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Disjoint unions of complete graphs characterized by their Laplacian spectrum *

Romain Boulet †

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Abstract

A disjoint union of complete graphs is in general not determined by its Laplacian spectrum. We show in this paper that if we only consider the family of graphs without isolated vertex then a disjoint union of complete graphs is determined by its Laplacian spectrum within this family. Moreover we show that the disjoint union of two complete graphs with \(a\) and \(b\) vertices, \(\frac{a}{2} > \frac{b}{2}\) and \(b > 1\) is determined by its Laplacian spectrum. A counter-example is given when \(\frac{a}{2} = \frac{b}{2}\).

Keywords: Graphs, Laplacian, complete graphs, graphs determined by its spectrum, strongly regular graphs.

AMS subject classifications: 05C50, 68R10.

1 Introduction and basic results

The Laplacian of a graph \(G\) is the matrix \(L\) defined by \(L = D - A\) where \(D\) is the diagonal matrix of the degrees of \(G\) and \(A\) is the adjacency matrix of \(G\). The Laplacian spectrum gives some informations about the structure of the graph but determining graphs characterized by their Laplacian spectrum remains a difficult problem [2].

In this paper we focus on the disjoint union of complete graphs. A complete graph on \(n\) vertices is denoted by \(K_n\) and the disjoint union of the graphs \(G\) and \(G'\) is denoted by \(G \cup G'\). The Laplacian spectrum of \(K_{k_1} \cup K_{k_2} \cup \ldots \cup K_{k_n}\) is

\[
\{k_1^{(k_1-1)}, k_2^{(k_2-1)}, \ldots, k_n^{(k_n-1)}, 0^{(n)}\}
\]

but in general the converse is not true: a disjoint union of complete graphs is not in general determined by its Laplacian spectrum. For instance [2] the disjoint union of the Petersen graph with 5 isolated vertices is \(L\)-cospectral with the disjoint union of the complete graph with five vertices and five complete graphs with two vertices, these graphs are depicted in figure 1.

In this paper we show in Section 2 that the disjoint union of complete graphs without isolated vertex is determined by its Laplacian spectrum in the family of graphs without isolated vertex. Then in Section 3 we study the disjoint union

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†Institut de Mathématiques de Toulouse, Université de Toulouse et CNRS (UMR 5219), France (boulet@univ-tlse2.fr).
Figure 1: The graph drawn on the left is a graph non-isomorphic to a disjoint union of complete graphs but Laplacian cospectral with the disjoint union of complete graphs drawn on the right.

of two complete graphs $K_a$ and $K_b$ and show that if $a > 5$ then $K_a \cup K_b$ is determined by its Laplacian spectrum.

To fix notations, the set of vertices of a graph $G$ is denoted by $V(G)$ and the set of edges is denoted by $E(G)$; for $v \in V(G)$, $d(v)$ denotes the degree of $v$. The complement of a graph $G$ is denoted by $\overline{G}$ and concerning the spectrum, \( \text{Sp}(G) = \{\mu_1^{(m_1)}, \ldots, \mu_k^{(m_k)}\} \) means that $\mu_i$ is $m_i$ times an eigenvalue of $L$ (the multiplicity of $\mu_i$ is at least $m_i$, we may allow $\mu_i = \mu_j$ for $i \neq j$).

We end this introduction with some known results about the Laplacian spectrum and strongly regular graphs.

**Theorem 1** [6] The multiplicity of the Laplacian eigenvalue 0 is the number of connected components of the graph.

**Theorem 2** [4, 6] Let $G$ be a graph on $n$ vertices whose Laplacian spectrum is $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \mu_n = 0$. Then:

1. $\mu_{n-1} \leq \frac{1}{n-1} \min\{d(v), v \in V(G)\}$.
2. If $G$ is not a complete graph then $\mu_{n-1} \leq \min\{d(v), v \in V(G)\}$.
3. $\mu_1 \leq \max\{d(u) + d(v), uv \in E(G)\}$.
4. $\mu_1 \leq n$.
5. $\sum \mu_i = 2|E(G)|$.
6. $\mu_1 \geq \frac{1}{n-1} \max\{d(v), v \in V(G)\} > \max\{d(v), v \in V(G)\}$.

**Theorem 3** Let $G$ be a graph on $n$ vertices, the Laplacian spectrum of $\overline{G}$ is:

$$\mu_i(\overline{G}) = n - \mu_{n-i}(G), \ 1 \leq i \leq n-1$$

**Corollary 1** Let $G$ be a graph on $n$ vertices, we have $\mu_1(G) \leq n$ with equality if and only if $G$ is a non-connected graph.

**Theorem 4** [2] A complete graph is determined by its Laplacian spectrum.

**Definition 1** [2] A graph $G$ is strongly regular with parameters $n, k, \alpha, \gamma$ if
• $G$ is not the complete graph or the graph without edges
• $G$ is $k$-regular
• Every two adjacent vertices have exactly $\alpha$ common neighbors
• Every two non-adjacent vertices have exactly $\gamma$ common neighbors

Theorem 5 [5] A regular connected graph is strongly regular if and only if it has exactly three distinct adjacency eigenvalues. A strongly regular non-connected graph is the disjoint union of $r$ complete graphs $K_{k+1}$ for a given $r$.

Theorem 6 [5] Let $G$ be a connected strongly regular graph with parameters $n, k, \alpha, \gamma$ and let $\theta, \tau$ the eigenvalues of its adjacency matrix. Then:
\[
\theta = \frac{\alpha - \gamma + \sqrt{\Delta}}{2}, \\
\tau = \frac{\alpha - \gamma - \sqrt{\Delta}}{2},
\]
where
\[
\Delta = (\alpha - \gamma)^2 + 4(k - \gamma) = (\theta - \tau)^2.
\]
Moreover, let $m_\theta$ (resp. $m_\tau$) the multiplicity of $\theta$ (resp. $\tau$), then:
\[
m_\theta = \frac{(n - 1)r + k}{\theta - \tau}, \\
m_\tau = \frac{(n - 1)\theta + k}{\theta - \tau}.
\]
That is:
\[
m_\theta = \frac{1}{2} \left( n - 1 - \frac{2k + (n - 1)(\alpha - \gamma)}{\sqrt{\Delta}} \right), \\
m_\tau = \frac{1}{2} \left( n - 1 + \frac{2k + (n - 1)(\alpha - \gamma)}{\sqrt{\Delta}} \right).
\]

2 Disjoint union of complete graphs

The aim of this section is to show that if we consider graphs without isolated vertex then the disjoint union of complete graphs is determined by its Laplacian spectrum.

We first state some results about disjoint union of complete graphs (including isolated vertices).

Proposition 1 The Laplacian spectrum of a graph $G$ with one and only one positive Laplacian eigenvalue $a$ is $\{a(ra-r), 0^{(r+p)}\}$ and $G$ is isomorphic to $K_a \cup K_1 \cup \cdots \cup K_a \cup K_1 \cup \cdots \cup K_1,$ $r$ times $p$ times.
We assume \( k \) of the Laplacian spectrum of \( G \) has and only one positive eigenvalue \( a \). If \( H \) is not a complete graph, then by Theorem 2 we have \( a \leq \min \{d(v), v \in V(G)\} \leq \max \{d(v), v \in V(G)\} < a \), contradiction. As a result \( H \) is a complete graph and \( H \) is isomorphic to \( K_a \) and there exists \( r \in \mathbb{N}^* \), \( p \in \mathbb{N} \) such that \( G \) is isomorphic to \( K_a \cup K_a \cup \cdots \cup K_a \cup \sum \{ K_i \cup K_j \cup \cdots \cup K_p \} \), \( r \) times \( p \) times.

\[ \square \]

**Theorem 7** There is no cospectral non-isomorphic disjoint union of complete graphs.

**Proof:** Let \( G = K_{k_1} \cup \cdots \cup K_{k_n} \) and \( G' = K_{k'_1} \cup \cdots \cup K_{k'_n} \), we have \( n = n' \) (same number of connected components). If \( G \) and \( G' \) are not isomorphic then there exists \( \lambda \in \mathbb{N} \setminus \{0, 1\} \) such that the number of connected components of \( G \) isomorphic to \( K_\lambda \) is different from the number of connected components of \( G' \) isomorphic to \( K_\lambda \). Therefore, the multiplicity of \( \lambda \) as an eigenvalue of the Laplacian spectrum of \( G \) is different from the multiplicity of \( \lambda \) as an eigenvalue of the Laplacian spectrum of \( G' \) and so \( G \) and \( G' \) are not cospectral.

\[ \square \]

**Theorem 8** Let \( G \) be a graph without isolated vertex. If the Laplacian spectrum of \( G \) is \( \{k_1^{(k_1-1)}, k_2^{(k_2-1)}, \ldots, k_n^{(k_n-1)}, 0^{(n)}\} \) with \( k_i \in \mathbb{N} \setminus \{0, 1\} \) then \( G \) is a disjoint union of complete graphs of order \( k_1, \ldots, k_n \).

**Proof:** The graph \( G \) has \( n \) connected components (Theorem 1) \( G_1, \ldots, G_n \) of order \( l_1, \ldots, l_n \). We denote by \( N \) the number of vertices of \( G \). We have

\[ N = \sum_{i=1}^{n} l_i = \sum_{i=1}^{n} k_i \]

Let \( k_j \) be an eigenvalue of \( G \), there exists \( i \) such that \( k_j \) is an eigenvalue of \( G_i \), so \( l_i \geq k_j \) (Theorem 2).

Let \( G_i \) be a connected component, as \( G \) does not have isolated vertices we have \( l_i > 1 \) and \( G_i \) possesses at least one eigenvalue different from 0, let \( k_j \) be this eigenvalue, we have \( l_i \geq k_j \). 

As a result

\[ \forall j \exists i : l_i \geq k_j \]

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We assume \( k_1 \geq k_2 \geq \ldots \geq k_n > 0 \) and \( l_1 \geq l_2 \geq \ldots \geq l_n > 1 \). We now show by induction on \( j \) that \( k_{n-j} \leq l_{n-j}, \forall j = 0...n-1 \).

- \( j = 0 \): we know that there exists \( j \) such that \( k_j \leq l_n, \) so \( k_n \leq l_n \).
- \( j = j_0 > 0 \). We assume that \( \forall j < j_0, k_{n-j} \leq l_{n-j} \) and let us show that \( k_{n-j_0} \leq l_{n-j_0} \) by contradiction. If \( k_{n-j_0} > l_{n-j_0} \) then \( k_{n-j_0} > l_{n-j} \), \( \forall j < j_0 \) so \( k_{n-j}, j \geq j_0 \), cannot be an eigenvalue of \( G_{n-j}, j < j_0 \). So

\[ \bigcup_{j<j_0} (\text{Sp}(G_{n-j}) \setminus \{0\}) \subset \bigcup_{j<j_0} \{k_{n-j}^{(k_{n-j}-1)}\}. \]
is a strongly regular graph with parameters $15$, $8$, $4$, $4$ is $\{a^{(a-1)}, b^{(b-1)}, 0^{(2)}\}$. The disjoint union of two complete graphs is not in general determined by its spectrum, here a counter-example. The Laplacian spectrum of the line graph of $K_a$ (which is a strongly regular graph with parameters $15$, $8$, $4$, $4$) is $\{10^{(9)}, 6^{(5)}, 0\}$, so the Laplacian spectrum of $L(K_6) \cup K_1$ is $\{10^{(9)}, 6^{(5)}, 0^{(2)}\}$ which is also the spectrum of $K_{10} \cup K_6$.

As the disjoint union $K_a \cup K_a \cup \ldots \cup K_a$ is determined by its spectrum [2], we assume $a \neq b$. The aim of this section is to show that a graph with Laplacian spectrum $\{a^{(a-1)}, b^{(b-1)}, 0^{(2)}\}$ with $a > \frac{b}{2}$ is the disjoint union of two complete graphs.

The paper [3] and the thesis [1] study graphs with few eigenvalues. We can in particular mention the following results:

**Theorem 9 [3]** Theorem 2.1 and Corollary 2.4] A $k$-regular connected graph with exactly two positive Laplacian eigenvalues $a$ and $b$ is strongly regular with parameters $n, k, \alpha, \gamma$, with $\gamma = \frac{a}{b}$. Moreover $k$ verifies $k^2 - k(a + b - 1) - \gamma + \gamma n = 0$.

### 3 Disjoint union of two complete graphs

In this section we consider the disjoint union of two complete graphs and we want to replace the condition “without isolated vertex” of Theorem 9 (this condition cannot be deduced from the spectrum) by a condition on the eigenvalues.

The spectrum of $K_a \cup K_b$, $a \geq b > 1$ is $\{a^{(a-1)}, b^{(b-1)}, 0^{(2)}\}$. The disjoint union of two complete graphs is not in general determined by its spectrum, here a counter-example. The Laplacian spectrum of the line graph of $K_a$ (which is a strongly regular graph with parameters $15$, $8$, $4$, $4$) is $\{10^{(9)}, 6^{(5)}, 0\}$, so the Laplacian spectrum of $L(K_6) \cup K_1$ is $\{10^{(9)}, 6^{(5)}, 0^{(2)}\}$ which is also the spectrum of $K_{10} \cup K_6$.

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5
Remark 1 In [3], the relation $\gamma = \frac{\alpha b}{n}$ and the equation $k^2 - k(a + b - 1) - \gamma + \gamma n = 0$ are given in the proof of Theorem 2.1.

Theorem 10 [3] Let $G$ be a non-regular graph with Laplacian spectrum $\{a^{(a-1)}, b^{(b-1)}, 0\}$ with $a > b > 1$, $a, b \in \mathbb{N}^*$. Then $G$ possesses exactly two different degrees $k_1$ and $k_2$ ($k_1 \geq k_2$) verifying: \[ k_1 + k_2 = a + b - 1 \quad \text{and} \quad k_2 \geq b \quad \text{if and only if} \quad G \text{ or } \overline{G} \text{ is not connected.} \]

Lemma 1 A regular graph with Laplacian spectrum $\{a^{(a-1)}, b^{(b-1)}, 0\}$ is a strongly regular graph with parameters $n, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}$.

Moreover we have $(a - b)^2 = a + b$.

**Proof:** Let $G$ be a regular graph with Laplacian spectrum $\{a^{(a-1)}, b^{(b-1)}, 0\}$, then according to Theorem 9, $G$ is strongly regular with parameters $n, k, \alpha, \gamma$. The spectrum of the adjacency matrix of $G$ is $\{(k-a)^{(a-1)}, (k-b)^{(b-1)}, 0\}$.

By Theorem 6 we have $k - b = \frac{\alpha - \gamma + \sqrt{\Delta}}{2}$ and $k - a = \frac{\alpha - \gamma - \sqrt{\Delta}}{2}$ where $\Delta = (\alpha - \gamma)^2 + 4(k - \gamma) = (a - b)^2$ and we have $\alpha - \gamma = 2k - a - b$ so (remind that $a + b - 1 = n$):

$$\alpha - \gamma = 2k - n - 1$$

(1)

Moreover Theorem 6 gives $b - 1 = \frac{1}{2} \left( n - 1 - \frac{2k + (a-1)(a-\gamma)}{\sqrt{\Delta}} \right)$ and $a - 1 = \frac{1}{2} \left( n - 1 + \frac{2k + (a-1)(a-\gamma)}{\sqrt{\Delta}} \right)$ so $a - b = \frac{2k + (a-1)(a-\gamma)}{\sqrt{\Delta}}$ and so

$$(a - b)^2 = 2k + (n - 1)(\alpha - \gamma)$$

(2)

But $ab = \gamma n$ (Theorem 9) so

$$(a - b)^2 = 2k + n\alpha - ab - (\alpha - \gamma)$$

(3)

Equations (1) and (2) give:

$$(a - b)^2 = 1 + n(\alpha + 1) - ab$$

(4)

As the mean of the degrees is $k$, we have $k = \frac{2|E|}{n}$ and $2|E|$ is the sum of the Laplacian eigenvalues, so $k = \frac{a^{(a-1)} + b^{(b-1)}}{n}$ i.e.

$$nk = a^2 + b^2 - n - 1$$

(5)

Equation (3) gives $a^2 + b^2 - ab - n - 1 = n\alpha$, using Equation (5) we have $nk - ab = n\alpha$ i.e. $n(k - \alpha) = ab$. But $ab = \gamma n$ so

$$\alpha + \gamma = k$$

(6)

Using $\Delta = (\alpha - \gamma)^2 + 4(k - \gamma)$ and $\Delta = (a - b)^2 = 2k + (n - 1)(\alpha - \gamma)$ (Equation 2) we obtain $(\alpha - \gamma)^2 + 4(k - \gamma) = 2k + (n - 1)(\alpha - \gamma)$ but $n - 1 = -\alpha + \gamma + 2k - 2$ (Equation 1) and $2k = 2\alpha + 2\gamma$ (Equation 3), so $(\alpha - \gamma)^2 + 4\alpha = 2\alpha + 2\gamma + (\alpha + 3\gamma - 2)(\alpha - \gamma)$ that is

$$(\alpha - \gamma)(4 - 4\gamma) = 0$$

(7)

As a result we have $\alpha = \gamma$ or $\gamma = 1$. Let us show that $\gamma = 1$ is impossible: $\gamma = 1$ implies $\alpha = k - 1$ and Equation (1) becomes $n = k + 1$ and so $G$ is the
complete graph with \( n \) vertices, which is impossible because \( \{a^{(a-1)}, b^{(b-1)}, 0\} \) with \( a > b > 1 \) is not the spectrum of a complete graph.

Finally \( \alpha = \gamma = \frac{1}{2} \) (Equation 15) and (Equation 14) \( k = \frac{n + 1}{2} \) and (Equation 2) \( (a - b)^2 = 2k = n + 1 = a + b. \)

\[ \square \]

**Theorem 11** Let \( G \) be a graph whose Laplacian spectrum is \( \{a^{(a-1)}, b^{(b-1)}, 0\} \) with \( a > b > 1, a, b \in \mathbb{N} \setminus \{0, 1\} \) and \( a > \frac{2}{3}b \). Then \( G \) is not regular.

**Proof**: Proof by contradiction. Let \( G \) be a graph whose Laplacian spectrum is \( \{a^{(a-1)}, b^{(b-1)}, 0\} \) with \( a > b > 1, a, b \in \mathbb{N} \setminus \{0, 1\} \) and \( a > \frac{2}{3}b \) and we assume that \( G \) is a \( k \)-regular graph. Then by the previous lemma we have that \( G \) is strongly regular and \( (a - b)^2 = a + b. \)

This implies \( a^2 - 2ab - a - b^2 < 0 \) so \( a - 2b - 1 < 0 \) and \( a \leq 2b. \) Then \( 3b \geq a + b = (a - b)^2 > \frac{2}{3}b^2 \) so \( b(\frac{4}{3}b - 3) < 0 \) which gives \( b \leq 6. \) We have \( \frac{2}{3}b < a \leq 2b \) and \( b \leq 6 \), by listing the different cases we show that \( (a - b)^2 \neq a + b \) for \( b \neq 3: \)

<table>
<thead>
<tr>
<th>( b )</th>
<th>( a )</th>
<th>( a + b )</th>
<th>( (a - b)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>11</td>
<td>9</td>
</tr>
<tr>
<td>8</td>
<td>12</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>9</td>
<td>14</td>
<td>16</td>
</tr>
<tr>
<td>10</td>
<td>15</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>11</td>
<td>17</td>
<td>25</td>
</tr>
<tr>
<td>12</td>
<td>18</td>
<td>36</td>
<td></td>
</tr>
</tbody>
</table>

If \( b = 3 \) then \( a = 6 \) and \( n + 1 = a + b = 9 \) but, according to Lemma 1, \( n + 1 \) is even, a contradiction.

\[ \square \]

**Remark 2** When \( b < a \leq \frac{2}{3}b, \) the equation \( (a - b)^2 = a + b \) admits an infinity of solutions; it is not difficult to show that the couples \( (a, b) = (u_i, u_i - 1) \) where \( u_0 = 3 \) and \( u_i = u_{i-1} + i + 2 \) are solutions.

**Lemma 2** There is no graph with Laplacian spectrum \( \{a^{(a-1)}, b^{(b-1)}, 0\} \) with \( a, b \in \mathbb{N} \setminus \{0, 1\} \) and \( a > 2b. \)

**Proof**: Let \( G \) be a graph with Laplacian spectrum \( \{a^{(a-1)}, b^{(b-1)}, 0\} \) with \( a, b \in \mathbb{N} \setminus \{0, 1\} \) and \( a > 2b, \) then \( G \) is not regular (Theorem 11) and by Theorem 10 we have that \( G \) possesses exactly two different degrees \( k_1 \) and \( k_2 \) verifying:

\[
\left\{ \begin{array}{l}
k_1 + k_2 = a + b - 1 \\
k_1k_2 = ab - \frac{ab}{n}
\end{array} \right.
\]

with \( k_2 \geq b + 1 \) because \( G \) and \( \overline{G} \) are connected (\( \overline{G} \) is disconnected if and only if the greatest eigenvalue of \( G \) is \( \vert G \vert \), but here \( \vert G \vert = a + b - 1 \neq a \).

We have \( \frac{ab}{n} \in \mathbb{N}^* \), but \( ab \neq n \) because \( a + b = n + 1 \) and \( a, b \geq 2 \Rightarrow ab \geq a + b. \) So \( \frac{ab}{n} \geq 2. \)
The integers \( k_1 \) and \( k_2 \) are solutions of the equation 
\[
x^2 - (a + b - 1)x + ab - \frac{ab}{n} = 0
\]
whose discriminant is \( \Delta = (a + b - 1)^2 - 4ab + 4\frac{ab}{n} \) and so 
\[
k_1 = \frac{a+b-1+\sqrt{\Delta}}{2}, \quad k_2 = \frac{a+b-1-\sqrt{\Delta}}{2}.
\]
On one hand we have:
\[
\Delta = (a + b)^2 - 2(a + b) + 1 - 4ab + 4\frac{ab}{n} = (a - b)^2 - 2(a + b) + 1 + 4\frac{ab}{n} \geq (a - b)^2 - 2(a + b) + 9
\]
and on the other hand we have (remind that \( k_2 \geq b + 1 \)):
\[
\Delta = (a + b - 1 - 2k_2)^2 \leq (a + b - 1 - 2(b + 1))^2 \leq (a - b - 3)^2 = (a - b)^2 - 6a + 6b + 9 = (a - b)^2 - 2a - 4a + 6b + 9 \text{ but } a > 2b \text{ i.e. } -4a < -8b < (a - b)^2 - 2a - 2b + 9 = (a - b)^2 - 2(a + b) + 9
\]
Contradiction.

\[\square\]

**Theorem 12** There is no graph with Laplacian spectrum \( \{a(a-1), b(b-1), 0\} \) with \( a, b \in \mathbb{N} \setminus \{0, 1\} \) and \( \frac{3}{2}b < a \).

**Proof:** Let \( G \) be a graph with Laplacian spectrum \( \{a(a-1), b(b-1), 0\} \) with \( a, b \in \mathbb{N} \setminus \{0, 1\} \) and \( \frac{3}{2}b < a \). By Lemma 2 we can assume \( a \leq 2b \). The graph \( G \) is not regular (Theorem 11) and by Theorem 10 we have that \( G \) possesses exactly two different degrees \( k_1 \) and \( k_2 \) verifying:
\[
\begin{cases}
k_1 + k_2 = a + b - 1 \\
k_1k_2 = ab - \frac{ab}{n}
\end{cases}
\]
with \( k_2 \geq b + 1 \).

First we show that we have \( b \geq 6 \); for that aim we use the relation \( \frac{3}{2}b < a \leq 2b \) and list the possible values of \( a \) if \( b < 6 \) and we show that \( n \) does not divide \( ab \). This is summed up into the following table.

<table>
<thead>
<tr>
<th>( b )</th>
<th>( a )</th>
<th>( n )</th>
<th>( ab )</th>
<th>Does ( n ) divide ( ab )?</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
<td>8</td>
<td>18</td>
<td>no</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>10</td>
<td>28</td>
<td>no</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
<td>13</td>
<td>45</td>
<td>no</td>
</tr>
<tr>
<td>10</td>
<td>14</td>
<td>32</td>
<td>50</td>
<td>no</td>
</tr>
</tbody>
</table>

Henceforth we assume \( b \geq 6 \). Let us show that the case \( k_2 = b + 1 \) is impossible. If \( k_2 = b + 1 \) then \( k_1 = a - 2 \). We denote by \( n_1 \) (resp. \( n_2 \)) the number of vertices of degree \( k_1 \) (resp. \( k_2 \)). The sum of the degrees is on one hand \( k_1n_1 + k_2n_2 \) and on the other hand \( a(a - 1) + b(b - 1) \) (sum of the eigenvalues) i.e. \( k_1(k_1 + 3) + k_2(k_2 - 3) + 4 \). So \( k_1n_1 + k_2n_2 = k_1(k_1 + 3) + k_2(k_2 - 3) + 4 \) i.e. \( k_1(n_1 - k_1 - 3) + k_2(n_2 - k_2 + 3) = 4 \). But \( (n_1 - k_1 - 3) + (n_2 - k_2 + 3) = 0 \) because \( n = k_1 + k_2 = n_1 + n_2 \) so \( (n_1 - k_1 - 3)(k_1 - k_2) = 4 \) i.e. \( (n_1 - k_1 - 3)(a - b - 3) = 4 \). As a result \( a - b - 3 \) divides 4.
• If $a - b - 3 = 1$ then $a = b + 4$ but $a > \frac{5}{2}b = b + \frac{3}{2}b > b + 4$ (because $b \geq 6$). This case is impossible.

• If $a - b - 3 = 2$ then $a = b + 5$ and $a > \frac{4}{3}b = b + \frac{2}{3}b \geq b + 5$ as soon as $b \geq 8$.
  - If $b = 6$ then $a = 11$ and $n = 16$, $ab = 66$ and 16 does not divide 66. This case is impossible.
  - If $b = 7$ then $a = 12$ and $n = 18$, $ab = 84$ and 18 does not divide 84. This case is impossible.

• If $a - b - 3 = 4$ then $a = b + 7$ and $a > \frac{3}{2}b = b + \frac{1}{2}b \geq b + 7$ as soon as $b \geq 11$. The cases $6 \leq b \leq 10$ are considered in the following table:

<table>
<thead>
<tr>
<th>$b$</th>
<th>$a$</th>
<th>$n$</th>
<th>$ab$</th>
<th>Does $n$ divide $ab$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>13</td>
<td>18</td>
<td>78</td>
<td>no</td>
</tr>
<tr>
<td>7</td>
<td>14</td>
<td>20</td>
<td>98</td>
<td>no</td>
</tr>
<tr>
<td>8</td>
<td>15</td>
<td>22</td>
<td>120</td>
<td>no</td>
</tr>
<tr>
<td>9</td>
<td>16</td>
<td>24</td>
<td>144</td>
<td>yes</td>
</tr>
<tr>
<td>10</td>
<td>17</td>
<td>26</td>
<td>170</td>
<td>no</td>
</tr>
</tbody>
</table>

For the case $b = 9$, $a = 16$ we have $k_1 = 14$ and $k_2 = 10$, $k_1k_2 = 140$ and $ab - \frac{ab}{n} = 138$, this case is impossible.

As a result we have $k_2 \geq b + 2$.

We have that $k_1$ and $k_2$ are solutions of the equation $x^2 - (a + b - 1)x + ab - \frac{ab}{n} = 0$ whose discriminant is $\Delta = (a + b - 1)^2 - 4ab + 4\frac{ab}{n}$.

Let $\overline{d}$ be the mean degree of $G$, we have $\overline{d} = \frac{a(a - 1) + b(b - 1)}{n} = \frac{-2ab + (a+b)(a+b-1)}{n} = -2\frac{ab}{n} + a + b$ so $\frac{ab}{n} = \frac{1}{2}(b + a - \overline{d}) \geq 4$ because $b \geq 6$ and $a > k_1 > \overline{d}$ gives $a - \overline{d} \geq 2$.

We have on one hand:

$$\Delta = (a + b)^2 - 2(a + b) + 1 - 4ab + 4\frac{ab}{n} \geq (a - b)^2 - 2(a + b) + 1 + 4\frac{ab}{n}$$

and on the other hand (remind that $a + b - 1 - 2k_2 = n - 2k_2 > 0$):

$$\Delta = (a + b - 1 - 2k_2)^2 \leq (a + b - 1 - 2(b + 2))^2$$
$$\leq (a - b - 5)^2 = (a - b)^2 - 10a + 10b + 25$$
$$= (a - b)^2 - 2(a + b) - 8a + 12b + 25 \text{ but } -8a < -\frac{40}{3}b$$
$$< (a - b)^2 - 2(a + b) - \frac{4}{3}b + 25 \leq (a - b)^2 - 2(a + b) + 17.$$

Contradiction.

\[\square\]

**Remark 3** If $b < a < \frac{5}{2}b$ then the system \(\begin{cases} k_1 + k_2 = a + b - 1 \\ k_1k_2 = ab - \frac{ab}{n} \end{cases}\) with $k_1, k_2, a, b, n \in \mathbb{N}^*$ admits solutions. If $(a - b)^2 = a + b$ then $\Delta = 1$ and the system admits a solution if $a + b$ is even. The following table shows solutions with $a, b \leq 1000$ and $(a - b)^2 \neq a + b$. 

9
Theorem 13 The graph $K_a \cup K_b$ with $a, b \in \mathbb{N} \setminus \{0, 1\}$ and $\frac{5}{3} b < a$ is determined by its Laplacian spectrum.

Proof: Let $G$ be a graph with Laplacian spectrum $\{a(a-1), b(b-1), 0^{(2)}\}$ with $a, b \in \mathbb{N}^*$ and $\frac{5}{3} b < a$ then $G$ has two connected components. If $G$ has an isolated vertex then the Laplacian spectrum of a connected component of $G$ is $\{a(a-1), b(b-1), 0\}$, which is impossible (Theorem 12). Consequently $G$ does not have isolated vertex and we apply Theorem 8.

□

The following corollary is straightforward thanks to Theorem 13.

Corollary 2 The complete bipartite graph $K_{a,b}$ with $a, b \in \mathbb{N} \setminus \{0, 1\}$ and $\frac{5}{3} b < a$ is determined by its Laplacian spectrum.

References


