall\, (\tilde{x},\tilde{y}) \in (\widetilde{G}_\varepsilon)^2, \quad \tilde{\kappa}_{t,\varepsilon}^{\tau}(\tilde{x},\tilde{y}) = \inf_{\substack{p(x)=\tilde{x}\\ p(y)=\tilde{y}}} \kappa_{t,\varepsilon}^{\tau}(x,y),$$

where $\kappa_{t,\varepsilon}^{\tau}(x,y)$ is the fully discrete cost function defined in (31).

We then define the following semi-groups: Let $\tilde{u} : \mathbb{T}^n \to \mathbb{R}$ and $(\varepsilon, \tau) \in \Lambda$, then

$$\forall\, \tilde{x} \in \widetilde{G}_\varepsilon, \quad \widetilde{T}^\tau_{t,\varepsilon}\tilde{u}(\tilde{x}) = \inf_{\tilde{y}\in\widetilde{G}_\varepsilon} \tilde{u}(\tilde{y}) + \tilde{c}^\tau_{t,\varepsilon}(\tilde{y}, \tilde{x}), \tag{38}$$

and the fully discrete semi-group

$$\forall\, \tilde{x} \in \widetilde{G}_\varepsilon, \quad \widetilde{\mathcal{T}}^\tau_{t,\varepsilon}\tilde{u}(\tilde{x}) = \inf_{\tilde{y}\in\widetilde{G}_\varepsilon} \tilde{u}(\tilde{y}) + \tilde{\kappa}^\tau_{t,\varepsilon}(\tilde{y}, \tilde{x}).$$

As in the previous section, we define

$$\widetilde{T}^{N\tau}_{t,\varepsilon} = \widetilde{T}^\tau_{t_{N-1},\varepsilon} \circ \cdots \circ \widetilde{T}^\tau_{t_1,\varepsilon} \circ \widetilde{T}^\tau_{t_0,\varepsilon}$$

and

$$\widetilde{\mathcal{T}}^{N\tau}_{t,\varepsilon} = \widetilde{\mathcal{T}}^\tau_{t_{N-1},\varepsilon} \circ \cdots \circ \widetilde{\mathcal{T}}^\tau_{t_1,\varepsilon} \circ \widetilde{\mathcal{T}}^\tau_{t_0,\varepsilon},$$

where $t_i = t + i\tau$.

We leave to the reader the verification that if $\tilde{u} : \mathbb{T}^n \to \mathbb{R}$ is a function and if $u : \mathbb{R}^n \to \mathbb{R}$ is its lift (which is then $\mathbb{Z}^n$ periodic), both functions $T^{N\tau}_{t,\varepsilon}u$ and $\mathcal{T}^{N\tau}_{t,\varepsilon}u$ are $\mathbb{Z}^n$ periodic and that the functions they respectively canonically induce on $\mathbb{T}^n$ are $\widetilde{T}^{N\tau}_{t,\varepsilon}\tilde{u}$ and $\widetilde{\mathcal{T}}^{N\tau}_{t,\varepsilon}\tilde{u}$. It comes from the fact that two infimums commute. Hence the previous convergence result Theorem 3.3 can be read equivalently on $\mathbb{T}^n$ or on the space of $\mathbb{Z}^n$ periodic functions on $\mathbb{R}^n$.

We can use the discrete weak KAM theorem to better understand the approximate semi-groups applied for a period 1 of time and obtain the following proposition.

Proposition 3.4. — *For any $(\varepsilon, \tau) \in \Lambda$, there exist unique constants $\overline{H}_{\varepsilon,\tau}$ and $\overline{\mathcal{H}}_{\varepsilon,\tau}$ such that there exist functions $u^*_{\varepsilon,\tau}, v^*_{\varepsilon,\tau} : \widetilde{G}_\varepsilon \to \mathbb{R}$ verifying:*

$$\widetilde{T}^1_{0,\varepsilon}u^*_{\varepsilon,\tau} = u^*_{\varepsilon,\tau} + \overline{H}_{\varepsilon,\tau},$$

and

$$\widetilde{\mathcal{T}}^1_{0,\varepsilon}v^*_{\varepsilon,\tau} = v^*_{\varepsilon,\tau} + \overline{\mathcal{H}}_{\varepsilon,\tau}.$$

Moreover, in u is any bounded initial datum on G_ε at $t = 0$, then we have in L^∞

$$\frac{1}{N\tau}T^{N\tau}_{0,\varepsilon}u \longrightarrow \overline{H}_{\varepsilon,\tau},$$

and

$$\frac{1}{N\tau}\mathcal{T}^{N\tau}_{0,\varepsilon}u \longrightarrow \overline{\mathcal{H}}_{\varepsilon,\tau},$$

as $N \to +\infty$.

Proof. — The first part is just a reformulation of the discrete weak KAM theorem (see for example the appendix of [**Zav12**] or [**BB07**]) while the second part is - as in the proof of Proposition 3.1 - a direct consequence of the fact that our approximation operators are weakly contracting for the infinity norm on bounded functions. □

Remark 3.5. — Note that the previous proposition can be interpreted in the (min,plus) framework: Equation (38) is the (min,plus) product of a vector $\tilde{u}$ by the matrix $\tilde{c}^{\tau}_{t\,e}$. Then, for $\tau = 1/\ell$, $\ell \in \mathbb{N}^*$, $\widetilde{T}^{1}_{0,\varepsilon}\tilde{u} = \widetilde{T}^{\ell\tau}_{0,\varepsilon}\tilde{u}$ is obtained by successive matrix multiplications with $\tilde{u}$. Hence there exists a matrix $C_{\varepsilon,\tau}$ such that $\widetilde{T}^{1}_{0,\varepsilon}\tilde{u}(\tilde{x}) = \inf_{\tilde{y}\in\tilde{G}_\varepsilon} \tilde{u}(\tilde{y}) + C_{\varepsilon,\tau}(\tilde{y},\tilde{x})$, with $C_{\varepsilon,\tau}(\tilde{y},\tilde{x}) < +\infty$ for all $\tilde{y},\tilde{x}$. The matrix $C_{\varepsilon,\tau}$ has a unique eigenvalue $H_{\varepsilon,\tau}$, and $\tilde{u}_{\varepsilon,\tau}$ it an eigenvector (see [**BCOQ92**] for details).

Let us recall that $\overline{H}$ is the effective Hamiltonian of H. It is obtained in homogenization theory by solving the cell problem ([**LPV87**]) and is also the constant found in the weak KAM theorem (3.1). Using the refined convergence result obtained in Theorem 3.3, we can estimate the error between the effective Hamiltonian and the discrete effective Hamiltonians defined in Proposition 3.4.

Theorem 3.6. — *With the notations of Theorem 3.3, let $(\varepsilon,\tau) \in \Lambda$ be such that $\varepsilon \leqslant \varepsilon_0$, $\tau \leqslant \tau_0$ and $\varepsilon/\tau \leqslant h_0$, then following inequalities hold:*

$$\left|\overline{H}_{\varepsilon,\tau} - \overline{H}\right| \leqslant M\Big(\frac{\varepsilon}{\tau} + \tau\Big),$$

and

$$\left|\overline{\mathcal{H}}_{\varepsilon,\tau} - \overline{H}\right| \leqslant M\Big(\frac{\varepsilon}{\tau} + \tau\Big),$$

where M is any constant coming from 3.3, and where $H_{\varepsilon,\tau}$ and $\mathcal{H}_{\varepsilon,\tau}$ are defined in Proposition 3.4.

Proof. — We will only prove the first inequality, the second being obtained in the same way. Start with a bounded and uniformly Lipschitz continuous function $u : \mathbb{R}^n \to \mathbb{R}$. By 3.3, the following inequality holds if $N\tau \geqslant T_0$ for some chosen $T_0 > 1$:

$$\left|(T_0^{N\tau}u)|_{G_\varepsilon} - T_{0,\varepsilon}^{N\tau}(u|_{G_\varepsilon})\right|_\infty \leqslant MN\tau\Big(\frac{\varepsilon}{\tau} + \tau\Big). \tag{39}$$

Dividing by $N\tau$ and letting N go to ∞ yields that

$$\left|\overline{H}_{\varepsilon,\tau} - \overline{H}\right| \leqslant M\Big(\frac{\varepsilon}{\tau} + \tau\Big).$$

□

Remark 3.7. — By Proposition 2.5, we always have the following additional information: $\overline{H}_{\varepsilon,\tau} \leqslant \overline{H}$.

Remark 3.8. — One may wonder what is the behavior of the quantity $(T_0^{N\tau}u)|_{G_\varepsilon} - T_{0,\varepsilon}^{N\tau}(u|_{G_\varepsilon})$ as $T = N\tau \to \infty$. The previous results show that it has a linear growth, of rate $\overline{H} - \overline{H}_{\varepsilon,\tau}$. Comparing with weak KAM solutions yields that the second error term is always bounded. However, in some cases more can be said. Indeed, in the autonomous case (L independant of t) Fathi proved the convergence of the Lax–Oleinik semigroup ([**Fat98**]), that is, for any initial condition u there exists a weak KAM solution u^* such that $(T_0^{N\tau}u) - N\tau\overline{H} \to u^*$ uniformly. Moreover, it can be proved that the iterated powers of a $(\min, +)$ matrix are periodic after a finite time. Therefore, for N big enough, the sequence $T_{0,\varepsilon}^{N\tau}(u|_{G_\varepsilon})$, is periodic after a certain time. In conclusion, in the autonomous case, one can write

$$(T_0^{N\tau}u)|_{G_\varepsilon} - T_{0,\varepsilon}^{N\tau}(u|_{G_\varepsilon}) = N\tau(\overline{H} - \overline{H}_{\varepsilon,\tau}) + w_N,$$

where w_N is asymptotic to a periodic sequence.

4. Fast (min,plus)-convolution

As we have seen in (32), the numerical scheme considered in this paper involves the computation of the (min,plus) convolution

$$\inf_{y \in G_\varepsilon} \left(u(y) + \tau K^*\Big(\frac{x-y}{\tau}\Big)\right), \quad x \in G_\varepsilon.$$

In dimension 1, if the grid G_ε is discretized by retaining N points only, the numerical cost is *a priori* of order N^2. As we will see now, we can use a fast (min,plus)-convolution algorithm that turns out to have a linear cost (i.e. proportional to N) in many situations.

In order to ease the presentation, we will not deal with functions defined on a grid, but on functions defined on finite and closed intervals.

Let $a, b \in \mathbb{R}$ with $a < b$. We write $f : [a, b] \to \mathbb{R}$ if f is such that

$$\begin{cases} f(x) < \infty & \text{if } x \in [a, b] \\ f(x) = \infty & \text{otherwise.} \end{cases}$$

For $f : [a, b] \to \mathbb{R}$, we say that f is respectively convex, concave, affine if $f_{|[a,b]}$ is respectively convex, concave, affine. Let $f : [a, b] \to \mathbb{R}$ and $g : [c, d] \to \mathbb{R}$. The (min,plus)-convolution (or convolution in the remaining of the paper) of f and g is defined by: $\forall x \in \mathbb{R}$,

$$f * g(x) = \inf_{y \in \mathbb{R}} f(y) + g(x - y). \tag{40}$$

Recall that $f * g = g * f$. As f and g are finite only on an interval, it is easy to see that $\forall x \in [a + c, b + d]$,

$$f * g(x) = \inf_{y \in [a,b]} f(y) + g(x - y),$$

and $\forall x \notin [a+c, b+d]$, $f * g(x) = \infty$.

We will only consider piecewise affine functions and decompose them according to their affine components: there exists $a_0 = a < a_i < \cdots < a_n = b$ such that

$$f = \min_{i \in \{0,\dots,n-1\}} f_i,$$

where $f_i : [a_i, a_{i+1}] \to \mathbb{R}$ is an affine function. For $i = 0, \dots, n-1$, we denote by $f_i' = (f(a_{i+1}) - f(a_i))/(a_{i+1} - a_i)$ the *slope* of f_i or the slope of f on $[a_i, a_{i+1}]$.

4.1. Convolution. — The fast algorithm to compute the convolution (40) is based on a decomposition of g $(= u)$ in piecewise convex and concave functions. As the function f $(= K^*(\cdot/\tau))$ considered will always be convex (see (6)), we thus see that we are led to compute separately the convolution of convex by convex functions, and concave by convex functions defined on finite intervals. As we will see, each block can be computed at a linear cost. In the end, the global cost of the algorithm thus depends on the number of convex and concave components on f, a number which might increase in the time evolution of the numerical solution of the Hamilton-Jacobi equation. We will come back later to this matter, but we emphasize that this procedure can be very easily implemented in parallel, each convolution block being calculated independently.

We start with the following result, the proof of which can be found for example [**BT08**].

Lemma 4.1 **(convolution of a convex function by an affine function)**
Let $f : [a, b] \to \mathbb{R}$ be a convex piecewise affine function and $g : [c, d] \to \mathbb{R}$ be an affine function of slope g'. Then $f * g : [a+c, b+d] \to \mathbb{R}$ is a convex piecewise affine function defined by

$$f * g(x) = \begin{cases} f(x-c) + g(c) & \text{if} \quad a + c \leqslant x \leqslant \alpha + c, \\ f(\alpha) + g(x - \alpha) & \text{if} \quad \alpha + c < x \leqslant \alpha + d, \\ f(x-d) + g(d) & \text{if} \quad \alpha + d < x \leqslant b + d, \end{cases}$$

where $\alpha = \min\{a_i$ in the decomposition of $f \mid f_i' \geqslant g'\}$.

Figure 1 illustrates this lemma. In the rest of the section, we will use a decomposition of such a convolution into three parts : $f * g = \min(g^1, g^c, g^2)$, where

(i) $g^1 = f * g_{|[c+a, c+\alpha]}$;
(ii) $g^c = f * g_{|[c+\alpha, d+\alpha]}$;
(iii) $g^2 = f * g_{|[d+\alpha, d+b]}$.

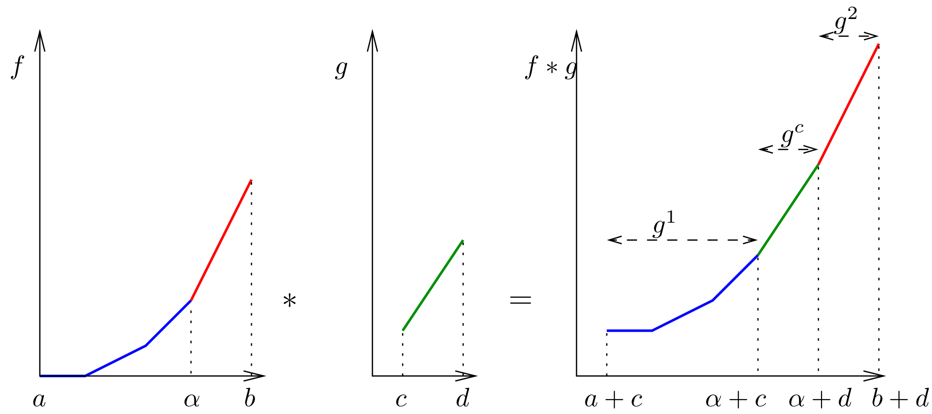

FIGURE 1. Convolution of a convex function by an affine function and decomposition into three functions.

In other words, g^1 is composed of the segments of f whose slope are strictly less than that of g, g^c corresponds to the segment g and g^2 is composed of the segments of f whose slope are greater than or equal to that of g. Note that g^c is also concave.

A direct consequence of this lemma is the following theorem, stated in [**LBT01**]. A complete proof is presented in [**BJT08**].

***Theorem 4.2* (convolution of a convex function by a convex function)**
If f and g are convex and piecewise affine, then $f * g$ is obtained by putting end-to-end the different linear pieces of f and g sorted by increasing slopes.

For sake of completeness, we give below Algorithm 1 for computing the (min,plus)-convolution of two convex piecewise affine functions.

We now turn to the case where f is convex and g is concave. We begin with the following lemma:

Lemma 4.3. — *Let $f : [a, b] \to \mathbb{R}$ be a convex piecewise affine function and $g : [c, d] \to \mathbb{R}$ be a concave piecewise affine function which decomposition is $g = \min_{j=1}^{m} g_j$. Then*

$$f * g = \min_{j=1}^{m} f * g_j.$$

Proof. — This is a direct consequence of the distributivity of $*$ over the minimum. □

Now, consider two functions $f : [a, b] \to \mathbb{R}$, convex, and $g : [c, d] \to \mathbb{R}$, concave, with respective decompositions in $f_i : [a_i, a_{i+1}] \to \mathbb{R}$, $i \in \{0, n-1\}$ and $g_j : [c_j, c_{j+1}] \to \mathbb{R}$, $j \in \{0, m-1\}$. The following lemma, that considers

Algorithm 1: Convolution of two convex functions

Data: $f : [0, n] \to \mathbb{R}$ a convex function with slopes (r_i), $g : [0, m] \to \mathbb{R}$ a convex function with slopes (ρ_i).
Result: $h = f * g$
begin
  $i \leftarrow 0$; $j \leftarrow 0$; $h(0) \leftarrow f(0) + g(0)$;
  while $i + j < n + m$ **do**
    if $i \neq n$ *and* $(r_i < \rho_j$ *or* $j = m)$ **then**
      $h(i + j + 1) \leftarrow h(i + j) + r_i$; $i \leftarrow i + 1$;
    else
      $h(i + j + 1) \leftarrow h(i + j) + \rho_j$; $j \leftarrow j + 1$;
end

two consecutive affine functions of g, leads to an efficient algorithm to compute the convolution of a convex function by a concave function.

Lemma 4.4. — *Consider the convolutions $f * g_{j-1}$ and $f * g_j$. Let*

$$\alpha_j = \min\{a_i \textit{ in the decomposition of } f \mid f_i' \geq g_j'\}$$

and

$$\alpha_{j-1} = \min\{a_i \textit{ in the decomposition of } f \mid f_i' \geq g_{j-1}'\}.$$

Then

– $\forall x \leqslant c_j + \alpha_j$, $f * g_j(x) \geqslant f * g_{j-1}(x)$;
– $\forall x \geqslant c_j + \alpha_{j-1}$, $f * g_{j-1}(x) \geqslant f * g_j(x)$.

Proof. — First, as f is convex and $g_{j-1}' > g_j'$, we have that $\alpha_{j-1} \geq \alpha_j$. From Lemma 4.1, for all $x \leqslant c_j + \alpha_j$,

$$f * g_j(x) = f(x - c_j) + g(c_j).$$

Either $x \leqslant c_{j-1} + \alpha_{j-1}$, then $f * g_{j-1}(x) = f(x - c_{j-1}) + g(c_{j-1})$ and

$$f * g_j(x) - f * g_{j-1}(x) = f(x - c_j) - f(x - c_{j-1}) + g(c_j) - g(c_{j-1});$$

as $x - c_{j-1} \leqslant \alpha_{j-1}$, then $f(x - c_{j-1}) - f(x - c_j) \leqslant g_{j-1}' \cdot (c_j - c_{j-1})$ and $f * g_j(x) - f * g_{j-1}(x) \geqslant 0$;

or $x > c_{j-1} + \alpha_{j-1}$, then $f * g_{j-1}(x) = f(\alpha_{j-1}) + g(x - \alpha_{j-1})$; as $c_{j-1} < x - \alpha_{j-1} \leqslant x - \alpha_j \leqslant c_j$, $g(c_j) - g(x - \alpha_{j-1}) = g_{j-1}' \cdot (c_j + \alpha_{j-1} - x)$ and $f(\alpha_{j-1}) - f(x - c_j) \leq g_{j-1}' \cdot (c_j + \alpha_{j-1} - x)$; then $f * g_j(x) - f * g_{j-1}(x) \geq 0$.

The second statement can be proved similarly. □

Another formulation of Lemma 4.4 is that $g_j^1 \geq f * g_{j-1}$ and that $g_{j-1}^2 \geq f * g_j$ and that the two fonctions intersect at least once. Hence g_j^1 and g_{j-1}^2 cannot appear in the minimum of $f * g_j$ and $f * g_{j-1}$. By transitivity, there is no need

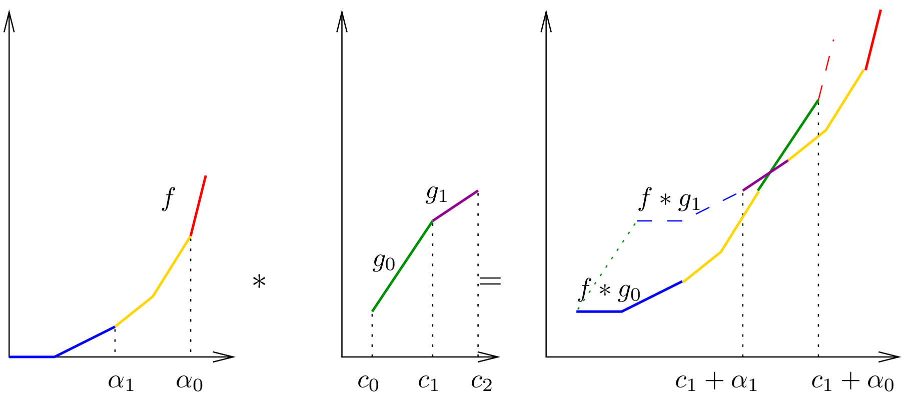

FIGURE 2. Convolution of a convex function by a concave function.

to compute entirely the convolution of the convex function by every affine component of the decomposition of the concave function. If there are more than two segments, successive applications of this lemma show that only the position of the segments of the concave function must be computed, except for the extremal segments.

The following lemma shows that $f * g_j$ and $f * g_{j-1}$ intersect in one and only one connected component, as for a given abscissa, the slope of $f * g_j$ is less than the one of $f * g_{j-1}$.

Lemma 4.5. — *Let $x \in \mathbb{R}$ where $f * g_j$ and $f * g_{j-1}$ are defined and differentiable. Then*

$$\frac{\mathrm{d}}{\mathrm{d}x} f * g_j(x) \leq \frac{\mathrm{d}}{\mathrm{d}x} f * g_{j-1}(x). \tag{41}$$

Proof. — For $x \in [a_0, \alpha_j]$, $f * g_j(x + c_j) = f(x - c_j) + g(c_j)$ and $f * g_{j-1}(x + c_{j-1}) = f(x) + g(c_{j-1})$. As f is convex and $c_j \geq c_{j-1}$, the result holds on $[a_0 + c_j, \alpha_j + c_j]$. Similarily, the result holds for $x \in [\alpha_{j-1} + c_j, a_n + c_j]$.

On $[c_j + \alpha_j, c_j + \alpha_{j-1}]$, $f * g_j$ is composed of segment g_j concatenated with the segments f_i, $i \in [\alpha_{j-1}, \alpha_j]$, possibly troncated on the right and $f * g_{j-1}$ is composed of segments f_i, $i \in [\alpha_{j-1}, \alpha_j]$ concatenated with g_{j-1}, possibly troncated on the left. As $\forall i \in [\alpha_{j-1}, \alpha_j]$, $g'_{j-1} \leq f'_i \leq g'_j$, the result holds. □

If one sets by convention $\frac{\mathrm{d}}{\mathrm{d}x} f * g_j(x) = -\infty$ for $x < c_{j-1} + a_0$ and $\frac{\mathrm{d}}{\mathrm{d}x} f * g_j(x) = +\infty$ for $x > c_j + a_n$, then the inequality always holds.

The intersection of $f * g_{j-1}$ and $f * g_j$ can then happen in one and only one of the four cases:

1. g^1_{j-1} and g^c_j intersect;
2. g^1_{j-1} and g^2_j intersect;
3. g^c_{j-1} and g^c_j intersect;
4. g^c_{j-1} and g^2_j intersect.

The following theorem is another consequence of these lemmas and is more precise about the shape on the convolution of a convex function by a concave function.

***Theorem 4.6* (convolution of a convex function by a concave function)**

The (min,plus)-convolution of a convex function by a concave function can be decomposed in three (possibly trivial) parts: a convex function, a concave function and a convex function.

Proof. — We use the notations defined in the former lemmas.

We now show by induction that the convolution of f by $\min_{k\leq j} g_j$, denoted h_j, is composed of

(i) a convex part h_j^1, which is the a restriction of $g_0(x) = f(x-c_0)+g(c_0)$ to $[c_0+a_0, \beta_j]$, with $\beta_j \leq c_0 + a_n$;

(ii) a concave part h_j^c, which is a minimum of some segments g_k, $k \leq j$ (up to some translation) defined on $[\beta_j, \gamma_j]$;

(iii) a convex part h_j^2, $g_j(x) = f(x-c_{j+1}) + g(c_{j+1})$ for $x \in [\gamma_j, a_n + c_{j+1}]$, $\gamma_j \geq a_0 + c_{j+1}$.

Note that with these conventions, the reals β_j and γ_j are uniquely determined at each step of the induction.

The case with $j=0$ is a direct consequence of Lemma 4.1. The case with $j=1$ is a consequence of Lemmas 4.4 and 4.1. The graphs of $f * g_1$ and $f * g_0$ intersect once and only once (where they are defined), and in $[c_1+\alpha_1, c_1+\alpha_0]$. Depending on when this intersection occurs, the concave part will be trivial, be made of only one (part of a) segment of g, or a minimum of the two segments g_0 and g_1.

Suppose now that the result holds for h_j and consider $h_{j+1} = \min(h_j, f * g_{j+1})$. The argument is exactly the same as for $j=1$: h_j and $f*g_{j+1}$ can only intersect once and only once. Indeed, h_j is the minimum of functions such that $\frac{\mathrm{d}}{\mathrm{d}x} f * g_k(x) \geq \frac{\mathrm{d}}{\mathrm{d}x} f * g_{j+1}(x)$, and then $\frac{\mathrm{d}}{\mathrm{d}x} h_j(x) \geq \frac{\mathrm{d}}{\mathrm{d}x} f * g_{j+1}(x)$. Note that this intersection has to occur after the point $c_{j+1} + \alpha_{j+1}$.

Moreover, as g_j^2 does not intersect g_{j+1}^2 and that h_j^2 is a part of g_j^2 (by the induction hypothesis), h_j^2 does not intersect g_{j+1}^2.

Therefore, only one of the four following cases may occur.

1. h_j^c intersects g_{j+1}^c and $h_{j+1}^1 = h_j^1$, $h_{j+1}^c = \min(h_j^c, g_{j+1}^c)$, $h_{j+1}^2 = g_{j+1}^2$, $\beta_{j+1} = \beta_j$ and $\gamma_{j+1} = \alpha_{j+1} + c_{j+2}$.
2. h_j^c intersects g_{j+1}^2 at y and $h_{j+1}^1 = h_j^1$, $h_{j+1}^c = h_j^c$, $h_{j+1}^2 = g_{j+1}^2$, $\beta_{j+1} = \beta_j$ and $\gamma_{j+1} = y$.
3. h_j^1 intersects g_{j+1}^c at y and $h_{j+1}^1 = h_j^1$, $h_{j+1}^c = h_j^c$, $h_{j+1}^2 = g_{j+1}^2$, $\beta_{j+1} = y$ and $\gamma_{j+1} = \alpha_{j+1} + c_{j+2}$.

4. h^1_j intersects g^2_{j+1} at y and $h^1_{j+1} = h^1_j$, h^c_{j+1} is trivial and $h^2_{j+1} = g^2_{j+1}$, $\beta_{j+1} = \gamma_{j+1} = y$.

□

If the concave function is composed of m segments and the convex function of n segments, then the convolution of those two functions can be computed in time $\mathcal{O}(n + m \log m)$. The $\log m$ term comes from the fact that one has to compute the minimum of m segments (see [BT08] for more details). If the functions are now defined on $\mathbb{N}$, then, as no intersection point has to be computed for the minimum, the time complexity is $O(n+m)$. The corresponding algorithm is given in Algorithm 2, where without loss of generality (the (min,plus)-convolution is shift-invariant), the functions f and g are defined on $\mathbb{N}$ and finite between respectively 0 and n, and 0 and m. The slopes of the functions are thus $f'_i = f(i) - f(i-1)$ and $g'_i = g(i) - g(i-1)$.

Algorithm 2: Convolution of a convex function by a concave function

Data: $f : [0, n] \to \mathbb{R}$ a convex function with slopes (r_i), $g : [0, m] \to \mathbb{R}$ a concave function with slopes (ρ_i).
Result: $h = f * g$

```
begin
    /* Initialization                                        */
    k ← 0;
    while k ⩽ m + n do h(k) ← +∞; k ← k + 1;
    i ← 0; j ← 0; h(0) ← f(0) + g(0);
    /* First convex part of the convolution                  */
    while f'_i ⩽ g'_0 do
        i ← i + 1; h(i) ← f(i) + g(0);
    /* Concave part of the convolution                       */
    j ← j + 1; h(i + j) ← f(i) + g(j);
    while j < m do
        while g'_j < f'_{i-1} do i ← i − 1;
        h(i + j) ← min(h(i + j), f(i) + g(j));
        h(i + j + 1) ← min(h(i + j + 1), f(i) + g(j + 1));
        j ← j + 1;
    /* Second convex part of the convolution                 */
    while i < n do
        i ← i + 1; h(i + m) ← min(h(i + m), f(i) + g(m));
end
```

4.2. Application. — We now go back to our initial problem. As already explained above, the computation of $\mathcal{T}^{\tau}_{t,\varepsilon}u(t,x)$ given in (32) is made of two steps:

- (min,plus)-convolution of u and $h : x \mapsto \tau K^*(\frac{x}{\tau})$;
- subtract $\tau V(t,x)$.

Note that the (min,plus)-convolution described above is here defined on functions that have a non-bounded support. But in the periodic case, u is 1-periodic and h is convex with a global minimum. Then, to compute the convolution, it is enough to compute it on a single period (the (min,plus) convolution preserves the periodicity), and replace h by its restriction on a support of size 2 centered on its minimum. If $\varepsilon = 1/N$ with the notation of the previous section, then both functions u and h are defined on grids of size N and $2N$ respectively.

The convolution of u and h can be efficiently computed following these steps:

1. Decompose u into convex and concave parts. This can be done in linear time: the three first points determine if a part is concave or convex. Then, this part is extended as much as possible while preserving the concavity or convexity and so on.
2. For each convex or concave part, perform the convolution with h using Algorithms 1 or 2.
3. Take the minimum of all these convolutions.

The complexity of this Algorithm is then $\mathcal{O}(cN)$, where c is the number of components in the decomposition of u into concave/convex parts.

4.3. Implementation issues. — The main issue with this algorithm is that c - the number of components in the decomposition of u - can become very large, and then lead to a quadratic time complexity, which is the complexity of a naive algorithm for computing the convolution. Experimentally, the reason for this is that, due to the discretization of u, nearly affine parts, after performing the convolution several times, are computed as fast alternations of convex and concave parts. As shown in Figure 3, one solution to make the computations more efficient would be to consider those parts are convex and use Algorithm 1.

To do this, we decompose u into convex and concave parts with a tolerance (we do not request for convex parts to have increasing increments, but the increments to have an increase more than $-\eta$). We will discuss this in the next section. The choice of an optimal tolerance η, as well as the comparison with parallel implementations, will be the subject of further studies.

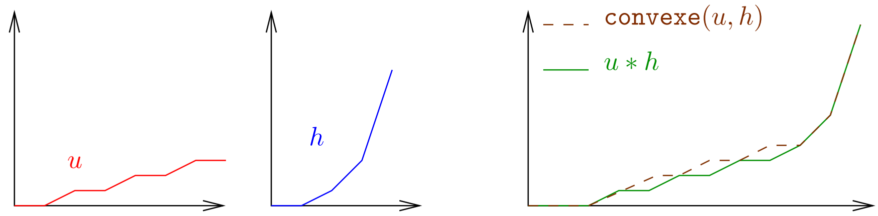

FIGURE 3. Approximation of the convolution: plain line shows the convolution of u and h, and the dashed line shows the function computed using Algorithm 1 when u is not convex, but has very small variations.

5. Numerical simulations

We take the Hamiltonian (3) with $P = 1$ and $V(t, x) = 1-\cos(x)$ on $[-\pi, \pi]$. We take $N = 600$ grid points, that is $\varepsilon = 1.7e - 3$, and $\tau = \sqrt{\varepsilon} = 0.04$. The comparisons are made with the fifth-order WENO algorithm (WENO5 see [JS96]) with 100 grid points, which will be considered here as the exact solution.

In a first simulation, we calculate the solution at times $t = 1, 2, 5$ and 15 with initial value $u(0, x) = \cos(2x)$. We see the very good agreement between the solution given by the WENO5 algorithm. However, the CPU time required by our algorithm is about 3 times the CPU time of the WENO5 algorithm (of order $2s$ to reach $t = 15$). In this case, for 600 grid points in $[-\pi, \pi]$, the number of convex/concave components c in the decomposition of u is of order 100 at each time step. The plot, rescaled such that $u(t, -\pi) = 0$, is shown in Figure 4. Note that after the time $t = 20$, the solution has converged and remains constant for larger times. According to Remark 3.8, the solution observed is thus very close to the weak-KAM solution $u^*(x)$.

In a second simulation, we use the approximated convolution with a tolerance $\eta = 10 \times \epsilon^2 = 2.7e - 5$. In this case, the number of convex/concave components c is always of order 10 (and equal to 3 - as expected from the shape of the solution - when the stationary state is attained), and the CPU time is reduced by a factor 20 when compared with $\eta = 0$ (and about 4 times quicker than the WENO5 algorithm), without any significant deterioration of the accuracy. We show the result in Figure 5.

Finally, we take $P = 2$ and the potential function $V(t, x) = \sin(t) \cos(2x)$, $\varepsilon = 0.01$ and $\tau = 0.1$. We take $\eta = 0$, and consider the initial data $u_0(x) = -\cos(3x)$. In Figure 6 we plot the evolution of $u(t, -\pi)$ with respect to the time. We observe the linear growth predicted by the result of the previous section, for a time interval $[0, 10000]$. Note that there are oscillations in the evolution of $u(t, -\pi)$, but at a scale too small to be visible on the plot.

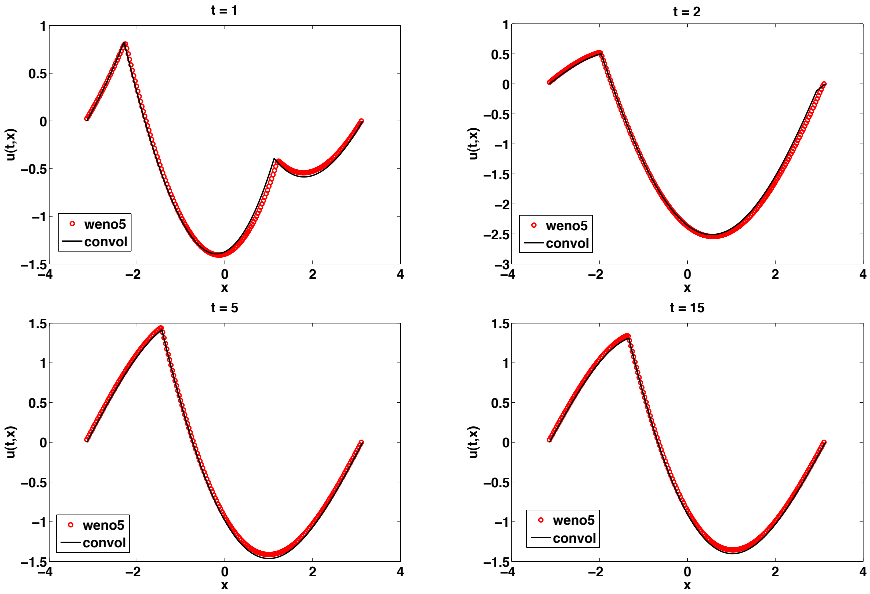

FIGURE 4. Solutions with $\eta = 0$. CPU $= 6s$ at $t = 15$

Appendix: Proof of the a priori compactness Proposition 3.2

We start with a lemma.

Lemma 5.1. — *Assume that the hypothesis* ***(i)*** *and* ***(iii)*** *are satisfied. Recall that* $L(t,x,v) = K^*(v) - V(t,x)$. *For any* $\Gamma > 1$, *there exists a constant* Γ' *such that for any* $x, y \in \mathbb{R}^n$ *and* $T > 0$ *and* $t > 1$, *if* $|x-y|/t < \Gamma$ *and* γ *that minimizes the quantity*

$$\inf_{\substack{\gamma(0)=x \\ \gamma(t)=y}} \int_0^t L\big(T+s, \gamma(s), \dot\gamma(s)\big)\mathrm{d}s,$$

then

$$\forall\, 0 \leqslant a \leqslant a+1 \leqslant t, \quad |\gamma(a) - \gamma(a+1)| < \Gamma'.$$

Proof. — Without loss of generality, we will assume that L is positive. Indeed, the potential V is bounded, and adding a constant doesn't change the minimizers. In this case, and under the hypothesis ***(i)***, there exists a nonnegative, increasing function α which tends to ∞ at ∞, such that

$$\forall t, x, v, \quad L(t,x,v) \geqslant \alpha(|v|)|v|.$$

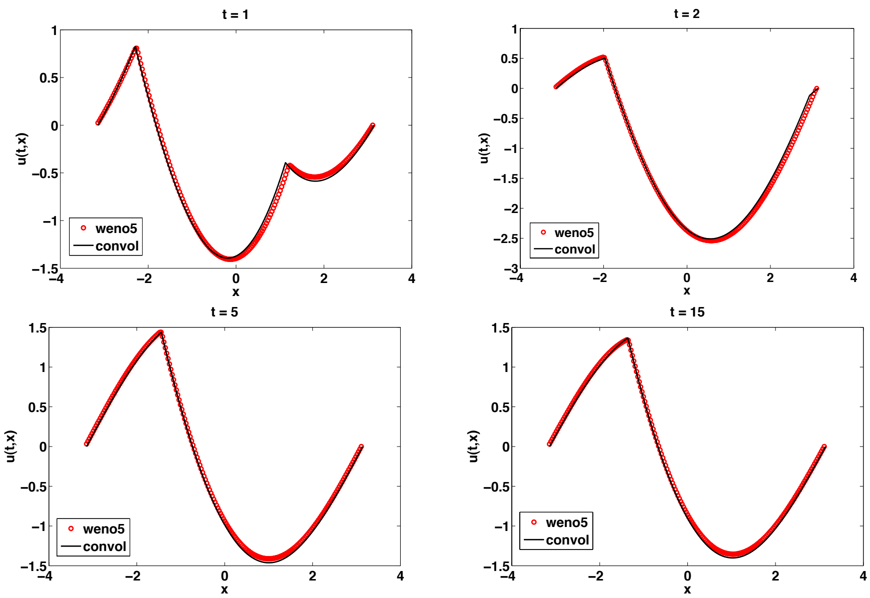

FIGURE 5. Solutions with $\eta = 1.7e-2$. CPU $= 0.3s$ at $t = 15$

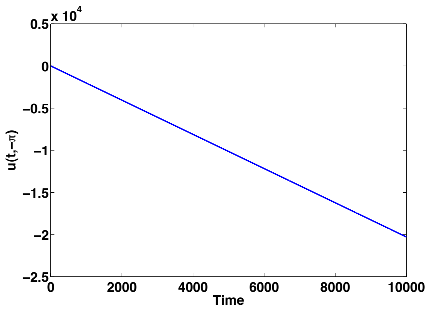

FIGURE 6. Evolution of $u(t, -\pi)$ over long times

The idea of the proof is that if γ at some point has a great velocity, then it must be slow later. It is then better to “slow down” the fast part and accelerate the “slow” one.

First, we set some notations. For all $(x, y, t, T) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}$, let

$$A_T^t(x, y) = \inf_{\substack{\gamma(0)=x \\ \gamma(t)=y}} \int_0^t L\big(T + s, \gamma(s), \dot{\gamma}(s)\big) \mathrm{d}s,$$

be the Lagrangian action.

We start by showing, that the superlinearity of L implies a superlinearity of A_T^t. As already done a few times, we may bound the action by comparing with a straight line, using that L is uniformly bounded on sets of the form $\mathbb{R}\times\mathbb{R}^n\times B(0,R)$, where $B(0,R)$ is the ball of radius $R>0$ in $\mathbb{R}^n$:

$$\begin{aligned} A_T^t(x,y) &\leqslant \int_0^t L\big(T+s,\frac{(t-s)x+sy}{t},\frac{y-x}{t}\big)\mathrm{d}s \\ &\leqslant tC^+\big(\frac{|x-y|}{t}\big)\max\Big(\frac{|x-y|}{t},1\Big), \end{aligned} \tag{42}$$

for some increasing function $z\mapsto C^+\big(z\big)$ defined on $\mathbb{R}^+$. Let γ realizing the infimum, and set

$$\mathcal{E}=\Big\{s\in[0,t]\quad\text{such that}\quad|\dot\gamma(s)|\geqslant\frac{|x-y|}{2t}\Big\}.$$

Then we get, using that $L>0$:

$$\begin{aligned} A_T^t(x,y) &= \int_0^t L\big(T+s,\gamma(s),\dot\gamma(s)\big)\mathrm{d}s \\ &\geqslant \int_{\mathcal{E}} L\big(T+s,\gamma(s),\dot\gamma(s)\big)\mathrm{d}s \\ &\geqslant \alpha\big(\frac{|x-y|}{2t}\big)\int_{\mathcal{E}}|\dot\gamma(s)|\mathrm{d}s\geqslant\alpha\big(\frac{|x-y|}{2t}\big)\frac{|x-y|}{2}. \end{aligned} \tag{43}$$

The last inequality comes from the fact that when γ is going at speed less than $|x-y|/2t$ for time t, it cannot travel more than $|x-y|/2$, therefore the integral is greater than $|x-y|/2$ by the triangular inequality. In other terms, equations (42) and (43) state that there are two positive functions C^+ and C^- which can be easily made increasing, such that

$$C^-\big(\frac{|x-y|}{t}\big)\frac{|x-y|}{t}\leqslant\frac{A_T^t(x,y)}{t}\leqslant C^+\big(\frac{|x-y|}{t}\big)\max\Big(\frac{|x-y|}{t},1\Big). \tag{44}$$

Moreover, thanks to the superlinearity of L those functions go to ∞ at ∞.

Now consider

- $\Gamma''>\Gamma$ such that $20C^+(\Gamma)<C^-(\Gamma'')$,
- $\Gamma'''>\Gamma''$ such that $\Gamma'''/\Gamma''\in\mathbb{N}^*$ and $30C^+(20\Gamma'')<C^-(\Gamma''')$,
- and finally $\Gamma'>\Gamma'''$ such that $40\Gamma'''/\Gamma''<C^-(\Gamma')/C^+(\Gamma)$.

Let us verify that Γ' satisfies the requirements of our lemma.

Assume by contradiction that for some $x,y\in\mathbb{R}^n$, $t,T\in\mathbb{R}_+$ such that $|x-y|<t\Gamma$ and γ realizing the action $A_T^t(x,y)$, there is an $a\in[0,t-1]$ such that $|\gamma(a)-\gamma(a+1)|\geqslant\Gamma'$. As γ is a minimizer, we have (using that $L>0$

and $\Gamma > 1$)

$$t\Gamma C^+(\Gamma) \geqslant A_T^t(x,y) \geqslant \int_a^{a+1} L\big(T+s,\gamma(s),\dot\gamma(s)\big)\mathrm{d}s \geqslant A_{T+a}^1\big(\gamma(a),\gamma(a+1)\big) \geqslant \Gamma' C^-(\Gamma'). \tag{45}$$

Hence we obtain

$$t \geqslant \frac{\Gamma' C^-(\Gamma')}{\Gamma C^+(\Gamma)} \geqslant \frac{40\Gamma'''}{\Gamma''}, \tag{46}$$

using the fact that $\Gamma' > \Gamma$ and the definition of Γ'.

We now assume that $a < t/2$, the other case may be treated similarly. Let $b \in [a, a+1]$ be the smallest such that $|\gamma(a) - \gamma(b)| = \Gamma'''$, and consider the sequence

$$c_i = b + 2i\frac{\Gamma'''}{\Gamma''}, \quad i \in \{0, \cdots, k\},$$

where k is greatest possible integer such that $c_k \leqslant t$. Note that using (46) and $a \leqslant t/2$, we have $k \geqslant 9$, and that for $i \in \{0, \cdots, k\}$, we have $c_{i+1} - c_i = 2\Gamma'''/\Gamma''$.

We claim that there exists an $i_0 \in \{0, \cdots, k\}$ such that

$$\frac{|\gamma(c_{i_0}) - \gamma(c_{i_0+1})|}{c_{i_0+1} - c_{i_0}} \leqslant \Gamma''.$$

Indeed, otherwise we would have, using (44)

$$\begin{aligned} A_T^t(x,y) &\geqslant \sum_{i=0}^{k-1} \int_{c_i}^{c_{i+1}} L\big(T+s,\gamma(s),\dot\gamma(s)\big)\mathrm{d}s \\ &\geqslant \sum_{i=0}^{k-1} \frac{|\gamma(c_{i_0}) - \gamma(c_{i_0+1})|}{c_{i_0+1} - c_{i_0}} C^-(\Gamma'') \\ &> \sum_{i=0}^{k-1} \frac{2\Gamma'''}{\Gamma''}\Gamma'' C^-(\Gamma''). \end{aligned}$$

By definition of k, we have that $(k+1) \times 2\Gamma'''/\Gamma'' \geqslant t/2 - 1$ while using (46), we have $t \geqslant 40\Gamma'''/\Gamma'' \geqslant 40$ and $2\Gamma'''/\Gamma'' \leqslant t/18$. Hence we deduce that $k \times 2\Gamma'''/\Gamma'' \geqslant t/3$. As $\Gamma'' \geqslant \Gamma$, the previous equation yields

$$A_T^t(x,y) > \frac{t}{3}\Gamma C^-(\Gamma'') \geqslant t\Gamma C^+(\Gamma),$$

which is absurd in view of (45).

Now we find a contradiction by constructing a curve δ which has an action less than γ. Let $[c,d] = [c_{i_0}, c_{i_0+1}]$. Recall that $N := \Gamma'''/\Gamma''$ is an integer. We define the curve δ as follows:

– $\delta(s) = \gamma(s)$ if $s \in [0, a] \cup [d, t]$;
– on $[a, b+N]$, δ coincides with the curve minimizing $A_{T+a}^{b+N-a}\big(\gamma(a), \gamma(b)\big)$;
– on $[b+N, c+N]$, δ is the translate of γ : $\delta(s) = \gamma(s-N)$;
– on $[c+N, d]$ (recall that $d = c+2N$) δ coincides with the curve minimizing $A_{T+c+N}^{N}\big(\gamma(c), \gamma(d)\big)$.

We now compute the difference of action between γ and δ, recalling that L is 1-periodic in time:

$$
\begin{aligned}
\int_0^t L\big(T+s,&\gamma(s), \dot\gamma(s)\big)\mathrm{d}s - \int_0^t L\big(T+s, \delta(s), \dot\delta(s)\big)\mathrm{d}s \\
&= \int_a^b L\big(T+s, \gamma(s), \dot\gamma(s)\big)\mathrm{d}s + \int_c^d L\big(T+s, \gamma(s), \dot\gamma(s)\big)\mathrm{d}s \\
&\quad - \int_a^{b+N} L\big(T+s, \delta(s), \dot\delta(s)\big)\mathrm{d}s - \int_{c+N}^d L\big(T+s, \delta(s), \dot\delta(s)\big)\mathrm{d}s \\
&\geqslant \Gamma''' C^-(\Gamma''') + \frac{2\Gamma'''}{\Gamma''}\Gamma'' C^-(\Gamma'') \\
&\quad - \frac{\Gamma'''}{\Gamma''}\Gamma'' C^+(\Gamma'') - \frac{\Gamma'''}{\Gamma''}(2\Gamma'')C^+(2\Gamma'') > 0.
\end{aligned}
$$

This contradicts the minimality of γ.

□

Remark 5.2. — In the previous proof, we only used the fact that L is periodic in time. In [**Itu96**], a similar result is proved when L is periodic in space (instead of in time). The idea of the proof is the same except that, when constructing the curve δ, instead of translating it in time (in third part of the construction), it is translated in space, while the “fast” part of γ between a and b is replaced by a geodesic (straight line) between $\gamma(a)$ and the closest point from $\gamma(a)$ in the grid $\gamma(b) + \mathbb{Z}^n$.

We now prove the lemma 3.2:

proof of lemma 3.2. — Recall now that L is periodic both in time and in space and that its Euler-Lagrange flow is complete. As in the previous lemma, assume $L > 0$. Let Γ and Γ' be as in the previous lemma, and γ be a minimizer such that $|\gamma(0) - \gamma(t)|/t \leqslant \Gamma$. The curve γ is then a trajectory of the Euler-Lagrange flow. Let moreover $0 \leqslant a \leqslant a+1 \leqslant t$. Finally, by superlinearity of L, let $A(1)$ be given by Equation (8), such that $L(t, x, v) \geqslant |v| - A(1)$. We therefore obtain that, with the notations used in the previous proof,

$$\int_0^1 |\dot\gamma(a+s)|\mathrm{d}s - A(1) \leqslant \int_0^1 L\big(T+a+s,\gamma(a+s),\dot\gamma(a+s)\big)\mathrm{d}s \leqslant C^+(\Gamma')\Gamma'.$$

Therefore, there is at least one point $s_0 \in [0,1]$ such that

$$|\dot\gamma(a+s_0)| \leqslant A(1) + C^+(\Gamma')\Gamma' := D.$$

By periodicity of the Lagrangian, and completeness of the Euler-Lagrange flow, there exists a constant D' depending only on D, such that $|\dot\gamma| \leqslant D'$ on $[a+s_0-1, a+s_0+1] \cap [0,t] \supset [a, a+1]$. Since a is arbitrary, this finishes the proof. □

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Anne Bouillard, Ecole Normale Supérieure, 45 rue d'Ulm, 75230 Paris Cedex 05, France.
E-mail : anne.bouillard@ens.fr

Erwan Faou, INRIA & Ecole Normale Supérieure de Cachan Bretagne, Avenue Robert Schumann 35170 Bruz, France • *E-mail :* Erwan.Faou@inria.fr

Maxime Zavidovique, Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie, Case 247 4, place Jussieu, 75252 Paris Cedex 05, France
E-mail : Maxime.Zavidovique@upmc.fr