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Submitted on 10 Oct 2012

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Simplicial simple-homotopy of flag complexes in terms of graphs

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Abstract

A flag complex can be defined as a simplicial complex whose simplices correspond to complete subgraphs of its 1-skeleton taken as a graph. In this article, by introducing the notion of s-dismantlability, we shall define the s-homotopy type of a graph and show in particular that two finite graphs have the same s-homotopy type if, and only if, the two flag complexes determined by these graphs have the same simplicial simple-homotopy type (Theorem 2.10, part 1). This result is closely related to similar results established by Barmak and Minian (2) in the framework of posets and we give the relation between the two approaches (theorems 3.5 and 3.7). We conclude with a question about the relation between the s-homotopy and the graph homotopy defined in [5].

Keywords: Barycentric subdivision, collapsibility, flag complexes, graphs, poset, simple-homotopy.

Introduction

Flag complexes are (abstract) simplicial complexes whose every minimal non-simplex has two elements (17,10,7); this means that a flag complex is completely determined by its 1-skeleton (all necessary definitions are recalled below). They constitute an important subset of the set of simplicial complexes; in particular, the barycentric subdivision of any simplicial complex is a flag complex and we know that a simplicial complex and its barycentric subdivision have the same simple-homotopy type.

Flag complexes arise naturally from the graph point of view and are also sometimes called clique complexes (5). Indeed, the 1-skeleton of a simplicial complex can be considered as a graph and it is easy to see that a simplicial complex $K$ is a flag complex if, and only if, we can write $K = \Delta_G(G)$ for some graph $G$ where, by definition, $\Delta_G(G)$ is the simplicial complex whose simplices are given by the complete subgraphs of $G$ (and this is sometimes taken as the definition of flag complexes, as in [12]).

In this paper, we are interested in the notion of simplicial simple-homotopy for flag complexes. We note that the determination of the simplicial simple-homotopy type is actually important not only for simplicial complexes but also for graphs because simplicial complexes arise in various constructions in graph theory. For example, this notion appears in the study of the clique graph ([14], [13]), in results about the polyhedral complex $\text{Hom}(G, H)$ introduced by Lovasz ([1], [11]) or in relation to evasiveness ([10]).

Simplicial simple-homotopy is defined by formal deformations themselves defined by the notion of elementary collapses consisting in the deletion of certain pairs of simplices (see §2). As the set of simplices of a flag complex is determined by its 1-skeleton seen as a graph, the aim of this paper is to relate formal deformations on flag complexes to certain operations on graphs. The key notion will be the one of s-dismantlability: the deletion or the addition of s-dismantlable vertices in a graph will play the role of elementary reductions or expansions ([6]) in a simplicial complex. More precisely, a vertex $g$ of a graph $G$ will be called s-dismantlable if its open neighborhood is a dismantlable graph and the deletion of an s-dismantlable vertex $g$ in $G$ is equivalent to the deletion of all simplices which contain $g$ in $\Delta_G(G)$.

In Section 1, we introduce the notion of s-dismantlability which allows us to define an equivalence relation for graphs; the equivalence class $[G]_s$ of a graph $G$ for this equivalence
relation will be called the \emph{s-homotopy type} of \( G \) and we give some properties related to these notions.

In Section \[2\] we study the correspondence between s-dismantlability in \( \mathcal{G} \) (the set of finite undirected graphs, without multiple edges) and simplicial simple-homotopy in \( \mathcal{K} \) (the set of finite simplicial complexes); we prove that two finite graphs \( G \) and \( H \) have the same s-homotopy type if, and only if, \( \Delta_{\mathcal{G}}(G) \) and \( \Delta_{\mathcal{G}}(H) \) have the same simple-homotopy type. Reciprocally, for a simplicial complex \( K \), \( \Gamma(K) \) is the graph whose vertices are the simplices of \( K \) and the edges are given by the inclusions; we show that two finite simplicial complexes \( K \) and \( L \) have the same simple-homotopy type if, and only if, \( \Gamma(K) \) and \( \Gamma(L) \) have the same s-homotopy type. We have to mention that these results need the introduction of barycentric subdivision in \( \mathcal{G} \), defined, for a graph \( G \), as the 1-skeleton of the usual barycentric subdivision (in \( \mathcal{K} \)) of \( \Delta_{\mathcal{G}}(G) \).

In Section \[3\] we consider the important class of flag complexes which results from posets: if \( P \) is a poset, \( \Delta_{\mathcal{P}}(P) \) is the flag complex whose simplices are given by chains of \( P \). In \( \{2\} \), Barmak and Minian define a notion of \emph{simple equivalence} in posets and show that there is a one-to-one correspondence between simple-homotopy types of finite simplicial complexes and simple equivalence classes of finite posets. As we have \( \Delta_{\mathcal{P}}(P) = \Delta_{\mathcal{G}}(\text{Comp}(P)) \) where \( \text{Comp}(P) \) is the comparability graph of \( P \), there is a close relation between this approach and our approach from graphs. We show that there is indeed a one-to-one correspondence between s-homotopy type in \( \mathcal{G} \) and simple homotopy type in \( \mathcal{P} \) and simple equivalence classes in the set \( \mathcal{P} \) of finite posets. Finally, we consider a triangle between finite graphs, posets and simplicial complexes recapitulating the close relations between s-homotopy type (in \( \mathcal{G} \)), simple type (in \( \mathcal{P} \)) and simple homotopy type (in \( \mathcal{K} \)).

In Section \[4\] we describe a weaker version of s-dismantlability on graphs which provides a closer connection with simplicial collapse for flag complexes (Proposition \[4,5\]).

Then we conclude in Section \[5\] with a question concerning the relation between s-homotopy and the graph homotopy defined in \[5\].

Some results have been set out in \[4\].

**Definitions, notations**

Let \( \mathcal{G} \) be the set of finite undirected graphs, without multiple edges. If \( G \in \mathcal{G} \), we have \( G = (V(G), E(G)) \) with \( E(G) \subseteq \{ \{g, g’\}, g, g’ \in V(G) \} \). For brevity, we write \( xy \in G \) or \( x \sim y \) for \( \{x, y\} \in E(G) \) and \( x \in G \) for \( x \in V(G) \). The closed neighborhood of \( g \) is \( N_G[g] := \{h \in G, g \sim h\} \cup \{g\} \) and \( N_G[g] := N_G[g] \setminus \{g\} \) is its open neighborhood. When no confusion is possible, a subset \( S \) of \( V(G) \) will also denote the subgraph of \( G \) induced by \( S \). We denote by \( G \setminus S \) the graph obtained from \( G \) by deleting \( S \) and all the edges adjacent to a vertex of \( S \). In particular, we use the notations \( G \setminus x \) and \( G \setminus xy \) to indicate the deletion of a vertex \( x \) or an edge \( xy \). The notation \( c = [g_1, \ldots, g_k] \) means that the subset \( \{g_1, \ldots, g_k\} \) of \( V(G) \) induces a complete subgraph of \( G \). We say that a graph \( G \) is a cone on a vertex \( g \in G \) if \( N_G[g] = G \). The notation \( pt \) will denote a graph reduced to a single vertex (looped or not looped).

Let \( \mathcal{K} \) be the set of (abstract) finite simplicial complexes. A simplicial complex \( K \) is a family of subsets of a finite set \( V(K) \) (the set of vertices of \( K \)) stable with respect to deletion of elements (if \( \sigma \in K \) and \( x \in \sigma \), then \( \sigma \setminus \{x\} \in K \)). An element \( \{x_0, x_1, \ldots, x_k\} \) of \( K \) is called a \( k \)-simplex and will be denoted by \( \lessdot x_0, x_1, \ldots, x_k \); the \( n \)-skeleton of \( K \) is the set \( K_n \) formed by all \( k \)-simplices of \( K \) with \( k \leq n \). The 0-skeleton is identified with the set \( V(K) \) of vertices of \( K \). A face of \( \sigma \) is \( \lessdot x_0, x_1, \ldots, x_k \) is any simplex included in \( \sigma \).

## 1 s-dismantlability and s-homotopy type in \( \mathcal{G} \)

### 1.1 Definitions

Let \( G \in \mathcal{G} \). We recall \([14, 3, 8]\) that a vertex \( g \) of \( G \) is called dismantlable if there is another vertex \( g’ \) of \( G \) which dominates \( g \) (i.e., \( g \neq g’ \) and \( N_G[g] \subseteq N_G[g’] \)); we note that this implies \( g \sim g’ \). A graph \( G \) is called dismantlable if it is reduced to a single vertex or if we can write \( V(G) = \{g_1, g_2, \ldots, g_n\} \) with \( g_i \) dismantlable in the subgraph induced by \( \{g_1, \ldots, g_i\} \), for \( 2 \leq i \leq n \). In particular, a dismantlable graph is necessarily non empty.
Definition 1.1 A vertex $g$ of a graph $G$ is called s-dismantlable in $G$ if $N_G(g)$ is dismantlable.

Let $H$ a subgraph of a graph $G$. We shall say that $G$ is dismantlable on $H$ if we can go from $G$ to $H$ by successive deletions of dismantlable vertices.

Example 1.2 No vertex of the graph $G_1$ of Fig. 2 is dismantlable but there are four s-dismantlable vertices, as the vertex $a$ in the picture ($N_{G_1}(a)$ is a dismantlable path).

![Figure 1: The vertex $a$ is s-dismantlable (and not dismantlable)](image)

Clearly, a dismantlable vertex is s-dismantlable (because the open neighborhood of a dismantlable vertex is a cone which is a dismantlable graph). When $g \in G$ is s-dismantlable, we shall write $G \leftarrow G \setminus \{g\}$ (elementary reduction); equivalently, $G \leftarrow G \cup \{x\}$ (elementary expansion) indicates the addition to $G$ of a vertex $x$ such that $N_{G \cup \{x\}}(x)$ is dismantlable. By analogy with the usual situation in $\mathcal{K}$, $G \leftarrow H$ (resp. $G \leftarrow H$) indicates that we can go from $G$ to $H$, by deleting (resp. adding) successively s-dismantlable vertices.

Definition 1.3 Two graphs $G$ and $H$ have the same s-homotopy type if there is a sequence $G = J_1, \ldots, J_k = H$ in $\mathcal{G}$ such that $G = J_1 \leftarrow J_2 \leftarrow \ldots \leftarrow J_{k-1} \leftarrow J_k = H$ where each arrow $\leftarrow$ represents the suppression or the addition of an s-dismantlable vertex.

This defines an equivalence relation in $\mathcal{G}$ and we shall denote by $[G]_s$ the equivalence class representing the s-homotopy type of a graph $G$. A graph $G$ will be called s-dismantlable if $[G]_s = [pt]_s$.

1.2 Properties

Lemma 1.4 Let $G, H \in \mathcal{G}$ such that $[G]_s = [H]_s$; then, there is a graph $W$ such that $G \leftarrow W \leftarrow H$.

proof: Let us suppose that $G$ and $H$ are two graphs such that $[G]_s = [H]_s$. It is sufficient to prove that an elementary reduction and an elementary expansion may be switched in the sequence of elementary operations from $G$ to $H$. So, let us suppose that in a graph $G'$, we have an elementary reduction followed by an elementary expansion:

$$(1) \quad G' \leftarrow G' \setminus g_1 \leftarrow (G' \setminus g_1) \cup \{g_2\}$$

This means that the graphs $N_{G'}(g_1)$ and $N_{G' \setminus g_1}(g_2)$ are dismantlable and, in particular, $g_2 \not\leftarrow g_1$. So we can adjoin $g_2$ to $G'$ by putting $N_{G'}(g_2) = N_{G' \setminus g_1}(g_2)$; of course, $N_{G'}(g_1) = N_{G'}(g_1)$ and the sequence (1) can be alternatively written

$$(2) \quad G' \leftarrow G' \cup \{g_2\} \leftarrow (G' \cup \{g_2\}) \setminus g_1$$

with isomorphic resultant graphs $(G' \setminus g_1) \cup \{g_2\}$ and $(G' \cup \{g_2\}) \setminus g_1$ and this proves that all the reductions can be pushed at the end of the sequence of elementary operations from $G$ to $H$. □

We recall that the suspension $SG$ of $G$ is the graph whose vertex set is $V(G) \cup \{x, y\}$ where $x$ and $y$ are two distinct vertices which are not in $V(G)$ and whose edge set is $E(G) \cup \{xy, g \in V(G)\}$; in the sequel, $SG$ will be also denoted by $G \cup \{x, y\}$. We shall need the following result:
Proposition 1.5 A graph $G$ is dismantlable if, and only if, its suspension $SG$ is dismantlable.

proof: Let $SG = G \cup \{x, y\}$. If $G$ is dismantlable, let $V(G) = \{g_1, g_2, \ldots, g_n\}$ with $g_i$ dismantlable in the subgraph induced by $\{g_1, \ldots, g_i\}$, for $2 \leq i \leq n$. Then we can write $V(SG) = \{g_1, x, y, g_2, \ldots, g_n\}$ with $g_i$ dismantlable in the subgraph induced by $\{g_1, x, y, \ldots, g_i\}$, for $2 \leq i \leq n$ and the subgraph of $SG$ induced by $\{g_1, x, y\}$ is of course dismantlable (it is a path). Let us now suppose that $SG$ is dismantlable and let $g$ be a dismantlable vertex in $SG$. If $g = x$ or $g = y$, this means that $G$ is a cone (because $N_{SG}(x) = N_{SG}(y) = G$ and a vertex $g'$ which dominates $g$ verifies $N_G(g') = G$). If $g \neq x$ and $g \neq y$, $g$ is also dismantlable in $G$ (because we have $\{x, y\} \in N_{SG}(g)$ and a vertex which dominates $g$ in $SG$ is necessarily different from $x$ and $y$ and, consequently, dominates $g$ in $G$). From this observation, it follows that when we delete a dismantlable vertex $g$ in $SG$, either $g \in G$ and $g$ is dismantlable in $G$, either $g \in \{x, y\}$ and this implies that $G$ is dismantlable (because it is a cone). By iteration of this procedure, we get that $G$ is a dismantlable graph. □

Lemma 1.6 (deletion of an edge) If $g$ and $g'$ are two distinct vertices of a graph $G$ such that $g \sim g'$ and $N_G(g) \cap N_G(g')$ is nonempty and dismantlable, then $[G]_{s} = [G \setminus gg']_{s}$. In other words, we can s-delete the edge $gg'$.

proof: We add to $G$ a vertex $x$ with edges $xz$ for every $z$ in $N_G(g) \setminus \{g'\}$ (it is an elementary expansion because $N_G[g] \setminus \{g'\}$ is a cone on $g$) and we write $G \cup x$ for the resulting graph. Let us verify that $g$ is s-dismantlable in $G \cup x$. We have $N_{G \cup x}(g) = N_G(g) \cup \{x\}$. If $N_G(g) \subseteq N_G[g']$, we can write $N_{G \cup x}(g) = (N_G(g) \cap N_G(g')) \cup x \cup g'$. If $N_G(g) \nsubseteq N_G[g']$, every $y$ in $N_G(g)$ which is not in $N_G[g']$ is dominated by $x$ in $N_{G \cup x}(g)$: so, $N_{G \cup x}(g)$ is dismantlable on its subgraph induced by the set of vertices $(N_G(g) \cap N_G(g')) \cup x \cup g'$. But $(N_G(g) \cap N_G(g')) \cup x \cup g'$ is the suspension of $N_G(g) \cap N_G(g')$ and, by Proposition [16], it is a dismantlable graph because $N_G(g) \cap N_G(g')$ is dismantlable. Thus, $g$ is s-dismantlable in $G \cup x$ and we can reduce $G \cup x$ on $(G \cup x) \setminus g$ which is clearly isomorphic to $G \setminus gg'$. □

Proposition 1.7 Let $G \in \mathcal{G}$ and $g \in V(G)$. If $N_G(g)$ is s-dismantlable, then $[G]_{s} = [G \setminus g]_{s}$.

proof: Let $g \in V(G)$ such that the graph $N_G(g)$ is s-dismantlable. Let us suppose first that $N_G(g) \nsubseteq pt$ and let $N_G(g) = \{y_1, y_2, \ldots, y_k\}$ with $y_i$ s-dismantlable in the subgraph of $N_G(g)$ induced by $\{y_1, \ldots, y_i\}$, for $2 \leq i \leq k$. Then, $N_G(y_i) \cap N_G(g)$ is dismantlable (it is the open neighborhood of $y_k$ in $N_G(g)$) and, by Lemma [14], $G$ is s-dismantlable on the graph $H_k = G \setminus y_kg$. Now, $N_{H_k}(y_i) = \{y_1, y_2, \ldots, y_{k-1}\}$ and $N_{H_k}(y_{k-1}) \cap N_{H_k}(y_i) = N_G(y_{k-1}) \cap N_G(y_i)$ is dismantlable in $H_k$ and, by the same argument, $H_k$ is s-dismantlable on $H_{k-1} = G \setminus \{y_{k-2}g, y_{k-1}g\}$. The iteration of this procedure shows that $G$ is s-dismantlable on $H_2 = G \setminus \{y_{k-2}g, y_{k-1}g, \ldots, y_3g, y_2g\}$. Of course, $g$ is s-dismantlable (in fact, dismantlable) in $H_2$ and this proves that $G$ is s-dismantlable on $G \setminus g$. If we don’t have $N_G(g) \nsubseteq pt$, we know by Lemma [14] that there is a graph $W$ such that $N_G(g) \nsubseteq W \nsubseteq pt$. Let $\{z_1, \ldots, z_m\}$ the set of vertices of $W$ which are not in $N_G(g)$. We define $H = G \cup \{z_1, \ldots, z_m\}$ with $N_H(z_i) = N_W(z_i) \cup \{g\}$. It is clear that $G \nsubseteq H$ and that $N_H(g) = W \nsubseteq pt$. Thus, by the previous discussion, we know that $H$ is s-dismantlable on $H \setminus g$ but $H \setminus g \nsubseteq G \setminus g$ (because of $W \nsubseteq N_G(g)$) and we can conclude that $[H]_{s} = [H \setminus g]_{s}$ (by $[G]_{s} = [H]_{s} = [H \setminus g]_{s} = [G \setminus g]_{s}$).

This result implies that the procedure of s-dismantlability can be done more rapidly by deleting a vertex whose open neighborhood is s-dismantlable. For example, we immediately get the following result:

Corollary 1.8 Let $G \in \mathcal{G}$. If $G$ is s-dismantlable, then $SG$ is also s-dismantlable.

proof: As $N_{SG}(x) = N_{SG}(y) = G$ and $G$ is s-dismantlable, we obtain by the previous proposition that $[SG]_{s} = [SG \setminus \{x, y\}]_{s}$, i.e. $[SG]_{s} = [G]_{s}$. □

Now, by analogy with the notion of collapsibility in simplicial complexes (see below), we introduce the following definition:
Definition 1.9 A graph $G$ is called s-collapsible if $G \not\supset pt$ (i.e. we can write $G = \{g_1, g_2, \ldots, g_n\}$ with $g_i$ s-dismantlable in the subgraph induced by $\{g_1, \ldots, g_i\}$, for $2 \leq i \leq n$).

Remark 1.10 Of course, a dismantlable graph is s-collapsible but the inclusion (of the family of dismantlable graphs in the family of s-collapsible graphs) is strict; for instance, the graph $G_1$ of example 1.2 is not dismantlable (because there is no dismantlable vertex) and s-collapsible: $G_1 \not\supset G_1 \setminus a$ because $a$ is s-dismantlable and it is easy to see that the graph $G_1 \setminus a$ is s-collapsible (even more: $G_1 \setminus a$ is dismantlable). Furthermore, it can be proved that this graph $G_1$ is minimal (in terms of number of vertices) in the family of s-collapsible graphs and without any dismantlable vertex.

Proposition 1.11 Let $G \in \mathcal{G}$. The graph $G$ is s-collapsible if, and only if, $SG$ is s-collapsible.

proof: Let $SG = G \cup \{x, y\}$. First, we observe that every s-dismantlable vertex $g$ in $G$ is also s-dismantlable in $SG$ because $NSG(g) = NG(g) \cup \{x, y\} = SN(c)(g)$ and the dismantlability of $SN(c)(g)$ is a consequence of the dismantlability of $NG(g)$ (Proposition 1.5).

Suppose now that $G \not\supset pt$ with $G = \{g_1, g_2, \ldots, g_n\}$ and $g_i$ s-dismantlable in the subgraph induced by $\{g_1, \ldots, g_i\}$, for $2 \leq i \leq n$. Then, by the previous observation, $g_i$ is s-dismantlable in the subgraph of $SG$ induced by $\{g_1, x, y, g_2, \ldots, g_i\}$, for $2 \leq i \leq n$. So, we have $SG \not\supset \{g_1, x, y\} \not\supset \{g_1\}$. Reciprocally, if $SG$ is s-collapsible, we shall prove that $G$ is also s-collapsible by induction on the number of vertices of $G$. If $|V(G)| = 1$, there is nothing to prove ($G$ and $SG$ are s-collapsible). Let us suppose that the s-collapsibleness of $SG$ implies the s-collapsibleness of $G$ if $|V(G)| = n$ for $n \geq 1$ and let $G$ with $|V(G)| = n + 1$ such that $SG$ is s-collapsible. Let $g \in V(SG)$ an s-dismantlable vertex. If $g \in \{x, y\}$, this means that $G$ is dismantlable (because $NSG(x) = NSG(g) = G$) and thus, s-collapsible. So, we can assume that $g \neq x$ and $g \neq y$. We have $NSG(g) = SN(c)(g)$; so, by Proposition 1.5 the dismantlability of $NSG(g)$ implies the dismantlability of $NG(g)$ and we have $G \not\supset G \setminus g$. Now, $SG \setminus g = S(G \setminus g)$ and we can apply the induction hypothesis to conclude that $G \setminus g$ is s-collapsible (and the same conclusion for $G$).

2 Relation with simple homotopy in $\mathcal{H}$

2.1 From $\mathcal{G}$ to $\mathcal{H}$

Let $K \in \mathcal{H}$: let us recall (3) that an elementary simplicial reduction (or collapse) in $K$ is the suppression of a pair of simplices $(\sigma, \tau)$ of $K$ such that $\tau$ is a proper maximal face of $\sigma$ and $\tau$ is not the face of another simplex (one says that $\tau$ is a free face of $K$).

![Figure 2: An elementary collapse](image)

This is denoted by $K \not\supset (K \setminus \{\sigma, \tau\})$ (and called elementary collapse or elementary reduction) or $(K \setminus \{\sigma, \tau\}) \not\supset K$ (elementary anticollapse or elementary expansion). More generally, a simplicial collapse $K \not\supset L$ (resp. anticollapse $K \not\supset L$) is a succession of elementary simplicial collapses (resp. elementary simplicial anticollapses) which transform $K$ into $L$. Collapses or anticollapses are called formal deformations. A simplicial complex $K$ is called collapsible if $K \not\supset pt$. Two simplicial complexes $K$ and $L$ have the same simple-homotopy type if there is (in $\mathcal{H}$) a finite sequence $K = M_1 \not\supset M_2 \not\supset \ldots \not\supset M_{k-1} \not\supset M_k = L$ where
each arrow \( \rightarrow \) represents a simplicial collapse or a simplicial anticollapse. We shall denote by \([K]_a\) the simple-homotopy type of \(K\).

Let us recall that for a simplex \( \sigma \in K\), \(\text{link}_K(\sigma) := \{ \tau \in K, \sigma \cap \tau = \emptyset \text{ and } \sigma \cup \tau \in K \}\) and \(\text{star}^a_K(\sigma)\) is the set of simplices of \(K\) containing \(\sigma\). Let us recall:

**Lemma 2.1** ([19, Lemma 2.7]) Let \(\sigma\) be a simplex of \(K \in \mathcal{K}\). If \(\text{link}_K(\sigma)\) is collapsible, then \(K \cong K \setminus \text{star}^a_K(\sigma)\).

We recall that the application \(\Delta_{\mathcal{G}}: \mathcal{G} \to \mathcal{K}\) is defined in the following way: if \(G \in \mathcal{G}\), \(\Delta_{\mathcal{G}}(G)\) is the simplicial complex whose simplices are the complete subgraphs of \(G\) (so we have \(V(\Delta_{\mathcal{G}}(G)) = V(G)\)). We note that if a graph \(G\) does not contain any triangle (i.e. a complete subgraph with three vertices), we can identify \(G\) and \(\Delta_{\mathcal{G}}(G)\) (we consider \(G\) either as a graph, or as a simplicial complex of dimension 1); it is the case of \(C_4\) in examples of Fig. 3.

**Lemma 2.2** ([14, Proposition 3.2]) Let \(G\) be a graph and \(g\) be a dismantlable vertex in \(G\); then \(\Delta_{\mathcal{G}}(G) \cong \Delta_{\mathcal{G}}(G \setminus g)\).

**proof**: Let \(a\) be a vertex which dominates \(g\) (i.e. \(N_G[g] \subseteq N_G[a]\) with \(a \neq g\)); for every maximal complete subgraph \(c\) of \(G\), we have \(g \in c \Rightarrow a \in c\). So, let \(c = [g,a,g_1,\ldots,g_n]\) a maximal complete subgraph of \(G\) which contains \(g\); then \(c' = [g,g_1,\ldots,g_n]\) is a free face of \(c\) (taken as a simplex of \(\Delta_{\mathcal{G}}(G)\)) and \(\Delta_{\mathcal{G}}(G) \setminus \Delta_{\mathcal{G}}(G) \setminus \{c,c'\}\). By iteration, \(\Delta_{\mathcal{G}}(G)\) collapses on the subcomplex formed by all simplices which do not contain \(g\), i.e. \(\Delta_{\mathcal{G}}(G) \setminus \Delta_{\mathcal{G}}(G \setminus g)\). \(\square\)

**Proposition 2.3** Let \(G, H \in \mathcal{G}\). Then, \(G \cong H \rightarrow \Delta_{\mathcal{G}}(G) \cong \Delta_{\mathcal{G}}(H)\).

**proof**: It suffices to prove that if \(g\) is s-dismantlable in \(G\), then \(\Delta_{\mathcal{G}}(G) \setminus g \cong \Delta_{\mathcal{G}}(G \setminus g)\). It follows from Lemma 2.2 that \(\Delta_{\mathcal{G}}(H)\) is collapsible for every dismantlable graph \(H\). Thus, by definition of s-dismantlability, \(\text{link}_{\Delta_{\mathcal{G}}(G)}(g)\) is collapsible when \(g\) is s-dismantlable (where \(g\) is the 0-simplex of \(\Delta_{\mathcal{G}}(G)\) determined by \(g\)). As \(\Delta_{\mathcal{G}}(G \setminus g) = \Delta_{\mathcal{G}}(G) \setminus \text{star}^a_{\Delta_{\mathcal{G}}(G)}(g)\), the conclusion follows from Lemma 2.1. \(\square\)

**Remark 2.4** The converse of Proposition 2.3 is not true; a counterexample is given by the graphs \(G\) and \(H\) of Fig. 4. Indeed, we have \(K \cong L = K \setminus \{a, b, c\} \cong L = \Delta_{\mathcal{G}}(H)\) and we don’t have \(G \cong H\) because there is no s-dismantlable vertex in \(G\).
2.2 From $\mathcal{K}$ to $\mathcal{G}$

We consider the application $\Gamma : \mathcal{K} \to \mathcal{G}$ whose definition is: if $K \in \mathcal{K}$, $\Gamma(K)$ is the graph whose vertices are the simplices of $K$ with edges $\{\sigma, \sigma'\}$ when $\sigma \subset \sigma'$ or $\sigma' \subset \sigma$.

If $\sigma$ is a simplex of $K \in \mathcal{K}$, we shall write $K[\sigma]$ for the simplicial subcomplex of $K$ formed by all faces of $\sigma$ $(K[\sigma] := \{\tau \in K, \tau \subset \sigma\})$; if $\tau$ is a maximal face of $\sigma$, $K[\sigma] \setminus \{\sigma, \tau\}$ is a simplicial complex.

In order to understand the relation of formal deformations to $s$-dismantlability, we have the following results:

**Lemma 2.5** If $\tau$ is a maximal face of $\sigma$ then $\Gamma(K[\sigma] \setminus \{\sigma, \tau\})$ is dismantlable.

**proof**: Let $\sigma = <a_0, a_1, \ldots, a_n>$ and $\tau = <a_1, \ldots, a_n>$. The vertices of $\Gamma(K[\sigma] \setminus \{\sigma, \tau\})$ are the simplices of $K[\sigma] \setminus \{\sigma, \tau\}$. These vertices can be written $<a_i, \ldots, a_{i_k}>$ with $0 \leq i_1 < i_2 < \ldots < i_k \leq n$ excepting $<a_0, a_1, \ldots, a_n>$ and $<a_1, \ldots, a_n>$; we shall say that such a vertex $<a_i, \ldots, a_{i_k}>$ contains $a$ if $a \in \{a_{i_1}, \ldots, a_{i_k}\}$. Every vertex $x = <a_i, \ldots, a_{i_n-1}>$ which does not contain $a_0$ is dismantlable (because it is dominated by $<a_0, a_1, \ldots, a_{i_n-1}>$ the unique $(n-1)$-simplex containing $<a_{i_1}, \ldots, a_{i_n-1}>$). Thus, $\Gamma(K[\sigma] \setminus \{\sigma, \tau\})$ can be dismantled on $\Gamma_{n-1}$ obtained by deleting all vertices corresponding to $(n-1)$-simplices which do not contain $a_0$. Next, every vertex $<a_{i_1}, \ldots, a_{i_{n-2}>}$ which does not contain $a_0$ is dismantlable in $\Gamma_{n-1}$ (because it is dominated by $<a_0, a_{i_1}, \ldots, a_{i_{n-2}>}$ the unique $(n-2)$-simplex containing $<a_{i_1}, \ldots, a_{i_{n-2}>}$). Thus, $\Gamma_{n-1}$ can be dismantled on $\Gamma_{n-2}$ obtained by deleting all vertices corresponding to $(n-2)$-simplices which do not contain $a_0$. The iteration of this procedure shows that $\Gamma(K[\sigma] \setminus \{\sigma, \tau\})$ is dismantlable on its subgraph induced by the vertices containing $a_0$. But this subgraph is a cone on $<a_0>$ and this shows that $\Gamma(K[\sigma] \setminus \{\sigma, \tau\})$ is dismantlable.

**Proposition 2.6** Let $K, L \in \mathcal{K}$. Then, $K \not\leq L \Rightarrow \Gamma(K) \not\leq \Gamma(L)$.

**proof**: It suffices to prove that if $\{\sigma, \tau\}$ is a collapsible pair in $K$, then $\Gamma(K) \not\leq \Gamma(K \setminus \{\sigma, \tau\})$. We note that $\Gamma(K \setminus \{\sigma, \tau\}) = \Gamma(K) \setminus \{\sigma, \tau\}$ and that the vertex $\tau$ is dismantlable in $\Gamma(K)$ (because it is dominated by $\sigma$); so, we have the reduction $\Gamma(K) \not\leq \Gamma(K) \setminus \{\sigma, \tau\}$. Now, we have $N_{\Gamma(K \setminus \{\sigma, \tau\}}(\sigma) = \Gamma(K[\sigma] \setminus \{\sigma, \tau\})$ (because $\sigma$ is a maximal simplex), and we conclude that $\sigma$ is $s$-dismantlable by the Lemma 2.5.

2.3 Barycentric subdivision

Let us recall the notion of barycentric subdivision in $\mathcal{K}$. If $K \in \mathcal{K}$, the $n$-simplices of the barycentric subdivision $\text{Bd}(K)$ (or $K'$) of $K$ are the $<\sigma_0, \sigma_1, \ldots, \sigma_n>$ composed of $n + 1$ simplices of $K$ such that $\sigma_0 \subset \sigma_1 \subset \ldots \subset \sigma_n$. Now, we define a similar notion in $\mathcal{G}$.

**Definition 2.7** If $G \in \mathcal{G}$, the barycentric subdivision $\text{Bd}(G)$ (or $G'$) of $G$ is the graph whose vertices are the complete subgraphs of $G$ and there is an edge between two vertices if, and only if, there is an inclusion between the two corresponding complete subgraphs.

**Remark 2.8** Each complete subgraph of cardinality at least two creates a new vertex in the barycentric subdivision (cf. Fig. 4). The equalities $\Gamma \circ \Delta_\mathcal{G} = \text{Bd}$ (in $\mathcal{G}$) and $\Delta_\mathcal{G} \circ \Gamma = \text{Bd}$ (in $\mathcal{K}$) follow directly from the definitions and will be useful in the following.
By the Proposition 2.6, we get
\[ \Delta \text{ and } [\Delta] \text{ of } G \in \{ g \}. \]

**Proof:** Let \( n \) be the cardinal of \( V(G) \); we choose to number the vertices of \( G \); thus, we have \( V(G) = \{ g_1, g_2, \ldots, g_n \} \). Let us recall that \( V(G') = \mathcal{C}(G) \), the set of complete subgraphs of \( G \). In what follows, every complete subgraph \( c \) is considered under its unique expression \( c = [g_{i_1}, \ldots, g_{i_k}] \) with \( 1 \leq i_1 < i_2 < \ldots < i_k \leq n \); we shall denote by \( i_k = \max(c) \). If \( g \in \{ g_{i_1}, \ldots, g_{i_k} \} \) and \( c = [g_{i_1}, \ldots, g_{i_k}] \in \mathcal{C}(G) \), we shall write \( g \in c \) and for \( c, d \in \mathcal{C}(G) \), we shall write \( c \subset d \) when \( c = [g_{i_1}, \ldots, g_{i_k}] \) and \( i_1, \ldots, i_k \in \{ j_1, \ldots, j_m \} \). We know that \( E(G') = \{ cd, c, d \in \mathcal{C}(G) \text{ with } c \subset d \text{ or } d \subset c \} \). To prove \( [G]_s = [G']_s \), we go from \( G \) to \( G' \) in two steps (addition and suppression of \( s \)-dismantlable vertices).

First step: For every \( c \in \mathcal{C}(G) \), we add a vertex \( \hat{c} \) to \( G \). We begin with complete subgraphs of cardinal 1, we proceed with complete subgraphs of cardinal 2, next with complete subgraphs of cardinal 3... until we have reached all complete subgraphs. When we add a vertex \( \hat{c} \) corresponding to the complete subgraph \( c = [g_{i_1}, \ldots, g_{i_k}] \) of cardinal \( k \), we add the edges \( \hat{c}g_{i_k} \) and \( \hat{c}g_j \) if \( j > i_k \) and \( c \cup g_j \in \mathcal{C}(G) \); this corresponds to the addition of an \( s \)-dismantlable vertex because the open neighborhood of \( \hat{c} \) (when we add it) is a cone on \( g_{i_k} \).

The graph \( H \) obtained at the end of the first step is such that \( V(H) = V(G) \cup V(G') \).

Second step: We note that \( g_1 \) is \( s \)-dismantlable in \( H \) (because \( N_H(g_1) = \{ \hat{g}_1 \} \cup N_G(g_1) \) is a cone on \( \hat{g}_1 \)) and, more generally, for \( 2 \leq i \leq n \), let us verify that the vertex \( g_i \) is \( s \)-dismantlable in \( H_i = H \setminus \{ g_1, \ldots, g_{i-1} \} \). We have \( W_i = N_{H_i}(g_i) = \{ \hat{c}, c \in \mathcal{C}(G), c \cup g_i \in \mathcal{C}(G) \text{ and } \max(c) \leq i \} \cup \{ g_j, g_j \in N_{G_i}(g_i) \text{ and } j > i \} \). Let us denote \( W'_i = \{ \hat{c}, c \in \mathcal{C}(G) \text{ and } \max(c) = i \} \cup \{ g_j, g_j \in N_{G_i}(g_i) \text{ and } j > i \} \), the cone on \( \hat{g}_1 \). We have either \( W_i = W'_i \), or \( W_i \) is dismantlable on \( W'_i \). Indeed, let us suppose \( W_i \neq W'_i \) and let \( \hat{c} \) such that \( c \cup g_i \in \mathcal{C}(G) \text{ and } \max(c) < i \). Then \( \hat{c}g_i \in W_i \) and \( N_{W_i}(\hat{c}) \subseteq N_{W'_i}(\hat{c}g_i) \); so, \( \hat{c} \) is dismantlable in \( W_i \) and more generally \( W_i \) is dismantlable on \( W'_i \). Consequently, \( W_i \) is dismantlable, i.e. \( g_i \) is \( s \)-dismantlable in \( H_i \). Thus, in \( H \), one can \( s \)-delete all vertices of \( G \) (in the following order: \( g_1, g_2, \ldots, g_n \)); the resultant graph is \( G' \).

\[ \square \]

2.4 Correspondence of homotopy classes

**Theorem 2.10** 1. Let \( G, H \in \mathcal{G} \); \( G \) and \( H \) have the same \( s \)-homotopy type if, and only if, \( \Delta_{\mathcal{G}}(G) \) and \( \Delta_{\mathcal{G}}(H) \) have the same simple-homotopy type:

\[ [G]_s = [H]_s \iff [\Delta_{\mathcal{G}}(G)]_s = [\Delta_{\mathcal{G}}(H)]_s \]

2. Let \( K, L \in \mathcal{K} \); \( K \) and \( L \) have the same \( s \)-homotopy type if, and only if, \( \Gamma(K) \) and \( \Gamma(L) \) have the same \( s \)-homotopy type:

\[ [K]_s = [L]_s \iff [\Gamma(K)]_s = [\Gamma(L)]_s \]

**Proof:** 1. \( \implies \): corollary of Proposition 2.9. \( \iff \): By the Proposition 2.10, we get \( [\Delta_{\mathcal{G}}(G)]_s = [\Delta_{\mathcal{G}}(H)]_s \implies [G]_s = [\Gamma(\Delta_{\mathcal{G}}(G))]_s = [\Gamma(\Delta_{\mathcal{G}}(H))]_s = [H']_s \) and we conclude with the Proposition 2.9.

---

**Figure 5:** Example of barycentric subdivision

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**Proposition 2.9** For every \( G \in \mathcal{G} \), \( G \) and \( G' \) have the same \( s \)-homotopy type (i.e. \( [G]_s = [G']_s \)).
2. corollary of Proposition 2.6 By using assertion 1 of the theorem, we obtain $[\Gamma(K)]_s = [\Gamma(L)]_s \implies [\Delta_\varphi(\Gamma(K))]_s = [\Delta_\varphi(\Gamma(L))]_s$. So, we get $[K']_s = [L']_s$ and we can conclude $[K]_s = [L]_s$ because it is well known that a simplicial complex and its barycentric subdivision have the same simple-homotopy type (F).

\textbf{Remark 2.11} It is clear that an $s$-collapsible graph is $s$-dismantlable (i.e., $G \not\triangleleft pt \implies [G]_s = [pt]_s$) and it follows from Proposition 2.6 and Theorem 2.10 that there exists $s$-dismantlable and not $s$-collapsible graphs. Indeed, there are well known examples of simplicial complexes $K$ (triangulations of the dunce hat (20) or of the Bing’s house (5), for instance) which are not collapsible ($K \not\triangleleft pt$) but have the same simple-homotopy type of a point ($[K]_s = [pt]_s$).

So, for example, the graph $D$ of Fig. 6 (with 17 vertices and 36 triangles) is $s$-dismantlable but not $s$-collapsible because $\Delta_\varphi(D)$ is a triangulation of the dunce hat. This graph $D$ is actually the comparability graph of a poset given in [13, Figure 2, p.380] and named here $P_d$.

An example of a graph $B$ such that $\Delta_\varphi(B)$ is a triangulation of the Bing’s house is given in [2, §5].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure6}
\caption{$\Delta_\varphi(D)$ is a triangulation of the dunce hat: $D = \text{Comp}(P_d)$}
\end{figure}

3 Relation with posets

3.1 From $\mathcal{P}$ to $\mathcal{G}$

Let $\mathcal{P}$ be the set of finite partially ordered sets or finite posets. In what follows, when $P \subseteq Q$ with $Q \in \mathcal{P}$, $P$ will be called subposet of $Q$ if, for every $x, y$ in $P$, $x \leq_P y$ if and only if, $x \leq_Q y$. If $P \in \mathcal{P}$, $\text{Comp}(P) \in \mathcal{G}$ is the comparability graph of $P$ (its vertices are the elements of $P$ with an edge $xy$ if, and only if, $x$ and $y$ are comparable).

Let $P \in \mathcal{P}$. For every $x$ in $P$, we define $P_{<x} := \{ y \in P, y < x \}$ and $P_{>x} := \{ y \in P, y > x \}$. We recall that $x$ is irreducible if either $P_{<x}$ has a maximum, or if $P_{>x}$ has a minimum. The poset $P$ is called dismantlable if we can write $P = \{ x_1, x_2, \ldots, x_n \}$ with $x_i$ irreducible in the subposet induced by $\{ x_1, \ldots, x_i \}$, for $2 \leq i \leq n$. Let us recall that a cone is a poset having a maximum or a minimum; if we can write $P = P_{\geq x}$ or $P = P_{\leq x}$ for some $x$ in $P$, $P$ will be called a cone on $x$. Cones are examples of dismantlable posets.

If $P, Q \in \mathcal{P}$, $P \ast Q$ is the poset whose elements are those of $P$ and $Q$ and with the relations $p \leq_P p', q \leq_Q q'$ and $p \leq q$ for all $p, p' \in P$ and $q, q' \in Q$. In particular, $P \ast \emptyset = \emptyset \ast P = P$ for all $P \in \mathcal{P}$.

\textbf{Lemma 3.1} Let $P, Q \in \mathcal{P}$; $P \ast Q$ is dismantlable if, and only if, $P$ or $Q$ is dismantlable.

\textbf{proof:} Let us suppose that $P \ast Q$ is dismantlable with $P \ast Q = \{ x_1, x_2, \ldots, x_N \}$ (where $N = |P| + |Q|$ and $x_i$ is irreducible in the subposet of $P \ast Q$ induced by $\{ x_1, \ldots, x_i \}$, for $2 \leq i \leq N$) and that $Q$ is a non dismantlable poset. We can write $P = \{ x_{i_1}, x_{i_2}, \ldots, x_{i_k} \}$ (where $k = |P|$) with $i_j < i_l$ for all $1 \leq j < l \leq k$. We shall verify that $P$ is dismantlable with $x_{i_k}$ irreducible in the subposet of $P \ast Q$ induced by $P \setminus \{ x_{i_1}, \ldots, x_{i_k} \} = \{ x_{i_1}, \ldots, x_{i_k} \}$, for $2 \leq l \leq k$. We know that $x_{i_k}$ is irreducible in $(P \ast Q) \setminus \{ x_{i_{l+1}}, x_{i_{l+2}}, \ldots, x_N \}$.

\footnote{It seems to be the most classical terminology ([15], [3], [8]); in [2], irreducible points are called (up or down) beat points.}
• First case: a maximum of \((P \ast Q) \setminus \{x_{i+1}, x_{i+2}, \ldots, x_N\}\) is also a maximum of \(P \setminus \{x_{i+1}, x_{i+2}, \ldots, x_N\}\) (this follows from \((P \ast Q) \setminus \{x_{i+1}, x_{i+2}, \ldots, x_N\}\) = \(P \setminus \{x_{i+1}, x_{i+2}, \ldots, x_N\}\).

• Second case: a minimum of \((P \ast Q) \setminus \{x_{i+1}, x_{i+2}, \ldots, x_N\}\) is also a minimum of \(P \setminus \{x_{i+1}, x_{i+2}, \ldots, x_N\}\). This follows from \((P \ast Q) \setminus \{x_{i+1}, x_{i+2}, \ldots, x_N\}\) = \(P \setminus \{x_{i+1}, x_{i+2}, \ldots, x_N\}\).

Let \(\text{Comp}\) be a dismantlable poset if, and only if, \(\text{Comp}\) is dismantlable in \(\text{Comp}\). By definition of \(\text{Comp}\), \(\text{Comp}\) is isomorphic to \(\text{Comp}\) for all \(p \neq b\). Indeed, \((P \ast Q) \setminus \{x_{i+1}, x_{i+2}, \ldots, x_N\}\) = \(Q\) would mean that \(Q\) is dismantlable (because \((P \ast Q) \setminus \{x_{i+1}, x_{i+2}, \ldots, x_N\}\) has a minimum and, thus, is dismantlable) which contradicts the fact that \(Q\) is non dismantlable (because \(Q\) is a subposet of \(Q\) obtained by suppression of irreducible elements).

In conclusion, \(x_i\) is irreducible in \(\{x_1, \ldots, x_i\}\), for \(2 \leq i \leq k\) and \(P\) is dismantlable.

Reciprocally, let us suppose that \(P\) is a dismantlable poset and that we have \(P = \{x_1, x_2, \ldots, x_n\}\) with \(x_i\) irreducible in the subposet induced by \(\{x_1, \ldots, x_i\}\), for \(2 \leq i \leq n\).

By the equalities \((P \ast Q) \supseteq P \setminus Q\) and \((P \ast Q) \supseteq Q\), we see that \(x_i\) is irreducible in the subposet of \(P \ast Q\) induced by \(\{x_1, \ldots, x_i\}\) if \(x_i\) is irreducible in the subposet of \(P\) induced by \(\{x_1, \ldots, x_i\}\). As a consequence, we can go from \(P \ast Q\) to \(\{x_1\} \ast Q\) by successive suppressions of irreducibles and this shows that \(P \ast Q\) is dismantlable (because \(\{x_1\} \ast Q\) is a cone). A similar argument yields that \(P \ast Q\) is dismantlable if we suppose \(Q\) dismantlable. □

In \cite{2}, Barmak and Minian introduce the notion of weak points in a poset: \(x \in P\) is a weak point if \(x\) is a weak point in \(P\) and only if, \(P_2 \supseteq P_1\) is dismantlable. So, by Lemma 3.1 \(x\) is a weak point if, and only if, \(P_2 \supseteq P_1\) is dismantlable. Now, it is well known (\cite{2,3}) that a poset \(P\) is a dismantlable poset if, and only if, \(\text{Comp}(P)\) is a dismantlable graph. As we have \(N_{\text{Comp}(P)}(x) = \text{Comp}(P_2 \ast P_1)\), we obtain the following result:

Proposition 3.2 Let \(P \in \mathcal{P}\) and \(x \in P\). Then, \(x\) is a weak point of \(P\) if, and only if, \(x\) is \(s\)-dismantlable in \(\text{Comp}(P)\).

The notation \(P \setminus \{x\}\) will mean that \(x\) is a weak point of \(P\) and we shall write \(P \setminus \{x\}\) if \(Q\) is a subposet of \(P\) obtained by successive deletions of weak points.

3.2 From \(\mathcal{G}\) to \(\mathcal{P}\)

If \(G \in \mathcal{G}\), \(C(G) \in \mathcal{P}\) is the poset whose elements are the complete subgraphs of \(G\) ordered by inclusion. Before establishing the relation between reduction by \(s\)-dismantlable vertices in \(\mathcal{G}\) and deletion of weak points in \(\mathcal{P}\), we recall that the poset product \(P \times Q\) of two posets \(P\) and \(Q\) is the set \(P \times Q\) ordered by \((p,q) \leq (p',q')\) if \(p \leq p'\) and \(q \leq q'\) for all \((p,q), (p',q') \in P \times Q\). In particular, \(P \times \{a,b\}\) is the poset formed by two copies of \(P\) (namely, \(P_a := P \times \{a\} = \{(p,a), p \in P\}\) and \(P_b := P \times \{b\} = \{(p,b), p \in P\}\)) with relations of \(P\) in the two copies \(P_a\) and \(P_b\) and the additional relations \((p,a) \leq (p',b)\) if \(p \leq p'\).

Lemma 3.3 Let \(P \in \mathcal{P}\) and \(S \in \mathcal{P}\) such that \(S\) contains \(W := P \times \{a,b\} \leq \{a,b\}\) as a subposet with the two following properties:

\[
\forall p \in P, \quad i) \ S_{< (p,b)} = W_{< (p,b)} \quad \text{and} \quad ii) \ S_{> (p,b)} = W_{> (p,b)}
\]

Let \(Q\) be the poset obtained from \(S\) by adding an element \(x\) (not in \(S\)) with the only relations \(x < (p,b)\) for all \(p \in P\).

If \(P\) is a dismantlable poset, then \(Q \setminus \{x\}\) with the only relations \(x < (p,b)\) for all \(p \in P\).

**proof**: By definition of \(Q\), we have \(x \leq Q y\) if, and only if, \(y \in P_b = \{(p,b), p \in P\}\). Of course, \(P_b\) is isomorphic to \(P\) and, if we suppose that \(P\) is dismantlable, this means that \(x\) is a weak point in \(Q\), i.e. \(Q \setminus \{x\} = Q \setminus \{x\}\).

Now, let \(p\) be an irreducible element in \(P\); we shall verify that \((p,b)\) is a weak point in \(S\).
• First case: we suppose that $P_{<p}$ has a maximum element $M$. We get

$$S_{<p, b} \overset{\text{i)}}{=} W_{<p, b} = \{ (p', b), p' < p \} \cup \{(p', a), p' \leq p \}$$

$$= \{ (p', b), p' < p \} \cup \{(p', a), p' \leq M \} \cup \{(p, a) \}$$

In $S_{<p, b}$, we have $y < (p, a) \Leftrightarrow y \leq (M, a)$; in other words, $(M, a)$ is a maximum element of $(S_{<p, b})_{<p, a}$ and this shows that $(p, a)$ is an irreducible point in $S_{<p, b}$.

Now, $S_{<p, b} \setminus \{(p, a)\} = \{ (p', b), p' < p \} \cup \{(p', a), p' \leq M \}$ is a cone on $(M, b)$ (because $y \leq (M, b)$ for all $y$ in $S_{<p, b} \setminus \{(p, a)\}$ and we can conclude that $S_{<p, b}$ is a dismantlable poset.

• Second case: we suppose that $P_{>p}$ has a minimum element $m$, then $S_{>p, b} \overset{\text{ii)}}{=} W_{>p, b}$ which is a poset isomorphic to $P_{>p}$; so, it is dismantlable (because it is a cone).

The conclusion of the two cases is that $(p, b)$ is a weak point in $S$; so, we have $S \setminus S \setminus \{(p, b)\}$.

Now, let us suppose that $P$ is dismantlable with $P = \{p_1, p_2, \ldots, p_n\}$ with $p_i$ irreducible in the subposet induced by $\{p_1, \ldots, p_i\}$, for $2 \leq i \leq n$. By iterating the preceding discussion we get

$$Q \overset{\text{s}}{\prec} Q \setminus \{x\} \overset{\text{s}}{\prec} Q \setminus \{x, (p_n, b)\} \overset{\text{s}}{\prec} Q \setminus \{x, (p_n, b), (p_{n-1}, b)\} \overset{\text{s}}{\prec} \ldots \overset{\text{s}}{\prec} Q \setminus \{x, (p_n, b), (p_{n-1}, b), \ldots, (p_2, b)\}$$

By condition i), we see that $y < (p_1, b)$ in $Q \setminus \{x, (p_n, b), (p_{n-1}, b), \ldots, (p_2, b)\}$ if, and only if, $y \leq (p_1, a)$; so, $(p_1, b)$ is a weak point in $Q \setminus \{x, (p_n, b), (p_{n-1}, b), \ldots, (p_2, b)\}$ (in fact, it is irreducible) and we have proved

$$Q \overset{\text{s}}{\prec} Q \setminus \{x, (p_n, b), \ldots, (p_2, b)\} \overset{\text{s}}{\prec} Q \setminus \{x, (p_n, b), \ldots, (p_2, b), (p_1, b)\}.$$

□

**Proposition 3.4** If $g$ is s-dismantlable in $G$, then $C(G) \setminus g \sim C(G \setminus g)$.

**proof**: We apply Lemma 3.3 with $Q = C(G)$ and $P = C(N_G(g))$. More precisely, with the notations of this lemma, we have $x = [g]$, $S = C(G) \setminus \{g\}$ and $W = C(N_G(g)) \setminus \{g\}$. If we denote by $g_i$ the elements of $N_G(g)$, the isomorphism between $W = C(N_G(g)) \setminus \{g\}$ and $P \times \{a, b, a < b\} = C(N_G(g)) \setminus \{g\}$ is given by identifying $([g_1, \ldots, g_n], a) \in P \times \{a, b, a < b\}$ with $[g_1, \ldots, g_n] \in W$ and $([g_1, \ldots, g_n], b) \in P \times \{a, b, a < b\}$ with $[g_1, \ldots, g_n] \in W$. Conditions i) and ii) are clearly verified.

By supposing $g$ s-dismantlable in $G$, we get that $N_G(g)$ is a dismantlable graph and $P = C(N_G(g))$ a dismantlable poset by 3.3 Lemma 2.2. So, by Lemma 3.3 we obtain

$$C(G) \overset{\text{s}}{\sim} C(G) \setminus \{\{g\} \cup \{g, g_1, \ldots, g_n\}, [g_1, \ldots, g_n] \in C(N_G(g))\} = C(G \setminus g)$$

□

### 3.3 Correspondence between s-homotopy and simple equivalence

A poset $P$ is said (2 Definition 3.4) simply equivalent to the poset $Q$ if we can transform $P$ to $Q$ by a finite sequence of additions or deletions of weak points. We denote by $[P]_s$ the equivalence class of $P$ for this relation (and call it the simple type of $P$).

**Theorem 3.5** 1. Let $P, Q \in \mathcal{P}$; $[P]_s = [Q]_s$ (in $\mathcal{P}$) $\iff$ $[\text{Comp}(P)]_s = [\text{Comp}(Q)]_s$ (in $\mathcal{G}$).

2. Let $G, H \in \mathcal{G}$; $[G]_s = [H]_s$ (in $\mathcal{G}$) $\iff$ $[C(G)]_s = [C(H)]_s$ (in $\mathcal{P}$).

**proof**: 1. The equivalence is a direct consequence of Proposition 3.2.

2. $\implies$: Follows from Proposition 3.4.

$\iff$: If $[C(G)]_s = [C(H)]_s$, we get $[\text{Comp}(C(G))]_s = [\text{Comp}(C(H))]_s$ by the first assertion of the theorem. As $\text{Comp} \circ C = \text{Bd}$, we have $[G']_s = [H']_s$ and the conclusion follows from Proposition 4.4. □
3.4 The triangle $(\mathcal{G}, \mathcal{P}, \mathcal{K})$

Let us recall that in $\mathcal{P}$ there is also a notion of barycentric subdivision $Bd : \mathcal{P} \to \mathcal{P}$ (for a poset $P$, $Bd(P) = P'$ is given by the chains of $P$ ordered by inclusion of underlying sets). There is also two classical applications, $\Delta : \mathcal{P} \to \mathcal{K}$ (the simplices of $\Delta(P)$, the order complex of $P$, are the $< x_0, x_1, \ldots, x_n >$ for every chain $x_0 < x_1 < \ldots < x_n$ of $P$) and $\Pi : \mathcal{K} \to \mathcal{P}$ (the elements of $\Pi(K)$, the face poset of $K$, are the simplices of $K$ ordered by inclusion). Thus, we get the triangle $(\mathcal{G}, \mathcal{P}, \mathcal{K})$ given in Fig. 7.

![Figure 7: The triangle $(\mathcal{G}, \mathcal{P}, \mathcal{K})$](image)

Let us list some easy properties of this triangle:

**Proposition 3.6**

1. $\Pi \circ \Delta = C \circ \text{Comp} = Bd$ (in $\mathcal{P}$), $\Delta \circ \Pi = \Delta \circ \Gamma = Bd$ (in $\mathcal{K}$), $\text{Comp} \circ C = \Gamma \circ \Delta = Bd$ (in $\mathcal{G}$).

2. We have the "commutative triangles": $\Delta = \Delta \circ \text{Comp}$, $\Delta = \Pi \circ \Delta$ and $\Gamma = \text{Comp} \circ \Pi$.

3. We have the "commutative triangles up to subdivision": $\Gamma \circ \Delta = Bd \circ \text{Comp}$, $\Delta \circ C = Bd \circ \Delta$, and $\Gamma \circ \text{Comp} = Bd \circ \Gamma$.

Now from Theorem 2.10 and Theorem 3.5, we get another proof of the Theorem (part of [2, First main Theorem 3.9]):

**Theorem 3.7**

1. Let $P, Q \in \mathcal{P}$. Then $P$ and $Q$ are simply equivalent if, and only if, $\Delta(P)$ and $\Delta(Q)$ have the same simple-homotopy type.

2. Let $K, L \in \mathcal{K}$. Then $K$ and $L$ have the same simple-homotopy type if, and only if, $\Pi(K)$ and $\Pi(L)$ are simply equivalent.

**Remark 3.8** The image of $\Delta$ is exactly the set of flag complexes but not all flag complexes are in the image of $\Delta$ (for example, the cyclic graph $C_5$ with 5 vertices may be considered as a flag complex and is not in the image of $\Delta$; equivalently, $C_5$ is not a comparability graph).

4 The weak-s-dismantlability

**Definition 4.1** Let $G \in \mathcal{G}$. An edge $gg'$ of $G$ will be called s-dismantlable if $N_G(g) \cap N_G(g')$ is nonempty and dismantlable.

We shall say that $G \Downarrow^s \downarrow H$ if we can go from $G$ to $H$ either by deleting s-dismantlable vertices or by deleting s-dismantlable edges.

**Definition 4.2** Two graphs $G$ and $H$ have the same ws-homotopy type if there is a sequence $G = J_1, \ldots, J_k = H$ in $\mathcal{G}$ such that $G = J_1 \downarrow^s J_2 \downarrow^s \ldots \downarrow^s J_{k-1} \downarrow^s J_k = H$ where each arrow $\downarrow^s$ represents either the suppression or the addition of an s-dismantlable vertex, or the suppression or the addition of an s-dismantlable edge.

Footnote: In fact, we can consider $\mathcal{G}$, $\mathcal{K}$ and $\mathcal{P}$ as categories (with obvious morphisms) and it is easy to verify that all applications in the triangle $(\mathcal{G}, \mathcal{P}, \mathcal{K})$ are covariant functors. The reader not acquainted with the notions of functors or categories may refer to book [12].
This defines an equivalence relation in \( \mathcal{G} \) and we shall denote by \( [G]_{ws} \) the equivalence class representing the ws-homotopy type of a graph \( G \). Of course, \( G \not\sim H \) implies \( G \not\sim ws H \) and the example given in remark \( 2.4 \) shows that the reverse implication is false in general; nevertheless, s-homotopy type and ws-homotopy type are the same:

**Proposition 4.3** For every \( G \in \mathcal{G} \), we have \( [G]_{s} = [G]_{ws} \).

**proof:** The inclusion \( [G]_{s} \subseteq [G]_{ws} \) follows from \( G \not\sim H \implies G \not\sim ws H \). Now, we have seen (Lemma \( 1.6 \)) that the deletion of an s-dismantlable edge corresponds to the addition of an s-dismantlable vertex followed by the suppression of an s-dismantlable vertex. This means that a sequence \( G = J_{1} \overset{ws}{\rightarrow} J_{2} \overset{ws}{\rightarrow} \ldots \overset{ws}{\rightarrow} J_{k-1} \overset{ws}{\rightarrow} J_{k} = H \) can be rewritten as a sequence from \( G \) to \( H \) using only suppressions and additions of s-dismantlable vertices and proves that \( [G]_{s} \supseteq [G]_{ws} \). \( \square \)

Actually, the weak-s-dismantlability behaves well with the map \( \Delta_{\mathcal{G}} \). The 1-skeleton of a simplicial complex can be considered as a graph (whose vertices are given by the 0-simplices and the edges are given by the 1-simplices). Following the notation of \([5]\), this defines a map \( sk : \mathcal{K} \rightarrow \mathcal{G} \)(if \( K \) a simplicial complex, \( sk(K) \) is its 1-skeleton taken as a graph). We note that \( sk(\Delta_{\mathcal{G}}(G)) = G \) for all \( G \in \mathcal{G} \). We have:

**Lemma 4.4** Let \( K, L \in \mathcal{K} \). Then, \( K \not\sim L \implies \left( sk(K) \not\sim ws sk(L) \text{ or } sk(K) = sk(L) \right) \).

**proof:** The simplicial collapse \( K \not\sim L \) says that we obtain \( L \) by deleting successively various pairs of simplices \( \{\sigma, \tau\} \) where \( \tau \) is a free face of \( \sigma \), so it is sufficient to prove that \( sk(K) \not\sim ws sk(K \setminus \{\sigma, \tau\}) \) for an elementary collapse \( K \not\sim (K \setminus \{\sigma, \tau\}) \). Of course, if \( \tau \) is a k-simplex, then \( \sigma \) is a \( (k + 1) \)-simplex and we consider the three cases \( k = 0, k = 1 \) and \( k \geq 2 \).

- If \( k = 0, \), \( \tau = \langle a, b \rangle \) is a vertex (or 0-simplex) belonging to a unique 1-simplex \( \sigma = \langle a, b \rangle \). In this case, we have \( sk(K) \not\sim sk(K \setminus \{a\}) \) because \( a \) is a vertex dominated by the vertex \( b \) in \( sk(K) \) and \( sk(K) \not\sim sk(K \setminus \{\sigma, \tau\}) = sk(K \setminus \{a\}) \) (also \( sk(K) \not\sim ws sk(K \setminus \{\sigma, \tau\}) = sk(K \setminus \{a\}) \)).

- If \( k = 1, \), \( \tau = \langle a, b \rangle \) is a 1-simplex such that there is a unique vertex \( c \) that is adjacent to \( a \) and \( b \); in other terms, \( N_{sk(K)}(a) \cap N_{sk(K)}(b) \) is reduced to the vertex \( c \) and this shows that the edge \( ab \) is s-dismantlable in \( sk(K) \). Now, it is clear that \( sk(K \setminus \{\sigma, \tau\}) = sk(K \setminus \{a, b, c\} = sk(K \setminus \{a, b\}) = sk(K \setminus \{a, b\}) \).

- If \( k \geq 2, \) the suppression of the pair \( \{\sigma, \tau\} \) in \( K \) does not affect the 1-skeleton of \( K \), i.e. \( sk(K) = sk(K \setminus \{\sigma, \tau\}) \). \( \square \)

**Proposition 4.5** Let \( G, H \in \mathcal{G} \). Then, \( G \not\sim ws H \iff \Delta_{\mathcal{G}}(G) \not\sim ws \Delta_{\mathcal{G}}(H) \).

**proof:** By replacing the 0-simplex \( < g \) by the 1-simplex \( < g, g' > \) in the proof of Proposition \( 2.3 \) we get \( \Delta_{\mathcal{G}}(G) \not\sim \Delta_{\mathcal{G}}(G \setminus gg') \) when the edge \( gg' \) is s-dismantlable in \( G \). This shows that \( G \not\sim ws H \) implies \( \Delta_{\mathcal{G}}(G) \not\sim ws \Delta_{\mathcal{G}}(H) \). The reverse inclusion follows from Lemma \( 1.4 \) and the fact that \( sk(\Delta_{\mathcal{G}}(G)) = G \) for all \( G \in \mathcal{G} \). \( \square \)

It is important to note that we can find a graph \( G \) whose vertices and edges are all non s-dismantlable and such that \( \Delta_{\mathcal{G}}(G) \) collapses on a strict subcomplex which does not admit any collapsible pair and which is not a flag subcomplex; the 6-regular graph given in appendix provides such an example.

## 5 Relation with graph homotopy of Chen, Yau and Yeh

In \([3]\), Ivashchenko introduces the notion of **contractible transformations** and calls contractible the trivial graph (the graph reduced to a point) and every graph obtained from the trivial graph by application of these contractible transformations. In what follows, to avoid any confusion, we call I-contractibility the contractibility in the sense of Ivashchenko. In a graph \( G \), the contractible transformations are the deletion of a vertex \( g \) if \( N_{G}(g) \) is I-contractible, the deletion of an edge \( gg' \) if \( N_{G}(g) \cap N_{G}(g') \) is I-contractible, the addition of a vertex \( x \) if \( N_{G}(x) \) is I-contractible and the addition of an edge between \( g \) and \( g' \) if \( g \not\approx g' \) and \( N_{G}(g) \cap N_{G}(g') \) is I-contractible. From these operations, in \([3]\), the authors introduce the graph homotopy type of a graph \( G \) that we shall call here I-homotopy type. Let us
say that a vertex \( g \) is I-dismantlable if \( N_G(g) \) is I-contractible. The \([5, \text{Lemma 3.4}]\) shows that we can reduce the four operations above to the two operations of deletion or addition of I-dismantlable vertices. Thus, one can define \( G \not\sim H \) as the passage from \( G \) to \( H \) by suppression of I-dismantlable vertices and \( G \not\simeq H \) as the passage from \( G \) to \( H \) by addition of I-dismantlable vertices. From this, we get the I-equivalence class of a graph (similarly to definition 1.3) we say that two graphs \( G \) and \( H \) have the same I-homotopy type if there is a sequence \( G = J_1, \ldots, J_k = H \) in \( \mathcal{G} \) such that \( G = J_1 \leftarrow J_2 \leftarrow \ldots \leftarrow J_{k-1} \leftarrow J_k = H \) where each arrow \( \leftarrow \) represents the suppression of a I-dismantlable vertex or the addition of a I-dismantlable vertex). So, \([G]_I \) denotes the I-homotopy type of a graph \( G \), i.e. the graph homotopy type of \( G \) in the terminology of \([5]\). In that way, \( G \) is I-contractible if, and only if, \([G]_I = [pt]_I \).

**Proposition 5.1** Let \( g \in G \).

1. If \( g \) is s-dismantlable, then \( g \) is I-dismantlable.
2. If \( G \) is s-dismantlable (i.e. \([G]_s = [pt]_s \)), then \( G \) is I-contractible.

**proof:** 1. Suppose that \( g \) is s-dismantlable in \( G \); then \( N_G(g) \) is dismantlable. But from the definition of I-contractibility, it is clear that “dismantlable \( \implies \) I-contractible” (the open neighborhood of a dismantlable vertex is a cone and this proves that a dismantlable vertex is I-dismantlable); thus \( N_G(g) \) is I-contractible, i.e. \( g \) is I-dismantlable.

2. It is a consequence of the assertion 1. \( \square \)

It also follows from the assertion 1 of Proposition 5.1 that if two graphs \( G \) and \( H \) have the same s-homotopy type, then they have the same I-homotopy type. We are unaware if the converse is true:

**Question:** Let \( G \in \mathcal{G} \). Are the s-homotopy type of \( G \) and the I-homotopy type of \( G \) identical?

**Acknowledgement**

The authors would like to thank Florian Marty for his pertinent indication to prove Proposition 5.1.

**Appendix: A particular non ws-reducible graph**

Some properties of the graph \( G \) of Fig. 8:

1. \( G \) is a 6-regular graph with \(|V(G)| = 10\) and \(|E(G)| = 30\).
2. For every vertex \( a \) of \( G \), the subgraph \( N_G(a) \) is not dismantlable because it is isomorphic to the graph shown in figure 8.b.
3. For every edge \( ab \) of \( G \), the subgraph \( N_G(a) \cap N_G(b) \) is not dismantlable because it is isomorphic to the (disconnected) graph shown in figure 8.c. So there is no s-dismantlable edge in \( G \).
4. The cliques (or maximal complete subgraphs) of \( G \) are of order 4 or 3. The cliques of order 3 are:

\[
[1, 2, 3] \quad [1, 5, 6] \quad [2, 4, 6] \quad [1, 8, 9] \quad [2, 7, 9] \quad [3, 4, 5] \quad [3, 7, 8] \quad [x, 5, 8] \quad [x, 6, 9] \quad [x, 4, 7] 
\]

and the cliques of order 4 are:

\[
c_1 = [1, 3, 5, 8] \quad c_2 = [1, 2, 6, 9] \quad c_3 = [2, 3, 4, 7] \quad c_4 = [x, 4, 5, 6] \quad c_5 = [x, 7, 8, 9] 
\]

5. In the simplicial complex \( \Delta_\mathcal{G}(G) \), there are five tetrahedra \( \sigma_i \) corresponding to the five 4-cliques \( c_i \). Corresponding to each tetrahedron \( \sigma_i \), each pair \( (\sigma_i, \tau) \) (with \( \tau \) being any maximal proper face of \( \sigma_i \)) is a collapsible pair.
6. Let \( K \) be a subcomplex obtained from \( \Delta_\mathcal{G}(G) \) after collapsing the five tetrahedras \( \tau_i \), \( 1 \leq i \leq 5 \).
Figure 8: $\Delta_{\varphi}(G)$ is non collapsible on a flag subcomplex

- There is no collapsible pair in $K$. Indeed, a collapsible pair in $K$ must be of the form $(\sigma', \tau')$ with $\sigma'$ a triangle (or 2-simplex) and $\tau'$ an edge of $\sigma'$ which is not the edge of another triangle. But we see from the lists of 3-cliques and 4-cliques that every edge of $G$ appears exactly once in the list of 3-cliques and exactly once in the list of 4-cliques; so, even after removing 5 collapsible pairs corresponding to the 5 tetrahedras, every edge appears in at least two triangles (and is not a free edge).
- $K$ is not a flag complex. For example, let $(\sigma, \tau)$ be a pair which has been collapsed in $\Delta_{\varphi}(G)$: $\sigma$ is a tetrahedron and $\tau$, a maximal proper face of $\sigma$, is of the form $<a,b,c>$. So $\tau$ is a non-simplex of $K$ with 3 vertices and every face of $\tau$ is a simplex of $K$.

In conclusion:
- $G$ is a non ws-reducible graph (i.e., there is not a strict subgraph $H$ of $G$ such that $G \not\cong_{ws} H$).
- We can find a strict subcomplex $K$ of $\Delta_{\varphi}(G)$ such that $\Delta_{\varphi}(G) \cong_{s} K$.
- Every strict subcomplex $K$ of $\Delta_{\varphi}(G)$ such that $\Delta_{\varphi}(G) \cong_{s} K$ is not a flag complex.

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