Spinorial representation of submanifolds in metric Lie groups

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SPINORIAL REPRESENTATION OF SUBMANIFOLDS IN METRIC LIE GROUPS

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Abstract. In this paper we give a spinorial representation of submanifolds of any dimension and codimension into Lie groups equipped with left invariant metrics.

Keywords: Spin geometry, metric Lie groups, isometric immersions, Weierstrass representation.

2000 Mathematics Subject Classification: 53C27, 53C40.

1. Introduction

sec preliminaries

2. Preliminaries

2.1. **Notation.** Let G be a Lie group, endowed with a left invariant metric $\langle .,.\rangle$, and $\mathcal{G}$ its Lie algebra: $\mathcal{G}$ is the space of the left invariant vector fields on G, equipped with the Lie bracket $[.,.]$ and is identified to the linear space tangent to G at the identity. The left multiplication induces a bundle isomorphism

TG trivial

$$TG \simeq G \times \mathcal{G} \tag{1}$$

which preserves the fibre metrics. We note that a vector field $X \in \Gamma(TG)$ is left invariant if, by (1), $X : G \to \mathcal{G}$ is a constant map. We consider ∇^G the Levi-Civita connection of $(G, \langle .,.\rangle)$ and the linear map

$$\begin{array}{cccc} \Gamma: & \mathcal{G} & \to & \Lambda^2\mathcal{G} \\ & X & \mapsto & \Gamma(X) \end{array}$$

such that, for all $X, Y \in \mathcal{G}$

def Gamma

$$\nabla^G_X Y = \Gamma(X)(Y). \tag{2}$$

By the Koszul formula, Γ is determined by the metric as follows: for all $X, Y, Z \in \mathcal{G}$,

koszul formula

$$\langle \Gamma(X)(Y), Z\rangle = \frac{1}{2}\langle [X,Y], Z\rangle + \frac{1}{2}\langle [Z,X], Y\rangle - \frac{1}{2}\langle [Y,Z], X\rangle. \tag{3}$$

Since ∇^G is without torsion, we have, for all $X, Y \in \mathcal{G}$,

nablaG without torsion

$$\Gamma(X)(Y) - \Gamma(Y)(X) = [X,Y]. \tag{4}$$

We note that the curvature of ∇^G is given by

curvature nablaG

$$R^G(X,Y) = [\Gamma(X), \Gamma(Y)] - \Gamma([X,Y]) \qquad \in \Lambda^2\mathcal{G} \tag{5}$$

for all $X, Y \in \mathcal{G}$. In the formula the first brackets stand for the commutator of the endomorphisms.

2.2. **The spinor bundle of G.** Let us denote by $Cl(\mathcal{G})$ the Clifford algebra of $\mathcal{G}$ with its scalar product, and let us consider the representation

$$\begin{array}{rccc} \rho: & Spin(\mathcal{G}) & \to & GL(Cl(\mathcal{G})) \\ & a & \mapsto & \xi \mapsto a\xi. \end{array}$$

This representation is not irreducible in general: it is a sum of irreducible representations [11]. By (1) the principal bundle Q_G of the positively oriented and orthonormal frames of G is also trivial

$$Q_G \simeq G \times SO(\mathcal{G}),$$

and we may consider the trivial spin structure

$$\tilde{Q}_G := G \times Spin(\mathcal{G})$$

and the corresponding spinor bundle

$$\Sigma := \tilde{Q}_G \times_\rho Cl(G) \simeq G \times Cl(\mathcal{G}).$$

A spinor field $\varphi \in \Gamma(\Sigma)$ is said to be left invariant if it is constant as a map $G \to Cl(\mathcal{G})$. The covariant derivative of a left invariant spinor field is

nabla spinor invariant

$$\nabla^G_X \varphi = \frac{1}{2}\Gamma(X) \cdot \varphi \tag{6}$$

where $\Gamma(X) \in \Lambda^2\mathcal{G} \subset Cl(\mathcal{G})$ and the dot "$\cdot$" stands for the Clifford product.

2.3. **The spin representation of $Spin(p)\times Spin(q)$.** Let us assume that $p+q=n$, and fix an orthonormal basis $e_1^o, e_2^o, \ldots, e_n^o$ of $\mathcal{G}$; this gives a splitting $\mathcal{G} = \mathbb{R}^p \oplus \mathbb{R}^q$ (the first factor corresponds to the first p vectors, and the second factor to the last q vectors of the basis) and a natural map

$$Spin(p) \times Spin(q) \to Spin(\mathcal{G})$$

associated to the isomorphism

$$Cl(\mathcal{G}) = Cl_p \hat{\otimes} Cl_q.$$

We thus also have a representation, still denoted by ρ,

rep spin p spin q

$$\begin{array}{rccc} \rho: & Spin(p) \times Spin(q) & \to & GL(Cl(\mathcal{G})) \\ & a & \mapsto & \xi \mapsto a\xi. \end{array} \tag{7}$$

on twisted spinor bundle

2.4. **The twisted spinor bundle.** We consider M a p-dimensional Riemannian manifold, $E \to M$ a bundle of rank q, with a fibre metric and a compatible connection. We assume that E and TM are oriented and spin, with given spin structures

$$\tilde{Q}_M \overset{2:1}{\to} Q_M \quad \text{and} \quad \tilde{Q}_E \overset{2:1}{\to} Q_E$$

where Q_M and Q_E are the bundles of positively oriented orthonormal frames of TM and E, and we set

$$\tilde{Q} := \tilde{Q}_M \times_M \tilde{Q}_E;$$

this is a $Spin(p) \times Spin(q)$ principal bundle. We define

$$\Sigma := \tilde{Q} \times_\rho Cl(\mathcal{G})$$

and

$$U\Sigma := \tilde{Q} \times_\rho Spin(\mathcal{G}) \quad \subset \ \Sigma$$

where ρ is the representation (7). The vector bundle Σ is equipped with the covariant derivative ∇ naturally associated to the spinorial connections on $\tilde{Q}_M$ and $\tilde{Q}_E$. Let us denote by $\tau : Cl(\mathcal{G}) \to Cl(\mathcal{G})$ the anti-automorphism of $Cl(\mathcal{G})$ such that

$$\tau(x_1 \cdot x_2 \cdots x_k) = x_k \cdots x_2 \cdot x_1$$

for all $x_1, x_2, \ldots, x_k \in \mathcal{G}$, and set

def brackets 1

$$\langle\langle ., . \rangle\rangle : \quad \begin{array}{rcl} Cl(\mathcal{G}) \times Cl(\mathcal{G}) & \to & Cl(\mathcal{G}) \\ (\xi, \xi') & \mapsto & \tau(\xi')\xi. \end{array} \tag{8}$$

This map is $Spin(\mathcal{G})-$invariant: for all $\xi, \xi' \in Cl(\mathcal{G})$ and $g \in Spin(\mathcal{G})$ we have

$$\langle\langle g\xi, g\xi' \rangle\rangle = \tau(g\xi')g\xi = \tau(\xi')\tau(g)g\xi = \tau(\xi')\xi = \langle\langle \xi, \xi' \rangle\rangle,$$

since $Spin(\mathcal{G}) \subset \{g \in Cl^0(\mathcal{G}) : \ \tau(g)g = 1\}$; this map thus induces a $Cl(\mathcal{G})-$valued map

def brackets 2

$$\langle\langle ., . \rangle\rangle : \quad \begin{array}{rcl} \Sigma \times \Sigma & \to & Cl(\mathcal{G}) \\ (\varphi, \varphi') & \mapsto & \langle\langle [\varphi], [\varphi'] \rangle\rangle \end{array} \tag{9}$$

where $[\varphi]$ and $[\varphi'] \in Cl(\mathcal{G})$ represent φ and φ' in some spinorial frame $\tilde{s} \in \tilde{Q}$.

Lemma 2.1. *The map $\langle\langle ., . \rangle\rangle : \Sigma \times \Sigma \to Cl(\mathcal{G})$ satisfies the following properties: for all $\varphi, \psi \in \Gamma(\Sigma)$ and $X \in \Gamma(TM)$,*

scalar product property1

$$\langle\langle \varphi, \psi \rangle\rangle = \tau \langle\langle \psi, \varphi \rangle\rangle \tag{10}$$

and

scalar product property2

$$\langle\langle X \cdot \varphi, \psi \rangle\rangle = \langle\langle \varphi, X \cdot \psi \rangle\rangle. \tag{11}$$

Proof. We have

$$\langle\langle \varphi, \psi \rangle\rangle = \tau[\psi]\ [\varphi] = \tau(\tau[\varphi]\ [\psi]) = \tau\langle\langle \psi, \varphi \rangle\rangle$$

and

$$\langle\langle X \cdot \varphi, \psi \rangle\rangle = \tau[\psi]\ [X][\varphi] = \tau([X][\psi])[\varphi] = \langle\langle \varphi, X \cdot \psi \rangle\rangle$$

where $[\varphi]$, $[\psi]$ and $[X]$ $\in Cl(\mathcal{G})$ represent φ, ψ and X in some given frame $\tilde{s} \in \tilde{Q}$. □

Lemma 2.2. *The connection ∇ is compatible with the product $\langle\langle ., . \rangle\rangle$:*

$$\partial_X \langle\langle \varphi, \varphi' \rangle\rangle = \langle\langle \nabla_X \varphi, \varphi' \rangle\rangle + \langle\langle \varphi, \nabla_X \varphi' \rangle\rangle$$

for all $\varphi, \varphi' \in \Gamma(\Sigma)$ and $X \in \Gamma(TM)$.

Proof. If $\varphi = [\tilde{s}, [\varphi]]$ is a section of $\Sigma = \tilde{Q} \times_\rho Cl(\mathcal{G})$, we have

$$\nabla_X \varphi = [\tilde{s}, \partial_X[\varphi] + \rho_*(\tilde{s}^*\alpha(X))([\varphi])], \qquad \forall X \in\ TM,$$

where ρ is the representation (7) and α is the connection form on $\tilde{Q}$; the term $\rho_*(\tilde{s}^*\alpha(X))$ is an endomorphism of $Cl(\mathcal{G})$ given by the multiplication on the left by an element belonging to $\Lambda^2\mathcal{G} \subset Cl(\mathcal{G})$, still denoted by $\rho_*(\tilde{s}^*\alpha(X))$. Such an element satisfies

$$\tau\left(\rho_*(\tilde{s}^*\alpha(X))\right) = -\rho_*(\tilde{s}^*\alpha(X)),$$

and we have

$$\begin{array}{rcl} \langle\langle \nabla_X \varphi, \varphi' \rangle\rangle + \langle\langle \varphi, \nabla_X \varphi' \rangle\rangle & = & \tau\{[\varphi']\}\left(\partial_X[\varphi] + \rho_*(\tilde{s}^*\alpha(X))[\varphi]\right) \\ & & +\tau\left\{\partial_X[\varphi'] + \rho_*(\tilde{s}^*\alpha(X))[\varphi']\right\}[\varphi] \\ & = & \tau\{[\varphi']\}\partial_X[\varphi] + \tau\left\{\partial_X[\varphi']\right\}[\varphi] \\ & = & \partial_X \langle\langle \varphi, \varphi' \rangle\rangle. \end{array}$$

□

We finally note that there is a natural action of $Spin(\mathcal{G})$ on $U\Sigma$, by right multiplication: for $\varphi = [\tilde{s}, [\varphi]] \in U\Sigma = \tilde{Q} \times_\rho Spin(\mathcal{G})$ and $a \in Spin(\mathcal{G})$ we set

def right action

$$\varphi \cdot a := [\tilde{s}, [\varphi] \cdot a] \ \in U\Sigma. \tag{12}$$

2.5. **The spin geometry of a submanifold of G.** We keep the notation of the previous section, assuming moreover here that M is a submanifold of a Lie group G and that $E \to M$ is its normal bundle. If we consider spin structures on TM and on E whose sum is the trivial spin structure of $TM \oplus E$ [13], we have

$$\Sigma = \tilde{Q} \times_\rho Cl(\mathcal{G}) \simeq M \times Cl(\mathcal{G}),$$

where the last bundle is the spinor bundle of G restricted to M. Two connections are thus defined on Σ, the connection ∇ and the connection ∇^G; they satisfy the following Gauss formula:

gauss formula

$$\nabla^G_X \varphi = \nabla_X \varphi + \frac{1}{2} \sum_{j=1}^{p} e_j \cdot B(X, e_j) \cdot \varphi \tag{13}$$

for all $\varphi \in \Gamma(\Sigma)$ and all $X \in \Gamma(TM)$, where $B : TM \times TM \to E$ is the second fundamental form of M into G. We refer to [1] for the proof (in a slightly different context). Since the covariant derivative of a left invariant spinor field is given by (6), the restriction to M of such a spinor field satisfies

gauss formula Gamma

$$\nabla_X \varphi = -\frac{1}{2} \sum_{j=1}^{p} e_j \cdot B(X, e_j) \cdot \varphi + \frac{1}{2} \Gamma(X) \cdot \varphi \tag{14}$$

for all $X \in TM$.

section main result

3. Main result

We consider M a p-dimensional Riemannian manifold, $E \to M$ a bundle of rank q, with a fibre metric and a compatible connection. We assume that E and TM are oriented and spin, with given spin structures. We suppose that a bilinear and symmetric map $B : TM \times TM \to E$ is given, and we moreover do the following two assumptions:

(1) There exists a bundle isomorphism

def bundle iso f

$$f : \ TM \oplus E \to M \times \mathcal{G} \tag{15}$$

which preserves the metrics; this mapping permits to define a bundle map

$$\Gamma : \ TM \oplus E \to \Lambda^2(TM \oplus E) \tag{16}$$

such that, for all $X, Y \in \Gamma(TM \oplus E)$,

def Gamma TM+E

$$f(\Gamma(X)(Y)) = \Gamma(f(X))(f(Y)) \tag{17}$$

(on the right hand side Γ is the map defined on $\mathcal{G}$ by (2)), together with the following notion: a section $Z \in \Gamma(TM \oplus E)$ will be said to be left invariant if $f(Z) : M \to \mathcal{G}$ is a constant map.

(2) The covariant derivative of a left invariant section $Z \in \Gamma(TM \oplus E)$ is given by

nabla field invariant

$$\nabla_X Z = \Gamma(X)(Z) - B(X, Z^T) + B^*(X, Z^N) \tag{18}$$

where $Z = Z^T + Z^N$ in $TM \oplus E$ and $B^* : TM \times E \to TM$ is the bilinear map such that for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(E)$

$$\langle B(X,Y), N\rangle = \langle Y, B^*(X,N)\rangle.$$

Remark 1. *These two assumptions are equivalent to the assumptions made in [10, 15]: they are necessary to write down the equations of Gauss, Codazzi and Ricci in a general metric Lie group, and to obtain a fundamental theorem for immersions in that context; see Section 4.*

rmk cond frame

Remark 2. *Sometimes it is convenient to write these assumptions in some local frames. For sake of simplicity, we assume that E is a trivial line bundle, oriented by a unit section ν. Let $(e_1^o, e_2^o, \ldots, e_n^o)$ be an orthonormal basis of $\mathcal{G}$ and $\Gamma_{ij}^k \in \mathbb{R}$, $1 \le i, j, k \le n$, be such that*

$$\Gamma(e_i^o)(e_j^o) = \sum_{k=1}^n \Gamma_{ij}^k\ e_k^o.$$

We set, for $i = 1, \ldots, n$, $\underline{e}_i \in \Gamma(TM \oplus E)$ such that $f(\underline{e}_i) = e_i^o$, and $f_i \in C^\infty(M)$, $T_i \in \Gamma(TM)$ such that $\underline{e}_i = T_i + f_i\nu$. Since f preserves the metrics, the vectors $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n$ are orthonormal, and we have

cond e_i orth

$$\langle T_i, T_j\rangle + f_i f_j = \delta_{ij} \tag{19}$$

for all $i, j = 1, \ldots, n$. The assumption (18) then reads as follows: for all $X \in TM$, $j = 1, \ldots, n$,

eqn 1 trad

$$\nabla_X T_j = \sum_{i,k} \Gamma_{ij}^k \langle X, T_i\rangle T_k + f_j S(X), \tag{20}$$

eqn 2 trad

$$df_j(X) = \sum_{i,k} \Gamma_{ij}^k f_k \langle X, T_i\rangle - h(X, T_j) \tag{21}$$

where $S(X) = B^*(X, \nu)$ and $h(X,Y) = \langle B(X,Y), \nu\rangle$. Conversely, if vector fields $T_i \in \Gamma(TM)$ and functions $f_i \in C^\infty(M)$, $1 \le i \le n$, are given such that (19), (20) and (21) hold, we may define a bundle isomorphism $f : TM \oplus E \to M \times \mathcal{G}$ preserving the metrics and such that (18) holds: setting $\underline{e}_i = T_i + f_i\nu$, we define f such that $f(\underline{e}_i) = e_i^o$, $i = 1, \ldots, n$.

We keep the notation of Section 2 and state the main result of the paper:

thm main result

Theorem 1. *We moreover assume that M is simply connected. The following statements are equivalent:*

(1) There exists a section $\varphi \in \Gamma(U\Sigma)$ such that

killing equation

$$\nabla_X \varphi = -\frac{1}{2}\sum_{j=1}^p e_j \cdot B(X, e_j) \cdot \varphi + \frac{1}{2}\Gamma(X) \cdot \varphi \tag{22}$$

for all $X \in TM$.

(2) There exists an isometric immersion $F : M \to G$ with normal bundle E and second fundamental form B.

More precisely, if φ is a solution of (22), replacing φ by $\varphi\cdot a$ for some $a\in Spin(\mathcal{G})$ if necessary, the formula $F=\int\xi$ where ξ is the $\mathcal{G}-$valued 1-form defined by

def xi

$$\xi(X):=\langle\langle X\cdot\varphi,\varphi\rangle\rangle \tag{23}$$

for all $X\in TM$, defines an isometric immersion with normal bundle E and second fundamental form B. Here $\int$ stands for the Darboux integral, i.e. $F=\int\xi:M\rightarrow G$ is such that $F^*\omega_G=\xi$, where $\omega_G\in\Omega^1(G,\mathcal{G})$ is the Maurer-Cartan form of G. Reciprocally, an isometric immersion $M\rightarrow G$ with normal bundle E and second fundamental form B may be written in that form.

The formula $F=\int\xi$ where ξ is defined by (23) may be regarded as a generalized Weierstrass representation formula.

This theorem generalizes the main result of [4] to a Lie group equipped with a left invariant metric.

Remark 3. *If φ is a solution of (22) and a belongs to $Spin(\mathcal{G})$, $\varphi':=\varphi\cdot a$ is also a solution of (22) (see (12) for the definition of $\varphi\cdot a$). Moreover the associated 1-forms ξ_φ and $\xi_{\varphi'}$ are linked by*

xi phi g

$$\xi_{\varphi'}\ =\ \tau(a)\ \xi_\varphi\ a=Ad(a^{-1})\circ\xi_\varphi. \tag{24}$$

Let us recall that a 1-form $\xi\in\Omega^1(M,\mathcal{G})$ is Darboux integrable if and only if it satisfies the structure equation $d\xi+[\xi,\xi]=0$ (M is simply connected). The theorem thus says that if φ is a solution of (22), it is possible to find an other solution φ' of this equation such that $\xi_{\varphi'}$ is Darboux integrable and $F=\int\xi_{\varphi'}$ is an immersion with normal bundle E and second fundamental form B. The proof of (1) $\Rightarrow$ (2) in the theorem will in fact follow these lines. See also Remark 5 below.

Remark 4. *We note that (22) implies the Dirac equation*

$$D\varphi=\left(\vec{H}+\gamma\right)\cdot\varphi$$

where the Dirac operator D is defined by

$$D\varphi=\sum_{j=1}^{p}e_j\cdot\nabla_{e_j}\varphi$$

and

$$\vec{H}=\frac{1}{2}\sum_{j=1}^{p}B(e_j,e_j)\ \in E\quad\text{ and }\quad\gamma=\frac{1}{2}\sum_{j=i}^{p}e_j\cdot\Gamma(e_j)\ \in Cl(TM\oplus E).$$

We now prove the theorem: (1) $\Rightarrow$ (2) will be a consequence of Propositions 3.1 and 3.3 below, and (2) $\Rightarrow$ (1) will be proved at the end of the section.

lem xi closed

Proposition 3.1. *Assume that $\varphi\in\Gamma(U\Sigma)$ is a solution of (22) and define ξ by (23). Then*

(1) ξ takes its values in $\mathcal{G}\subset Cl(\mathcal{G})$;
(2) there exists $T\in SO(\mathcal{G})$ such that $\xi=T\circ f$;
(3) replacing φ by $\varphi\cdot a$ where $a\in Spin(\mathcal{G})$ is such that $Ad(a)=T$, we have $\xi=f$, and ξ satisfies the structure equation

xi structure equation

$$d\xi+[\xi,\xi]=0. \tag{25}$$

Proof. 1- By the very definition of ξ, we have

$$\xi(X)=\tau[\varphi][X][\varphi]$$

for all $X\in TM$, where $[X]$ and $[\varphi]$ represent X and φ in a given frame $\tilde{s}$ of $\tilde{Q}$. Since $[X]$ belongs to $\mathcal{G}\subset Cl(\mathcal{G})$ and $[\varphi]$ is an element of $Spin(\mathcal{G})$, $\xi(X)$ belongs to $\mathcal{G}$.

2- This amounts to show that for every left invariant section $Z\in\Gamma(TM\oplus E)$, the map $\xi(Z):M\to\mathcal{G}$ is constant. Indeed, assuming that this property holds, if $(e_1^o,\ldots,e_n^o)$ is a fixed orthonormal basis of $\mathcal{G}$ and denoting by $\underline{e}_1,\ldots,\underline{e}_n$ the left invariant sections of $TM\oplus E$ such that $f(\underline{e}_i)=e_i^o$, $i=1,\ldots,n$, we have, for all section $Z=\sum_i Z_i\underline{e}_i\in\Gamma(TM\oplus E)$,

$$\xi(Z)=\sum_{i=1}^n Z_i\ \xi(\underline{e}_i)$$

where $(\xi(\underline{e}_1),\ldots,\xi(\underline{e}_n))$ is a constant orthonormal basis of $\mathcal{G}$. Considering the orthogonal transformation $T:\mathcal{G}\to\mathcal{G}$ such that $T(e_i^o)=\xi(\underline{e}_i)$, $i=1,\ldots,n$, we get

$$\xi(Z)=\sum_{i=1}^n Z_i\ T(e_i^o)=T\left(\sum_{i=1}^n Z_ie_i^o\right)=T(f(Z)),$$

i.e. $\xi=T\circ f$. We thus assume that $Z\in\Gamma(TM\oplus E)$ is left invariant, and compute, for $X\in TM$,

$$\begin{aligned}\partial_X\ \xi(Z) &= \langle\langle\nabla_XZ\cdot\varphi,\varphi\rangle\rangle+(id+\tau)\langle\langle Z\cdot\varphi,\nabla_X\varphi\rangle\rangle\\ &= \langle\langle\left\{\Gamma(X)(Z)-B(X,Z^T)+B^*(X,Z^N)\right\}\cdot\varphi,\varphi\rangle\rangle\\ &\quad+\frac{1}{2}(id+\tau)\langle\langle Z\cdot\varphi,(-\sum_{j=1}^p e_j\cdot B(X,e_j)+\Gamma(X))\cdot\varphi\rangle\rangle.\end{aligned}$$

This expression is zero, as a consequence of the following formulas:

lemma proof xi=f **Lemma 3.2.** *For all $X\in TM$ and $Z\in TM\oplus E$,*

proof xi=f id 1
$$\langle\langle\Gamma(X)(Z)\cdot\varphi,\varphi\rangle\rangle=-\frac{1}{2}(id+\tau)\langle\langle Z\cdot\varphi,\Gamma(X)\cdot\varphi\rangle\rangle \tag{26}$$

and

proof xi=f id 2
$$\langle\langle\left\{B(X,Z^T)-B^*(X,Z^N)\right\}\cdot\varphi,\varphi\rangle\rangle=-\frac{1}{2}(id+\tau)\langle\langle Z\cdot\varphi,\sum_{j=1}^p e_j\cdot B(X,e_j)\cdot\varphi\rangle\rangle. \tag{27}$$

Proof. We first prove (26): we have

$$\begin{aligned}\frac{1}{2}(id+\tau)\langle\langle Z\cdot\varphi,\Gamma(X)\cdot\varphi\rangle\rangle &= -\langle\langle\varphi,[\Gamma(X),Z]\cdot\varphi\rangle\rangle\\ &= -\langle\langle\Gamma(X)(Z)\cdot\varphi,\varphi\rangle\rangle,\end{aligned}$$

since $\tau_{|\mathcal{G}}=id$, $\tau_{|\Lambda^2\mathcal{G}}=-id$ and by Lemma A.1. The proof of (27) is similar:

$$\begin{aligned}\frac{1}{2}(id+\tau)\langle\langle Z\cdot\varphi,\sum_{j=1}^p e_j\cdot B(X,e_j)\cdot\varphi\rangle\rangle &= -\langle\langle\varphi,\left[\sum_{j=1}^p e_j\cdot B(X,e_j),Z\right]\cdot\varphi\rangle\rangle\\ &= -\langle\langle\left\{B(X,Z^T)-B^*(X,Z^N)\right\}\cdot\varphi,\varphi\rangle\rangle\end{aligned}$$

by Lemma A.3. □

3- For all $a \in Spin(\mathcal{G})$ and $X \in TM$, we have

$$\begin{aligned} \langle\langle X\cdot(\varphi\cdot a),\varphi\cdot a\rangle\rangle &= \tau([\varphi]a)[X][\varphi]a \\ &= \tau(a)\ \langle\langle X\cdot\varphi,\varphi\rangle\rangle\ a \\ &= Ad(a^{-1})(\xi(X)) \\ &= Ad(a^{-1})(T\circ f(X)); \end{aligned}$$

thus, replacing φ by $\varphi\cdot a$ where $a\in Spin(\mathcal{G})$ is such that $Ad(a)=T$ we get $\xi=f$.
We compute, for $X,Y\in\Gamma(TM)$ such that $\nabla X=\nabla Y=0$ at x_0,

$$\begin{aligned} \partial_X\ \xi(Y) &= \langle\langle Y\cdot\nabla_X\varphi,\varphi\rangle\rangle+\langle\langle Y\cdot\varphi,\nabla_X\varphi\rangle\rangle \\ &= (id+\tau)\langle\langle Y\cdot\varphi,\nabla_X\varphi\rangle\rangle \\ &= (id+\tau)\langle\langle\varphi,-\frac{1}{2}\sum_{j=1}^{p}Y\cdot e_j\cdot B(X,e_j)\cdot\varphi+\frac{1}{2}Y\cdot\Gamma(X)\cdot\varphi\rangle\rangle \end{aligned}$$

and

$$\begin{aligned} d\xi(X,Y) &= \partial_X\ \xi(Y)-\partial_Y\ \xi(X) \\ &= (id+\tau)\langle\langle\varphi,\mathcal{C}\cdot\varphi\rangle\rangle+\frac{1}{2}(id+\tau)\langle\langle\varphi,\{Y\cdot\Gamma(X)-X\cdot\Gamma(Y)\}\cdot\varphi\rangle\rangle \end{aligned} \tag{28}$$

dxi interm

with

$$\mathcal{C}:=-\frac{1}{2}\sum_{j=1}^{p}\left\{Y\cdot e_j\cdot B(X,e_j)-X\cdot e_j\cdot B(Y,e_j)\right\}.$$

Now, for $X=\sum_{1\leq k\leq p}x_ke_k$ and $Y=\sum_{1\leq k\leq p}y_ke_k$,

$$\sum_{j=1}^{p}X\cdot e_j\cdot B(Y,e_j)=-B(Y,X)+\sum_{j=1}^{p}\sum_{k\neq j}x_ke_k\cdot e_j\cdot B(Y,e_j)$$

and

$$\sum_{j=1}^{p}Y\cdot e_j\cdot B(X,e_j)=-B(X,Y)+\sum_{j=1}^{p}\sum_{k\neq j}y_ke_k\cdot e_j\cdot B(X,e_j),$$

which yields the formula

$$\mathcal{C}=-\frac{1}{2}\sum_{j=1}^{p}\sum_{k\neq j}e_k\cdot e_j\cdot(y_kB(X,e_j)-x_kB(Y,e_j)).$$

Since a Clifford product of three pairwise orthogonal vectors is changed to its opposite by τ, we deduce that $\tau[\mathcal{C}]=-[\mathcal{C}]$; this implies

$$\tau\langle\langle\varphi,\mathcal{C}\cdot\varphi\rangle\rangle=\tau(\tau[\varphi]\tau[\mathcal{C}][\varphi])=-\tau[\varphi]\tau[\mathcal{C}][\varphi]=-\langle\langle\varphi,\mathcal{C}\cdot\varphi\rangle\rangle.$$

Thus the first term in (28) is zero and

$$\begin{aligned} d\xi(X,Y) &= \frac{1}{2}(id+\tau)\langle\langle\varphi,\{Y\cdot\Gamma(X)-X\cdot\Gamma(Y)\}\cdot\varphi\rangle\rangle \\ &= \frac{1}{2}\langle\langle\varphi,\{Y\cdot\Gamma(X)-\Gamma(X)\cdot Y+X\cdot\Gamma(Y)-\Gamma(Y)\cdot X\}\cdot\varphi\rangle\rangle \end{aligned}$$

since $\tau_{|\mathcal{G}}=id$ and $\tau_{|\Lambda^2\mathcal{G}}=-id$. We finally notice that

$$\frac{1}{2}\left\{Y\cdot\Gamma(X)-\Gamma(X)\cdot Y+X\cdot\Gamma(Y)-\Gamma(Y)\cdot X\right\}=-\Gamma(X)(Y)+\Gamma(Y)(X)$$

(Lemma A.1), which yields

$$\begin{aligned} d\xi(X,Y) &= -\xi(\Gamma(X)(Y)-\Gamma(Y)(X)) \\ &= -[\xi(X),\xi(Y)], \end{aligned}$$

since $\xi = f$, Γ satisfies (17), and by (4). □

We keep the notation of Proposition 3.1, and moreover assume that M is simply connected; we consider

$$F : M \to G$$

such that $F^*\omega_G = \xi$ (assuming that φ is chosen in such a way that ξ satisfies the structure equation (25)). The next proposition follows from the properties of the Clifford product:

lem F isometry

Proposition 3.3. *1. The map $F : M \to G$ is an isometry.*
2. The map

$$\begin{aligned} \Phi_E : \quad E &\to M \times \mathcal{G} \\ X \in E_m &\mapsto (F(m), \xi(X)) \end{aligned}$$

is an isometry between E and the normal bundle of $F(M)$ into G, preserving connections and second fundamental forms. Here, for $X \in E$, $\xi(X)$ still stands for the quantity $\langle\langle X \cdot \varphi, \varphi\rangle\rangle$.

Proof. For $X, Y \in \Gamma(TM \oplus E)$, we have

$$\begin{aligned} \langle \xi(X), \xi(Y)\rangle &= -\frac{1}{2}\left(\xi(X)\xi(Y)+\xi(Y)\xi(X)\right) \\ &= -\frac{1}{2}\left(\tau[\varphi][X][\varphi]\tau[\varphi][Y][\varphi]+\tau[\varphi][Y][\varphi]\tau[\varphi][X][\varphi]\right) \\ &= -\frac{1}{2}\tau[\varphi]\left([X][Y]+[Y][X]\right)[\varphi] \\ &= \langle X, Y\rangle, \end{aligned}$$

since $[X][Y] + [Y][X] = -2\langle [X], [Y]\rangle = -2\langle X, Y\rangle$. This implies that F is an isometry, and that Φ_E is a bundle map between E and the normal bundle of $F(M)$ into G which preserves the metrics of the fibres. Let us denote by B_F and ∇'^F the second fundamental form and the normal connection of the immersion F; the aim is now to prove that

preserves ff connection

$$\xi(B(X,Y)) = B_F(\xi(X),\xi(Y)) \quad \text{and} \quad \xi(\nabla'_X N) = \nabla'^F_{\xi(X)}\xi(N) \tag{29}$$

for $X, Y \in \Gamma(TM)$ and $N \in \Gamma(E)$. First,

$$B_F(\xi(X),\xi(Y)) = (\nabla^G_{\xi(X)}\xi(Y))^N = \{\partial_X\, \xi(Y) + \Gamma(\xi(X))(\xi(Y))\}^N$$

where the superscript N means that we consider the component of the vector which is normal to the immersion. We fix a point $x_0 \in M$, assume that $\nabla Y = 0$ at x_0, and compute:

$$\begin{aligned} \partial_X\, \xi(Y) &= \langle\langle Y \cdot \nabla_X\varphi, \varphi\rangle\rangle + \langle\langle Y\cdot\varphi, \nabla_X\varphi\rangle\rangle \\ &= (id+\tau)\langle\langle \varphi, Y\cdot\nabla_X\varphi\rangle\rangle \\ &= \frac{1}{2}(id+\tau)\langle\langle \varphi, Y\cdot\{-\sum_{j=1}^{p} e_j\cdot B(X,e_j)+\Gamma(X)\}\cdot\varphi\rangle\rangle. \end{aligned}$$

We showed in the proof of Proposition 3.1 that

$$Y\cdot\sum_{j=1}^{p}e_j\cdot B(X,e_j)=-B(X,Y)+\mathcal{D}$$

where $\mathcal{D}$ is a term which satisfies $\tau\mathcal{D}=-\mathcal{D}$. Since moreover $\tau[B(X,Y)]=[B(X,Y)]$ and $\tau[Y\cdot\Gamma(X)]=-[\Gamma(X)\cdot Y]$, we get

$$\begin{array}{rcl}\partial_X\ \xi(Y) & = & \xi(B(X,Y))^N+\langle\langle\varphi,\frac{1}{2}(Y\cdot\Gamma(X)-\Gamma(X)\cdot Y)\cdot\varphi\rangle\rangle^N\\ & = & \xi(B(X,Y))-\langle\langle\varphi,\Gamma(X)(Y)\cdot\varphi\rangle\rangle^N,\end{array}$$

since $\xi(B(X,Y))$ is normal to the immersion and by Lemma A.1. This implies that

$$\begin{array}{rcl}B_F(\xi(X),\xi(Y)) & = & \xi(B(X,Y))-\langle\langle\varphi,\Gamma(X)(Y)\cdot\varphi\rangle\rangle^N+\Gamma(\xi(X))(\xi(Y))^N\\ & = & \xi(B(X,Y))\end{array}$$

since

$$\begin{array}{rcl}\langle\langle\varphi,\Gamma(X)(Y)\cdot\varphi\rangle\rangle & = & \langle\langle\Gamma(X)(Y)\cdot\varphi,\varphi\rangle\rangle\qquad(\text{since }\tau\ [\Gamma(X)(Y)]=[\Gamma(X)(Y)]\in\mathcal{G})\\ & = & \xi(\Gamma(X)(Y))\\ & = & f(\Gamma(X)(Y))\\ & = & \Gamma(f(X))(f(Y))\qquad(\text{by definition of }\Gamma\text{ on }TM\oplus E)\\ & = & \Gamma(\xi(X))(\xi(Y)).\end{array}$$

We finally show the second identity in (29): we have

$$\begin{array}{rcl}\nabla'^F_{\xi(X)}\xi(N) & = & (\nabla^G_{\xi(X)}\xi(N))^N\\ & = & (\partial_X\ \xi(N)+\Gamma(\xi(X))(\xi(N)))^N\\ & = & \langle\langle\nabla'_XN\cdot\varphi,\varphi\rangle\rangle^N+\langle\langle N\cdot\nabla_X\varphi,\varphi\rangle\rangle^N+\langle\langle N\cdot\varphi,\nabla_X\varphi\rangle\rangle^N\\ & & +\Gamma(\xi(X))(\xi(N)))^N.\end{array}$$

The first term in the right hand side is $\xi(\nabla'_XN)$, and we only need to show that

lem F isometry expr

$$\langle\langle N\cdot\nabla_X\varphi,\varphi\rangle\rangle^N+\langle\langle N\cdot\varphi,\nabla_X\varphi\rangle\rangle^N+\Gamma(\xi(X))(\xi(N)))^N=0. \tag{30}$$

We have

$$\begin{array}{rcl}\langle\langle N\cdot\nabla_X\varphi,\varphi\rangle\rangle+\langle\langle N\cdot\varphi,\nabla_X\varphi\rangle\rangle & = & (id+\tau)\langle\langle N\cdot\nabla_X\varphi,\varphi\rangle\rangle\\ & = & \frac{1}{2}(id+\tau)\langle\langle\{\sum_{j=1}^{p}e_j\cdot N\cdot B(X,e_j)+N\cdot\Gamma(X)\}\cdot\varphi,\varphi\rangle\rangle\\ & = & \frac{1}{2}(id+\tau)\langle\langle\sum_{j=1}^{p}e_j\cdot N\cdot B(X,e_j)\cdot\varphi,\varphi\rangle\rangle\\ & & -\langle\langle\Gamma(X)(N)\cdot\varphi,\varphi\rangle\rangle\end{array}$$

since $\tau\ [N\cdot\Gamma(X)]=-[\Gamma(X)\cdot N]$ and by Lemma A.1. Taking into account that

$$\langle\langle\Gamma(X)(N)\cdot\varphi,\varphi\rangle\rangle=\Gamma(\xi(X))(\xi(N))$$

(see the first part of the proof above), the identity (30) will be proved if we show that the vector

$$\frac{1}{2}(id+\tau)\langle\langle\sum_{j=1}^{p}e_j\cdot N\cdot B(X,e_j)\cdot\varphi,\varphi\rangle\rangle$$

is tangent to the immersion. We have

$$\begin{aligned}\sum_{j=1}^{p} e_j\cdot N\cdot B(X,e_j) &= -\sum_{j=1}^{p} e_j\cdot B(X,e_j)\cdot N-2\sum_{j=1}^{p} e_j\langle B(X,e_j),N\rangle\\ &= -\sum_{j=1}^{p} B(X,e_j)\cdot N\cdot e_j-2B^*(X,N)\\ &= -\tau\sum_{j=1}^{p} e_j\cdot N\cdot B(X,e_j)-2B^*(X,N);\end{aligned}$$

thus

$$\frac{1}{2}(id+\tau)\langle\langle\sum_{j=1}^{p} e_j\cdot N\cdot B(X,e_j)\cdot\varphi,\varphi\rangle\rangle=-\langle\langle B^*(X,N)\cdot\varphi,\varphi\rangle\rangle,$$

which is a vector tangent to the immersion since $B^*(X,N)$ belongs to TM; (30) follows, which finishes the proof. □

We finally show the converse statement (2) ⇒ (1) : we suppose that $F:M\to G$ is an isometric immersion with normal bundle E and second fundamental form B, we consider the orthonormal frame $s_o=\mathit{1}_{SO(\mathcal{G})}$ of $\mathcal{G}$, and the spinor frame $\tilde{s}_o=\mathit{1}_{Spin(\mathcal{G})}$ (recall that $Q_G=G\times SO(\mathcal{G})$ and $\tilde{Q}_G=G\times Spin(\mathcal{G})$; see Section 2). The spinor field $\varphi=[\tilde{s}_o,\mathit{1}_{Cl(\mathcal{G})}]$ satisfies (22) as a consequence of the Gauss formulas (13)-(14); moreover, its associated 1-form is, for all $X\in TM$,

$$\xi(X)=\langle\langle F_*X\cdot\varphi,\varphi\rangle\rangle=\tau[\varphi]\ [F_*X]\ [\varphi]=[F_*X],$$

where $[F_*X]\in\mathcal{G}$ represents F_*X in s_o, that is $[F_*X]=\omega_G(F_*X)$ ($\omega_G\in\Omega^1(G,\mathcal{G})$ is the Maurer-Cartan form of G). Thus $\xi=F^*\omega_G$, that is $F=\int\xi$.

rmk congruence

Remark 5. *We proved that if $\varphi\in\Gamma(U\Sigma)$ is a solution of (22) such that ξ_φ satisfies the structure equation (25) then $F=\int\xi_\varphi$ is an immersion with normal bundle E and second fundamental form B. By (24) it is clear that if $a\in Spin(\mathcal{G})$ is such that $Ad(a^{-1}):\mathcal{G}\to\mathcal{G}\ \in SO(\mathcal{G})$ is an automorphism of Lie algebra, then $\xi_{\varphi\cdot a}$ satisfies the structure equation too; in fact, the corresponding immersions $F_\varphi=\int\xi_\varphi$ and $F_{\varphi\cdot a}=\int\xi_{\varphi\cdot a}$ are linked by the following formula: if $\Phi_a:G\to G$ is the automorphism of G such that $d(\Phi_a)_e=Ad(a^{-1})$, then Φ_a is also an isometry for the left invariant metric, and*

expr F phi g

$$F_{\varphi\cdot a}=L_b\circ\Phi_a\circ F_\varphi \tag{31}$$

for some b belonging to G. This relies on the following formula: if $\Phi:G\to G$ is an automorphism, $\omega_G\in\Omega^1(G,\mathcal{G})$ is the Maurer-Cartan form of G and $F:M\to G$ is a smooth map, then

$$(\Phi\circ F)^*\omega_G=d(\Phi)_e\circ(F^*\omega_G).$$

This formula applied to $\Phi=\Phi_a$ and $F=F_\varphi$ shows that $\Phi_a\circ F_\varphi$ is a solution of the Darboux equation associated to the form $\xi_{\varphi\cdot a}$; thus, by uniqueness of a solution of the Darboux equation, (31) holds for some b belonging to G.

Remark 6. *uniqueness of immersions + correspondence between immersions and solutions belonging to a sub-bundle.*

4. An application: the fundamental theorem for immersions in a metric Lie group

tion fundamental theorem

We now show that the equations of Gauss, Ricci and Codazzi on B are exactly the integrability conditions of (22). We recall these equations for immersions in the metric Lie group G: if R^G denotes the curvature tensor of $(G,\langle .,.\rangle)$, and if R^T and R^N stand for the curvature tensors of the connections on TM and on E (M is a submanifold of G and E is its normal bundle), then we have, for all $X,Y,Z\in\Gamma(TM)$ and $N\in\Gamma(E)$,

(1) the Gauss equation

Gauss equation G

$$(R^G(X,Y)Z)^T=R^T(X,Y)Z-B^*(X,B(Y,Z))+B^*(Y,B(X,Z)), \tag{32}$$

(2) the Ricci equation

Ricci equation G

$$(R^G(X,Y)N)^N=R^N(X,Y)N-B(X,B^*(Y,N))+B(Y,B^*(X,N)), \tag{33}$$

(3) the Codazzi equation

Codazzi equation G

$$(R^G(X,Y)Z)^N=\tilde\nabla_XB(Y,Z)-\tilde\nabla_YB(X,Z); \tag{34}$$

in the last equation, $\tilde\nabla$ denotes the natural connection on $T^*M\otimes T^*M\otimes E$.

These equations make sense if M is an abstract manifold and $E\rightarrow M$ is an abstract bundle as in Section 3, if we assume the existence of the bundle map f in (15), since f permits to define Γ on $TM\oplus E$ by (17), and R^G may be written in terms of Γ only (see (4)-(5)). We prove the following:

Proposition 4.1. *We assume that M is simply connected. There exists $\varphi\in\Gamma(U\Sigma)$ solution of (22) if and only if $B:TM\times TM\rightarrow E$ satisfies the Gauss, Ricci and Codazzi equations.*

Proof. We first prove that the Gauss, Ricci and Codazzi equations are necessary if we have a non-trivial solution of (22). We assume that $\varphi\in\Gamma(U\Sigma)$ is a solution of (22) and compute the curvature

$$R(X,Y)\varphi=\nabla_X\nabla_Y\varphi-\nabla_Y\nabla_X\varphi-\nabla_{[X,Y]}\varphi.$$

We fix a point $x_0\in M$, and assume that $\nabla X=\nabla Y=0$ at x_0. We have

$$\begin{aligned}\nabla_X\nabla_Y\varphi &= -\frac12\sum_{j=1}^p e_j\cdot\left(\tilde\nabla_XB(Y,e_j)\cdot\varphi+B(Y,e_j)\cdot\nabla_X\varphi\right)\\ &\quad+\frac12\left(\nabla_X\Gamma(Y)\cdot\varphi+\Gamma(Y)\cdot\nabla_X\varphi\right)\\ &= -\frac12\sum_{j=1}^p e_j\cdot\tilde\nabla_XB(Y,e_j)\cdot\varphi-\frac14\sum_{j,k=1}^p e_j\cdot e_k\cdot B(Y,e_j)\cdot B(X,e_k)\\ &\quad-\frac14\sum_{j=1}^p e_j\cdot B(Y,e_j)\cdot\Gamma(X)\cdot\varphi+\frac12\nabla_X\Gamma(Y)\cdot\varphi-\frac14\Gamma(Y)\cdot\sum_{j=1}^p e_j\cdot B(X,e_j)\cdot\varphi\\ &\quad+\frac14\Gamma(Y)\cdot\Gamma(X)\cdot\varphi.\end{aligned}$$

Thus

R function B

$$
\begin{aligned}
R(X,Y)\varphi \quad = \quad & -\frac{1}{2}\sum_{j=1}^{p} e_j\cdot\left(\tilde{\nabla}_X B(Y,e_j)-\tilde{\nabla}_Y B(X,e_j)\right)\cdot\varphi \\
& +\frac{1}{4}\sum_{j\neq k} e_j\cdot e_k\cdot\left(B(X,e_j)\cdot B(Y,e_k)-B(Y,e_j)\cdot B(X,e_k)\right)\cdot\varphi \\
& -\frac{1}{4}\sum_{j=1}^{p}\left(B(X,e_j)\cdot B(Y,e_j)-B(Y,e_j)\cdot B(X,e_j)\right)\cdot\varphi \\
& \underbrace{+\frac{1}{2}\left[\sum_{j=1}^{p} e_j\cdot B(X,e_j),\Gamma(Y)\right]\cdot\varphi}_{\mathcal{C}_1}+\underbrace{-\frac{1}{2}\left[\sum_{j=1}^{p} e_j\cdot B(Y,e_j),\Gamma(X)\right]\cdot\varphi}_{\mathcal{C}_2} \\
& \underbrace{+\frac{1}{2}\left(\nabla_X\Gamma(Y)-\nabla_Y\Gamma(X)\right)\cdot\varphi}_{\mathcal{C}_3}+\underbrace{-\frac{1}{2}[\Gamma(X),\Gamma(Y)]\cdot\varphi}_{\mathcal{C}_4}.
\end{aligned}
\tag{35}
$$

We computed the second and the third terms in [4]; we only recall the result here:

computation AB

Lemma 4.2. [4] *Let us set*

$$\mathcal{A}:=\frac{1}{4}\sum_{j\neq k} e_j\cdot e_k\cdot\left(B(X,e_j)\cdot B(Y,e_k)-B(Y,e_j)\cdot B(X,e_k)\right)$$

and

$$\mathcal{B}:=-\frac{1}{4}\sum_{j}\left(B(X,e_j)\cdot B(Y,e_j)-B(Y,e_j)\cdot B(X,e_j)\right).$$

We have

$$\mathcal{A}=\frac{1}{2}\sum_{j<k}\left\{\langle B^*(X,B(Y,e_j)),e_k\rangle-\langle B^*(Y,B(X,e_j)),e_k\rangle\right\} e_j\cdot e_k$$

and

$$\mathcal{B}=\frac{1}{2}\sum_{k<l}\langle B(X,B^*(Y,n_k))-B(Y,B^*(X,n_k)),n_l\rangle\, n_k\cdot n_l.$$

We now compute the other terms in (35). We first compute the covariant derivative of Γ, considering Γ as a map

$$\Gamma: \quad TM\oplus E\to End(TM\oplus E).$$

emma formula nabla gamma

Lemma 4.3. *If $X,Y\in TM$ and $Z\in TM\oplus E$,*

$$
\begin{aligned}
(\nabla_X\Gamma)(Y)Z \quad = \quad & [\Gamma(X),\Gamma(Y)]Z-B(X,(\Gamma(Y)Z)^T)+B^*(X,(\Gamma(Y)Z)^N) \\
& +\Gamma(Y)(B(X,Z^T))-\Gamma(Y)(B^*(X,Z^N))-\Gamma(\Gamma(X)Y)(Z) \\
& +\Gamma(B(X,Y))(Z).
\end{aligned}
$$

Here the brackets stand for the commutator of the endomorphisms.

Proof. Since the expression is tensorial, we may assume that $X,Y,Z\in\Gamma(TM\oplus E)$ are left invariant vector fields. By definition,

edo1_nabla

$$\nabla_X\Gamma(Y)Z=\nabla_X(\Gamma(Y)Z)-\Gamma(\nabla_X Y)Z-\Gamma(Y)(\nabla_X Z). \tag{36}$$

Since X, Y and Z are left invariant vector fields, so are $\Gamma(Y)Z$, $\nabla_X Y$ and $\nabla_X Z$, and, by (18),

$$\nabla_X(\Gamma(Y)Z) = \Gamma(X)(\Gamma(Y)Z) - B(X,(\Gamma(Y)Z)^T) + B^*(X,(\Gamma(Y)Z)^N),$$

$$\Gamma(Y)(\nabla_X Z) = \Gamma(Y)(\Gamma(X)Z) - \Gamma(Y)B(X,Z^T) + \Gamma(Y)B^*(X,Z^N)$$

and

$$\Gamma(\nabla_X Y)(Z) = \Gamma(\Gamma(X)Y)Z - \Gamma(B(X,Y^T))Z + \Gamma(B^*(X,Y^N))Z.$$

Plugging these formulas in (36), we get the result. □

We now regard Γ as a map

$$\Gamma : \quad TM \oplus E \to \Lambda^2(TM \oplus E) \subset Cl(TM \oplus E),$$

and compute the term $\mathcal{C}_3$ in (35).

Lemma 4.4. *If $X, Y \in TM$,*

$$\begin{aligned}\frac{1}{2}\left((\nabla_X\Gamma)(Y) - (\nabla_Y\Gamma)(X)\right) \;=\;& [\Gamma(X),\Gamma(Y)] - \frac{1}{2}\Gamma([\Gamma(X),Y] - [\Gamma(Y),X]) \\ & -\frac{1}{2}\left[\sum_{j=1}^{p} e_j\cdot B(X,e_j),\Gamma(Y)\right] + \frac{1}{2}\left[\sum_{j=1}^{p} e_j\cdot B(Y,e_j),\Gamma(X)\right].\end{aligned}$$

Here the brackets stand for the commutator in $Cl(TM \oplus E)$.

Proof. We first note that the linear maps $Z \mapsto [\Gamma(X),\Gamma(Y)]Z$, $Z \mapsto \Gamma(\Gamma(X)Y)Z$ and $Z \mapsto \Gamma(B(X,Y))Z$ appearing in Lemma 4.3 are respectively represented by the bivectors $[\Gamma(X),\Gamma(Y)]$, $\Gamma([\Gamma(X),Y])$ and $\Gamma(B(X,Y))$ (Lemmas A.1 and A.2 in the appendix). Moreover, by Lemma A.4 applied to the linear maps $B(X,.) : TM \to E$ and $\Gamma(Y) : TM \oplus E \to TM \oplus E$, the map

$$Z \mapsto -B^*(X,(\Gamma(Y)Z)^N) + \Gamma(Y)(B^*(X,Z^N)) + B(X,(\Gamma(Y)Z)^T) - \Gamma(Y)(B(X,Z^T))$$

is represented by the bivector

$$\left[\sum_{j=1}^{p} e_j\cdot B(X,e_j),\Gamma(Y)\right] \quad \in Cl(TM \oplus E).$$

The result follows. □

We now deduce the sum of the last four terms in (35):

computation other terms

Lemma 4.5. *Let us set, for $X, Y \in TM$,*

$$R^G(X,Y) = [\Gamma(X),\Gamma(Y)] - \Gamma([\Gamma(X),Y] - [\Gamma(Y),X]) \ \in \Lambda^2(TM \oplus E)$$

(note that R^G is the curvature tensor of G, pulled-back to $TM \oplus E$ by the bundle isomorphism f introduced in (15)). Then

$$\mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 + \mathcal{C}_4 = \frac{1}{2}R^G(X,Y).$$

We thus get the formula

R function B 2

$$R(X,Y)\varphi = -\frac{1}{2}\sum_{j=1}^{p} e_j\cdot\left(\tilde{\nabla}_X B(Y,e_j)-\tilde{\nabla}_Y B(X,e_j)\right)\cdot\varphi + \mathcal{A}\cdot\varphi+\mathcal{B}\cdot\varphi+\frac{1}{2}R^G(X,Y)\cdot\varphi \tag{37}$$

where $\mathcal{A}$ and $\mathcal{B}$ are computed in Lemma 4.2 and R^G may be conveniently written in the form

$$\begin{aligned} R^G(X,Y) &= \sum_{1\leq j<k\leq p}\langle R^G(X,Y)(e_j),e_k\rangle e_j\cdot e_k \\ &\quad+\sum_{j=1}^{p}\sum_{r=1}^{q}\langle R^G(X,Y)(e_j),n_r\rangle e_j\cdot n_r \\ &\quad+\sum_{1\leq l<r\leq q}\langle R^G(X,Y)(n_r),n_l\rangle n_l\cdot n_r. \end{aligned}$$

On the other hand, the curvature of the spinorial connection is given by

R function RT RN

$$R(X,Y)\varphi = \frac{1}{2}\left(\sum_{1\leq j<k\leq p}\langle R^T(X,Y)(e_j),e_k\rangle\ e_j\cdot e_k + \sum_{1\leq k<l\leq q}\langle R^N(X,Y)(n_k),n_l\rangle\ n_k\cdot n_l\right)\cdot\varphi. \tag{38}$$

We now compare the expressions (37) and (38): since in a given frame $\tilde{s}$ belonging to $\tilde{Q}$, φ is represented by an element which is invertible in $Cl(\mathcal{G})$ (it is in fact represented by an element belonging to $Spin(\mathcal{G})$), we may identify the coefficients and get

$$\langle R^T(X,Y)(ej),e_k\rangle=\langle B^*(X,B(Y,e_j)),e_k\rangle-\langle B^*(Y,B(X,e_j)),e_k\rangle+\langle R^G(X,Y)(e_j),e_k\rangle,$$

$$\langle R^N(X,Y)(n_k),n_l\rangle=\langle B(X,B^*(Y,n_k)),n_l\rangle-\langle B(Y,B^*(X,n_k)),n_l\rangle+\langle R^G(X,Y)(n_k),n_l\rangle$$

and

$$\langle\tilde{\nabla}_X B(Y,e_j)-\tilde{\nabla}_Y B(X,e_j),n_r\rangle=\langle R^G(X,Y)(e_j),n_r\rangle$$

for all the indices. These equations are the equations of Gauss, Ricci and Codazzi.

We now prove that the equations of Gauss, Ricci and Codazzi are also sufficient to get a solution of (22). The calculations above show that the connection on Σ defined by

def nabla prime

$$\nabla'_X\varphi:=\nabla_X\varphi+\frac{1}{2}\sum_{j=1}^{p}e_j\cdot B(X,e_j)\cdot\varphi-\frac{1}{2}\Gamma(X)\cdot\varphi \tag{39}$$

for all $\varphi\in\Gamma(\Sigma)$ and $X\in\Gamma(TM)$ is flat if and only if the equations of Gauss, Ricci and Codazzi hold. But if this connection is flat there exists a solution $\varphi\in\Gamma(U\Sigma)$ of (22); this is because ∇' may be also interpreted as a connection on $U\Sigma$ regarded as a principal bundle (of goup $Spin(\mathcal{G})$, acting on the right): indeed, ∇ defines such a connection (since it comes from a connection on $\tilde{Q}$), and the right hand side term in (39) defines a linear map

$$\begin{array}{rcl} TM & \rightarrow & \chi_V^{inv}(U\Sigma) \\ X & \mapsto & \varphi \mapsto \frac{1}{2}\sum_{j=1}^{p} e_j \cdot B(X,e_j)\cdot\varphi - \frac{1}{2}\Gamma(X)\cdot\varphi \end{array}$$

from TM to the vector fields on $U\Sigma$ which are vertical and invariant under the action of the group (these vector fields are of the form $\varphi\mapsto\eta\cdot\varphi$, $\eta\in\Lambda^2(TM\oplus E)\subset Cl(TM\oplus E)$). Assuming that the equations of Gauss, Codazzi and Ricci hold, we thus get a solution $\varphi\in\Gamma(U\Sigma)$ of (22). □

The considerations above give a spinorial proof of the fundamental theorem of submanifold theory in the metric Lie group G :

Corollary 1. *We keep the hypotheses and notation of Section 2, and moreover assume that M is simply connected and that $B: TM\times TM\rightarrow E$ is bilinear, symmetric and satisfies the equations of Gauss, Codazzi and Ricci. Then there is an isometric immersion of M into G with normal bundle E and second fundamental form B. The immersion is unique up to a rigid motion in G, that is up to a transformation of the form*

rigid motion

$$L_b\circ\Phi_a:\quad \begin{array}{rcl} G & \rightarrow & G \\ g & \mapsto & b\Phi_a(g) \end{array} \tag{40}$$

where $a\in Spin(\mathcal{G})$ is such that $Ad(a):\ \mathcal{G}\rightarrow\mathcal{G}$ is an automorphism of Lie algebra, $\Phi_a: G\rightarrow G$ is the group automorphism such that $d(\Phi_a)_e = Ad(a)$, and b belongs to G.

Proof. The equations of Gauss, Codazzi and Ricci are the integrability conditions of (22). We thus get a solution $\varphi\in\Gamma(U\Sigma)$ of (22); with such a spinor field at hand, $F=\int\xi$ where ξ is defined in (23) is the immersion. Finally, a solution of (22) is unique up to the right action of an element of $Spin(\mathcal{G})$; the right multiplication of φ by $a\in Spin(\mathcal{G})$ and the left multiplication by $b\in G$ in the last integration give also an immersion, if $Ad(a):\mathcal{G}\rightarrow\mathcal{G}$ is moreover an automorphism of Lie algebra. This immersion is obtained form the immersion defined by φ by a rigid motion, as described in (40). □

Remark 7. *In $\mathbb{R}^n$, a rigid motion as in (40) is a transformation of the form*

$$\begin{array}{rcl} \mathbb{R}^n & \rightarrow & \mathbb{R}^n \\ x & \mapsto & ax+b, \end{array}$$

with $a\in SO(n)$ and $b\in\mathbb{R}^n$.

section specialcases section Rn

5. Special cases

5.1. **Submanifolds in $\mathbb{R}^n$.** If the metric Lie group is $\mathbb{R}^n$ with its natural metric, we recover the main result of [4]. We suppose that M is a p-dimensional Riemannian manifold, $E\rightarrow M$ a bundle of rank q, with a fibre metric and a compatible connection. We assume that TM and E are oriented and spin with given spin structures, and that $B: TM\times TM\rightarrow E$ is bilinear and symmetric.

theorem section Rn

Theorem 2. [4] *We moreover assume that M is simply connected. The following statements are equivalent:*

(1) *There exists a section* $\varphi\in\Gamma(U\Sigma)$ *such that*

killing equation Rn (41)
$$\nabla_X\varphi=-\frac{1}{2}\sum_{j=1}^{p}e_j\cdot B(X,e_j)\cdot\varphi$$

for all $X\in TM$.

(2) *There exists an isometric immersion* $F:\ M\to\mathbb{R}^n$ *with normal bundle* E *and second fundamental form* B.
Moreover, $F=\int\xi$ *where* ξ *is the* $\mathbb{R}^n-$*valued 1-form defined by*

def xi Rn (42)
$$\xi(X):=\langle\langle X\cdot\varphi,\varphi\rangle\rangle$$

for all $X\in TM$.

Proof. We only prove (1) $\Rightarrow$ (2). This will be a consequence of Theorem 1 if we may define a bundle map f as in (15) such that (18) holds. We assume that φ is a solution of (41), and set

$$\begin{array}{rccc} f: & TM\oplus E & \to & M\times\mathbb{R}^n\\ & Z & \mapsto & \langle\langle Z\cdot\varphi,\varphi\rangle\rangle.\end{array}$$

The map Γ defined by (17) is $\Gamma=0$. We now show that (18) is satisfied for every $Z\in\Gamma(TM\oplus E)$ such that $f(Z):M\to\mathbb{R}^n$ is a constant map: for all $X\in TM$, we have $\partial_X\{f(Z)\}=0$, which reads

$$\langle\langle\nabla_XZ\cdot\varphi,\varphi\rangle\rangle+\langle\langle Z\cdot\nabla_X\varphi,\varphi\rangle\rangle+\langle\langle Z\cdot\varphi,\nabla_X\varphi\rangle\rangle=0.$$

But, by (41) and (11),

$$\begin{array}{rcl}\langle\langle Z\cdot\nabla_X\varphi,\varphi\rangle\rangle+\langle\langle Z\cdot\varphi,\nabla_X\varphi\rangle\rangle &=& \langle\langle\left[\sum_{j=1}^{p}e_j\cdot B(X,e_j),Z\right]\cdot\varphi,\varphi\rangle\rangle\\ &=& \langle\langle\{B(X,Z^T)-B^*(X,Z^N)\}\cdot\varphi,\varphi\rangle\rangle,\end{array}$$

where we use Lemma A.3 in the last step. Thus

$$\langle\langle\nabla_XZ\cdot\varphi,\varphi\rangle\rangle=\langle\langle\{-B(X,Z^T)+B^*(X,Z^N)\}\cdot\varphi,\varphi\rangle\rangle$$

and

$$\nabla_XZ=-B(X,Z^T)+B^*(X,Z^N),$$

which is (18) with $\Gamma=0$. □

5.2. **Submanifolds in** $\mathbb{H}^n$**.** Spinor representations of submanifolds in $\mathbb{H}^n$ with its natural metric were already given in [14, 3, 4]. We give here an other representation using the group structure of $\mathbb{H}^n$, with an arbitrary left invariant metric. Let us set

$$\mathbb{H}^n=\{a=(a',a_n)\ \in\mathbb{R}^n:\ a_n>0\},$$

and, for $a\in\mathbb{H}^n$, the similarity of $\mathbb{R}^{n-1}$ (by a similarity we mean an homothety composed by a translation)

$$\begin{array}{rccc}\varphi_a: & \mathbb{R}^{n-1} & \to & \mathbb{R}^{n-1}\\ & x & \mapsto & a_nx+a'.\end{array}$$

The similarities of $\mathbb{R}^{n-1}$ naturally form a group under composition, and the bijection

$$\begin{array}{rccc}\varphi: & \mathbb{H}^n & \to & \{\text{similarities } \mathbb{R}^{n-1}\to\mathbb{R}^{n-1}\}\\ & a & \mapsto & \varphi_a\end{array}$$

induces a group structure on $\mathbb{H}^n$: it is such that

prod Hn

$$ab = (a_n b' + a', a_n b_n) \tag{43}$$

for all $a, b \in \mathbb{H}^n$; the neutral element is $e = (0, 1) \in \mathbb{H}^n$. Let us denote by $(e_1^o, e_2^o, \ldots, e_n^o)$ the canonical basis of $T_e\mathbb{H}^n = \mathbb{R}^n$ and keep the same letters to denote the corresponding left invariant vector fields on $\mathbb{H}^n$. The Lie bracket may be easily seen to be given by

$$[e_i^o, e_j^o] = 0 \quad \text{and} \quad [e_n^o, e_i^o] = e_i^o$$

for $i, j = 1, \ldots, n-1$. This may also be written in the form

def bracket Rn

$$[X, Y] = l(X)Y - l(Y)X \tag{44}$$

for all $X, Y \in \mathbb{R}^n$, where $l : \mathbb{R}^n \to \mathbb{R}$ is the linear form such that $l(e_i^o) = 0$ if $i \leq n-1$ and $l(e_n^o) = 1$. This property implies that every left invariant metric on $\mathbb{H}^n$ has constant negative curvature $-|l|^2$ [13, 12].

We suppose that a left invariant metric $\langle ., . \rangle$ is given on $\mathbb{H}^n$, and consider the vector $U_o \in T_e\mathbb{H}^n$ such that $l(X) = \langle U_o, X \rangle$ for all $X \in T_e\mathbb{H}^n$. We have $|U_o| = |l|$, and, by the Koszul formula (3),

expr Gamma Hn

$$\Gamma(X)(Y) = -\langle Y, U_o \rangle X + \langle X, Y \rangle U_o \tag{45}$$

for all $X, Y \in T_e\mathbb{H}^n$.

We keep the hypotheses made at the beginning of Section 5.1. We suppose moreover that $U \in \Gamma(TM \oplus E)$ is given such that $|U| = |l|$ and, for all $X \in TM$,

nabla U Hn

$$\nabla_X U = -|U|^2 X + \langle X, U \rangle U - B(X, U^T) + B^*(X, U^N). \tag{46}$$

We set, for $X \in TM$ and $Y \in TM \oplus E$,

def Gamma Hn

$$\Gamma(X)(Y) = -\langle Y, U \rangle X + \langle X, Y \rangle U. \tag{47}$$

rem U solution eqn Z

Remark 8. *Equation (46) implies the following:*

1. *U is a solution of (18), with the definition (47) of Γ.*
2. *The norm of U is constant, since, by a straightforward computation,*

$$d|U|^2(X) = 2\langle \nabla_X U, U \rangle = 0$$

for all $X \in TM$. The additional hypothesis $|U| = |l|$ is thus not very restrictive.

We note that it is not necessary to assume the existence of U solution of (46) to get a spinor representation of a submanifold in $\mathbb{H}^n$ if $\mathbb{H}^n$ is regarded as the set of unit vectors in Minkowski space $\mathbb{R}^{n,1}$ [14, 3, 4]. Nevertheless, this hypothesis seems necessary if we consider $\mathbb{H}^n$ as a group, since the group structure introduces an anisotropy: the vector $e_n \in T_e\mathbb{H}^n$ is indeed a special direction for the group structure.

Let us construct the spinor bundles Σ and $U\Sigma$ on M as in Section 2.4 with here $\mathcal{G} = T_e\mathbb{H}^n$.

Theorem 3. *We assume that M is simply connected. The following statements are equivalent:*

1. *There exists a spinor field $\varphi \in \Gamma(U\Sigma)$ solution of (22) where Γ is defined by (47).*

(2) There exists an isometric immersion $M \to \mathbb{H}^n$ with normal bundle E and second fundamental form B.

Proof. We assume that $\varphi \in \Gamma(U\Sigma)$ is a solution of (22) where Γ is defined by (47), and define $f : TM \oplus E \to M \times T_e\mathbb{H}^n$ by

$$f(Z) = \langle\langle Z \cdot \varphi, \varphi \rangle\rangle$$

for all $Z \in TM \oplus E$. Let us first observe that if Z is a vector field solution of (18), then $f(Z)$ is constant: we have, for all $X \in TM$,

$$\partial_X f(Z) = \langle\langle \nabla_X Z \cdot \varphi, \varphi \rangle\rangle + (id + \tau)\langle\langle Z \cdot \nabla_X \varphi, \varphi \rangle\rangle;$$

this is 0, by (18), (22) and formulas (26)-(27) in Lemma 3.2. Since U is a solution of (18) (see Remark 8), we deduce that $f(U) \in T_e\mathbb{H}^n$ is a constant, and, since $|f(U)| = |U| = |U_o|$, replacing φ by $\varphi \cdot a$ for some $a \in Spin(T_e\mathbb{H}^n)$ if necessary, we may suppose that $f(U) = U_o$. Since Γ is defined on $T_e\mathbb{H}^n$ by (45) and on $TM \oplus E$ by (47), and since f preserves the metrics, it is straightforward to see that $f(\Gamma(X)(Y)) = \Gamma(f(X))(f(Y))$ for all $X, Y \in TM \oplus E$. Finally, (18) holds for all $Z \in \Gamma(TM \oplus E)$ such that $f(Z)$ is constant: this is the same argument as in the proof of Theorem 2 in Section 5.1, just adding the term Γ. The result then follows from Theorem 1. □

5.3. **Hypersurfaces in a metric Lie group.** We assume that G is a simply connected n-dimensional metric Lie group, M is a p-dimensional Riemannian manifold, $n = p+1$, and E is the trivial line bundle on M, oriented by a unit section $\nu \in \Gamma(E)$. We moreover suppose that M is simply connected and that $h : TM \times TM \to \mathbb{R}$ is a given symmetric bilinear form, and that the hypotheses (1) and (2) of Section 3 with $B = h\nu$ hold. According to Theorem 1, an isometric immersion of M into G with second fundamental form h is equivalent to a section φ of $\Gamma(U\Sigma)$ solution of the Killing equation (22). Note that $Q_E \simeq M$ and the double covering

$$\tilde{Q}_E \to Q_E$$

is trivial, since M is assumed to be simply connected. Fixing a section $\tilde{s}_E$ of $\tilde{Q}_E$ we get an injective map

$$\begin{array}{ccc} \tilde{Q}_M & \to & \tilde{Q}_M \times_M \tilde{Q}_E =: \tilde{Q} \\ \tilde{s}_M & \mapsto & (\tilde{s}_M, \tilde{s}_E). \end{array}$$

Using

$$Cl_p \simeq Cl^0_{p+1} \subset Cl_{p+1}$$

(induced by the Clifford map $\mathbb{R}^p \to Cl_{p+1}$, $X \mapsto X \cdot e_{p+1}$), we deduce a bundle isomorphism

identif spineurs

$$(48) \qquad \begin{array}{ccc} \tilde{Q}_M \times_\rho Cl_p & \to & \tilde{Q} \times_\rho Cl^0_{p+1} \quad \subset \Sigma \\ \psi & \mapsto & \psi^*. \end{array}$$

It satisfies the following properties: for all $X \in TM$ and $\psi \in \tilde{Q}_M \times_\rho Cl_p$,

properties spinors M G

$$(49) \qquad (X \cdot_M \psi)^* = X \cdot \nu \cdot \psi^* \quad \text{and} \quad \nabla_X(\psi^*) = (\nabla_X \psi)^*.$$

To write down the Killing equation (22) in the bundle $\tilde{Q}_M \times_\rho Cl_p$, we need to decompose the Clifford action of $\Gamma(X)$ into its tangent and its normal parts:

Lemma 5.1. *Recall the notation introduced in Remark 2. Then, for all $X \in TM$,*

$$\Gamma(X) = \sum_i \langle X, T_i\rangle \sum_{j<k} \Gamma_{ij}^k \left(\frac{1}{2}(T_j \cdot T_k - T_k \cdot T_j) + (f_k T_j - f_j T_k) \cdot \nu\right). \tag{50}$$

Proof. We have

$$X = \sum_{i=1}^n \langle X, \underline{e}_i\rangle \underline{e}_i = \sum_{i=1}^n \langle X, T_i\rangle \underline{e}_i,$$

$$\begin{aligned} \Gamma(X)(\underline{e}_j) &= \sum_{i=1}^n \langle X, T_i\rangle \Gamma(\underline{e}_i)(\underline{e}_j) \\ &= \sum_{i=1}^n \langle X, T_i\rangle \sum_{k=1}^n \Gamma_{ij}^k \underline{e}_k \\ &= \sum_{1\le i,k\le n} \Gamma_{ij}^k \langle X, T_i\rangle (T_k + f_k\nu), \end{aligned}$$

and thus

$$\begin{aligned} \Gamma(X) &= \frac{1}{2}\sum_{j=1}^n \underline{e}_j \cdot \Gamma(X)(\underline{e}_j) \\ &= \frac{1}{2}\sum_{j=1}^n (T_j + f_j\nu) \cdot \sum_{1\le i,k\le n} \Gamma_{ij}^k \langle X, T_i\rangle (T_k + f_k\nu) \\ &= \frac{1}{2}\sum_{1\le i,j,k\le n} \Gamma_{ij}^k \langle X, T_i\rangle (T_j + f_j\nu) \cdot (T_k + f_k\nu). \end{aligned}$$

Now

$$(T_j + f_j\nu) \cdot (T_k + f_k\nu) = T_j \cdot T_k + f_k T_j \cdot \nu - f_j T_k \cdot \nu - f_j f_k,$$

and the result follows since $\Gamma_{ij}^k = -\Gamma_{ik}^j$. □

The section $\varphi \in \Gamma(U\Sigma)$ solution of (22) thus identifies to a section ψ of $\tilde{Q}_M \times_\rho Cl_p$ solution of

$$\begin{aligned} \nabla_X \psi &= -\frac{1}{2}\sum_{j=1}^p h(X, e_j) e_j \cdot_M \psi + \frac{1}{2}\tilde{\Gamma}(X) \cdot_M \psi \\ &= -\frac{1}{2} S(X) \cdot_M \psi + \frac{1}{2}\tilde{\Gamma}(X) \cdot_M \psi \end{aligned}$$

for all $X \in TM$, where

def Gamma tilde

$$\tilde{\Gamma}(X) = \sum_i \langle X, T_i\rangle \sum_{j<k} \Gamma_{ij}^k \left(\frac{1}{2}(T_j \cdot_M T_k - T_k \cdot_M T_j) + (f_k T_j - f_j T_k)\right). \tag{51}$$

and $S : TM \to TM$ is the symmetric operator associated to h. We deduce the following result:

thm hypersurfaces

Theorem 4. *Let $S : TM \to TM$ be a symmetric operator. The following two statements are equivalent:*

(1) there exists an isometric immersion of M into G with shape operator S;

(2) there exists a normalized spinor field $\psi\in\Gamma(\tilde{Q}_M\times_\rho Cl_p)$ solution of

equation psi (52)
$$\nabla_X\psi=-\frac{1}{2}S(X)\cdot_M\psi+\frac{1}{2}\tilde{\Gamma}(X)\cdot_M\psi$$

for all $X\in TM$, where $\tilde{\Gamma}$ is defined in (51).

Here, a spinor field $\psi\in\Gamma(\tilde{Q}_M\times_\rho Cl_p)$ is said to be normalized if it is represented in some frame $\tilde{s}\in\tilde{Q}_M$ by an element $[\psi]\in Cl_p\simeq Cl^0_{p+1}$ belonging to $Spin(p+1)$.

We will see below explicit representation formulas in the cases of the dimensions 3 and 4.

5.4. **Surfaces in a 3-dimensional metric Lie group.** Since $Cl_2\simeq\Sigma_2$ we have
$$\tilde{Q}_M\times_\rho Cl_2\simeq\Sigma M,$$
and φ is equivalent to a spinor field $\psi\in\Gamma(\Sigma M)$ solution of (52) and such that $|\psi|=1$. Moreover, the explicit representation formula $F=\int\xi$ may be written in terms of ψ : it may be proved by a computation that

cit representation dim 3 (53)
$$\langle\langle X\cdot\varphi,\varphi\rangle\rangle=i2\mathcal{R}e\langle X\cdot\psi^+,\psi^-\rangle+j\left(\langle X\cdot\psi^+,\alpha(\psi^+)\rangle-\langle X\cdot\psi^-,\alpha(\psi^-)\rangle\right)$$

where the brackets $\langle .,.\rangle$ stand here for the natural hermitian product on Σ_2 and $\alpha:\Sigma_2\to\Sigma_2$ is the natural quaternionic structure. If $G=\mathbb{R}^3$, this is the explicit representation formula given in [6] (see also [3]).

We also note that the expression (51) of $\tilde{\Gamma}$ simplifies if the Lie group is 3-dimensional:

lem Gamma dim 3 **Lemma 5.2.** *If (j,k,l) is a permutation of $\{1,2,3\}$ and $\epsilon_{jkl}=\pm1$ denotes its sign, then, for all $\psi\in\Gamma(\Sigma M)$,*
$$\left(\frac{1}{2}\left(T_j\cdot_M T_k-T_k\cdot_M T_j\right)+\left(f_kT_j-f_jT_k\right)\right)\cdot\psi=\epsilon_{jkl}(f_l-T_l)\cdot\omega\cdot\psi.$$

Proof. Keeping the notation introduced above, we note that
$$\underline{e}_j\cdot\underline{e}_k\cdot\underline{e}_l=\epsilon_{jkl}\ \omega\cdot\nu,$$
which yields
$$\underline{e}_j\cdot\underline{e}_k=-\epsilon_{jkl}\ \omega\cdot\nu\cdot\underline{e}_l.$$
Thus
$$\begin{array}{rcl}T_j\cdot T_k+(f_kT_j-f_jT_k)\cdot\nu-f_jf_k&=&-\epsilon_{jkl}\ \omega\cdot\nu\cdot(T_l+f_l\nu)\\&=&\epsilon_{jkl}(f_l-T_l\cdot\nu)\cdot\omega\end{array}$$
since $T_l\cdot\nu=-\nu\cdot T_l$, $T_l\cdot\omega=-\omega\cdot T_l$ and $\omega\cdot\nu=\nu\cdot\omega$. Switching the indices j and k we also get
$$T_k\cdot T_j+(f_jT_k-f_kT_j)\cdot\nu-f_kf_j=\epsilon_{kjl}(f_l-T_l\cdot\nu)\cdot\omega=-\epsilon_{jkl}(f_l-T_l\cdot\nu)\cdot\omega$$
and deduce that
$$\frac{1}{2}\left(T_j\cdot T_k-T_k\cdot T_j\right)+\left(f_kT_j-f_jT_k\right)\cdot\nu=\epsilon_{jkl}(f_l-T_l\cdot\nu)\cdot\omega.$$
The result is then a consequence of the first property in (49). □

5.4.1. *The metric Lie group S^3.* A spinor representation of a surface immersed in S^3 was already given in [14] (see also [3, 4]). We give here a spinor representation relying on the group structure; it appears that it coincides with the result in [14].

We regard the sphere S^3 as the set of the unit quaternions, with its natural group structure. The Lie algebra of S^3 identifies to $\mathbb{R}^3$, with the bracket $[X,Y]=2X\times Y$ for all $X,Y\in\mathbb{R}^3$ ($\times$ is the usual cross product). By the Koszul formula (3), for all $X,Y\in\mathbb{R}^3$,

$$\Gamma(X)(Y)=X\times Y.$$

As a bivector, for all $X=X_1e_1^o+X_2e_2^o+X_3e_3^o\in\mathbb{R}^3$,

$$\begin{aligned}\Gamma(X) &= \frac{1}{2}\left(e_1^o\cdot\Gamma(X)(e_1^o)+e_2^o\cdot\Gamma(X)(e_2^o)+e_3^o\cdot\Gamma(X)(e_3^o)\right)\\ &= X_1e_2^o\cdot e_3^o+X_2e_3^o\cdot e_1^o+X_3e_1^o\cdot e_2^o\\ &= -X\cdot(e_1^o\cdot e_2^o\cdot e_3^o).\end{aligned}$$

Thus, if $\varphi\in\tilde{Q}\times_\rho Cl_3^0$ represents an immersion of an oriented surface M in S^3 and if $\psi\in\Gamma(\Sigma M)$ is such that $\varphi=\psi^*$, then, for all $X\in TM$,

$$\begin{aligned}\Gamma(X)\cdot\varphi &= -X\cdot(e_1^o\cdot e_2^o\cdot e_3^o)\cdot\varphi\\ &= -X\cdot\omega\cdot\nu\cdot\varphi\\ &= (X\cdot\nu)\cdot\omega\cdot\varphi\\ &= (X\cdot\omega\cdot\psi)^*\end{aligned}$$

where ω is the area form of M, and ν is the vector normal to M in S^3. Since $\varphi\in\Gamma(U\Sigma)$ is a solution of (22), $\psi\in\Gamma(\Sigma M)$ is a solution of

$$\nabla_X\psi=-\frac{1}{2}S(X)\cdot\psi+\frac{1}{2}X\cdot\omega\cdot\psi$$

and satisfies $|\psi|=1$. Taking the trace, we get

$$\begin{aligned}D\psi &= e_1\cdot\nabla_{e_1}\psi+e_2\cdot\nabla_{e_2}\psi\\ &= H\psi-\omega\cdot\psi\end{aligned}$$

where (e_1,e_2) is a positively oriented and orthonormal basis of TM. Now, setting $\overline{\psi}=\psi^+-\psi^-$ and since $\omega\cdot\psi=-i\overline{\psi}$ (recall that $i\omega$ acts as the identity on Σ^+M and as -identity on Σ^-M), we get

$$D\psi=H\psi-i\overline{\psi},$$

which is also the spinor characterization given by Morel in [14].

Ce n'est sans doute pas une coincidence, mais je ne me l'explique pas complètement.

5.4.2. *Surfaces in the 3-dimensional metric Lie groups $E(\kappa,\tau)$, $\tau\neq 0$.* We recover here a spinor characterization of immersions in the 3-dimensional homogeneous spaces $E(\kappa,\tau)$; this result was obtained by one of the authors in [16], using a characterization of immersions in these spaces by Daniel [5]. We give here an independent proof, and rather obtain the result of Daniel as a corollary.

The metric Lie group $E(\kappa,\tau)$, $\tau\neq 0$, is defined as follows: its Lie algebra is $\mathcal{G}=\mathbb{R}^3$, with the bracket defined on the vectors e_1^o,e_2^o,e_3^o of the canonical basis by

$$[e_1^o,e_2^o]=2\tau e_3^o,\qquad [e_2^o,e_3^o]=\sigma e_1^o,\qquad [e_3^o,e_1^o]=\sigma e_2^o$$

where $\sigma = \frac{\kappa}{2\tau}$. The metric on $\mathcal{G}$ is the canonical metric, ie the metric such that the basis (e_1^o, e_2^o, e_3^o) is orthonormal. The Levi-Civita connection is then given by

Gamma E k t

$$\Gamma(X)(Y) = \{\tau(X - \langle X, e_3^o\rangle e_3^o) + (\sigma - \tau)\langle X, e_3^o\rangle e_3^o\} \times Y \tag{54}$$

for $X, Y \in \mathcal{G}$; see e.g. [5].

Let $S : TM \to TM$ be a symmetric operator. We assume that a vector field $T \in \Gamma(TM)$ and a function $f \in C^\infty(M, \mathbb{R})$ are given such that

eqn T f E k t

$$|T|^2 + f^2 = 1, \tag{55}$$

eqn T E k t

$$\nabla_X T = f(S(X) - \tau JX) \tag{56}$$

and

eqn f E k t

$$df(X) = -\langle S(X) - \tau JX, T\rangle \tag{57}$$

for all $X \in TM$.

Theorem 5. [16] *If M is simply connected, the following two statements are equivalent:*

(1) There exists $\psi \in \Gamma(\Sigma M)$ such that $|\psi| = 1$ and

eqn spinor E k t

$$\nabla_X \psi = -\frac{1}{2}S(X) \cdot \psi + \frac{1}{2}\{(2\tau - \sigma)\langle X, T\rangle\,(T - f) - \tau X\} \cdot \omega \cdot \psi \tag{58}$$

for all $X \in TM$.

(2) There exists an isometric immersion of M into $E(\kappa, \tau)$, with shape operator S.

Proof. We consider the trivial line bundle $E = \mathbb{R}\nu$, where ν is a unit section. The bundle $TM \oplus E$ is of rank 3, and is assumed to be oriented by the orientation of TM and by ν. We suppose that it is endowed with the natural product metric. Let us denote by $\times$ the natural cross product in the fibers. We set

$$e_3 = T + f\nu,$$

and, for all $X, Y \in TM \oplus E$,

def Gamma TM+E E k t

$$\Gamma(X)(Y) = \{\tau(X - \langle X, e_3\rangle e_3) + (\sigma - \tau)\langle X, e_3\rangle e_3\} \times Y. \tag{59}$$

Defining $B : TM \times TM \to E$ and its adjoint $B^* : TM \times E \to TM$ by

def B E k t

$$B(X, Y) = \langle S(X), Y\rangle\nu \qquad \text{and} \qquad B^*(X, \nu) = S(X) \tag{60}$$

for all $X, Y \in TM$, the equations (56) and (57) are equivalent to the single equation

eqn e3 invariant

$$\nabla_X e_3 = \Gamma(X)(e_3) - B(X, e_3^T) + B^*(X, e_3^N) \tag{61}$$

for all $X \in TM$, where ∇ is the sum of the Levi-Civita connection on TM and the trivial connection on E. This is (18) for $Z = e_3$. We will need the following expression for Γ :

lemma Gamma E k t

Lemma 5.3. *For all $X \in TM$, the linear map $\Gamma(X) : TM \oplus E \to TM \oplus E$ is represented by the bivector*

$$\Gamma(X) = \{(2\tau - \sigma)\,\langle X, T\rangle\,(T \cdot \nu - f) - \tau X \cdot \nu\} \cdot \omega.$$

Proof. The linear map $\Gamma(X)$ defined by (59) is represented by the bivector

$$\Gamma(X)=\frac{1}{2}\left(e_1\cdot\Gamma(X)(e_1)+e_2\cdot\Gamma(X)(e_2)+e_3\cdot\Gamma(X)(e_3)\right)$$

where e_1,e_2 are such that e_1,e_2,e_3 is a positively oriented and orthonormal basis of $TM\oplus E$ (see Lemma A.1); thus, a straightforward computation shows that $\Gamma(X)$ is represented by the bivector

lem Gamma E k t

$$\Gamma(X)=-\tau(X\times e_3)\cdot e_3+(\sigma-\tau)\langle X,e_3\rangle\ e_1\cdot e_2. \tag{62}$$

The following formula may be checked by a direct computation: for $X,Y\in TM\oplus E$,

$$X\times Y=-\left(X\cdot Y+\langle X,Y\rangle\right)e_1\cdot e_2\cdot e_3;$$

this gives

$$\begin{array}{rcl}(X\times e_3)\cdot e_3&=&-\left(X\cdot e_3+\langle X,e_3\rangle\right)e_1\cdot e_2\cdot e_3\cdot e_3\\ &=&\left(X-\langle X,e_3\rangle e_3\right)e_1\cdot e_2\cdot e_3\\ &=&\left(X-\langle X,T\rangle\left(T+f\nu\right)\right)\cdot\omega\cdot\nu\\ &=&\left(X\cdot\nu-\langle X,T\rangle\left(T\cdot\nu-f\right)\right)\cdot\omega.\end{array}$$

Moreover,

$$\begin{array}{rcl}\langle X,e_3\rangle\ e_1\cdot e_2&=&\langle X,T\rangle\left(-e_1\cdot e_2\cdot e_3\cdot e_3\right)\\ &=&\langle X,T\rangle\left(-\omega\cdot\nu\cdot(T+f\nu)\right)\\ &=&-\langle X,T\rangle\left(T\cdot\nu-f\right)\cdot\omega.\end{array}$$

Plugging these two formulas in (62) we get the result. □

We deduce the following key lemma:

prop phi equiv psi

Lemma 5.4. *A spinor field $\varphi\in\Gamma(U\Sigma)$ solution of (22) is equivalent to a spinor field $\psi\in\Gamma(\Sigma M)$ solution of (58).*

Proof. We use the identification $\psi\in\Gamma(\Sigma M)\mapsto\psi^*\in\Gamma(\Sigma)$ described at the beginning of the section; we recall that, for all $X\in TM$,

properties ident psi phi

$$\left(\nabla_X\psi\right)^*=\nabla_X(\psi^*)\qquad\text{and}\qquad\left(X\cdot\psi\right)^*=X\cdot\nu\cdot(\psi^*). \tag{63}$$

Thus, if $\varphi\in\Gamma(U\Sigma)$ is a solution of (22) and if $\psi\in\Gamma(\Sigma M)$ is such that $\psi^*=\varphi$, using (63), the formula

$$\sum_{j=1}^{p}e_j\cdot B(X,e_j)=\sum_{j=1}^{p}e_j\cdot\langle S(X),e_j\rangle\nu=S(X)\cdot\nu$$

and Lemma 5.3, we get:

$$\begin{array}{rcl}\left(\nabla_X\psi\right)^*&=&\nabla_X\varphi\\ &=&-\frac{1}{2}S(X)\cdot\nu\cdot\varphi+\frac{1}{2}\left\{(2\tau-\sigma)\left\langle X,T\right\rangle\left(T\cdot\nu-f\right)-\tau X\cdot\nu\right\}\cdot\omega\cdot\varphi\\ &=&\left(-\frac{1}{2}S(X)\cdot\psi+\frac{1}{2}\left\{(2\tau-\sigma)\left\langle X,T\right\rangle\left(T-f\right)-\tau X\right\}\cdot\omega\cdot\psi\right)^*.\end{array}$$

This gives (58). Reciprocally, if ψ is a solution of (58), the spinor field $\varphi=\psi^*$ solves (22). This proves the lemma. □

Instead of $\psi \in \Gamma(\Sigma M)$ solution of (58) we may thus consider $\varphi \in \Gamma(U\Sigma)$ solution of (22). The theorem will thus be a consequence of Theorem 1 if we can define a bundle isomorphism $f : TM \oplus E \to M \times \mathcal{G}$ such that (17) and (18) hold. Let us set

$$f(Z) = \langle\langle Z \cdot \varphi, \varphi \rangle\rangle.$$

We first observe that $f(e_3)$ is constant: indeed, for all $X \in TM$,

$$\partial_X(f(e_3)) = \langle\langle \nabla_X e_3 \cdot \varphi, \varphi \rangle\rangle + (id + \tau)\langle\langle e_3 \cdot \nabla_X \varphi, \varphi \rangle\rangle = 0$$

in view of (61), (22) and identities (26)-(27) in Lemma 3.2. Moreover, since f preserves the norm of the vectors, $f(e_3)$ is a unit vector. Replacing φ by $\varphi \cdot a$ for some $a \in Spin(\mathcal{G})$ if necessary, we may thus assume that $f(e_3) = e_3^o$. We now check (17): since the map f is an orientation preserving isometry and using $f(e_3) = e_3^o$, we have, for all $X, Y \in TM$,

$$\begin{aligned}
f(\Gamma(X)(Y)) &= f\left(\{\tau(X - \langle X, e_3\rangle e_3) + (\sigma - \tau)\langle X, e_3\rangle e_3\} \times Y\right) \\
&= \{\tau(f(X) - \langle f(X), f(e_3)\rangle f(e_3)) + (\sigma - \tau)\langle f(X), f(e_3)\rangle f(e_3)\} \times f(Y) \\
&= \{\tau(f(X) - \langle f(X), e_3^o\rangle e_3^o) + (\sigma - \tau)\langle f(X), e_3^o\rangle e_3^o\} \times f(Y) \\
&= \Gamma(f(X))(f(Y)).
\end{aligned}$$

Finally, the proof of (18) is very similar to the proof of this identity made in Section 5.1 for $G = \mathbb{R}^n$: we only have to add the term involving Γ which appears in the expression (22) of the covariant derivative of φ; we leave the details to the reader. □

Remark 9. *We also get an explicit representation formula: the immersion is given by the Darboux integral of $\xi : X \mapsto \langle\langle X \cdot \varphi, \varphi \rangle\rangle$, which may be written in terms of ψ by the formula (53).*

We deduce the following result, first obtained by Daniel in [5] using the moving frame method:

Corollary 2. *If S, T, f, κ y τ satisfy (55)-(57), the Gauss equation*

Gauss equation E k t

$$K = \det S + \tau^2 + \left(\kappa - 4\tau^2\right) f^2 \tag{64}$$

and the Codazzi equation

Codazzi equation E k t

$$\nabla_X(SY) - \nabla_Y(SX) - S([X, Y]) = (\kappa - 4\tau^2) f(\langle Y, T\rangle X - \langle X, T\rangle Y), \tag{65}$$

then there exists an isometric immersion of M into $E(\kappa, \tau)$ with shape operator S. Moreover the immersion is unique up to a global isometry of $E(\kappa, \tau)$ preserving the orientations.

Proof. The equations (64) and (65) are equivalent to the Gauss and Codazzi equations (32) and (34) where B is defined by (60). □

Peut-on en déduire la transformation de Lawson ?

5.4.3. *The last 3-dimensional riemannian homogeneous space: the metric Lie group Sol_3.* We describe here the special case of a surface in Sol_3 : this achieves the spinor representation of immersions of surfaces into 3-dimensional riemannian homogeneous spaces [16].

Let us recall that Sol_3 is the only metric Lie group whose isometry group is 3-dimensional. It is defined as follows: its Lie algebra is $\mathcal{G} = \mathbb{R}^3$, with the bracket defined on the canonical basis (e_1^o, e_2^o, e_3^o) by

$$[e_1^o, e_2^o] = 0, \qquad [e_2^o, e_3^o] = -e_2^o, \qquad [e_3^o, e_1^o] = -e_1^o.$$

The metric on $\mathcal{G}$ is the canonical metric, i.e., the metric such that the basis (e_1^o, e_2^o, e_3^o) is orthonormal. By the Koszul formula, the Levi-Civita connection is then such that

Gamma ijk sol3

$$\Gamma_{11}^3 = -\Gamma_{13}^1 = -1, \qquad \Gamma_{22}^3 = -\Gamma_{23}^2 = 1 \tag{66}$$

and $\Gamma_{ij}^k = 0$ for the other indices.

Let us consider an oriented riemannian surface M, and a symmetric operator $S : TM \to TM$. We suppose that there exist tangent vectors fields $T_i \in \Gamma(TM)$ and functions $f_i \in C^\infty(M)$ for $1 \leqslant i \leqslant 3$ satisfying

sol1

$$\langle T_i, T_j \rangle + f_i f_j = \delta_i^j \tag{67}$$

for all $1 \leq i, j \leq 3$, and, for all $X \in TM$,

sol2

$$\begin{aligned} \nabla_X T_i &= (-1)^i \langle X, T_i \rangle T_3 + f_i S(X), \\ df_i(X) &= (-1)^i \langle X, T_i \rangle f_3 - \langle SX, T_i \rangle \end{aligned} \tag{68}$$

for $1 \leqslant i \leqslant 2$,

sol3

$$\begin{aligned} \nabla_X T_3 &= \sum_{j=1}^{2} (-1)^{j+1} \langle X, T_j \rangle T_j + f_3 S(X), \\ df_3(X) &= \sum_{j=1}^{2} (-1)^{j+1} \langle X, T_j \rangle f_j - \langle S(X), T_3 \rangle. \end{aligned} \tag{69}$$

The equations (68) and (69) are the equations (20) and (21) in Remark 2, with the coefficients Γ_{ij}^k given by (66). According to (51) together with the simplification in Lemma 5.2, we set

$$\tilde{\Gamma}(X) = -\{\langle X, T_1 \rangle (T_2 - f_2) + \langle X, T_2 \rangle (T_1 - f_1)\} \cdot \omega \tag{70}$$

for all $X \in TM$. Theorem 4 then yields the following result:

Theorem 6. *If M is simply connected, the following two statements are equivalent:*

(1) there exists an isometric immersion of M into Sol_3 with shape operator S;
(2) there exists $\psi \in \Gamma(\Sigma M)$ such that $|\psi| = 1$ and

killing equation sol3

$$\nabla_X \psi = -\frac{1}{2} S(X) \cdot \psi + \frac{1}{2} \tilde{\Gamma}(X) \cdot \psi \tag{71}$$

for all $X \in TM$.

Remark 10. *This theorem implies the result of Lodovici [10] concerning isometric immersions in Sol_3 : by the considerations in Section 4, the equation (71) is solvable if and only if the equations of Gauss and Codazzi hold. Compléter ?*

Appendix A. Skew-symmetric operators and bivectors

We consider $\mathbb{R}^n$ endowed with its canonical scalar product. A skew-symmetric operator $u:\mathbb{R}^n\to\mathbb{R}^n$ naturally identifies to a bivector $\underline{u}\in\Lambda^2\mathbb{R}^n$, which may in turn be regarded as belonging to the Clifford algebra $Cl_n(\mathbb{R})$. We precise here the relations between the Clifford product in $Cl_n(\mathbb{R})$ and the composition of endomorphisms. If a and b belong to the Clifford algebra $Cl_n(\mathbb{R})$, we set

$$[a,b]=\frac{1}{2}\left(a\cdot b-b\cdot a\right),$$

where the dot $\cdot$ is the Clifford product. We denote by $(e_1,\ldots,e_n)$ the canonical basis of $\mathbb{R}^n$.

lem1 ap1

Lemma A.1. *Let $u:\mathbb{R}^n\to\mathbb{R}^n$ be a skew-symmetric operator. Then the bivector*

biv rep u

$$\underline{u}=\frac{1}{2}\sum_{j=1}^n e_j\cdot u(e_j)\quad\in\ \Lambda^2\mathbb{R}^n\subset Cl_n(\mathbb{R}) \tag{72}$$

represents u, and, for all $\xi\in\mathbb{R}^n$,

$$[\underline{u},\xi]=u(\xi).$$

In the paper, and for sake of simplicity, we will use the same letter u to denote $\underline{u}$.

Proof. For $i<j$, we consider the linear map

$$u:\qquad e_i\mapsto e_j,\qquad e_j\mapsto -e_i,\qquad e_k\mapsto 0\quad\text{if}\quad k\neq i,j;$$

it is skew-symmetric and corresponds to the bivector $e_i\wedge e_j\in\Lambda^2\mathbb{R}^n$; it is thus naturally represented by $\underline{u}=e_i\cdot e_j=\frac{1}{2}\left(e_i\cdot e_j-e_j\cdot e_i\right)$, which is (72). We then compute, for $k=1,\ldots,n$,

$$[\underline{u},e_k]=\frac{1}{2}\left(e_i\cdot e_j\cdot e_k-e_k\cdot e_i\cdot e_j\right)$$

and easily get

$$[\underline{u},e_k]=e_j\quad\text{if}\quad k=i,\quad -e_i\quad\text{if}\quad k=j,\quad 0\quad\text{if}\quad k\neq i,j.$$

The result follows by linearity. □

lem2 ap1

Lemma A.2. *Let $u:\mathbb{R}^n\to\mathbb{R}^n$ and $v:\mathbb{R}^n\to\mathbb{R}^n$ be two skew-symmetric operators, represented in $Cl_n(\mathbb{R})$ by*

$$u=\frac{1}{2}\sum_{j=1}^n e_j\cdot u(e_j)\qquad\textit{and}\qquad v=\frac{1}{2}\sum_{j=1}^n e_j\cdot v(e_j)$$

respectively. Then $[u,v]\in\Lambda^2\mathbb{R}^n\subset Cl_n(\mathbb{R})$ represents $u\circ v-v\circ u$.

Proof. For $\xi\in\mathbb{R}^n$, the Jacobi equation yields

$$[[u,v],\xi]=[u,[v,\xi]]-[v,[u,\xi]].$$

Thus, using Lemma A.1 repeatedly, $[u,v]$ represents the map

$$\begin{array}{rcl}\xi\ \mapsto\ [[u,v],\xi] &=& [u,[v,\xi]]-[v,[u,\xi]]\\ &=& [u,v(\xi)]-[v,u(\xi)]\\ &=& (u\circ v-v\circ u)(\xi),\end{array}$$

and the result follows. □

We now assume that $\mathbb{R}^n=\mathbb{R}^p\oplus\mathbb{R}^q$, $p+q=n$.

lem3 ap1

Lemma A.3. *Let us consider a linear map $u : \mathbb{R}^p \to \mathbb{R}^q$ and its adjoint $u^* : \mathbb{R}^q \to \mathbb{R}^p$. Then the bivector*

$$\underline{u} = \sum_{j=1}^{p} e_j \cdot u(e_j) \quad \in \ \Lambda^2\mathbb{R}^n \subset Cl_n(\mathbb{R})$$

represents

$$\begin{pmatrix} 0 & -u^* \\ u & 0 \end{pmatrix} : \qquad \mathbb{R}^p \oplus \mathbb{R}^q \to \mathbb{R}^p \oplus \mathbb{R}^q,$$

we have

biv u u*

$$\underline{u} = \frac{1}{2}\left(\sum_{j=1}^{p} e_j \cdot u(e_j) + \sum_{j=p+1}^{n} e_j \cdot (-u^*(e_j))\right) \tag{73}$$

and, for all $\xi = \xi_p + \xi_q \in \mathbb{R}^n$,

$$[\underline{u}, \xi] = u(\xi_p) - u^*(\xi_q).$$

As above, we will simply denote $\underline{u}$ by u.

Proof. In view of Lemma A.1, $\underline{u}$ represents the linear map $\xi \mapsto [\underline{u}, \xi]$. We compute, for $\xi \in \mathbb{R}^p$,

$$\begin{aligned}
[\underline{u}, \xi] &= \frac{1}{2}\left(\sum_{j=1}^{p} e_j \cdot u(e_j) \cdot \xi - \xi \cdot \sum_{j=1}^{p} e_j \cdot u(e_j)\right) \\
&= -\frac{1}{2}\sum_{j=1}^{p} (e_j \cdot \xi + \xi \cdot e_j) \cdot u(e_j) \\
&= \sum_{j=1}^{p} \langle \xi, e_j \rangle \ u(e_j) \\
&= u(\xi),
\end{aligned}$$

and, for $\xi \in \mathbb{R}^q$,

$$\begin{aligned}
[\underline{u}, \xi] &= \frac{1}{2}\left(\sum_{j=1}^{p} e_j \cdot u(e_j) \cdot \xi - \xi \cdot \sum_{j=1}^{p} e_j \cdot u(e_j)\right) \\
&= \frac{1}{2}\sum_{j=1}^{p} e_j \cdot (u(e_j) \cdot \xi + \xi \cdot u(e_j)) \\
&= -\sum_{j=1}^{p} e_j \ \langle u(e_j), \xi \rangle \\
&= -\sum_{j=1}^{p} e_j \ \langle e_j, u^*(\xi) \rangle \\
&= -u^*(\xi).
\end{aligned}$$

Finally,

$$\begin{aligned}\underline{u} &= \sum_{j=1}^{p} e_j \cdot u(e_j) \\ &= \frac{1}{2}\left(\sum_{j=1}^{p} e_j \cdot u(e_j) + \sum_{j=1}^{p} -u(e_j) \cdot e_j\right)\end{aligned}$$

with

$$\begin{aligned}\sum_{j=1}^{p} -u(e_j) \cdot e_j &= -\sum_{i=p+1}^{p+q}\sum_{j=1}^{p}\langle u(e_j), e_i\rangle\ e_i \cdot e_j \\ &= \sum_{i=p+1}^{p+q} e_i \cdot \left(-\sum_{j=1}^{p}\langle e_j, u^*(e_i)\rangle\ e_j\right) \\ &= \sum_{i=p+1}^{p+q} e_i \cdot (-u^*(e_i)),\end{aligned}$$

which gives (73). □

lem4 ap1

Lemma A.4. *Let us consider two linear maps $u : \mathbb{R}^p \to \mathbb{R}^q$ and $v : \mathbb{R}^n \to \mathbb{R}^n$, with v skew-symmetric, and the associated bivectors*

$$\underline{u} = \sum_{j=1}^{p} e_j \cdot u(e_j), \qquad \underline{v} = \frac{1}{2}\sum_{j=1}^{n} e_j \cdot v(e_j).$$

Then $[\underline{u}, \underline{v}] \in \Lambda^2\mathbb{R}^n$ represents the map

$$\xi = \xi_p + \xi_q \mapsto -u^*(v(\xi)_q) + v(u^*(\xi_q)) + u(v(\xi)_p) - v(u(\xi_p)),$$

where the sub-indices p and q mean that we take the components of the vectors in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. In view of Lemma A.1, this may also be written in the form

$$[[\underline{u}, \underline{v}], \xi] = -u^*(v(\xi)_q) + v(u^*(\xi_q)) + u(v(\xi)_p) - v(u(\xi_p))$$

for all $\xi \in \mathbb{R}^n$.

Proof. From Lemmas A.2 and A.3, the bivector $[\underline{u}, \underline{v}] \in \Lambda^2\mathbb{R}^n$ represents

$$\begin{pmatrix} 0 & -u^* \\ u & 0 \end{pmatrix} \circ v - v \circ \begin{pmatrix} 0 & -u^* \\ u & 0 \end{pmatrix},$$

that is the map

$$\begin{aligned}\xi \ \mapsto\ & \begin{pmatrix} 0 & -u^* \\ u & 0 \end{pmatrix}\begin{pmatrix} v(\xi)_p \\ v(\xi)_q \end{pmatrix} - v\begin{pmatrix} 0 & -u^* \\ u & 0 \end{pmatrix}\begin{pmatrix} \xi_p \\ \xi_q \end{pmatrix} \\ &= \begin{pmatrix} -u^*(v(\xi)_q) + v(u^*(\xi_q)) \\ u(v(\xi)_p) - v(u(\xi_p)) \end{pmatrix},\end{aligned}$$

which gives the result. □

References

[1] C. Bär, *Extrinsic bounds for the eigenvalues of the Dirac operator*, Ann. Glob. Anal. Geom. **16** (1998) 573-596.

[2] P. Bayard, *On the spinorial representation of spacelike surfaces into 4-dimensional Minkowski space*, J. Geom. Phys. **74** (2013) 289-313.

[3] P. Bayard, M.A. Lawn & J. Roth, *Spinorial representation of surfaces in four-dimensional Space Forms*, Ann. Glob. Anal. Geom., **44:4** (2013) 433-453.

[4] P. Bayard, M.A. Lawn & J. Roth, *Spinorial representation of submanifolds in Riemannian space forms*, arXiv...

[5] B. Daniel

[6] T. Friedrich, *On the spinor representation of surfaces in Euclidean* 3*-space*, J. Geom. Phys. **28** (1998) 143-157.

[7] M.A. Lawn, *A spinorial representation for Lorentzian surfaces in* $\mathbb{R}^{2,1}$, J. Geom. Phys. **58:6** (2008) 683-700.

[8] M.A. Lawn and J. Roth, *Spinorial characterization of surfaces in pseudo-Riemannian space forms*, Math. Phys. Anal. and Geom. **14:3** (2011) 185-195.

[9] H.B. Lawson, *The global behaviour of minimal surfaces in* S^n, Ann. Math. **92:2** (1970) 224-237.

[10] Lodovici, *An isometric immersion theorem in* Sol_3, Matemática Contemporânea **30** (2006) 109-123.

[11] P. Lounesto, *Clifford Algebras and Spinors*,

[12] Meeks & Perez

[13] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Adv. Math. **21** (1976) 293-329.

[14] B. Morel, *Surfaces in* $\mathbb{S}^3$ *and* $\mathbb{H}^3$ *via spinors*, Actes du séminaire de théorie spectrale, Institut Fourier, Grenoble, **23** (2005) 9-22.

[15] Piccione, Tausk

[16] J. Roth, *Spinorial characterizations of surfaces into 3-homogeneous manifolds*, J. Geom. Phys. **60** (2010) 1045-1061.