The Spinc Dirac operator on hypersurfaces and applications

Roger Nakad, Julien Roth

The Spin^c Dirac Operator on Hypersurfaces and Applications

Roger NAKAD* and **Julien ROTH†**

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Abstract

We extend to the eigenvalues of the hypersurface Spin^c Dirac operator well known lower and upper bounds. Examples of limiting cases are then given. Futhermore, we prove a correspondence between the existence of a Spin^c Killing spinor on homogeneous 3-dimensional manifolds $\mathbb{E}^*(\kappa, \tau)$ with 4-dimensional isometry group and isometric immersions of $\mathbb{E}^*(\kappa, \tau)$ into the complex space form $\mathbb{M}^4(c)$ of constant holomorphic sectional curvature $4c$, for some $c \in \mathbb{R}^*$. As applications, we show the non-existence of totally umbilic surfaces in $\mathbb{E}^*(\kappa, \tau)$ and we give necessary and sufficient geometric conditions to immerse a 3-dimensional Sasaki manifold into $\mathbb{M}^4(c)$.

Key words. Spin^c structures, isometric immersions, spectrum of the Dirac operator, parallel and Killing spinors, manifolds with boundary and boundary conditions, Sasaki and Kähler manifolds.

1 Introduction

It is well known that the spectrum of the Dirac operator on hypersurfaces of a Spin manifold detects informations on the geometry of such manifolds and their hypersurfaces ([3, 4, 5, 16, 18, 19]). For example, O. Hijazi, S. Montiel and X. Zhang [16] proved that on the compact boundary M^n of a Riemannian compact Spin manifold

*Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany. E-mail: nakad@mpim-bonn.mpg.de

†Laboratoire d'Analyse et de Mathématiques appliquées, Université Paris-Est Marne-la-Vallée, 77454 Marne-la-Vallée. E-mail: julien.roth@univ-mlv.fr

$\mathcal{Z}^{n+1}$ of dimension $n+1$ and with nonnegative scalar curvature, the first positive eigenvalue λ_1 of the Dirac operator satisfies

$$\lambda_1 \geq \frac{n}{2} \inf_M H, \tag{1}$$

where H denotes the mean curvature of M, assumed to be nonnegative. Equality holds if and only if H is constant and every eigenspinor associated with λ_1 is the restriction to M of a parallel spinor on $\mathcal{Z}$ (and so $\mathcal{Z}$ is Ricci-flat). As application of the limiting case, they gave an elementary proof of the famous Alexandrov theorem [16]: *the only compact embedded hypersurface in $\mathbb{R}^{n+1}$ of constant mean curvature is the sphere $\mathbb{S}^n$ of dimension n.*

Assume now that M^n is a closed hypersurface of $\mathcal{Z}^{n+1}$. Evaluating the Rayleigh quotient applied to a parallel or Killing spinor field coming from $\mathcal{Z}$, C. Bär [5] derived an upper bound for the eigenvalues of the Dirac operator on M by using the min-max principle. More precisely, there are at least μ eigenvalues $\lambda_1, \cdots, \lambda_\mu$ of the Dirac operator on M satisfying

$$\lambda_j^2 \leq n^2\alpha^2 + \frac{n^2}{4\,\mathrm{vol}(M)} \int_M H^2 dv, \tag{2}$$

where $\mathrm{vol}(M)$ is the volume of M, dv is the volume form of the manifold M, α is the Killing number ($\alpha = 0$ if the ambient spinor field is parallel) and μ is the dimension of the space of parallel or Killing spinors.

Recently, $\mathrm{Spin^c}$ geometry became a field of active research with the advent of Seiberg-Witten theory [22, 35, 31]. Applications of the Seiberg-Witten theory to 4-dimensional geometry and topology are already notorious ([9, 24, 25, 13]). From an intrinsic point of view, Spin, almost complex, complex, Kähler, Sasaki and some classes of CR manifolds have a canonical $\mathrm{Spin^c}$ structure. The complex projective space $\mathbb{C}P^m$ is always $\mathrm{Spin^c}$ but not Spin if m is even. Nowadays, and from the extrinsic point of vue, it seems that it is more natural to work with $\mathrm{Spin^c}$ structures rather than Spin structures. Indeed, O. Hijazi, S. Montiel and F. Urbano [20] constructed on Kähler-Einstein manifolds with positive scalar curvature, $\mathrm{Spin^c}$ structures carrying Kählerian Killing spinors. The restriction of these spinors to minimal Lagrangian submanifolds provides topological and geometric restrictions on these submanifolds. In [30, 29], and via $\mathrm{Spin^c}$ spinors, the authors gave an elementary proof for a *Lawson type correspondence* between constant mean curvature surfaces of 3-dimensional homogeneous manifolds with 4-dimensional isometry group. We point out that, using Spin spinors, we cannot prove this *Lawson type correspondence*. Moreover, they characterized isometric immersions of a 3-dimensional almost contact metric manifold M into the complex space form by the existence of a $\mathrm{Spin^c}$ structure on M carrying a special spinor called a generalized Killing spinor.

In the first part of this paper and using the Spin^c Reilly inequality, we extend the lower bound (1) to the first positive eigenvalue of the Dirac operator defined on the compact boundary of a Spin^c manifold. Examples of the limiting case are then given where the equality case cannot occur if we consider the Spin Dirac operator on these examples. Also, by restriction of parallel and Killing Spin^c spinors, we extend the upper bound (2) to the eigenvalues of the Dirac operator defined on a closed hypersurface of Spin^c manifolds. Examples of the limiting case are also given.

In the second part, we study Spin^c structures on 3-dimensional homogeneous manifolds $\mathbb{E}^*(\kappa, \tau)$ with 4-dimensional isometry group. It is well known that the only complete simply connected Spin^c manifolds admitting real Killing spinor other than the Spin manifolds are the non-Einstein Sasakian manifolds endowed with their canonical or anti-canonical Spin^c structure [27]. Since $\mathbb{E}^*(\kappa, \tau)$ are non-Einstein Sasakian manifolds [7], the canonical and the anti-canonical Spin^c structure carry real Killing spinors. In [30], the authors proved that this canonical (resp. this anti-canonical) Spin^c structure on $\mathbb{E}^*(\kappa, \tau)$ is the lift of the canonical (resp. the anti-canonical) Spin^c structure on $\mathbb{M}^2(\kappa)$ via the submersion $\mathbb{E}^*(\kappa, \tau) \longrightarrow \mathbb{M}^2(\kappa)$, where $\mathbb{M}^2(\kappa)$ denotes the simply connected 2-dimensional manifold with constant curvature κ . Moreover, they proved that the Killing constant of the real Killing spinor field is equal to $\frac{\tau}{2}$. Here, we reprove the existence of a Killing spinor for the canonical and the anti-canonical Spin^c structure. This proof is based on the existence of an isometric embedding of $\mathbb{E}^*(\kappa, \tau)$ into the complex projective space or the complex hyperbolic space (see Proposition 4.1). Conversely, from the existence of a Killing spinor on $\mathbb{E}^*(\kappa, \tau)$, we prove the existence of an isometric immersion of $\mathbb{E}^*(\kappa, \tau)$ into the complex space form $\mathbb{M}^4(c)$ of constant holomorphic sectional curvature $4c$, for some $c \in \mathbb{R}^*$ (see Proposition 4.2). Since every non-Einstein Sasaki manifold has a Spin^c structure with a Killing spinor, it is natural to ask if this last result remains true for any 3-dimensional Sasaki manifold. Indeed, every simply connected non-Einstein Sasaki manifold can be immersed into $\mathbb{M}^4(c)$ for some $c \in \mathbb{R}^*$, providing that the scalar curvature is constant (see Theorem 4.3). Finally, we make use of the existence of a Killing spinor on $\mathbb{E}^*(\kappa, \tau)$ to calculate some eigenvalues of Berger spheres endowed with differents Spin^c structures. By restriction of this Killing spinor to any surface of $\mathbb{E}^*(\kappa, \tau)$, we give a Spin^c proof for the non-existence of totally umbilic surfaces in $\mathbb{E}^*(\kappa, \tau)$ (see Theorem 4.4) proved already by R. Souam and E. Toubiana [32].

2 Preliminaries

In this section, we briefly introduce basic notions concerning the Dirac operator on Spin^c manifolds (with or without boundary) and their hypersurfaces. Details can be found in [10], [26], [23], [15] and [5].

The Dirac operator on Spin^c manifolds. We consider an oriented Riemannian manifold (M^n, g) of dimension n with or without boundary and denote by $\mathrm{SO}M$ the SO_n-principal bundle over M of positively oriented orthonormal frames. A Spin^c structure of M is is given by an $\mathbb{S}^1$-principal bundle $(\mathbb{S}^1 M, \pi, M)$ of some Hermitian line bundle L and a Spin_n^c-principal bundle $(\mathrm{Spin}^c M, \pi, M)$ which is a 2-fold covering of the $\mathrm{SO}_n \times \mathbb{S}^1$-principal bundle $\mathrm{SO}M \times_M \mathbb{S}^1 M$ compatible with the group covering

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}_n^c = \mathrm{Spin}_n \times_{\mathbb{Z}_2} \mathbb{S}^1 \longrightarrow \mathrm{SO}_n \times \mathbb{S}^1 \longrightarrow 0.$$

The bundle L is called the auxiliary line bundle associated with the Spin^c structure. If $A : T(\mathbb{S}^1 M) \longrightarrow i\mathbb{R}$ is a connection 1-form on $\mathbb{S}^1 M$, its (imaginary-valued) curvature will be denoted by F_A, whereas we shall define a real 2-form Ω on $\mathbb{S}^1 M$ by $F_A = i\Omega$. We know that Ω can be viewed as a real valued 2-form on M [10, 21]. In this case, $i\Omega$ is the curvature form of the auxiliary line bundle L [10, 21].

Let $\Sigma M := \mathrm{Spin}^c M \times_{\rho_n} \Sigma_n$ be the associated spinor bundle where $\Sigma_n = \mathbb{C}^{2^{[\frac{n}{2}]}}$ and $\rho_n : \mathrm{Spin}_n^c \longrightarrow \mathrm{End}(\Sigma_n)$ the complex spinor representation [10, 23, 29]. A section of ΣM will be called a spinor field. This complex vector bundle is naturally endowed with a Clifford multiplication, denoted by "$\cdot$", $\cdot : \mathbb{C}l(TM) \longrightarrow \mathrm{End}(\Sigma M)$ which is a fiber preserving algebra morphism and with a natural Hermitian scalar product $< .,. >$ compatible with this Clifford multiplication [26, 10, 15]. If n is even, $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$ can be decomposed into positive and negative spinors by the action of the complex volume element [10, 26, 15, 29]. If such data are given, one can canonically define a covariant derivative ∇ on ΣM given, for all $X \in \Gamma(TM)$, by [10, 23, 15, 29]:

$$\nabla_X \psi = X(\psi) + \frac{1}{4} \sum_{j=1}^{n} e_j \cdot \nabla_X e_j \cdot \psi + \frac{i}{2} A(s_*(X))\psi, \tag{3}$$

where the second ∇ is the Levi-Civita connection on M, $\psi = [\widetilde{b \times s}, \sigma]$ is a locally defined spinor field, $b = (e_1, \cdots, e_n)$ is a local oriented orthonormal tangent frame, $s : U \longrightarrow \mathbb{S}^1 M$ is a local section of $\mathbb{S}^1 M$, $\widetilde{b \times s}$ is the lift of the local section $b \times s : U \longrightarrow \mathrm{SO}M \times_M \mathbb{S}^1 M$ to the 2-fold covering and $X(\psi) = [\widetilde{b \times s}, X(\sigma)]$. For any other connection A' on $\mathbb{S}^1 M$, there exists a real 1-form α on M such that $A' = A + i\alpha$ [10]. If we endow the $\mathbb{S}^1$-principal fiber bundle $\mathbb{S}^1 M$ with the connection A', there exists on ΣM a covariant derivative ∇' given by

$$\nabla'_X \psi = \nabla_X \psi + \frac{i}{2} \alpha(X)\psi, \tag{4}$$

for all $X \in \Gamma(TM)$ and $\psi \in \Gamma(\Sigma M)$. Moreover, the curvature 2-form of A' is given by $F_{A'} = F_A + id\alpha$. But F_A (resp. $F_{A'}$) can be viewed as an imaginary 2-form on M denoted by $i\Omega$ (resp. $i\Omega'$). Thus, $i\Omega$ (resp. $i\Omega'$) is the curvature of the auxiliary line bundle associated with the $\mathbb{S}^1$-principal fiber bundle $\mathbb{S}^1 M$ endowed with the connection A (resp. $A^{'}$) and we have $i\Omega^{'} = i\Omega + id\alpha$.

The Dirac operator, acting on $\Gamma(\Sigma M)$, is a first order elliptic operator locally given by $D = \sum_{j=1}^{n} e_j \cdot \nabla_{e_j}$, where $\{e_j\}_{j=1,\cdots,n}$ is any orthonormal local basis tangent to M. An important tool when examining the Dirac operator on $\mathrm{Spin}^{\mathrm{c}}$ manifolds is the Schrödinger-Lichnerowicz formula [10, 23]:

$$D^2 = \nabla^*\nabla + \frac{1}{4} S \,\mathrm{Id}_{\Gamma(\Sigma M)} + \frac{i}{2}\Omega\cdot, \tag{5}$$

where S is the scalar curvature of M, ∇^* is the adjoint of ∇ with respect to the L^2-scalar product and $\Omega\cdot$ is the extension of the Clifford multiplication to differential forms. The Ricci identity is given, for all $X \in \Gamma(TM)$, by

$$\sum_{j=1}^{n} e_j \cdot \mathcal{R}(e_j, X)\psi = \frac{1}{2}\mathrm{Ric}(X) \cdot \psi - \frac{i}{2}(X \lrcorner \Omega) \cdot \psi, \tag{6}$$

for any spinor field ψ. Here Ric (resp. $\mathcal{R}$) denotes the Ricci tensor of M (resp. the $\mathrm{Spin}^{\mathrm{c}}$ curvature associated with the connection ∇) and $\lrcorner$ the interior product.

A Spin structure can be seen as a $\mathrm{Spin}^{\mathrm{c}}$ structure with trivial auxiliary line bundle L endowed with the trivial connection. Every almost complex manifold (M^{2m}, g, J) of complex dimension m has a canonical $\mathrm{Spin}^{\mathrm{c}}$ structure. In fact, the complexified cotangent bundle $T^*M \otimes \mathbb{C} = \Lambda^{1,0}M \oplus \Lambda^{0,1}M$ decomposes into the $\pm i$-eigenbundles of the complex linear extension of the complex structure. Thus, the spinor bundle of the canonical $\mathrm{Spin}^{\mathrm{c}}$ structure is given by

$$\Sigma M = \Lambda^{0,*}M = \oplus_{r=0}^{m} \Lambda^{0,r}M,$$

where $\Lambda^{0,r}M = \Lambda^r(\Lambda^{0,1}M)$ is the bundle of r-forms of type $(0,1)$. The auxiliary line bundle of this canonical $\mathrm{Spin}^{\mathrm{c}}$ structure is given by $L = (K_M)^{-1} = \Lambda^m(\Lambda^{0,1}M)$, where K_M is the canonical bundle of M [10, 26, 29]. Let $\ltimes$ be the Kähler form defined by the complex structure J, i.e. $\ltimes(X, Y) = g(X, JY)$ for all vector fields $X, Y \in \Gamma(TM)$. The auxiliary line bundle $L = (K_M)^{-1}$ has a canonical holomorphic connection induced from the Levi-Civita connection whose curvature form is given by $i\Omega = i\rho$, where ρ is the Ricci 2-form given by $\rho(X, Y) = \mathrm{Ric}(X, JY)$. For any other $\mathrm{Spin}^{\mathrm{c}}$ structure the spinorial bundle can be written as [10, 20]:

$$\Sigma M = \Lambda^{0,*}M \otimes \mathcal{L},$$

where $\mathcal{L}^2 = K_M \otimes L$ and L is the auxiliary bundle associated with this $\mathrm{Spin}^{\mathrm{c}}$ structure. In this case, the 2-form $\ltimes$ can be considered as an endomorphism of ΣM via Clifford multiplication and we have the well-known orthogonal splitting $\Sigma M = \oplus_{r=0}^{m} \Sigma_r M$, where $\Sigma_r M$ denotes the eigensubbundle corresponding to the eigenvalue $i(m - 2r)$

of $\ltimes$, with complex rank $\binom{m}{k}$. The bundle $\Sigma_r M$ correspond to $\Lambda^{0,r}M \otimes \mathcal{L}$. For the canonical $\mathrm{Spin^c}$ structure, the subbundle $\Sigma_0 M$ is trivial. Hence and when M is a Kähler manifold, this $\mathrm{Spin^c}$ structure admits parallel spinors (constant functions) lying in $\Sigma_0 M$ [27]. Of course, we can define another $\mathrm{Spin^c}$ structure for which the spinor bundle is given by $\Lambda^{*,0}M = \oplus_{r=0}^m \Lambda^r(T^*_{1,0}M)$ and the auxiliary line bundle by K_M. This $\mathrm{Spin^c}$ structure will be called the anti-canonical $\mathrm{Spin^c}$ structure.

Spin^c hypersurfaces and the Gauss formula. Let (M^n, g) be an n-dimensional oriented hypersurface isometrically immersed in a Riemannian $\mathrm{Spin^c}$ manifold $(\mathcal{Z}^{n+1}, g_{\mathcal{Z}})$. The hypersurface M inherts a $\mathrm{Spin^c}$ structure from that on $\mathcal{Z}$, and we have [26, 5, 29, 28]:

$$\begin{cases} \Sigma\mathcal{Z}_{|_M} & \simeq \Sigma M \quad \text{if } n \text{ is even,} \\ \Sigma^+\mathcal{Z}_{|_M} & \simeq \Sigma M \quad \text{if } n \text{ is odd.} \end{cases}$$

Moreover Clifford multiplication by a vector field X, tangent to M, is given by

$$X \bullet \varphi = (X \cdot \nu \cdot \psi)_{|_M},$$

where $\psi \in \Gamma(\Sigma\mathcal{Z})$ (or $\psi \in \Gamma(\Sigma^+\mathcal{Z})$ if n is odd), φ is the restriction of ψ to M, "$\cdot$" is the Clifford multiplication on $\mathcal{Z}$, "$\bullet$" that on M and ν is the unit normal vector. When n is odd, we can also get $\Sigma^-\mathcal{Z}_{|_M} \simeq \Sigma M$. In this case, the Clifford multiplication by a vector field X tangent to M is given by $X \bullet \varphi = -(X \cdot \nu \cdot \psi)_{|_M}$ and we have $\Sigma\mathcal{Z}_{|_M} \simeq \Sigma M \oplus \Sigma M$. The connection 1-form defined on the restricted $\mathbb{S}^1$-principal bundle $(\mathbb{S}^1 M =: \mathbb{S}^1\mathcal{Z}_{|_M}, \pi, M)$, is given by

$$A = A^{\mathcal{Z}}{}_{|_M} : T(\mathbb{S}^1 M) = T(\mathbb{S}^1\mathcal{Z})_{|_M} \longrightarrow i\mathbb{R}.$$

Then the curvature 2-form $i\Omega$ on the $\mathbb{S}^1$-principal bundle $\mathbb{S}^1 M$ is given by $i\Omega = i\Omega^{\mathcal{Z}}{}_{|_M}$, which can be viewed as an imaginary 2-form on M and hence as the curvature form of the line bundle L, the restriction of the line bundle $L^{\mathcal{Z}}$ to M. We denote by $\nabla^{\mathcal{Z}}$ the spinorial Levi-Civita connection on $\Sigma\mathcal{Z}$ and by ∇ that on ΣM. For all $X \in \Gamma(TM)$ and for every spinor field $\psi \in \Gamma(\Sigma\mathcal{Z})$ (or $\psi \in \Gamma(\Sigma^+\mathcal{Z})$ if n is odd), we consider $\varphi = \psi_{|_M}$ and we get the following $\mathrm{Spin^c}$ Gauss formula [26, 5, 28]:

$$(\nabla^{\mathcal{Z}}_X \psi)_{|_M} = \nabla_X \varphi + \frac{1}{2} II(X) \bullet \varphi, \tag{7}$$

where II denotes the Weingarten map with respect to ν. Moreover, Let $D^{\mathcal{Z}}$ and D be the Dirac operators on $\mathcal{Z}$ and M, after denoting by the same symbol any spinor and it's restriction to M, we have

$$\widetilde{D}\varphi = \frac{n}{2} H\varphi - \nu \cdot D^{\mathcal{Z}}\varphi - \nabla^{\mathcal{Z}}_\nu \varphi, \tag{8}$$

$$\widetilde{D}(\nu \cdot \varphi) = -\nu \cdot \widetilde{D}\varphi, \tag{9}$$

where $H = \frac{1}{n}\mathrm{tr}(II)$ denotes the mean curvature and $\widetilde{D} = D$ if n is even and $\widetilde{D} = D \oplus (-D)$ if n is odd.

Homogeneous 3-dimensional manifolds with 4-dimensional isometry group. We denote a 3-dimensional homogeneous manifold with 4-dimensional isometry group by $\mathbb{E}(\kappa,\tau)$, $\kappa - 4\tau^2 \neq 0$. It is a Riemannian fibration over a simply connected 2-dimensional manifold $\mathbb{M}^2(\kappa)$ with constant curvature κ and such that the fibers are geodesic. We denote by τ the bundle curvature, which measures the default of the fibration to be a Riemannian product. Precisely, we denote by ξ a unit vertical vector field, that is tangent to the fibers. If $\tau \neq 0$, the vector field ξ is a Killing field and satisfies for all vector field X,

$$\nabla_X \xi = \tau X \wedge \xi,$$

where ∇ is the Levi-Civita connection and $\wedge$ is the exterior product. In this case $\mathbb{E}(\kappa,\tau)$ is denoted by $\mathbb{E}^*(\kappa,\tau)$. When τ vanishes, we get a product manifold $\mathbb{M}^2(\kappa) \times \mathbb{R}$. If $\tau \neq 0$, these manifolds are of three types: they have the isometry group of the Berger spheres if $\kappa > 0$, of the Heisenberg group Nil_3 if $\kappa = 0$ or of $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$ if $\kappa < 0$. Note that if $\tau = 0$, then $\xi = \frac{\partial}{\partial t}$ is the unit vector field giving the orientation of $\mathbb{R}$ in the product $\mathbb{M}^2(\kappa) \times \mathbb{R}$. The manifold $\mathbb{E}^*(\kappa,\tau)$ admits a local direct orthonormal frame $\{e_1, e_2, e_3\}$ with $e_3 = \xi$, and such that the Christoffel symbols $\Gamma_{ij}^k = \langle \nabla_{e_i} e_j, e_k \rangle$ are given by

$$\begin{cases} \Gamma_{12}^3 = \Gamma_{23}^1 = -\Gamma_{21}^3 = -\Gamma_{13}^2 = \tau, \\ \\ \Gamma_{32}^1 = -\Gamma_{31}^2 = \tau - \frac{\kappa}{2\tau}, \\ \\ \Gamma_{ii}^i = \Gamma_{ij}^i = \Gamma_{ji}^i = \Gamma_{ii}^j = 0, \quad \forall\, i,j \in \{1,2,3\}. \end{cases} \tag{10}$$

We call $\{e_1, e_2, e_3 = \xi\}$ the canonical frame of $\mathbb{E}^*(\kappa,\tau)$. Except the Berger spheres and with $\mathbb{R}^3$, $\mathbb{H}^3$, $\mathbb{S}^3$ and the solvable group Sol_3, the manifolds $\mathbb{E}(\kappa,\tau)$ define the geometry of Thurston. The authors [30] proved that there exists on $\mathbb{E}^*(\kappa,\tau)$ a $\mathrm{Spin^c}$ structure (the canonical $\mathrm{Spin^c}$ structure) carrying a Killing spinor field ψ of Killing constant $\frac{\tau}{2}$, i.e., a spinor field ψ satisfying

$$\nabla_X \psi = \frac{\tau}{2} X \cdot \psi,$$

for all $X \in \Gamma(T\mathbb{E}^*(\kappa,\tau))$. Moreover, $\xi \cdot \psi = -i\psi$ and the curvature of the auxiliary line bundle is given by

$$i\Omega(e_1, e_2) = -i(\kappa - 4\tau^2) \quad \text{and} \quad i\Omega(e_k, e_j) = 0, \tag{11}$$

elsewhere in the canonical frame $\{e_1, e_2, \xi\}$. There exists also another $\mathrm{Spin^c}$ structure (the anti-canonical $\mathrm{Spin^c}$ structure) carrying a Killing spinor field ψ of Killing constant $\frac{\tau}{2}$ such that $\xi \cdot \psi = i\psi$ and the curvature of the auxiliary line bundle is given by

$$i\Omega(e_1, e_2) = i(\kappa - 4\tau^2) \quad \text{and} \quad i\Omega(e_k, e_j) = 0, \tag{12}$$

elsewhere in the canonical frame $\{e_1, e_2, \xi\}$.

3 Lower and upper bounds for the eigenvalues of the hypersurface Dirac operator

We will extend the lower bound (1) and the upper bound (2) to the eigenvalues of the hypersurface $\mathrm{Spin^c}$ Dirac operator $\widetilde{D}$. Examples of the limiting cases are then given.

3.1 Lower bounds for the eigenvalues of the hypersurface Dirac operator

We assume that the manifold $\mathcal{Z}^{n+1}$ is a $\mathrm{Spin^c}$ manifold having a compact domain $\mathbb{D}$ with compact boundary $M = \partial\mathbb{D}$. Using suitable boundary conditions for the Dirac operator $D^{\mathcal{Z}}$, we extend the lower bound (1) to the first positive eigenvalue of the extrinsic hypersurface Dirac operator $\widetilde{D}$ on M endowed with the induced $\mathrm{Spin^c}$ structure.

Since M is compact, the Dirac operator $\widetilde{D}$ has a discrete spectrum and we denote by $\pi_+ : \Gamma(\Sigma M) \longrightarrow \Gamma(\Sigma M)$ the projection onto the subspace of $\Gamma(\Sigma M)$ spanned by eigenspinors corresponding to the nonnegative eigenvalues of $\widetilde{D}$. This projection provides an Atiyah-Patodi-Singer type boundary conditions for the Dirac operator $D^{\mathcal{Z}}$ of the domain $\mathbb{D}$. It has been proved that this is a global self-adjoint elliptic condition [17, 16].

It is not difficult to extend the Spin Reilly inequality (see [17], [16], [18], [19]) to $\mathrm{Spin^c}$ manifolds. Indeed, for all spinor fields $\psi \in \Gamma(\Sigma\mathbb{D})$, we have

$$\int_{\partial\mathbb{D}} \Big(< \widetilde{D}\varphi, \varphi > -\frac{n}{2} H|\varphi|^2 \Big) ds \geq \int_{\mathbb{D}} \Big(\frac{1}{4} S^{\mathcal{Z}} |\psi|^2 \;+\; < \frac{i}{2}\Omega^{\mathcal{Z}} \cdot \psi, \psi > \; - \; \frac{n}{n+1} |D^{\mathcal{Z}}\psi|^2 \Big) dv, \quad (13)$$

where dv (resp. ds) is the Riemannian volume form of $\mathbb{D}$ (resp. $\partial\mathbb{D}$). Moreover equality occurs if and only if the spinor field ψ is a twistor-spinor, i.e., if and only if it satisfies $\mathrm{P}^{\mathcal{Z}}\psi = 0$, where $\mathrm{P}^{\mathcal{Z}}$ is the twistor operator acting on $\Sigma\mathcal{Z}$ locally given, for all $X \in \Gamma(T\mathcal{Z})$, by $\mathrm{P}^{\mathcal{Z}}_X\psi = \nabla^{\mathcal{Z}}_X\psi + \frac{1}{n+1} X \cdot D^{\mathcal{Z}}\psi$. Now, we can state the main theorem of this section:

Theorem 3.1 *Let $(\mathcal{Z}^{n+1}, g_{\mathcal{Z}})$ be a Riemannian $\mathrm{Spin^c}$ manifold such that the operator $S^{\mathcal{Z}} + 2i\Omega^{\mathcal{Z}}\cdot$ is nonnegative. We consider M^n a compact hypersurface with nonnegative* *mean curvature H and bounding a compact domain $\mathbb{D}$ in $\mathcal{Z}$. Then, the first positive eigenvalue λ_1 of $\widetilde{D}$ satisfies*

$$\lambda_1 \geqslant \frac{n}{2} \inf_M H. \tag{14}$$

Equality holds if and only if H is constant and the eigenspace corresponding to λ_1 consists of the restrictions to M of parallel spinors on the domain $\mathbb{D}$.

Proof. Let φ be an eigenspinor on M corresponding to the first positive eigenvalue $\lambda_1 > 0$ of $\widetilde{D}$, i.e., $\widetilde{D}\varphi = \lambda_1\varphi$ and $\pi_+\varphi = \varphi$. The following boundary problem has a unique solution (see [17], [16], [18] and [19])

$$\begin{cases} D^{\mathcal{Z}}\psi = 0 & \text{on } \mathbb{D} \\ \pi_+\psi = \pi_+\varphi = \varphi & \text{on } M = \partial\mathbb{D}. \end{cases}$$

From the Reilly inequality (13), we get

$$\int_M (\lambda_1 - \frac{n}{2}H)|\psi|^2 ds \geq \int_{\mathbb{D}} (\frac{1}{4}S^{\mathcal{Z}}|\psi|^2 + \frac{i}{2} < \Omega^{\mathcal{Z}} \cdot \psi, \psi >) dv \geq 0,$$

which implies (14). If the equality case holds in (14), then ψ is a harmonic spinor and a twistor spinor, hence parallel. Since $\pi_+\psi = \varphi$ along the boundary, ψ is a non-trivial parallel spinor and $\lambda_1 = \frac{n}{2}H$. Futhermore, since ψ is parallel, we deduce by (8) that $\widetilde{D}\varphi = \frac{n}{2}H\varphi$. Hence we have $\varphi = \pi_+\psi = \psi$. Conversely if H is constant, the fact that the restriction to M of a parallel spinor on $\mathbb{D}$ is an eigenspinor with eigenvalue $\frac{n}{2}H$ is a direct consequence of (8).

Examples 3.1 *A complete simply connected Riemannian* $\mathrm{Spin^c}$ *manifold $\mathcal{Z}^{n+1}$ carrying a parallel spinor field is isometric to the Riemannian product of a simply connected Kähler manifold $\mathcal{Z}_1^{n_1}$ of complex dimension m_1 ($n_1 = 2m_1$) and a simply connected* Spin *manifold $\mathcal{Z}_2^{n_2}$ of dimension n_2 ($n+1 = n_1 + n_2$) carrying a parallel spinor and the* $\mathrm{Spin^c}$ *structure of $\mathcal{Z}$ is the product of the canonical* $\mathrm{Spin^c}$ *structure of $\mathcal{Z}_1$ and the* Spin *structure of $\mathcal{Z}_2$ [27]. Moreover, if we assume that $\mathcal{Z}_1$ is Einstein, then*

$$i\Omega^{\mathcal{Z}}(X,Y) = i\rho^{\mathcal{Z}_1}(X_1,Y_1) = i\mathrm{Ric}^{\mathcal{Z}_1}(X_1, JY_1) = i\frac{S^{\mathcal{Z}^1}}{n_1} \ltimes (X_1,Y_1), \tag{15}$$

for every $X = X_1 + X_2, Y = Y_1 + Y_2 \in \Gamma(T\mathcal{Z})$ and where J denotes the complex structure on $\mathcal{Z}_1$. Moreover, if the Einstein manifold $\mathcal{Z}_1$ is of positive scalar curvature, we have, for any spinor field $\psi \in \Gamma(\Sigma\mathcal{Z})$,

$$\begin{aligned} S^{\mathcal{Z}}|\psi|^2 + 2i < \Omega^{\mathcal{Z}} \cdot \psi, \psi > &= S^{\mathcal{Z}_1}|\psi|^2 + \frac{i}{m_1} S^{\mathcal{Z}_1} < \ltimes \cdot \psi, \psi > \\ &= S^{\mathcal{Z}_1} \sum_{r=0}^{m_1} (1 - \frac{m_1 - 2r}{m_1})|\psi_r|^2 = S^{\mathcal{Z}_1} \sum_{r=0}^{m_1} \frac{2r}{m_1} |\psi_r|^2 \geq 0. \end{aligned}$$

Finally, the first positive eigenvalue of the Dirac operator $\widetilde{D}$ of any compact hypersurface with nonnegative constant mean curvature H and bounding a compact domain $\mathbb{D}$ in $\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2$ satisfies the equality case in (14) for the restricted Spin^c *structure. Next, we will give some explicit examples. The Alexandrov theorem for $\mathbb{S}^2_+ \times \mathbb{R}$ says that the only embedded compact surface with constant mean curvature $H > 0$ in $\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2 = \mathbb{S}^2_+ \times \mathbb{R}$ is the standard rotational sphere described in [1, 2, 8]. Hence, the first positive eigenvalue of the Dirac operator $\widetilde{D}$ on the rotational sphere satisfies the equality case in (14). We consider the complex projective space $\mathbb{C}P^m$ ($\mathcal{Z}_2 = \{\emptyset\}$) endowed with the Einstein Fubini-Study metric and the canonical* Spin^c *structure. The first positive eigenvalue of the Dirac operator $\widetilde{D}$ of any compact hypersurface M with nonnegative constant mean curvature H and bounding a compact domain $\mathbb{D}$ in $\mathbb{C}P^m$ satisfies the equality case in (14). Compact embedded hypersurfaces in $\mathbb{C}P^m$ are examples of manifolds viewed as a boundary of some enclosed domain in $\mathbb{C}P^m$. As an example, we know that there exists an isometric embedding of $\mathbb{E}^*(\kappa, \tau)$ into $\mathbb{M}^4(\frac{\kappa}{4} - \tau^2)$ of constant mean curvature $H = \frac{\kappa - 16\tau^2}{12\tau}$ [34]. Here $\mathbb{M}^4(\frac{\kappa}{4} - \tau^2)$ denotes the complex space form of constant holomorphic sectional curvature $\kappa - 4\tau^2$. We choose $\kappa > 16\tau^2$ and $\tau > 0$, then H is positive. In this case, $\mathbb{E}^*(\kappa, \tau)$ are Berger spheres (compact) and $\mathbb{M}^4$ is the complex projective space $\mathbb{C}P^2$ of constant holomorphic sectional curvature $\kappa - 4\tau^2 > 0$. The canonical* Spin^c *structure on $\mathbb{M}^4$ carries a parallel spinor and hence the equality case in (14) is satisfied for the first positive eigenvalue of $\widetilde{D}$ defined on Berger spheres. Finally, we recall that $\mathbb{S}^2_+ \times \mathbb{R}$ and $\mathbb{C}P^m$ (when m is odd), have also a unique* Spin *structure. Hence, Inequality (14) holds for the first positive eigenvalue of the* Spin *Dirac operator $\widetilde{D}$ defined on the rotational sphere or on any compact embedded hypersurface in $\mathbb{C}P^m$ (when m is odd). But, equality cannot occur since this unique* Spin *structure on $\mathbb{S}^2_+ \times \mathbb{R}$ and on $\mathbb{C}P^m$ does not carry a parallel spinor.*

3.2 Upper bounds for the eigenvalues of the Dirac operator

A spinor field ψ on a Riemannian Spin^c manifold $\mathcal{Z}^{n+1}$ is called a real Killing spinor with Killing constant $\alpha \in \mathbb{R}$ if

$$\nabla^{\mathcal{Z}}_X \psi = \alpha\, X \cdot \psi, \tag{16}$$

for all $X \in \Gamma(T\mathcal{Z})$. When $\alpha = 0$, the spinor field ψ is a parallel spinor. We define

$$\mu = \mu(\mathcal{Z}, \alpha) := \dim_{\mathbb{C}}\{\psi, \psi \text{ is a Killing spinor on } \mathcal{Z} \text{ with Killing constant } \alpha\}$$

Theorem 3.2 *Let M be an n-dimensional closed oriented hypersurface isometrically immersed in a Riemannian* Spin^c *manifold $\mathcal{Z}$. We endow M with the induced* Spin^c *structure. For any $\alpha \in \mathbb{R}$, there are at least $\mu(\mathcal{Z}, \alpha)$ eigenvalues $\lambda_1, \ldots, \lambda_\mu$ of the Dirac operator $\widetilde{D}$ on M satisfying*

$$\lambda_j^2 \leq n^2\alpha^2 + \frac{n^2}{4\mathrm{vol}(M)} \int_M H^2 dv, \tag{17}$$

where H denotes the mean curvature of M. If equality holds, then H is constant.

Proof. First, note that the set of Killing spinors with Killing constant α is a vector space. Moreover, linearly independent Killing spinors are linearly independent at every point, the space of restrictions of Killing spinors on $\mathcal{Z}$ to M, i.e.,

$$\{\psi|_M,\ \psi \text{ is a spinor on } \mathcal{Z} \text{ satisfying } \nabla^{\mathcal{Z}}_X \psi = \alpha\, X\cdot\psi, \quad \forall X \in \Gamma(T\mathcal{Z})\}$$

is also μ-dimensional. Now let ψ be a Killing spinor on $\mathcal{Z}$ with Killing constant $\alpha \in \mathbb{R}$. Killing spinors have constant length so we can assume that $|\psi| \equiv 1$. By definition, we have $D^{\mathcal{Z}}\psi = -(n+1)\alpha\ \psi$, and hence using (8) we get $\widetilde{D}\varphi = n\alpha\ \nu\cdot\varphi + \frac{n}{2}H\varphi$. We denote by $(.,.) = \mathrm{Re}\int_M <.,.>$ the real part of the L^2-scalar product. Now, we compute the Rayleigh quotient of $\widetilde{D}^2$

$$\frac{(\widetilde{D}^2\psi,\psi)}{(\psi,\psi)} = \frac{(\widetilde{D}\psi,\widetilde{D}\psi)}{\mathrm{vol}(M)} = \frac{(n\alpha\ \nu\cdot\psi + \frac{n}{2}H\psi, n\alpha\ \nu\cdot\psi + \frac{n}{2}H\psi)}{\mathrm{vol}(M)} = n^2\alpha^2 + \frac{n^2}{4}\frac{\int_M H^2}{\mathrm{vol}(M)},$$

i.e., the Rayleigh quotient of $\widetilde{D}^2$ is bounded by $n^2\alpha^2 + \frac{n^2}{4}\frac{\int_M H^2}{\mathrm{vol}(M)}$ on a μ-dimensional space of spinors on M. Hence, the Min-Max principle implies the assertion. If equality holds, then the restriction to M of every Killing spinor ψ of Killing constant α satisfies $\widetilde{D}^2\varphi = \lambda_1^2\varphi$. But, it is known that [11]

$$\widetilde{D}^2\varphi = \hat{D}^2\varphi + \frac{n}{2}dH\cdot\nu\cdot\varphi + \frac{n^2H^2}{4}\varphi, \tag{18}$$

where $\hat{D}$ is the Dirac-Witten operator given by $\hat{D} = \sum_{j=1}^n e_j\cdot\nabla^{\mathcal{Z}}_{e_j}$. Hence, using that $\hat{D}\psi = -n\alpha\psi$ and (18), we get

$$\lambda_1^2\varphi = n^2\alpha^2\varphi + \frac{n}{2}dH\cdot\nu\cdot\varphi + \frac{n^2}{4}H^2\varphi.$$

Considering the real part of the scalar product of the last equality by φ implies that $\lambda_1^2 = n^2\alpha^2 + \frac{n^2H^2}{4}$. Hence, H is constant.

Examples 3.2 *Simply connected complete Riemannian* $\mathrm{Spin^c}$ *manifolds carrying parallel spinors were described in Examples 3.1. The only* $\mathrm{Spin^c}$ *structures on an irreducible Kähler not Ricci-flat manifold $\mathcal{Z}$ which carry parallel spinors are the canonical and the anti-canonical one. In both cases, $\mu(\mathcal{Z},0) = 1$ [27]. Hence, Inequality (17) holds for the first eigenvalue of the Dirac operator $\widetilde{D}$ defined on any compact Riemannian hypersurface endowed with the restricted* $\mathrm{Spin^c}$ *structure. The complex projective space $\mathbb{C}P^m$ or the complex hyperbolic space $\mathbb{C}H^m$ with the Fubini-Study metric are examples of irreducible Kähler not Ricci-flat manifolds. From Examples 3.1, the equality case in (14) is achieved for the first positive eigenvalue of the Dirac operator $\widetilde{D}$ defined on Berger spheres embedded into $\mathbb{C}P^2$. Hence, Inequality (17) is* *also an equality in this case. Also, for rotational constant mean curvature H spheres embedded into $\mathbb{S}^2 \times \mathbb{R}$, Inequality (17) is an equality because in this case, Inequality (14) is an equality. The only complete simply connected* $\mathrm{Spin^c}$ *manifolds admitting real Killing spinors other than the* Spin *manifolds are the non-Einstein Sasakian manifolds endowed with their canonical or anti-canonical* $\mathrm{Spin^c}$ *structure [27]. The manifolds* $\mathbb{E}^*(\kappa,\tau)$ *are examples of* $\mathrm{Spin^c}$ *manifolds carrying a Killing spinor ψ of Killing constant $\frac{\tau}{2}$.*

4 Spin^c structures on $\mathbb{E}^*(\kappa,\tau)$ and applications

In this section, we make use of the existence of a $\mathrm{Spin^c}$ Killing spinor to immerse $\mathbb{E}^*(\kappa,\tau)$ into complex space forms, to calculate some eigenvalues of the Dirac operator on Berger spheres and to prove the non-existence of totally umbilic surfaces in $\mathbb{E}^*(\kappa,\tau)$.

4.1 Isometric immersions of $\mathbb{E}^*(\kappa,\tau)$ into complex space forms

From the existence of an isometric embedding of $\mathbb{E}^*(\kappa,\tau)$ into $\mathbb{M}^4(\frac{\kappa}{4}-\tau^2)$, we reprove that the only $\mathrm{Spin^c}$ structures on $\mathbb{E}^*(\kappa,\tau)$ carrying a Killing spinor are the canonical and the anti-canonical one. Conversely, the existence of a $\mathrm{Spin^c}$ Killing spinor allows to immerse $\mathbb{E}^*(\kappa,\tau)$ in $\mathbb{M}^4(\frac{\kappa}{4}-\tau^2)$. More generally, we give necessary and sufficient geometric conditions to immerse any 3-dimensional Sasaki manifold into $\mathbb{M}^2(c)$ for some $c \in \mathbb{R}^*$.

Proposition 4.1 *The only* $\mathrm{Spin^c}$ *structures on* $\mathbb{E}^*(\kappa,\tau)$ *carrying a real Killing spinor are the canonical and the anti-canonical one. Moreover, the Killing constant is given by $\frac{\tau}{2}$.*

Proof. It is known that there exists an isometric embedding of $\mathbb{E}^*(\kappa,\tau)$ into $\mathbb{M}^4(\frac{\kappa}{4}-\tau^2)$ of constant mean curvature $H = \frac{\kappa-16\tau^2}{12\tau}$ [34]. Moreover, the second fundamental form is given by

$$II(X) = -\tau X - \frac{4\tau^2-\kappa}{\tau} g_{\mathbb{M}^4}(X,\xi)\xi,$$

for every $X \in \Gamma(T\mathbb{E}^*(\kappa,\tau))$. Here, we recall that the normal vector of the immersion is given by $\nu := J\xi$ and $\{e_1, e_2, \xi, \nu = J\xi\}$ is a local orthonormal basis tangent to $\mathbb{M}^4$ where $\{e_1, e_2, \xi\}$ is the canonical frame of $\mathbb{E}^*(\kappa,\tau)$. We denote by η the real 1-form associated with ξ, i.e., $\eta(X) = g(X,\xi)$ for any $X \in \Gamma(T\mathbb{E}^*(\kappa,\tau))$. The restriction of the canonical $\mathrm{Spin^c}$ structure on $\mathbb{M}^4(\frac{\kappa}{4}-\tau^2)$ induces a $\mathrm{Spin^c}$ structure on $\mathbb{E}^*(\kappa,\tau)$ and by the $\mathrm{Spin^c}$ Gauss formula (7), the restriction of the parallel spinor on $\mathbb{M}^4(\frac{\kappa}{4}-\tau^2)$

induces a spinor field φ on $\mathbb{E}^*(\kappa,\tau)$ satisfying, for all $X \in \Gamma(T\mathbb{E}^*(\kappa,\tau))$,

$$\nabla_X \varphi = \frac{\tau}{2} X \bullet \varphi + \frac{4\tau^2 - \kappa}{8\tau} \eta(X) \xi \bullet \varphi.$$

Moreover, the spinor field φ satisfies $\xi \bullet \varphi = -i\varphi$ [30, Theorem 3] and the curvature of the auxiliary line bundle L associated with the induced $\mathrm{Spin^c}$ structure is given by [30, Theorem 3]

$$i\Omega(e_1, e_2) = -6i(\frac{\kappa}{4} - \tau^2), \quad \text{and} \quad i\Omega(e_i, e_j) = 0, \tag{19}$$

elsewhere in the basis $\{e_1, e_2, \xi\}$. We deduce that, for all $X \in \Gamma(T\mathbb{E}^*(\kappa,\tau))$,

$$\nabla_X \varphi = \frac{\tau}{2} X \bullet \varphi - i \frac{4\tau^2 - \kappa}{8\tau} g(X, \xi) \varphi.$$

The connection A on the $\mathbb{S}^1$-principal fiber bundle $\mathbb{S}^1(\mathbb{E}^*(\kappa,\tau))$ associated with the induced $\mathrm{Spin^c}$ structure is the restriction to $\mathbb{E}^*(\kappa,\tau)$ of the connection on the $\mathbb{S}^1$-principal fiber bundle $\mathbb{S}^1\mathbb{M}^4$ associated with the canonical $\mathrm{Spin^c}$ structure on $\mathbb{M}^4(\frac{\kappa}{4} - \tau^2)$, i.e., the connection A on $\mathbb{S}^1(\mathbb{E}^*(\kappa,\tau))$ is the restriction to $\mathbb{E}^*(\kappa,\tau)$ of the connection on $\mathbb{S}^1(\mathbb{M}^4(\frac{\kappa}{4} - \tau^2))$ defined by the Levi-Civita connection. Let α be the real 1-form on $\mathbb{E}^*(\kappa,\tau)$ defined by

$$\alpha(X) = \frac{4\tau^2 - \kappa}{4\tau} g(X, \xi),$$

for any $X \in \Gamma(T\mathbb{E}^*(\kappa,\tau))$. We endow the $\mathbb{S}^1$-principal fiber bundle $\mathbb{S}^1(\mathbb{E}^*(\kappa,\tau))$ with the connection $A' = A + i\alpha$. From (4), there exists on $\Sigma\mathbb{E}^*(\kappa,\tau)$ a covariant derivative ∇' such that

$$\nabla'_X \varphi = \nabla_X \varphi + \frac{i}{2}\alpha(X)\varphi = \frac{\tau}{2} X \bullet \varphi,$$

for all $X \in \Gamma(T\mathbb{E}^*(\kappa,\tau))$. Hence, we obtain a $\mathrm{Spin^c}$ structure on $\mathbb{E}^*(\kappa,\tau)$ carrying a Killing spinor field and whose $\mathbb{S}^1$-principal fiber bundle $\mathbb{S}^1(\mathbb{E}^*(\kappa,\tau))$ has a connection given by A'. Now, we should prove that this $\mathrm{Spin^c}$ structure is the canonical one. First, we calculate the curvature $i\Omega' = i\Omega + id\alpha$ of A'. It is easy to check that $\xi \lrcorner d\alpha = 0$ and $d\alpha(e_1, e_2) = -\frac{4\tau^2 - \kappa}{2}$. Hence, using (19), we get

$$\Omega'(e_1, e_2) = -(\kappa - 4\tau^2) \text{ and } \xi \lrcorner \Omega' = 0.$$

The curvature form $i\Omega'$ is the same as the curvature form associated with the connection on the auxiliary line bundle of the canonical $\mathrm{Spin^c}$ structure on $\mathbb{E}^*(\kappa,\tau)$. Since $\mathbb{E}^*(\kappa,\tau)$ is a simply connected manifold, we deduce that the $\mathbb{S}^1$-principal fiber bundle $\mathbb{S}^1(\mathbb{E}^*(\kappa,\tau))$ endowed with the connection A' is the auxiliary line bundle of the canonical $\mathrm{Spin^c}$ structure on $\mathbb{E}^*(\kappa,\tau)$. Hence, we have on $\mathbb{E}^*(\kappa,\tau)$ two $\mathrm{Spin^c}$ structures with the same auxiliary line bundle (the canonical one and the one obtained by restriction of the canonical one on $\mathbb{M}^4$). But, on a Riemannian manifold M, $\mathrm{Spin^c}$ structures having the same auxiliary line bundle are parametrized by $H^1(M,\mathbb{Z}_2)$ [26], which is trivial in our case since $\mathbb{E}^*(\kappa,\tau)$ is simply connected. To get the anti-canonical $\mathrm{Spin^c}$ structure on $\mathbb{E}^*(\kappa,\tau)$, we restrict the anti-canonical $\mathrm{Spin^c}$ structure on $\mathbb{M}^4$. In this case, $\xi\bullet\varphi = i\varphi$, $\Omega(e_1,e_2) = 6(\frac{\kappa}{4}-\tau^2)$, $\xi\lrcorner\Omega = 0$ and we choose the real 1-form α to be $\alpha(X) = -\frac{4\tau^2-\kappa}{4\tau}g(X,\xi)$.

Next, we want to prove the converse. Indeed, we have

Proposition 4.2 *The manifolds $\mathbb{E}^*(\kappa,\tau)$ are isometrically immersed into $\mathbb{M}^4(c)$ for some c. Moreover, $\mathbb{E}^*(\kappa,\tau)$ are of constant mean curvature and η-umbilic.*

Proof. We recall that the 3-dimensional homogeneous manifolds $\mathbb{E}^*(\kappa,\tau)$ have a $\mathrm{Spin^c}$ structure (the canonical $\mathrm{Spin^c}$ structure) carrying a Killing spinor field φ of Killing constant $\frac{\tau}{2}$. Moreover $\xi\bullet\varphi = -i\varphi$ and

$$\Omega(e_1,e_2) = -(\kappa-4\tau^2) \quad\text{and}\quad \Omega(e_i,e_j) = 0, \tag{20}$$

in the basis $\{e_1,e_2,e_3=\xi\}$. We denote by A the connection on the auxiliary line bundle defining the canonical $\mathrm{Spin^c}$ structure. Let α be the real 1-form on $\mathbb{E}^*(\kappa,\tau))$ defined by $\alpha(X) = -\frac{4\tau^2-\kappa}{4\tau}g(X,\xi)$, for any $X\in\Gamma(T\mathbb{E}^*(\kappa,\tau))$. We endow the $\mathbb{S}^1$-principal fiber bundle $\mathbb{S}^1(\mathbb{E}^*(\kappa,\tau))$ with the connection $A^{'} = A + i\alpha$. Then, there exists on $\Sigma\mathbb{E}^*(\kappa,\tau)$ a covariant derivative $\nabla^{'}$ such that

$$\begin{aligned}\nabla^{'}_X\varphi &= \frac{\tau}{2}X\bullet\varphi + \frac{i}{2}\alpha(X)\varphi\\ &= \frac{\tau}{2}X\bullet\varphi + \frac{4\tau^2-\kappa}{8\tau}\eta(X)\xi\bullet\varphi,\end{aligned} \tag{21}$$

for all $X\in\Gamma(T\mathbb{E}^*(\kappa,\tau))$. Hence, we obtain a $\mathrm{Spin^c}$ structure on $\mathbb{E}^*(\kappa,\tau)$ carrying a spinor field φ satisfying (21) and whose $\mathbb{S}^1$-principal fiber bundle $\mathbb{S}^1(\mathbb{E}^*(\kappa,\tau))$ has a connection given by $A^{'}$. We calculate the curvature $i\Omega^{'} = i\Omega + id\alpha$ of $A^{'}$. It is easy to check that $\xi\lrcorner d\alpha = 0$ and $d\alpha(e_1,e_2) = \frac{4\tau^2-\kappa}{2}$. Hence,

$$\Omega^{'}(e_1,e_2) = -6(\frac{\kappa}{4}-\tau^2) \quad\text{and}\quad \xi\lrcorner\Omega^{'} = 0.$$

Since $\mathbb{E}^*(\kappa,\tau)$ are Sasakian, by [30, Theorem 4], we get an isometric immersion of $\mathbb{E}^*(\kappa,\tau)$ into $\mathbb{M}^4(c)$ for $c = \frac{\kappa}{4}-\tau^2$. Moreover, $\mathbb{E}^*(\kappa,\tau)$ are of constant mean curvature and η-umbilic (see [30]).

More general, we have:

Theorem 4.3 *Every simply connected non-Einstein 3-dimensional Sasaki manifold M^3 of constant scalar curvature can be immersed into $\mathbb{M}^4(c)$ for some $c\in\mathbb{R}^*$. Moreover, M is η-umbilic and of constant mean curvature.*

Proof. We recall that a Sasaki structure on a 3-dimensional manifold M^3 is given by a Killing vector field ξ of unit length such that the tensors $\mathfrak{X} := \nabla\xi$ and $\eta := g(\xi, \cdot)$ are related by

$$\mathfrak{X}^2 = -\mathrm{Id} + \eta \otimes \xi.$$

We know that a non-Einstein Sasaki manifold has a $\mathrm{Spin^c}$ structure carrying a Killing spinor field φ of Killing constant β. By rescaling the metric, we can assume that $\beta = -\frac{1}{2}$. Moreover, the Killing vector field ξ defining the Sasaki structure satisfies $\xi \bullet \varphi = -i\varphi$ (see [27]). The Ricci tensor on M is given by

$$\mathrm{Ric}(e_j) = \frac{S-2}{2} e_j, j = 1, 2 \quad \text{and} \quad \mathrm{Ric}(\xi) = 2\xi,$$

where S denotes the scalar curvature of M and $\{e_1, e_2, \xi\}$ a local orthonormal frame of M. Because we assumed that M is non-Einstein, we have $S \neq 6$ and hence we can find $c \in \mathbb{R}^*$ such that $S = 8c + 6$. The Ricci identity (6) in $X = \xi$ gives that $\xi \lrcorner \Omega = 0$ and by the Schrödinger-Lichnerowicz formula, it follows that $\Omega(e_1, e_2) = \frac{6-S}{2}$. Let α be the real 1-form on M defined by $\alpha(X) = -cg(X, \xi)$, for any $X \in \Gamma(TM)$. We endow the $\mathbb{S}^1$-principal fiber bundle $\mathbb{S}^1 M$ with the connection $A^{'} = A + i\alpha$, where A denotes the connection on $\mathbb{S}^1 M$ whose curvature form is given by $i\Omega$. From (4), there exists on ΣM a covariant derivative $\nabla^{'}$ such that

$$\nabla^{'}_X \varphi = -\frac{1}{2} X \bullet \varphi - \frac{i}{2} c g(X, \xi)\varphi.$$

Now, we calculate the curvature $i\Omega^{'} = i\Omega + i d\alpha$ of $A^{'}$. It is easy to check that $\xi \lrcorner d\alpha = 0$ and $d\alpha(e_1, e_2) = -2c$. Hence,

$$\xi \lrcorner \Omega^{'} = 0, \qquad \Omega^{'}(e_1, e_2) = \frac{6-S}{2} - 2c = -6c.$$

By [30, Theorem 4], M is immersed into $\mathbb{M}^4(c)$. Additionally, M is η-umbilic and of constant mean curvature.

4.2 Totally umbilic surfaces in $\mathbb{E}^*(\kappa, \tau)$

By restriction of the Killing spinor of Killing constant $\frac{\tau}{2}$ on $\mathbb{E}^*(\kappa, \tau)$ to a surface M^2, the authors characterized isometric immersions into $\mathbb{E}(\kappa, \tau)$ by the existence of a $\mathrm{Spin^c}$ structure carrying a special spinor field [30]. More precisely, consider (M^2, g) a Riemannian surface. We denote by E a field of symmetric endomorphisms of TM, with trace equal to $2H$. The vertical vector field can be written as $\xi = dF(T) + f\nu$, where ν is the unit normal vector to the surface, f is a real function on M and T the tangential part of ξ. The isometric immersion of (M^2, g) into $\mathbb{E}(\kappa, \tau)$ with shape operator E, mean curvature H is characterized by a $\mathrm{Spin^c}$ structure on M carrying a non-trivial spinor field φ satisfying, for all $X \in \Gamma(TM)$,

$$\nabla_X \varphi = -\frac{1}{2} EX \bullet \varphi + i\frac{\tau}{2} X \bullet \overline{\varphi}.$$

Moreover, the auxiliary bundle has a connection of curvature given, in any local orthonormal frame $\{t_1, t_2\}$, by $i\Omega(t_1, t_2) = -i(\kappa - 4\tau^2)f = -i(\kappa - 4\tau^2)\frac{<\varphi,\overline{\varphi}>}{|\varphi|^2}$. The vector T is given by

$$g(T, t_1) =< it_2 \bullet \varphi, \frac{\varphi}{|\varphi|^2} > \quad \text{and} \quad g(T, t_2) = - < it_1 \bullet \varphi, \frac{\varphi}{|\varphi|^2} > .$$

Here and also by restriction of the Killing spinor, we gave an elementary Spin^c proof of the following result proved by R. Souam and E. Toubiana in [32].

Theorem 4.4 *There are no totally umbilic surfaces in* $\mathbb{E}^*(\kappa, \tau)$.

Proof. Assume that M is a totally umbilical surface in $\mathbb{E}^*(\kappa, \tau)$, i.e. $E = H$ Id. Then $d^\nabla E(e_1, e_2) = (\nabla_{t_1} E)t_2 - (\nabla_{t_2} E)t_1 = J(dH)$. The Spin^c curvature $\mathcal{R}$ on the spinor field φ is given by [30]:

$$\mathcal{R}(t_1, t_2)\varphi = -\frac{1}{2}J(dH) \bullet \varphi + i\frac{H^2}{2}\overline{\varphi} + i\frac{\tau^2}{2}\overline{\varphi}.$$

The Spin^c Ricci identity (6) on the surface M implies

$$t_1 \bullet \mathcal{R}(t_1, t_2)\varphi = \frac{1}{2}\mathrm{Ric}(t_2) \bullet \varphi - \frac{i}{2}(t_2 \lrcorner \Omega) \bullet \varphi$$

Hence,

$$-\frac{1}{2}t_1 \bullet J(dH) \bullet \varphi + \frac{i}{2}H^2 t_1 \bullet \overline{\varphi} + \frac{i}{2}\tau^2 t_1 \bullet \overline{\varphi} = \frac{1}{2}\mathrm{Ric}(t_2) \bullet \varphi + \frac{i}{2}\Omega(t_1, t_2)t_1 \bullet \varphi$$

Consider the real part of the scalar product of the last identity by φ, we get

$$g(t_1, J(dH)) = \Omega(t_1, t_2) < it_1 \bullet \varphi, \frac{\varphi}{|\varphi|^2} >= -\Omega(t_1, t_2)g(T, t_2).$$

Finally, $-g(t_2, dH) = (\kappa - 4\tau^2)fg(T, t_2)$. The same holds for t_1. Then,

$$dH = -(\kappa - 4\tau^2)fT,$$

which gives the contradiction. The last identity is the same obtained by R. Souam and E. Toubiana in [32].

4.3 Spectrum of the Spin^c Dirac operator on Berger spheres

In this subsection, we apply a method of C. Bär [6, 12] to find explicitly some eigenvalues of the Spin^c Dirac operator on Berger spheres, i.e., on $\mathbb{E}^*(\kappa, \tau)$ with $\kappa > 0$.

Lemma 4.5 *Let (M^n, g) be a Riemannian* $\mathrm{Spin^c}$ *manifold carrying a Killing spinor φ of Killing number $\alpha \in \mathbb{R}^*$. Then, $(\lambda_k(\triangle) + (\frac{n-1}{2})^2)_{k\in\mathbb{N}}$ are some eigenvalues of $(D + \frac{\alpha}{2}\mathrm{Id})^2$. Here $\lambda_k(\triangle)$, $k = 0, 1, ...$ denote the eigenvalue of the Laplacian $\triangle$.*

Proof: We have $D\varphi = -\frac{n\alpha}{2}\varphi$. For every $f \in C^\infty(M, \mathbb{R})$, we can easily check that

$$D^2(f\varphi) = (\frac{n^2}{4} - \frac{n}{2})f\varphi - \alpha D(f\varphi) + (\triangle f)\varphi,$$

Hence, $(D + \frac{\alpha}{2}\mathrm{Id})^2(f\varphi) = (\triangle f + (\frac{n-1}{2})^2 f)\varphi$. Now, if $\{f_k\}_{k\in\mathbb{N}}$ denotes a L^2-orthonormal basis of eigenfunctions of $\triangle$ of M, then for every $k \in \mathbb{N}$, we get

$$(D + \frac{\alpha}{2}\mathrm{Id})^2(f_k\varphi) = (\lambda_k(\triangle) + (\frac{n-1}{2})^2)f_k\varphi,$$

where $\lambda_k(\triangle)$ is the eigenvalue of $\triangle$ whose eigenfunction is f_k. So, $(\lambda_k(\triangle) + (\frac{n-1}{2})^2)_{k\in\mathbb{N}}$ are some eigenvalues of $(D + \frac{\alpha}{2}\mathrm{Id})^2$.

Spectrum of Berger spheres endowed with the canonical Spin^c structure. We consider Berger spheres with Berger metrics $g_{\kappa,\tau}$, $\kappa > 0$ and $\tau \neq 0$ defined by

$$g_{(\kappa,\tau)}(X, Y) = \frac{\kappa}{4}\Big(g(X, Y) + (\frac{4\tau^2}{\kappa} - 1)g(X, \xi)g(Y, \xi)\Big),$$

where g is the standard metric on $\mathbb{S}^3$ of constant curvature 1. For simplicity, we can assume that $\kappa = 4$ ($\tau \neq \pm 1$). For any function f, the Laplacian $\triangle_{4,\tau}$ with respect to $g_{4,\tau}$ is related to the Laplacian $\triangle$ with respect to g by [33]

$$\triangle_{4,\tau} f = \triangle f - (1 - \tau^{-2})\xi(\xi(f)).$$

It is known that each eigenfunction f_k of $\triangle$ corresponding to $\lambda_k(\triangle) = k(2 + k)$ ($k \in \mathbb{N}$) is also an eigenfunction of $\triangle_{4,\tau}$ [33] corresponding to

$$\lambda_k(\triangle) - (1 - \tau^{-2})(k - 2p)^2, \quad 0 \leq p \leq [\frac{k}{2}].$$

Moreover, each eigenvalue of $\triangle_{4,\tau}$ takes the above form. We recall that the eigenspace of $\triangle$ corresponding to $\lambda_k(\triangle)$ is the restriction to the sphere $\mathbb{S}^3$ of the set of harmonic homogeneous polynomial on $\mathbb{R}^4$ of degree k. When we consider Berger spheres endowed with the canonical $\mathrm{Spin^c}$ structure, we get by Lemma 4.5

$$\Big(D + \frac{\tau}{2}\mathrm{Id}\Big)^2(f_k\varphi) = \Big[2 + k(2 + k) - (1 - \tau^{-2})(k - 2p)\Big]f_k\varphi,$$

where φ is the Killing spinor field of Killing constant $\frac{\tau}{2}$. Hence,

$$\mu_{k,p} = -\frac{\tau}{2} \pm \sqrt{2 + k(2 + k) - (1 - \tau^{-2})(k - 2p)}$$

are some eigenvalues of the Dirac operator on Berger spheres with $-1 < \tau < 1$ and endowed with the canonical Spin^c structure.

Spectrum of Berger spheres endowed with the Spin^c structure induced from the canonical one on $\mathbb{M}^4(1-\tau^2)$. On Berger spheres, we have shown that the Spin^c structure induced from the canonical one on $\mathbb{M}^4(1-\tau^2)$ carries a spinor field φ satisfying

$$\nabla_X\varphi = \frac{\tau}{2}X \bullet \varphi - i\frac{\tau^2-1}{2\tau}g(X,\xi)\varphi = \nabla'_X\varphi - i\frac{\tau^2-1}{2\tau}g(X,\xi)\varphi$$

Then, denoting by D (resp. D') the Dirac operator associated with the restricted Spin^c structure (resp. with the canonical Spin^c structure), we get $D\varphi = D'\varphi - \frac{\tau^2-1}{2\tau}\varphi$. for any function f, we have

$$D(f\varphi) = \mathrm{grad} f \cdot \varphi + fD\varphi = D'(f\varphi) - fD'\varphi + fD\varphi = D'(f\varphi) - (\frac{\tau^2-1}{2\tau})f\varphi.$$

Hence, we have $D(f_k\varphi) = \Big(\mu_{k,p} - \frac{\tau^2-1}{2\tau}\Big)f_k\varphi$ and $\mu_{k,p} - \frac{\tau^2-1}{2\tau}$ are some eigenvalues of the Dirac operator on Berger spheres endowed with the Spin^c structure induced from the canonical one on $\mathbb{M}^4(1-\tau^2)$, $-1 < \tau < 1$.

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