On Carleman estimates with two large parameters

Jérôme Le Rousseau

HAL Id: hal-00738717
https://hal.science/hal-00738717v4

ON CARLEMAN ESTIMATES WITH TWO LARGE PARAMETERS

JÉRÔME LE ROUSSEAU

ABSTRACT. A Carleman estimate for a differential operator P is a weighted energy estimate with a weight of exponential form $\exp(\tau\varphi)$ that involves a large parameter, $\tau > 0$. The function φ and the operator P need to fulfill some sub-ellipticity properties that can be achieved for instance by choosing $\varphi = \exp(\alpha\psi)$, involving a second large parameter, $\alpha > 0$, with ψ satisfying some geometrical conditions. The purpose of this article is to give the framework to keep explicit the dependency upon the two large parameters in the resulting Carleman estimates. The analysis is absed on the introduction of a proper Weyl-Hörmander calculus for pseudo-differential operators. Carleman estimates of various strengths are considered: estimates under the (strong) pseudo-convexity condition and estimates under the simple characteristic property. In each case the associated geometrical conditions for the function ψ is proven necessary and sufficient. In addition some optimality results with respect to the power of the large parameters are proven.

KEYWORDS: Carleman estimate; Weyl-Hörmander calculus with parameters; pseudo-convexity.

AMS 2000 SUBJECT CLASSIFICATION: 35A02; 35B45 ; 35S05.

CONTENTS

1. INTRODUCTION

1.1. **Setting and results.** Carleman estimates are an important tool in subjects in analysis of partial differential equations (PDEs) such as unique continuation, control theory and inverse problems. They are weighted

---

L^2-norm estimates of the solution of a PDE where the weight takes an exponential form

$$\tau^\gamma \|e^{\tau\varphi}u\|_{L^2} \lesssim \|e^{\tau\varphi}Pu\|_{L^2}, \qquad \tau \geq \tau_0, \ u \in \mathscr{C}_c^\infty(X),$$

with X a bounded open set and some $\gamma \in \mathbb{R}$. Here P is a differential operator, φ is the weight function. The exponential weight involves a parameter τ that can be taken as large as needed. Additional terms in the l.h.s., involving derivatives of u, can be obtained depending on the order of P and on the *joint* properties of P and φ on X. For instance for a second-order operator P such an estimate can take the form

$$(1.1) \qquad \tau^3 \|e^{\tau\varphi}u\|_{L^2}^2 + \tau \|e^{\tau\varphi}\nabla_x u\|_{L^2}^2 \lesssim \|e^{\tau\varphi}Pu\|_{L^2}^2, \qquad \tau \geq \tau_0, \ u \in \mathscr{C}_c^\infty(X).$$

This type of estimate was used for the first time by T. Carleman [Car39] to achieve uniqueness properties for the Cauchy problem of an elliptic operator. Later, A.-P. Calderón and L. Hörmander further developed Carleman's method [Cal58, Hör58]. To this day, Carleman estimates remain an essential method to prove unique continuation properties; see for instance [Zui83] for an overview. On such questions more recent advances have been concerned with differential operators with singular potentials, starting with the contribution of D. Jerison and C. Kenig [JK85]. The reader is also referred to [Sog89, KT01, KT02]. In more recent years, the field of applications of Carleman estimates has gone beyond the original domain. They are also used in the study of inverse problems (see e.g. [BK81, Isa98, IIY03, KSU07]) and control theory for PDEs. Through unique continuation properties, they are used for the exact controllability of hyperbolic equations [BLR92]. They also yield the null controllability of linear parabolic equations [LR95] and the null controllability of classes of semi-linear parabolic equations [FI96, Bar00, FCZ00].

The work of L. Hörmander in [Hör58, Hör63] provided large classes of operators for which such estimates can be derived. He introduced the notion of pseudo-convexity and strong pseudo-convexity that provides a sufficient condition to achieve such estimates. In particular, choosing the weight function of the form $\varphi = \exp(\alpha\psi)$, with ψ satisfying the strong pseudo-convexity condition and $\alpha > 0$ chosen large, yields for an operator of order m an estimate of the form:

$$\sum_{|\beta|<m} \tau^{2(m-|\beta|)-1} \|e^{\tau\varphi}D_x^\beta u\|_{L^2}^2 \leq C\|e^{\tau\varphi}Pu\|_{L^2}^2, \qquad \tau \geq \tau_0, \ u \in \mathscr{C}_c^\infty(X).$$

In this type of estimate the parameter τ plays the same rôle as a differentiation. One may notice that the number of such "differentiations" in the l.h.s. amounts to $m - \frac{1}{2}$. For such an inequality one usually speaks of an estimate with a loss of a half derivative. This is connected to the terminology used in the study of sub-ellipticity; in fact the study of the conjugated operator $P_\varphi = e^{\tau\varphi}Pe^{-\tau\varphi}$ is central in the derivation of such an estimate and one precisely exploits its sub-elliptic property induced by the strong pseudo-convexity of the function ψ.

The parameter α can be viewed as a convexification parameter. As shown in Proposition 28.3.3 in [Hör85a] this allows one to obtain the proper sub-ellipticity condition on the conjugated operator P_φ from the strong pseudo-convexity of the function ψ.

With this choice of weight function, $\varphi = e^{\alpha\psi}$, one introduces a second large parameter, $\alpha > 0$. Several authors have derived Carleman estimates for some operators in which the dependency upon the second large parameters is explicit. See for instance [FI96]. Such result can be very useful to address applications such as inverse problems. On such questions see for instance [Ell00, EI00, IK08, BY12]. The type of operators for which such estimates have been obtained remains limited. In the present article we provide large classes of operators for which such estimates can be achieved.

For a second order estimate the resulting Carleman estimate can take the form (compare with (1.1)):

$$(1.2) \qquad (\alpha\tau)^3 \|\varphi^{3/2}e^{\tau\varphi}u\|_{L^2}^2 + \alpha\tau\|\varphi^{1/2}e^{\tau\varphi}\nabla_x u\|_{L^2}^2 \lesssim \|e^{\tau\varphi}Pu\|_{L^2}^2, \qquad \tau \geq \tau_0, \ \alpha \geq \alpha_0, \ u \in \mathscr{C}_c^\infty(X).$$

In the present article, we provide a general framework for the analysis and the derivation of Carleman estimates with two large parameters. For that purpose we introduce a pseudo-differential calculus of the Weyl-Hörmander type that resembles the semi-classical calculus and takes intro account the two large parameters τ and α as well as the weight function $\varphi = \exp(\alpha\psi)$. We introduce Sobolev spaces associated with this calculus and

provide boundedness results for pseudo-differential operators in this calculus. The notions of pseudo-convexity and strong pseudo-convexity are revisited in this framework. We prove how estimates of the form of (1.2) follow from these properties of the function ψ used to build the weight function. In the proof, positivity is obtained throught the Fefferman-Phong inequality as in [Ler85, Hör85a]. In the case of an operator of order m the estimate that we obtain under strong pseudo-convexity condition is of the general form

$$(1.3) \qquad \sum_{|\beta|<m} \left(\tau\alpha\right)^{2(m-|\beta|)-1}\|\varphi^{m-|\beta|-\frac{1}{2}}e^{\tau\varphi}D_x^\beta u\|_{L^2}^2 \lesssim \|e^{\tau\varphi}Pu\|_{L^2}^2.$$

Moreover, we prove that they are *necessary and sufficient* for estimates of the form (1.2) to hold. This result is in contrast with the existing results in the literature: see [Hör63, Chapter 8] and [Hör85a, Chapter 28].

If one consider an operator such as the Laplace operator the associated Carleman estimate with two large parameters is given by

$$(1.4) \qquad \alpha^4\tau^3\|\varphi^{3/2}e^{\tau\varphi}u\|_{L^2}^2 + \alpha^2\tau\|\varphi^{1/2}e^{\tau\varphi}\nabla_x u\|_{L^2}^2 \lesssim \|e^{\tau\varphi}\Delta u\|_{L^2}^2, \qquad \tau \geq \tau_0,\ \alpha \geq \alpha_0,\ u \in \mathscr{C}_c^\infty(X).$$

Here we simply require $\psi' \neq 0$ in a neighborhood of $\overline{X}$ and α_0 and τ_0 to be sufficiently large. This estimate is still characterized by the loss of a half derivative, yet we have an additional power of the parameter α as compared to (1.2)–(1.3). In Section 1.4 below, we show that such a stronger estimate can turn out to be useful for unique continuation considerations. We thus investigate this type of estimate. Under conditions stronger than the strong pseudo-convexity condition on the operator and the weight function we show that such Carleman estimates can indeed be achieved. The condition we put forward concerns the simplicity of the characteristics of the conjugated operator $e^{\tau\varphi}Pe^{-\tau\varphi}$. For an operator of order m the estimate takes the form

$$(1.5) \qquad \alpha \sum_{|\beta|<m} \left(\tau\alpha\right)^{2(m-|\beta|)-1}\|\varphi^{m-|\beta|-\frac{1}{2}}e^{\tau\varphi}D_x^\beta u\|_{L^2}^2 \lesssim \|e^{\tau\varphi}Pu\|_{L^2}^2.$$

We moreover prove that this simple-characteristic property is *necessary and sufficient* for such a stronger estimate to hold.

Finally one may wonder if stronger estimates can be obtained. In particular, can greater powers of the parameter α be obtained? We provided answers to this question of optimality in connexion with the notions of (strong) pseudo-convexity and the simple characteristic property.

1.2. **Further perspectives.** The results we present here only concern local Carleman estimates, i.e., applied to smooth functions with compact supports. In particular we do not address boundary problems. Considering such questions require a specialization in the type of operators to be considered, which we chose not to carry out here. However, the calculus framework we introduce here can be used when tackling the problem of deriving Carleman estimates for boundary problems or transmission problems. In the case of elliptic boundary problems this calculus is in fact used to derive Carleman estimates with two large parameters in [BL13]. On can also hope to extend the techniques and results of [LR10, LR11, LL] and obtain Carleman estimates with two large parameters for elliptic and parabolic transmission problems across a smooth interface.

As mentioned above, we consider Carleman estimates with the loss of a half derivative here. It would be interesting to carry out a similar analysis for estimates with a larger loss of derivatives. such estimates can be very important in some classes of inverse problems See for instance [KSU07, DSFKSU09].

The case of quasi-homogeneous operators, as studied in [Deh84, Isa93] is also of interest for an extension of the results presented here.

Here we focus our attentions on weight function of the form $\varphi = e^{\alpha\psi}$. Other convexification procedures can be carried out, for instance by choosing $\varphi = \psi + \frac{1}{2}\alpha\psi^2$. An analysis similar to the present one would be of interest.

1.3. **Notation.** Here, Ω denotes an open subset of $\mathbb{R}^n$. Estimates will be derived for function in $\mathscr{C}_c^\infty(X)$ with X an open subset of Ω such that $X \Subset \Omega$, i.e., X has a compact closure contained in Ω.

We recall the definition of the Poisson bracket of two functions in phase-space:

$$\{f, g\} = \sum_j \left(\partial_{\xi_j} f \partial_{x_j} g - \partial_{x_j} f \partial_{\xi_j} g \right).$$

It is associated with the symbol of the commutator of two (pseudo-)differential operators (see e.g. (A.4)), which will be used at many places in the present article. A review of some aspect of pseudo-differential calculus is provided in Appendix A.

We shall use the standard notation $A \lesssim B$ that stand for $A \leq CB$ with a positive constant C that does not depend on the parameters involved in the analysis. Moreover, when the constant C is used, it refers to a constant that is independent of those parameters. Its value may however change from one line to another. If we want to keep track of the value of a constant we shall use another letter.

1.4. **A motivating example.** Consider the Laplace operator $A = -\Delta$ and the bi-Laplace operator $B = \Delta^2$ in $\Omega \subset \mathbb{R}^n$, for $n \geq 2$.

Here we consider the unique continuation property for B. The result we present is well-known since the work of Calderón [Cal58] (see also [Zui83, Section 3.3]). Yet we show how introducing a second large parameter in the Carleman estimate for A yields a fairly simple proof of this property.

For the Laplace operator, for a properly chosen weight function $\varphi(x)$ (see e.g. [LL12] or Theorem 4.16 below) we can obtain the following Carleman estimate: for an open subset $X \Subset \Omega$ there exist $C > 0$ and $\tau_0 > 0$ such that

$$(1.6) \qquad \sum_{|\beta| \leq 2} \tau^{3-2|\beta|} \|e^{\tau\varphi} D_x^\beta u\|_{L^2}^2 \leq C \|e^{\tau\varphi} Au\|_{L^2}^2, \qquad u \in \mathscr{C}_c^\infty(X), \quad \tau \geq \tau_0.$$

Applying this estimate twice we can then write,

$$\sum_{|\beta'| \leq 2} \tau^{3-2|\beta'|} \sum_{|\beta| \leq 2} \tau^{3-2|\beta|} \|e^{\tau\varphi} D_x^{\beta+\beta'} u\|_{L^2}^2 \lesssim \sum_{|\beta'| \leq 2} \tau^{3-2|\beta'|} \|e^{\tau\varphi} A D_x^{\beta'} u\|_{L^2}^2 = \sum_{|\beta'| \leq 2} \tau^{3-2|\beta'|} \|e^{\tau\varphi} D_x^{\beta'} Au\|_{L^2}^2$$

$$\lesssim \|e^{\tau\varphi} Bu\|_{L^2}^2,$$

which reads

$$(1.7) \qquad \sum_{|\beta| \leq 4} \tau^{6-2|\beta|} \|e^{\tau\varphi} D_x^\beta u\|_{L^2}^2 \lesssim \|e^{\tau\varphi} Bu\|_{L^2}^2.$$

If you compare with results that can be obtained for some other elliptic operators of order 4 this is a much weaker estimate. For the operator $P = D_{x_1}^4 + D_{x_2}^4$ in $\mathbb{R}^2$ we have

$$(1.8) \qquad \sum_{|\beta| \leq 4} \tau^{7-2|\beta|} \|e^{\tau\varphi} D_x^\beta u\|_{L^2}^2 \lesssim \|e^{\tau\varphi} Pu\|_{L^2}^2.$$

for a properly chosen weight function (see Examples 4.2 and Theorem 4.16 below). The powers in the large parameter τ are one order lower. Estimate (1.8) is characterized by a loss of a half derivative, where as (1.7) exhibits the loss of a full derivative. This situation cannot be improved because of the following result.

Proposition 1.1. *Let $n \geq 2$. Let $\psi \in \mathscr{C}^\infty(\Omega)$ be a weight function and X be an open subset such that $X \Subset \Omega$. Assume that there exist $C > 0$, $\tau_0 > 0$, $\gamma_j \geq 6$, and $j \in \{0, \ldots, 4\}$ such that*

$$(1.9) \qquad \sum_{j=0}^4 \sum_{|\beta|=j} \tau^{\gamma_j - 2j} \|e^{\tau\varphi} D_x^\beta u\|_{L^2}^2 \leq C \|e^{\tau\varphi} Bu\|_{L^2}^2,$$

for all $u \in \mathscr{C}_c^\infty(X)$ and $\tau \geq \tau_0$. We then have $\gamma_j = 6$, for all $j \in \{0, \ldots, 4\}$.

We refer to Appendix B.2 for a proof.

Let us now consider a nonlinear problem of the form

$$(1.10) \qquad Bu = g(u, u', u^{(2)}, u^{(3)}) \text{ in } \Omega,$$

with the nonlinear function g satisfying the estimation

$$(1.11) \qquad |g(y_0, y_1, y_2, y_3)| \lesssim \sum_{j=0}^{3} |y_j|.$$

Let $x_0 \in \Omega$ and let f be a smooth function such that $f'(x_0) \neq 0$. A unique continuation problem is then:

Does (1.10) and $u = 0$ in $\{x; \ f(x) \geq f(x_0)\}$ imply that u vanishes in the vicinity of x_0?

Because of the nonlinearity involving the third-order derivative of the solution u, and observing that the power in τ for the derivative of third order in (1.7) is zero, a fact that cannot be repaired according to Proposition 1.1, we may face a difficulty to a direct proof of the unique continuation property. We shall however see that replacing the Carleman estimate (1.6) for A by its counterpart estimate with two large parameters allows one to obtain a straightforward proof.

By Theorem 4.16 below, for any $r \in \mathbb{R}$ (see Corollary 3.10 and the remark that follows), there exist $C > 0$, $\tau_0 > 0$, and $\alpha_0 > 0$ such that

$$(1.12) \qquad \alpha \sum_{|\beta| \leq 2} (\alpha\tau)^{3-2|\beta|} \|\varphi^{\frac{3}{2}-|\beta|+r} e^{\tau\varphi} D_x^{\beta} u\|_{L^2}^2 \leq C \|\varphi^r e^{\tau\varphi} Au\|_{L^2}^2, \qquad u \in \mathscr{C}_c^{\infty}(X).$$

if $\tau \geq \tau_0$ and $\alpha \geq \alpha_0$, for $\varphi = e^{\alpha\psi(x)}$, with ψ smooth such that $\psi > 0$ and $|\psi'| > 0$ in Ω, with $X \Subset \Omega$. The parameter α quantifies the convexity of the weight function. Applying this estimate twice, we then obtain

$$\alpha^2 \sum_{|\beta'| \leq 2} (\alpha\tau)^{3-2|\beta'|} \sum_{|\beta| \leq 2} (\alpha\tau)^{3-2|\beta|} \|\varphi^{3-|\beta|+|\beta'|} e^{\tau\varphi} D_x^{\beta+\beta'} u\|_{L^2}^2$$

$$\lesssim \alpha \sum_{|\beta'| \leq 2} (\alpha\tau)^{3-2|\beta'|} \|e^{\tau\varphi} \varphi^{\frac{3}{2}-|\beta'|} A D_x^{\beta'} u\|_{L^2}^2$$

$$= \alpha \sum_{|\beta'| \leq 2} (\alpha\tau)^{3-2|\beta'|} \|e^{\tau\varphi} \varphi^{\frac{3}{2}-|\beta'|} D_x^{\beta'} Au\|_{L^2}^2 \lesssim \|e^{\tau\varphi} Bu\|_{L^2}^2,$$

which reads

$$(1.13) \qquad \alpha^2 \sum_{|\beta| \leq 4} (\alpha\tau)^{6-2|\beta|} \|\varphi^{3-|\beta|} e^{\tau\varphi} D_x^{\beta} u\|_{L^2}^2 \lesssim \|e^{\tau\varphi} Bu\|_{L^2}^2.$$

The explicit dependency upon the parameter α including a gain of the factor α^2 on the l.h.s. of (1.13) as compared to (1.7) allows us to simply conclude to the unique continuation problem stated above, as we shall see now.

Proposition 1.2. *Let $u \in H^4(\Omega)$ be such that $Bu = g(u, u', u^{(2)}, u^{(3)})$ and such that $u = 0$ in $\{x; \ f(x) \geq f(x_0)\}$, for some $x_0 \in \Omega$ and f smooth such that $f'(x_0) \neq 0$. Then, there exists a neighborhood B_0 of x_0 such that $u_{|B_0} \equiv 0$.*

We insist again here on the fact that this is not a new result. Yet Carleman estimates with a second large parameter yield a simple proof. For further existing results, the reader is referred to [Hör85a, Section 28.3] and [Zui83] where a wide range of unique continuation results are available.

Proof. We pick a function ψ whose gradient does not vanish near a neighborhood V of x_0 and that satisfies $\langle \nabla f(x_0), \nabla\psi(x_0) \rangle > 0$ and is such that $f - \psi$ reaches a strict local minimum at x_0 as one moves along the level set $\{x \in V; \ \psi(x) = \psi(x_0)\}$. For instance, we may choose $\psi(x) = f(x) - c|x - x_0|^2 + C_0$. The constant C_0 is chosen such that ψ is locally positive. We then set $\varphi = e^{\alpha\psi}$. In the neighborhood V the geometrical situation is illustrated in Figure 1.
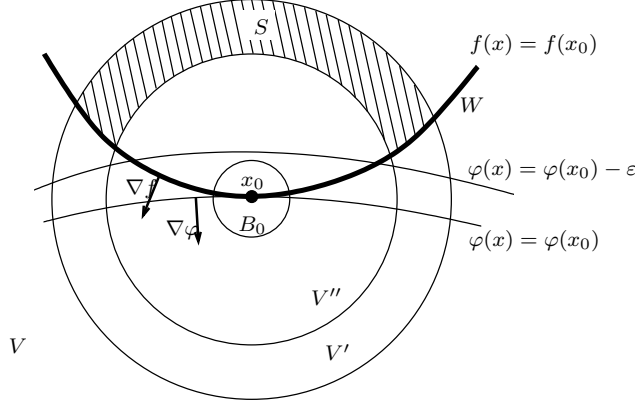
FIGURE 1. Local geometry for the unique continuation problem. The striped region contains the support of $[B, \chi]u$.

We call W the region $\{x \in V; \ f(x) \geq f(x_0)\}$ (region beneath $\{f(x) = f(x_0)\}$ in Figure 1). We choose V' and V'' neighborhoods of x_0 such that $V'' \Subset V' \Subset V$ and we pick a function $\chi \in \mathscr{C}_c^\infty(V')$ such that $\chi = 1$ in V''. We set $v = \chi u$ and we then have $v \in H_0^4(V)$. Observe that the weak Carleman estimate (1.13) applies to v by a density argument. We have

$$Bv = B(\chi u) = \chi \, Bu + [B, \chi]u,$$

where the commutator is a third-order differential operator. For $\tau \geq \tau_0 > 0$ and $\alpha \geq \alpha_0 > 0$ we thus obtain

$$\alpha^2 \sum_{|\beta| \leq 3} (\alpha\tau)^{6-2|\beta|} \|\varphi^{3-|\beta|} e^{\tau\varphi} D_x^\beta(\chi u)\|_{L^2}^2 \lesssim \|e^{\tau\varphi} \chi g(u, u', u^{(2)}, u^{(3)})\|_{L^2}^2 + \|e^{\tau\varphi}[B, \chi]u\|_{L^2}^2$$

$$\lesssim \sum_{|\beta| \leq 3} \|e^{\tau\varphi} \chi D_x^\beta u\|_{L^2}^2 + \|e^{\tau\varphi}[B, \chi]u\|_{L^2}^2$$

$$\lesssim \sum_{|\beta| \leq 3} \|e^{\tau\varphi} D_x^\beta(\chi u)\|_{L^2}^2 + \sum_{j \in J} \|e^{\tau\varphi} Q_j u\|_{L^2}^2,$$

with J finite and each Q_j is a differential operator of order less than 3, whose coefficients have support in $\operatorname{supp}(\chi')$. For α sufficiently large we find[1]

$$\alpha^2 \sum_{|\beta| \leq 3} (\alpha\tau)^{6-2|\beta|} \|\varphi^{3-|\beta|} e^{\tau\varphi} D_x^\beta(\chi u)\|_{L^2}^2 \lesssim \sum_{j \in J} \|e^{\tau\varphi} Q_j u\|_{L^2}^2,$$

As $\chi = 1$ in V'' we then write

$$\alpha^2 \sum_{|\beta| \leq 3} (\alpha\tau)^{6-2|\beta|} \|\varphi^{3-|\beta|} e^{\tau\varphi} D_x^\beta u\|_{L^2(V'')}^2 \lesssim \sum_{j \in J} \|e^{\tau\varphi} Q_j u\|_{L^2(S)}^2,$$

where $S = V' \setminus (V'' \cup W)$, since the supports of $Q_j u$, $j \in J$, are confined in the region where χ varies and u does not vanish (see the striped region in Figure 1).

For all $\varepsilon \in \mathbb{R}$, we set $V_\varepsilon = \{x \in V; \ \varphi(x) \leq \varphi(x_0) - \varepsilon\}$. There exists $\varepsilon > 0$ such that $S \Subset V_\varepsilon$. We then choose a ball B_0 with center x_0 such that $B_0 \subset V'' \setminus V_\varepsilon$ and obtain, for $\tau \geq \tau_0$,

$$e^{\tau \inf_{B_0} \varphi} \|u\|_{H^3(B_0)} \lesssim e^{\tau \sup_S \varphi} \|u\|_{H^3(S)},$$

Since $\inf_{B_0} \varphi > \sup_S \varphi$, letting τ go to $+\infty$, we obtain $u = 0$ in B_0. ∎

---

[1]This is the precise point where estimate (1.7) is not sufficient to carry on with the simple proof we present.

Estimate (1.13) is a weak form of Carleman estimate, with the loss of a full derivative. We improved upon (1.7) with the introduction of the second parameter α. It is then natural to question the optimality of the power of α in (1.13). The same question holds for the Carleman estimate for the Laplace operator (1.12). The following two propositions provide positive answers to these questions.

Proposition 1.3. *Let $n \geq 2$. Let $\psi \in \mathscr{C}^\infty(\Omega)$ be a weight function and X be an open subset such that $X \Subset \Omega$. Assume that there exist $C > 0$, $\tau_0 > 0$, $\alpha_0 > 0$, and $\gamma \geq 1$ such that*

$$\alpha^\gamma \sum_{|\beta| \leq 2} (\alpha\tau)^{3-2|\beta|} \|\varphi^{\frac{3}{2}-|\beta|} e^{\tau\varphi} D_x^\beta u\|_{L^2}^2 \leq C \|e^{\tau\varphi} Au\|_{L^2}^2, \tag{1.14}$$

where $\varphi = e^{\alpha\psi}$, for all $u \in \mathscr{C}_c^\infty(X)$, $\tau \geq \tau_0$ and $\alpha \geq \alpha_0$. We then have $\gamma = 1$.

This result is proven in a more general setting in Proposition 5.11. An explicit proof similar to that of the following proposition can also be carried out. To ease the reading of this introductory section we have placed this proof in Appendix B.3.

Proposition 1.4. *Let $n \geq 2$. Let $\psi \in \mathscr{C}^\infty(\Omega)$ be a weight function and X be an open subset such that $X \Subset \Omega$. Assume that there exist $C > 0$, $\tau_0 > 0$, $\alpha_0 > 0$, and $\gamma \geq 2$ such that*

$$\alpha^\gamma \sum_{|\beta| \leq 4} (\alpha\tau)^{6-2|\beta|} \|\varphi^{3-|\beta|} e^{\tau\varphi} D_x^\beta u\|_{L^2}^2 \leq C \|e^{\tau\varphi} Bu\|_{L^2}^2, \tag{1.15}$$

where $\varphi = e^{\alpha\psi}$, for all $u \in \mathscr{C}_c^\infty(\Omega)$, $\tau \geq \tau_0$ and $\alpha \geq \alpha_0$. We then have $\gamma = 2$.

We refer to Appendix B.3 for a proof.

1.5. **Outline.** In Section 2 we introduce the Weyl-Hörmander pseudo-differential calculus with two parameters and the associated Sobolev function spaces. Section 3 is devoted to the study of Carleman estimates with two large parameters under strong pseudoconvexity assumptions such as the estimate presented in (1.3). In Section 4 we investigate conditions that lead to stronger estimates. The simple-characteristic property leads to estimates of the form of (1.4). We also investigate the impact of the ellipticity of the operator on the Carleman estimate, either under the pseudo-convexity condition or under the simple-characteristic condition. In Section 5 we prove that that strong pseudo-convexity is necessary and sufficient for a Carleman estimate of the form of (1.3) to hold. We also prove that the simple-characteristic property is necessary and sufficient for a Carleman estimate of the form of (1.5) to hold. Finally, we discuss the optimality of the power of the parameter α in the different types of estimates presented here. In Appendix A we present some elements of pseudo-differential calculus. In Appendix B we collected some technical computations and proofs.

2. A PSEUDO-DIFFERENTIAL CALCULUS WITH TWO LARGE PARAMETERS

We set $W = \mathbb{R}^n \times \mathbb{R}^n$, often referred to as phase-space. A typical element of W will be $X = (x, \xi)$, with $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$.

Let $\psi \in \mathscr{C}^\infty(\mathbb{R}^n; \mathbb{R})$ be such that

$$\psi \geq C > 0, \quad \text{and} \quad \|\psi'\|_\infty < \infty. \tag{2.1}$$

We then set

$$\varphi(x) = e^{\alpha\psi(x)}, \quad \text{with} \quad \alpha \geq 1.$$

We make the following further assumption on the function ψ.

Assumption 2.1. *There exists $k > 0$ such that*

$$\sup_{\mathbb{R}^n} \psi \leq (k+1) \inf_{\mathbb{R}^n} \psi.$$

As a consequence we find

$$\forall x, y \in \mathbb{R}^n, \quad \varphi(y) \leq \varphi(x)^{k+1}. \tag{2.2}$$

2.1. **Metric and order function on phase-space.** We consider the metric on phase-space:

$$(2.3) \qquad g = \alpha^2 |dx|^2 + \frac{|d\xi|^2}{\mu^2}, \qquad \text{with } \mu^2 = \mu^2(x, \xi; \tau, \alpha) = \big(\tau\alpha\varphi(x)\big)^2 + |\xi|^2, \quad \text{and } \tau \geq 1, \ \alpha \geq 1.$$

We shall refer to μ as to the order function below. The explicit dependency of μ upon the parameter τ and α is dropped to ease notation.

The first result of this section shows that this metric on W defines a Weyl-Hörmander pseudo-differential calculus.

Proposition 2.2. *The metric g and the order function μ are admissible, in the sense that,*

 (1) *g satisfies the uncertainty principle, with $\lambda_g = h_g^{-1} = \alpha^{-1}\mu$.*
 (2) *μ and g are slowly varying;*
 (3) *μ and g are temperate.*

For a presentation of the Weyl-Hörmander calculus we refer to [Ler10], [Hör85b, Sections 18.4–6] and [Hör79].

Proof. The dual quadratic form of g is

$$g^\sigma = \mu^2 |dx|^2 + \frac{|d\xi|^2}{\alpha^2}.$$

We then have

$$\lambda_g(X) = \big(h_g\big)^{-1}(X) = \inf_{\substack{T \in W \\ T \neq 0}} \Big(g_X^\sigma(T)/g_X(T)\Big)^{\frac{1}{2}} = \alpha^{-1}\mu(X) \geq \tau\varphi(x) \geq 1.$$

The uncertainty principle is thus fulfilled.

We now prove the slow variations of g and μ, viz., there exist $K > 0$, $r > 0$, such that

$$\forall X, Y, T \in W, \quad g_X(Y - X) \leq r^2 \quad \Rightarrow \quad \begin{cases} g_Y(T) \leq Kg_X(T), \\ K^{-1} \leq \frac{\mu(X)}{\mu(Y)} \leq K. \end{cases}$$

We thus assume that $g_X(Y - X) \leq r^2$, with $0 < r < 1$ to be chosen below. With $X = (x, \xi)$ and $Y = (y, \eta)$, this gives

$$\alpha|x - y| + \frac{|\xi - \eta|}{\mu(X)} \leq Cr.$$

Under this condition, we observe that we have

$$(2.4) \qquad \varphi(x) = e^{\alpha\psi(x)} = \varphi(y)e^{\alpha\big(\psi(x) - \psi(y)\big)} \leq \varphi(y)e^{\alpha|x-y| \ \|\psi'\|_\infty} \leq \varphi(y)e^{Cr\|\psi'\|_\infty} \lesssim \varphi(y).$$

Similarly we have

$$(2.5) \qquad\qquad\qquad\qquad\qquad \varphi(y) \lesssim \varphi(x).$$

We also have

$$(2.6) \qquad\qquad\qquad\qquad |\eta| \leq |\eta - \xi| + |\xi| \leq Cr\mu(X) + |\xi| \lesssim \mu(X).$$

Next, we write

$$|\xi| \leq |\eta - \xi| + |\eta| \leq Cr\mu(X) + |\eta| \leq Cr\big(\tau\alpha\varphi(x) + |\xi|\big) + |\eta|.$$

Hence, for r sufficiently small, with (2.4), we have

$$(2.7) \qquad\qquad\qquad\qquad\qquad |\xi| \lesssim \tau\alpha\varphi(x) + |\eta| \lesssim \mu(Y).$$

With (2.4) and (2.7), resp. (2.5) and (2.6), we find

$$\mu(X) \lesssim \mu(Y), \quad \text{resp. } \mu(Y) \lesssim \mu(X).$$

Then if $T = (t, \theta) \in W$ we find

$$\frac{|\theta|^2}{\mu(Y)^2} \lesssim \frac{|\theta|^2}{\mu(X)^2} \lesssim \frac{|\theta|^2}{\mu(Y)^2},$$

and this gives $g_Y(T) \lesssim g_X(T) \lesssim g_Y(T)$.

We now prove the temperance of g and μ, viz., there exist $K > 0$, $N > 0$, such that

$$\forall X, Y, T \in W, \quad \frac{g_X(T)}{g_Y(T)} \leq C\big(1 + g_X^\sigma(X - Y)\big)^N,$$

$$\forall X, Y \in W, \quad \frac{\mu(X)}{\mu(Y)} \leq C\big(1 + g_X^\sigma(X - Y)\big)^N.$$

For $X = (x, \xi)$ and $Y = (y, \eta)$ we have

$$g_X^\sigma(X - Y) = \mu(X)^2|x - y|^2 + \frac{|\xi - \eta|^2}{\alpha^2}.$$

We note that

(2.8) $$\qquad |\xi| \leq |\eta| + |\xi - \eta| \leq |\eta| + \frac{|\xi - \eta|}{\alpha}\tau\alpha\varphi(y) \lesssim \big(1 + g_X^\sigma(X - Y)^{\frac{1}{2}}\big)\mu(Y).$$

First, if $\alpha|x - y| \leq 1$, then $\varphi(x) \lesssim \varphi(y)$, arguing as in (2.4). We thus have

$$\tau\alpha\varphi(x) \lesssim \mu(Y).$$

Second, if $\alpha|x - y| \geq 1$ we write

$$\tau\alpha\varphi(x) \lesssim \mu(X) \leq \mu(X)\frac{\mu(Y)}{\alpha} \leq |x - y|\mu(X)\mu(Y) \lesssim \big(1 + g_X^\sigma(X - Y)^{\frac{1}{2}}\big)\mu(Y).$$

In any case, we have $\tau\alpha\varphi(x) \lesssim \big(1 + g_X^\sigma(X - Y)^{\frac{1}{2}}\big)\mu(Y)$ and along with (2.8) we obtain the temperance of μ:

$$\mu(X) \lesssim \big(1 + g_X^\sigma(X - Y)^{\frac{1}{2}}\big)\mu(Y) \lesssim \big(1 + g_X^\sigma(X - Y)\big)\mu(Y).$$

For the temperance of g we need to prove

$$\alpha|t| + \frac{|\theta|}{\mu(X)} \lesssim \big(1 + g_X^\sigma(X - Y)\big)^N\Big(\alpha|t| + \frac{|\theta|}{\mu(Y)}\Big), \qquad T = (t, \theta) \in W.$$

To conclude it suffices to prove

$$\mu(Y) \lesssim \big(1 + g_X^\sigma(X - Y)\big)^N\mu(X).$$

We have

(2.9) $$\qquad |\eta| \leq |\xi| + |\xi - \eta| \leq |\xi| + \frac{|\xi - \eta|}{\alpha}\tau\alpha\varphi(x) \lesssim \big(1 + g_X^\sigma(X - Y)^{\frac{1}{2}}\big)\mu(X).$$

It thus remains to prove

(2.10) $$\qquad \tau\alpha\varphi(y) \lesssim \big(1 + g_X^\sigma(X - Y)\big)^N\mu(X).$$

First, if $\alpha|x - y| \leq 1$, then $\varphi(y) \lesssim \varphi(x)$, arguing as in (2.5). Estimate (2.10) is then clear. Second, if $\alpha|x - y| \geq 1$, with Assumption 2.1 and (2.2) we write

$$\tau\alpha\varphi(y) \leq \tau\alpha\varphi(x)^{k+1} \lesssim \frac{\mu(X)^{k+1}}{(\tau\alpha)^k} \lesssim \Big(\frac{\mu(X)}{\tau\alpha}\Big)^k\mu(X) \lesssim \Big(|x - y|\frac{\mu(X)}{\tau}\Big)^k\mu(X) \lesssim \big(1 + g_X^\sigma(X - Y)^{\frac{1}{2}}\big)^k\mu(X),$$

since $\tau \geq 1$. In any case, we thus have

$$\tau\alpha\varphi(y) \lesssim \big(1 + g_X^\sigma(X - Y)^{\frac{1}{2}}\big)^k\mu(X),$$

which concludes the proof. $\blacksquare$

2.2. **Sobolev Spaces.** We shall define Sobolev spaces associated with the calculus defined by the metric g. Note that the semi-classical setting of the metric g allows us to introduce such spaces without relying on the more intricate analysis of [BC94]. The proofs of all the results listed below can be found in Appendix B.

We set $\tilde{\tau}(x) = \tau \alpha \varphi(x)$.

Lemma 2.3. *Let* $k, s \in \mathbb{R}$. *For* τ *sufficiently large,* $H = \mathrm{Op^w}(\tilde{\tau}^{-s}\mu^{-k})\,\mathrm{Op^w}(\tilde{\tau}^s\mu^k)$ *is an homeomorphism of* $L^2(\mathbb{R}^n)$ *onto itself.*

We now define, for $k, s \in \mathbb{R}$,

$$\mathscr{H}_{k,s}(\mathbb{R}^n) = \Big\{ \mathrm{Op^w}(\tilde{\tau}^{-s}\mu^{-k})v;\ v \in L^2(\mathbb{R}^n) \Big\}.$$

Note that because of the boundedness of the function ψ (see Assumption 2.1) this space is in fact algebraically equal to the usual Sobolev space $H^k(\mathbb{R}^n)$.

For $u \in \mathscr{H}_{k,s}(\mathbb{R}^n)$ we set

$$\|u\|_{k,s} = \|\mathrm{Op^w}(\tilde{\tau}^s\mu^k)u\|_{L^2(\mathbb{R}^n)}.$$

With Lemma 2.3 we see that $\|.\|_{k,s}$ is a norm on $\mathscr{H}_{k,s}$. Moreover, for $u \in \mathscr{H}_{k,s}(\mathbb{R}^n)$, if $v \in L^2(\mathbb{R}^n)$ is such that $u = \mathrm{Op^w}(\tilde{\tau}^{-s}\mu^{-k})v$, then v is uniquely defined.

Lemma 2.4. *Let* $k, s \in \mathbb{R}$. *There exist* $C > 0$ *and* $\tau_1(k, s) > 0$ *such that for all* $u \in \mathscr{S}(\mathbb{R}^n)$ *there exists* $v \in L^2(\mathbb{R}^n)$ *such that* $u = \mathrm{Op^w}(\tilde{\tau}^{-s}\mu^{-k})v$ *and*

$$1/C\|u\|_{k,s} \leq \|v\|_{L^2} \leq C\|u\|_{k,s}, \qquad \tau \geq \tau_1(k, s).$$

Proposition 2.5. *Let* $k, s \in \mathbb{R}$ *and* $\tau \geq \tau_1(k, s)$.

   (1) *The function space* $\mathscr{H}_{k,s}(\mathbb{R}^n)$ *equipped with* $\|.\|_{k,s}$ *is a Hilbert space with* $\mathscr{S}(\mathbb{R}^n)$ *as a dense subspace.*
   (2) *There exists* $C > 0$ *such that if* $u \in \mathscr{H}_{k,s}$ *and* $u = \mathrm{Op^w}(\tilde{\tau}^{-s}\mu^{-k})v$, *with* $v \in L^2(\mathbb{R}^n)$, *we have*

$$1/C\|u\|_{k,s} \leq \|v\|_{L^2} \leq C\|u\|_{k,s}$$

   (3) *Let* $k, k', s, s' \in \mathbb{R}$. *For* $a \in S(\tilde{\tau}^s\mu^k, g)$ *there exist* $C > 0$ *and* $\tau_1 > 0$ *such that for all* $\tau \geq \tau_1$, *we have*

$$\|\mathrm{Op^w}(a)u\|_{k',s'} \leq C\|u\|_{k+k',s+s'}, \qquad u \in \mathscr{H}_{k+k',s+s'}(\mathbb{R}^n).$$

In particular, if $k' \leq k$ and $s' \leq s$ we have

$$\mathscr{S}(\mathbb{R}^n) \subset \mathscr{H}_{k,s}(\mathbb{R}^n) \subset \mathscr{H}_{k',s'}(\mathbb{R}^n).$$

We finally sharpen the result of Lemma 2.4.

Lemma 2.6. *Let* $k, k', s, s' \in \mathbb{R}$. *There exist* $C > 0$ *and* $\tau_1(k, k', s, s') > 0$ *such that, for* $\tau \geq \tau_1(k, k', s, s') > 0$ *we have:*

   (1) *For all* $u \in \mathscr{S}(\mathbb{R}^n)$ *there exists a unique* $v \in L^2(\mathbb{R}^n)$ *such that* $u = \mathrm{Op^w}(\tilde{\tau}^{-s}\mu^{-k})v$;
   (2) *Moreover,* $v \in \mathscr{H}_{k',s'}(\mathbb{R}^n)$ *and*

$$C^{-1}\|u\|_{k+k',s+s'} \leq \|v\|_{k',s'} \leq C\|u\|_{k+k',s+s'}.$$

Lemma 2.7. *Let* $s, k \in \mathbb{R}$ *and* $a \in S(\tilde{\tau}^s\mu^k, g)$. *There exists* $C > 0$ *such that, for* τ *sufficiently large,*

$$\big|(\mathrm{Op^w}(a)u, u)\big| \leq C\|u\|^2_{\frac{k}{2},\frac{s}{2}}, \quad u \in \mathscr{S}(\mathbb{R}^n).$$

Proposition 2.8. *Let* $s, k \in \mathbb{R}$. *There exists* $C > 0$ *such that, for* τ *sufficiently large,*

$$\big(\mathrm{Op^w}(\tilde{\tau}^s\mu^k)u, u\big) \geq C\|u\|^2_{\frac{k}{2},\frac{s}{2}}, \qquad u \in \mathscr{S}(\mathbb{R}^n).$$
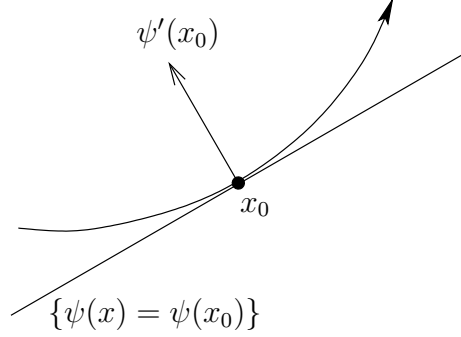
FIGURE 2. Geometry of the bicharacteristics in the case the function ψ is pseudo-convex.

3. CARLEMAN ESTIMATES UNDER STRONG PSEUDOCONVEXITY ASSUMPTIONS

Let $P(x, D_x)$ be a differential operator of order m, with homogeneous principal symbol $p(x, \xi)$.

Definition 3.1 (Principal normality [Hör85a, Definition 28.2.4])**.** The operator $P(x, D_x)$ is said to be principally normal on Ω if for all open subset $X \Subset \Omega$, there exists some $C > 0$

$$(3.1) \qquad \left|\{\overline{p}, p\}(x, \xi)\right| \le C |p(x, \xi)| \, |\xi|^{m-1}, \quad x \in \overline{X}, \ \xi \in \mathbb{R}^n.$$

Elliptic operators and operators with real coefficients in the principal part are typical examples of principally normal operators.

We shall now revisit some consequences of the pseudo-convexity and strong pseudo-convexity properties.

3.1. **Pseudo-convexity properties and symbol estimates.** Let $\psi \in \mathscr{C}^\infty(\Omega, \mathbb{R})$. We recall the following definitions [Hör63].

Definition 3.2 (pseudo-convexity)**.** We say that ψ is pseudo-convex at $x \in \Omega$ w.r.t. $P(x, D_x)$ if $\psi'(x) \neq 0$ and if

$$(\Psi \mathrm{c}) \qquad \forall \xi \in \mathbb{R}^n \backslash 0, \ \ p(x, \xi) = 0 \text{ and } \{p, \psi\}(x, \xi) = 0 \ \ \Rightarrow \ \ \mathrm{Re}\,\{\overline{p}, \{p, \psi\}\}(x, \xi) > 0.$$

The function ψ is said to be pseudo-convex w.r.t. Ω and p if $\psi' \neq 0$ in Ω and $(\Psi \mathrm{c})$ is valid for all $x \in \Omega$.

In the case of a real principal symbol, we have $\{p, \psi\} = H_p \psi$ and $\{p, \{p, \psi\}\} = H_p^2 \psi$, i.e., first and second derivatives of the function ψ along the bicharacteristics associated with p. Then the pseudoconvexity implies that if the function $\psi(x)$ goes through an extremum along the bicharacteristics then it needs to be a non-degenerate minimum. This implies a convexity for the bicharacteristics as illustrated in Figure 2.

We note that

$$(3.2) \qquad \{p, \psi\}(x, \xi) = \langle p_\xi'(x, \xi), \psi'(x) \rangle, \quad \text{and} \quad \mathrm{Re}\,\{\overline{p}, \{p, \psi\}\}(x, \xi) = \theta_{p, \psi}(x, \xi),$$

with

$$(3.3) \qquad \theta_{p, \psi}(x, \xi) = \sum_{j,k} \partial^2_{x_j x_k} \psi(x)\, \partial_{\xi_k} \overline{p}(x, \xi)\, \partial_{\xi_j} p(x, \xi) + \mathrm{Re} \sum_j \partial_{x_j} \psi(x)\, \{\overline{p}, \partial_{\xi_j} p\}(x, \xi).$$

Observe that $\theta_{p, \psi}(x, \xi)$ is homogeneous of degree $2m - 2$ in ξ.

Definition 3.3 (strong pseudo-convexity)**.** We say that ψ is strongly pseudo-convex at $x \in \Omega$ w.r.t. p if

    (1) ψ is pseudo-convex at x w.r.t. p;

(2) if for all $\xi \in \mathbb{R}^n$ and $\tau > 0$,

(s-Ψc)

$$p(x, \xi + i\tau\psi'(x)) = 0 \text{ and } \{p, \psi\}(x, \xi + i\tau\psi'(x)) = 0 \quad \Rightarrow \quad \frac{1}{2i}\{\overline{p}(x, \xi - i\tau\psi'(x)), p(x, \xi + i\tau\psi'(x))\} > 0.$$

The function ψ is said to be strongly pseudo-convex w.r.t. Ω and p if items (1) and (2) are valid for all $x \in \Omega$.

Remark 3.4. Note that the pseudo-convexity and strong pseudo-convexity properties actually depend on the level sets of ψ (since $\psi' \neq 0$ rather than on the function ψ itself. They are of geometrical nature. In particular they do not depend on the coordinate system used.

We note that

$$(3.4) \quad \frac{1}{2i}\{\overline{p}(x, \xi - i\tau\psi'(x)), p(x, \xi + i\tau\psi'(x))\} = \{\operatorname{Re} p(x, \xi + i\tau\psi'(x)), \operatorname{Im} p(x, \xi + i\tau\psi'(x))\} = \Theta_{p,\psi}(x, \xi, \tau),$$

with

$$(3.5) \qquad\qquad \Theta_{p,\psi}(x, \xi, \tau) = \tau \sum_{j,k} \partial^2_{x_j x_k}\psi(x) \, \partial_{\xi_j} p(x, \xi + i\tau\psi'(x)) \, \partial_{\xi_k}\overline{p}(x, \xi - i\tau\psi'(x))$$

$$+ \operatorname{Im} \sum_j \partial_{x_j} p(x, \xi + i\tau\psi'(x)) \, \partial_{\xi_j}\overline{p}(x, \xi - i\tau\psi'(x)).$$

Observe that $\Theta_{p,\psi}(x, \xi, \tau)$ is homogeneous of degree $2m - 1$ in (ξ, τ).

Remark 3.5. Note that there are operators for which no weight function can be strongly pseudo-convex. This is for instance the case of the bi-Laplace operator $P = \Delta^2$ in dimension greater than or equal to two. In fact, observe that with $p(\xi) = |\xi|^4$ then

$$p(x, \xi + i\tau\psi') = \Big(\sum_j (\xi_j + i\tau\psi'_j)^2\Big)^2, \qquad \{p, \psi\}(x, \xi + i\tau\psi') = 4\Big(\sum_j (\xi_j + i\tau\psi'_j)^2\Big)\langle\xi + i\tau\psi', \psi'\rangle,$$

and

$$\Theta_{p,\psi}(x, \xi, \tau) = 16\tau p(x, \xi + i\tau\psi') \sum_{j,k} \partial^2_{x_j x_k}\psi(x)\big(\xi_j + i\tau\psi'_j\big)\big(\xi_k + i\tau\psi'_k\big).$$

Then if $\xi \perp \psi'(x)$ and $|\xi| = |\tau\psi'(x)|$ we have $p(x, \xi + i\tau\psi') = \{p, \psi\}(x, \xi + i\tau\psi') = 0$ and yet $\Theta_{p,\psi}(x, \xi, \tau) = 0$. In fact, this obstruction makes sense: a Carleman estimate of the form of (3.10) in Theorem 3.8 below cannot be achieved for the bi-Laplace operator because of the optimality result of Proposition 1.1.

The following result sharpens the estimate of Proposition 28.3.3 in [Hör85a] in the case of a weight function of the form $\varphi = e^{\alpha\psi}$.

Proposition 3.6. *Let P be principally normal on Ω and let ψ be a strongly pseudo-convex function w.r.t. Ω and P. We set $\varphi = e^{\alpha\psi}$ and*

$$\zeta = \zeta(x, \xi, \tau) = \xi + i\tau\varphi'(x) = \xi + i\tilde{\tau}(x)\psi'(x), \quad \tilde{\tau}(x) = \tau\alpha\varphi(x).$$

Let X be an open subset such that $X \Subset \Omega$. There exist $C > 0$, $\tau_0 \geq 1$, $\alpha_0 > 0$ such that we have

$$(3.6) \qquad \tilde{\tau}(x)\mu^{2(m-1)} \leq C\Big(|p(x, \zeta)|^2 + \frac{1}{2i}\{\overline{p(x, \zeta)}, p(x, \zeta)\}\Big), \quad \tau \geq \tau_0, \; \alpha \geq \alpha_0, \; (x, \xi) \in \overline{X} \times \mathbb{R}^n.$$

Remark 3.7. The following computations will be useful in the proof and at various places in the article. With (3.3) we find

$$\theta_{p,\varphi}(x, \xi) = \sum_{j,k} \partial^2_{x_j x_k}\varphi(x) \, \partial_{\xi_k}\overline{p}(x, \xi) \, \partial_{\xi_j} p(x, \xi) + \operatorname{Re} \sum_j \partial_{x_j}\varphi(x) \{\overline{p}, \partial_{\xi_j} p\}(x, \xi),$$

which, as $\varphi = e^{\alpha\psi}$ yields

$$(3.7) \qquad\qquad \theta_{p,\varphi}(x, \xi) = \alpha\varphi\theta_{p,\psi}(x, \xi) + \alpha^2\varphi|\langle p'_\xi(x, \xi), \psi'(x)\rangle|^2.$$

With (3.4) we find

$$\frac{1}{2i}\big\{\overline{p}(x,\overline{\zeta}),p(x,\zeta)\big\} = \Theta_{p,\varphi}(x,\xi,\tau) = \tau\sum_{j,k}\partial^2_{x_jx_k}\varphi(x)\,\partial_{\xi_j}p(x,\zeta)\,\partial_{\xi_k}\overline{p}(x,\overline{\zeta}) + \mathrm{Im}\sum_j\partial_{x_j}p(x,\zeta)\,\partial_{\xi_j}\overline{p}(x,\overline{\zeta}).$$

which yields

(3.8)
$$\frac{1}{2i}\big\{\overline{p}(x,\overline{\zeta}),p(x,\zeta)\big\} = \tilde{\tau}(x)\sum_{j,k}\partial^2_{x_jx_k}\psi(x)\,\partial_{\xi_j}p(x,\zeta)\,\partial_{\xi_k}\overline{p}(x,\overline{\zeta}) + \tilde{\tau}(x)\alpha|\langle p'_\xi(x,\zeta),\psi'(x)\rangle|^2$$
$$+ \mathrm{Im}\sum_j\partial_{x_j}p(x,\zeta)\,\partial_{\xi_j}\overline{p}(x,\overline{\zeta}).$$
$$= \Theta_{p,\psi}(x,\xi,\tilde{\tau}(x)) + \tilde{\tau}(x)\alpha|\langle p'_\xi(x,\zeta),\psi'(x)\rangle|^2.$$

Proof of Proposition 3.6. The proof is along the lines of that of Proposition 28.3.3 in [Hör85a].

On the compact set $\{(x,\xi)\in \overline{X}\times\mathbb{R}^n;\ |\xi|=1\}$ the pseudo-convexity of ψ, with (3.2) and (3.3), implies

$$\nu\Big(|p(x,\xi)| + |\langle p'_\xi(x,\xi),\psi'(x)\rangle|\Big) + \theta_{p,\psi}(x,\xi) \geq C > 0,$$

for ν sufficiently large. By homogeneity, with (3.3), we find

$$\nu\Big(|p(x,\xi)|\,|\xi|^{m-2} + |\langle p'_\xi(x,\xi),\psi'(x)\rangle|\,|\xi|^{m-1}\Big) + \theta_{p,\psi}(x,\xi) \gtrsim |\xi|^{2m-2}, \qquad x\in\overline{X}, \quad \xi\in\mathbb{R}^n.$$

We first consider the case $\tilde{\tau}(x)\ll|\xi|$. With the form of $\theta_{p,\psi}$ in (3.3) we then obtain

$$\nu\Big(|p(x,\zeta)|\,|\zeta|^{m-2} + |\langle p'_\xi(x,\zeta),\psi'(x)\rangle|\,|\zeta|^{m-1}\Big) + \sum_{j,k}\partial^2_{x_jx_k}\psi(x)\,\partial_{\xi_k}\overline{p}(x,\overline{\zeta})\,\partial_{\xi_j}p(x,\zeta)$$
$$+ \mathrm{Re}\sum_j\partial_{x_j}\psi(x)\,\big\{\overline{p},\partial_{\xi_j}p\big\}(x,\xi) \gtrsim |\zeta|^{2m-2}.$$

Since $p(x,\xi)$ is principally normal, Lemma 28.2.5 in [Hör85a], with $N=\psi'(x)$ and $\tilde{\tau}(x)$ in place of τ, yields

$$\Big|\tilde{\tau}(x)^{-1}\mathrm{Im}\sum_j\partial_{x_j}p(x,\zeta)\partial_{\xi_j}\overline{p}(x,\overline{\zeta}) - \mathrm{Re}\sum_j\partial_{x_j}\psi(x)\big\{\overline{p},\partial_{\xi_j}p\big\}(x,\xi)\Big|$$
$$\leq C\Big(\tilde{\tau}(x)|\psi'(x)|\,|\zeta|^{2m-3} + \tilde{\tau}(x)^{-1}|p(x,\zeta)|\,|\zeta|^{m-1} + \langle p'_\xi(x,\zeta),\psi'(x)\rangle\,|\zeta|^{m-1}\Big).$$

Using that $\tilde{\tau}(x)\ll|\xi|$, we then obtain

$$\tilde{\tau}(x)^{-1}|p(x,\zeta)|\,|\zeta|^{m-1} + |\langle p'_\xi(x,\zeta),\psi'(x)\rangle|\,|\zeta|^{m-1} + \sum_{j,k}\partial^2_{x_jx_k}\psi(x)\,\partial_{\xi_k}\overline{p}(x,\overline{\zeta})\,\partial_{\xi_j}p(x,\zeta)$$
$$+ \tilde{\tau}(x)^{-1}\mathrm{Im}\sum_j\partial_{x_j}p(x,\zeta)\partial_{\xi_j}\overline{p}(x,\overline{\zeta}) \gtrsim |\zeta|^{2m-2}.$$

With the Young inequality, for $\tilde{\tau}(x)$ and α sufficiently large, i.e., for τ and α sufficiently large, we obtain

(3.9)
$$\tilde{\tau}(x)^{-1}|p(x,\zeta)|^2 + \alpha|\langle p'_\xi(x,\zeta),\psi'(x)\rangle|^2 + \tilde{\tau}(x)^{-1}\Theta_{p,\psi}(x,\xi,\tilde{\tau}(x)) \gtrsim |\zeta|^{2m-2}.$$

With (3.9) and (3.8) we thus obtain (3.6)

$$|p(x,\zeta)|^2 + \frac{1}{2i}\big\{\overline{p}(x,\overline{\zeta}),p(x,\zeta)\big\} \gtrsim \tilde{\tau}(x)|\zeta|^{2(m-1)} \gtrsim \tilde{\tau}(x)\mu^{2(m-1)},$$

using that $\tilde{\tau}(x)\ll|\xi|$.

We now treat the case $|\xi|\leq\delta\tilde{\tau}(x)$, for some fixed δ. We introduce $\hat{\tau}>0$ and place ourselves on the compact set

$$\mathscr{C} = \{(x,\xi,\hat{\tau});\ \hat{\tau}^2 + |\xi|^2 = 1,\ \ |\xi|\leq\delta\hat{\tau},\ \ x\in\overline{X}\}.$$

As ψ is strongly pseudo-convex, we have

$$\alpha\Big(\hat{\tau}^{-1}|p(x,\xi+i\hat{\tau}\psi'(x))|^2 + \hat{\tau}|\langle p'_\xi(x,\xi+i\hat{\tau}\psi'(x)),\psi'(x)\rangle|^2\Big) + \Theta_{p,\psi}(x,\xi,\hat{\tau}) \geq C, \quad (x,\xi,\hat{\tau})\in\mathscr{C},$$

for α sufficiently large. By homogeneity we find

$$\alpha\Big(\hat{\tau}^{-1}|p(x,\xi+i\hat{\tau}\psi'(x))|^2 + \hat{\tau}|\langle p'_\xi(x,\xi+i\hat{\tau}\psi'(x)),\psi'(x)\rangle|^2\Big) + \Theta_{p,\psi}(x,\xi,\hat{\tau}) \gtrsim (|\hat{\tau}|+|\xi|)^{2m-1},$$

for all $x \in \overline{X}$, $\hat{\tau} > 0$ and ξ such that $|\xi| \leq \delta\hat{\tau}$.

We apply this last inequality to $\hat{\tau} = \tilde{\tau}(x)$ in the case $|\xi| \leq \delta\tilde{\tau}(x)$. As $\alpha/\tilde{\tau}(x) = 1/(\tau\varphi(x)) \leq 1$, this yields

$$|p(x,\zeta)|^2 + \tilde{\tau}(x)\alpha|\langle p'_\xi(x,\zeta),\psi'(x)\rangle|^2 + \Theta_{p,\psi}(x,\xi,\tilde{\tau}(x)) \gtrsim |\zeta|^{2m-1},$$

We have $|\zeta|^{2m-1} \gtrsim \tilde{\tau}(x)\mu^{2(m-1)}$. Hence, recalling (3.8) we also obtain (3.6) in this second case. ∎

3.2. **Carleman estimate.** Based on the pseudo-convexity properties we presented above and the pseudo-differential calculus of Section 2, we shall now prove a Carleman estimate with two large parameters.

Theorem 3.8. *Let $P = P(x,D_x)$ be a principally normal operator in Ω of order m and $\psi \in \mathscr{C}^\infty(\Omega)$, $\psi \geq C > 0$ on Ω, be a strongly pseudo-convex function w.r.t. Ω and P. We set $\varphi = e^{\alpha\psi}$. If X is an open subset, $X \Subset \Omega$, there exist $C > 0$, $\alpha_1 > 0$, and $\tau_1 \geq 1$ such that*

$$\tag{3.10} \sum_{|\beta|<m} (\tau\alpha)^{2(m-|\beta|)-1}\|\varphi^{m-|\beta|-\frac{1}{2}}e^{\tau\varphi}D_x^\beta u\|_{L^2}^2 \leq C\|e^{\tau\varphi}Pu\|_{L^2}^2,$$

for all $u \in \mathscr{C}_c^\infty(X)$, $\alpha \geq \alpha_1$, and $\tau \geq \tau_1$.

Proof. Let Y be an open subset of Ω such that $X \Subset Y \Subset \Omega$. Proposition 3.6 applies in Y.

Carleman estimates of the form (3.10) can be handled locally and can be patched together (see [Hör63]). Hence, there is no loss of generality in restricting ourselves to a small neighborhood V of a point $x_0 \in X$, with $V \Subset Y$. We also choose a small neighborhood W such that $V \Subset W \Subset Y$.

There, the function ψ can be used as a coordinate function. We can then modify the function ψ outside W so that it satisfies Assumption 2.1. This allows us to use the calculus that we introduced in Section 2. We assume $u \in \mathscr{C}_c^\infty(V)$.

We denote by $p = p(x,\xi)$ the principal symbol of P. We set $P_\varphi = e^{\tau\varphi}Pe^{-\tau\varphi}$ and $v = e^{\tau\varphi}u$. We have $P_\varphi = P(x,D_x + i\tau\varphi'(x)) \in \Psi(\mu^m,g)$. Its principal symbol in $S(\mu^m,g)/S(\alpha\mu^{m-1},g)$ is given by

$$p_\varphi = p_\varphi(x,\xi,\tau) = p(x,\xi+i\tau\varphi'(x)).$$

By Lemma 2.4, there exists $w \in L^2(\mathbb{R}^n)$ such that $v = \mathrm{Op}^{\mathrm{w}}(\mu^{1-m})w$. We set $Q = \mathrm{Op}^{\mathrm{w}}(p_\varphi)\mathrm{Op}^{\mathrm{w}}(\mu^{1-m}) \in \Psi(\mu,g)$ (we consider only the principal part of P_φ at first). We then write

$$\|Qw\|_{L^2}^2 = (Q^*Qw,w)_{L^2},$$

where the Weyl symbol of $Q^*Q \in \Psi(\mu^2,g)$ is given by

$$\tag{3.11} \rho = \mu^{2-2m}\Big(|p_\varphi|^2 + \frac{1}{2i}\{\overline{p_\varphi},p_\varphi\}\Big) \mod S(\alpha^2,g).$$

by (A.2) and (A.3).

By Proposition 3.6, for τ and α sufficiently large, we have $\rho - C\tau\alpha\varphi \geq 0$ in W. Let $\chi \in \mathscr{C}_c^\infty(W)$ be such that, $0 \leq \chi \leq 1$ and $\chi = 1$ in a neighborhood of V. We then write

$$\rho = \hat{\rho} + r, \qquad \hat{\rho} = \rho\chi + \mu^2(1-\chi), \quad r = (\rho-\mu^2)(1-\chi).$$

Then $\hat{\rho} \in S(\mu^2,g)$ and $\hat{\rho}-C\tau\alpha\varphi \geq 0$ in the whole phase-space. Recalling that $\lambda_g^2 = \mu^2/\alpha^2$, the Fefferman-Phong inequality (see [FP78] and Theorem A.4 below) then yields,

$$\tag{3.12} \|Qw\|_{L^2}^2 \geq (\tau\alpha\varphi w,w)_{L^2} - C\|\alpha w\|_{L^2}^2 + (\mathrm{Op}^{\mathrm{w}}(r)w,w)_{L^2}.$$

As $\varphi = e^{\alpha\psi}$ with $\psi \geq C > 0$ on Ω we find, for α sufficiently large,

$$\|Qw\|_{L^2}^2 \geq C\|\tilde{\tau}^{\frac{1}{2}}w\|_{L^2}^2 + (\mathrm{Op}^{\mathrm{w}}(r)w,w)_{L^2}.$$

Lemma 3.9. *For any $N \in \mathbb{N}$, we have*

$$\| \operatorname{Op}^{\mathrm{w}}(r)w\|_{L^2} \le C_N \tau^{-N} \|v\|_{L^2}$$

See Appendix B.10 for a proof.

For τ sufficiently large, with Lemma 2.6 we thus obtain

$$\| \operatorname{Op}^{\mathrm{w}}(p_\varphi)v\|_{L^2} = \|Qw\|_{L^2} \gtrsim \| \operatorname{Op}^{\mathrm{w}}(\tilde{\tau}^{\frac{1}{2}}\mu^{m-1})v\|_{L^2} = \|v\|_{m-1,\frac{1}{2}}.$$

As $P_\varphi - \operatorname{Op}^{\mathrm{w}}(p_\varphi) \in \Psi(\alpha\mu^{m-1}, g)$, for α sufficiently large we obtain

$$\|P_\varphi v\|_{L^2} \gtrsim \|v\|_{m-1,\frac{1}{2}}.$$

For any β, with $|\beta| \le m - 1$, observe that

$$\left(\tau\alpha\varphi\right)^{(m-|\beta|)-\frac{1}{2}} D_x^\beta = \tilde{\tau}^{\frac{1}{2}}\tilde{\tau}^{(m-|\beta|)-1} D_x^\beta \in \Psi(\tilde{\tau}^{\frac{1}{2}}\mu^{m-1}, g),$$

and thus by Proposition 2.5 we have

$$\|P_\varphi v\|_{L^2} \gtrsim \sum_{|\beta|<m} \left(\tau\alpha\right)^{(m-|\beta|)-\frac{1}{2}} \|\varphi^{(m-|\beta|)-\frac{1}{2}} D_x^\beta v\|_{L^2}.$$

The conclusion of the proof is then classical. ■

Note that the power of the function φ that appears in the estimate can be shifted easily, as stated in the following corollary.

Corollary 3.10. *Let $r \in \mathbb{R}$. Under the same assumption as in Theorem 3.8 there exist $C > 0$, $\alpha_1 > 0$, and $\tau_1 \ge 1$ such that*

$$(3.13) \qquad \sum_{|\beta|<m} \left(\tau\alpha\right)^{2(m-|\beta|)-1} \|\varphi^{(m-|\beta|)-\frac{1}{2}+r} e^{\tau\varphi} D_x^\beta u\|_{L^2}^2 \le C\|\varphi^r e^{\tau\varphi} Pu\|_{L^2}^2,$$

for all $u \in \mathscr{C}_c^\infty(X)$, $\alpha \ge \alpha_1$, and $\tau \ge \tau_1$.

Note that similar results can be achieved for the Carleman estimates proven in the section below. We shall omit to write these results below.

Proof. We assume $r \ne 0$. Let $s \in \mathbb{R}^*$ and let $Q = q(x, D_x)$ be a differential operator of order $\ell \in \mathbb{N}^*$. The symbol of the commutator $[Q, \varphi^s]$ in the standard quantization is

$$q \circ \varphi^s - \varphi^s q = \sum_{1 \le |\gamma| \le \ell} \frac{i^{|\gamma|}}{\gamma!} \partial_\xi^\gamma q(x, \xi) \partial_x^\gamma (\varphi^s).$$

(See Appendix A where notation related to the pseudo-differential calculus is presented.) Observing that $|\partial_x^\gamma(\varphi^s)| \lesssim \alpha^{|\gamma|}\varphi^s$ we find that

$$(3.14) \qquad [Q, \varphi^s] = \sum_{\gamma=0}^{\ell-1} a_\gamma(x) D_x^\gamma, \quad \text{with } |a_\gamma(x)| \lesssim \alpha^{\ell-|\gamma|}\varphi^s.$$

Let $w = \varphi^r u$. Starting from the Carleman estimate of Theorem 3.8 for w we derive (3.13).

For $|\beta| \le m - 1$ we write

$$\|\varphi^{m-\frac{1}{2}-|\beta|+r} e^{\tau\varphi} D_x^\beta u\|_{L^2} \lesssim \|\varphi^{m-\frac{1}{2}-|\beta|} e^{\tau\varphi} D_x^\beta w\|_{L^2} + \|\varphi^{m-\frac{1}{2}-|\beta|+r} e^{\tau\varphi} [D_x^\beta, \varphi^{-r}]w\|_{L^2},$$

which by (3.14) gives

$$(\tau\alpha)^{2m-1-2|\beta|}\|\varphi^{m-\frac{1}{2}-|\beta|+r}e^{\tau\varphi}D_x^\beta u\|_{L^2}^2 \lesssim (\tau\alpha)^{2m-1-2|\beta|}\Big(\|\varphi^{m-\frac{1}{2}-|\beta|}e^{\tau\varphi}D_x^\beta w\|_{L^2}^2$$
$$+ \sum_{|\gamma|\le|\beta|-1} \alpha^{2|\beta|-2|\gamma|}\|\varphi^{m-\frac{1}{2}-|\beta|}e^{\tau\varphi}D_x^\gamma w\|_{L^2}^2\Big)$$
$$\lesssim \text{l.h.s. } (3.10),$$

if $\alpha > 0$ and $\tau \ge 1$. We thus obtain

$$(3.15) \qquad \sum_{|\beta|<m} \big(\tau\alpha\big)^{2(m-|\beta|)-1}\|\varphi^{m+r-|\beta|-\frac{1}{2}}e^{\tau\varphi}D_x^\beta u\|_{L^2}^2 \le C\|e^{\tau\varphi}Pw\|_{L^2}^2,$$

With (3.14) we obtain

$$\|e^{\tau\varphi}Pw\|_{L^2}^2 \lesssim \|\varphi^r e^{\tau\varphi}Pu\|_{L^2}^2 + \|e^{\tau\varphi}[P,\varphi^r]u\|_{L^2}^2$$
$$\lesssim \|\varphi^r e^{\tau\varphi}Pu\|_{L^2}^2 + \sum_{|\gamma|\le m-1} \alpha^{2(m-|\gamma|)}\|\varphi^r e^{\tau\varphi}D_x^\gamma u\|_{L^2}^2.$$

As $\alpha \ll \varphi(x)$ for α large we can "absorb" the sum on the r.h.s. of the previous estimate by the l.h.s. of (3.15). ∎

We finish this section with some remarks.

Remark 3.11. (1) As we pointed out in the introductory section, Carleman estimates are central tools for the quantification of the unique continuation property. See [Zui83] for manifolds results. A natural question concerns the necessity of the pseudo-convexity conditions with respect to the unique continuation property. Answers to such question can be found in [Ali83], where for example it is proven that if the pseudo-convexity condition of Definition 3.2 is strongly violated then unique continuation does not hold. A mild violation of the pseudo-convexity condition as presented in [LR85] (see also [Hör85a, Section 28.4]) can yet preserve the unique continuation property with an additional compactness property.

(2) Here, we chose to use the notion of principal normality introduced by L. Hörmander and N. Lerner [Hör63, Ler85, Hör85a]. In [CDSZ96b] the authors propose a generalization of this notion. That work was motivated by the counter-example to uniqueness of [CDS95] in which the principal normality is replaced by the (P) condition of Treves-Nirenberg. In [CDSZ96b] Carleman estimates are derived for operators satisfying the generalized principal normality property and unique continuation results are obtained. Their derivation of the Carleman estimate is based on a generalization of the Fefferman-Phong inequality [CDSZ96a], a technical point that we preferred to avoid here. Note also that the uniqueness result that is proven in [CDSZ96b] corresponds to a stronger condition on the weight function than the pseudo-convexity condition we use here. It coincides with the simple-characteristic property presented in Section 4.1. Note finally that nonuniqueness may also result of a strong violation of their generalized principal normality condition with a zeroth-order perturbation of the operator.

4. Cases of stronger estimates

In this section we present classes of operators for which stronger Carleman estimates with two large parameters can be derived as compared to the result of Theorem 3.8.

As in the previous section $p(x,\xi)$ denotes the *homogeneous* principal part of the differential operator $P = P(x,D_x)$.

4.1. Simple characteristics. We introduce the map

$$(4.1) \qquad \begin{aligned} \rho_{x,\xi} : \mathbb{R}^+ &\to \mathbb{C}, \\ \hat\tau &\mapsto p(x,\xi+i\hat\tau\psi'(x)), \end{aligned}$$

where $x \in \Omega$ and $\xi \in \mathbb{R}^n$.

Definition 4.1. Given a weight function ψ and an operator P we say that the simple-characteristic property is satisfied in Ω if, for all $x \in \Omega$, we have $\xi = 0$ and $\hat{\tau} = 0$ when the map $\rho_{x,\xi}$ has a double root.

Note that the case $\xi = 0$ is particular, as the root $\hat{\tau} = 0$ has of course multiplicity m. Note also that we have

$$(4.2) \qquad \rho'_{x,\xi}(\hat{\tau}) = i\langle p'_\xi(x, \xi + i\hat{\tau}\psi'(x)), \psi'(x)\rangle = i\{p, \psi\}(x, \xi + i\hat{\tau}\psi'(x)).$$

We can thus formulate the condition of Definition 4.1 as

$$(4.3) \qquad p(x, \xi + i\hat{\tau}\psi'(x)) = \{p, \psi\}(x, \xi + i\hat{\tau}\psi'(x)) = 0 \quad \Rightarrow \quad \xi = 0, \ \hat{\tau} = 0.$$

Note that this implies that the hypersurface $\{\psi = \mathrm{Cst}\}$ is not characteristics, that is $p(x, \psi'(x)) \neq 0$. Otherwise choosing ξ collinear to $\psi'(x)$ we find $\rho_{x,\xi}$ identically zero. In particular this implies that $|\psi'| \neq 0$.

With the simple-characteristic property we shall obtain below a Carleman estimate with an additional power in the second large parameter α.

Examples 4.2. If $P(x, D_x)$ is a first-order operator then $\rho_{x,\xi}(\hat{\tau})$ is a first-order polynomial in $\hat{\tau}$. If the leading coefficient does not vanish then the simple-characteristic property is clearly fulfilled.

For a second-order elliptic operators then $\rho_{x,\xi}(\hat{\tau})$ is a second-order polynomial in $\hat{\tau}$ where the leading coefficient does not vanish if $\psi'(x) \neq 0$. In dimension $n > 2$, property (4.3) holds. Indeed, if ξ and $\psi'(x)$ are not colinear the equation $p(x, \xi + \sigma\psi'(x)) = 0$ admits two roots, σ_1 and σ_2, such that $\mathrm{Im}\,\sigma_1 > 0$ and $\mathrm{Im}\,\sigma_2 < 0$ (see [LM68, proof of Proposition 1.1, Chapter 2] or [Hör83]). If ξ and $\psi'(x)$ are colinear then there is a double real root. In any case there cannot be a double root for $\hat{\tau} \mapsto \rho_{x,\xi}(\hat{\tau})$ apart from $\hat{\tau} = 0$, which implies $\xi = 0$. In dimension $n = 2$ the property (4.3) holds for second-order elliptic operators with real coefficients. With complex coefficients it may not hold: consider for instance $P = (D_{x_1} + iD_{x_2})^2$. Then any real $\hat{\tau}$ is a double root for $\hat{\tau} \mapsto \rho_{x,\xi}(\hat{\tau})$ for $\xi_1 = \hat{\tau}\partial_{x_2}\psi(x)$ and $\xi_2 = -\hat{\tau}\partial_{x_1}\psi(x)$.

Evidently, if $p(x, \xi)$ and $q(x, \xi)$ are such that their respective maps (4.1) satisfy property (4.3), with different roots, then their product will also satisfy this property.

Note that elliptic operators with real coefficients of order higher than two may however not satisfy this property, e.g. the bi-Laplace operator Δ^2 in $\Omega \Subset \mathbb{R}^n$ with $n \geq 2$, independently of the choice of the weight function. An example of an elliptic operator of order greater than two satisfying the simple-characteristic property (4.3) is $D_{x_1}^4 + D_{x_2}^4$ in $\Omega \subset \mathbb{R}^2$ for any function ψ whose gradient does not vanish in Ω.

An example of a second-order operator that is not elliptic satisfying the simple-characteristic property (4.3) is $\frac{1}{2}D_{x_1}^2 + D_{x_1}D_{x_2}$ in $\Omega \subset \mathbb{R}^2$, with $\psi(x) = \frac{1}{2}(x_1 - a)^2$, where a is such that $|\psi'| \neq 0$ in Ω, e.g., if $\Omega \subset \{x_1 > a\}$. Another example is $P = D_{x_1}D_{x_2}$ in $\Omega \subset \mathbb{R}^2$, with $\psi(x) = x_1 + x_2$. We shall go back to this example at the end of Section 5.

Note that the examples we have just given are principally normal (see Definition 3.1) as they are either elliptic or have real coefficients.

Remark 4.3. The simple-characteristic property (4.3) implies that ψ is strongly pseudo-convex with respect to Ω and P. However there exist operators and weight functions that satisfy the strong pseudo-convexity conditions without fulfilling the simple-characteristic property (4.3). Such an example is given by $P(D_x) = D_{x_1}$ in $\Omega \subset \mathbb{R}^2$ with the weight function $\psi(x) = x_1^2/2 + x_2$.

Details on some of the examples and remark above can be found in Appendix B.11.

Remark 4.4. The simple-characteristic property is independent of the notion of principal normality. Consider the operator $P(x, D_x) = D_{x_1} + i(x_1 D_{x_2} + D_{x_3})$ with symbol $p(x, \xi) = \xi_1 + i(x_1\xi_2 + \xi_3)$. With $\psi(x) = x_1$, the simple-characteristics property is fulfilled. However the operator is not principally normal. Observe that we have

$$\{\bar{p}, p\}(x, \xi) = 2i\{\mathrm{Re}\,p, \mathrm{Im}\,p\} = \{\xi_1, x_1\xi_2 + \xi_3\} = \xi_2,$$

which implies that (3.1) cannot be achieved if $x_1 = \xi_1 = \xi_3 = 0$.

Proposition 4.5. *Let P be principally normal on Ω and let ψ be such that the simple-characteristic property (4.3) is fulfilled. We set $\varphi = e^{\alpha\psi}$ and*

$$\zeta = \zeta(x,\xi,\tau) = \xi + i\tau\varphi'(x) = \xi + i\tilde{\tau}(x)\psi'(x), \quad \tilde{\tau}(x) = \tau\alpha\varphi(x).$$

Let X be an open subset such that $X \Subset \Omega$. There exist $C > 0$, $\tau_0 \geq 1$, $\alpha_0 > 0$, and ν_0 such that we have

$$(4.4) \quad \tilde{\tau}(x)^2 \mu^{2(m-1)} \leq C\Big(\nu|p(x,\zeta)|^2 + \frac{\tau\varphi(x)}{2i}\{\overline{p(x,\zeta)}, p(x,\zeta)\}\Big), \quad \tau \geq \tau_0, \; \alpha \geq \alpha_0, \; \nu \geq \nu_0, \; (x,\xi) \in \overline{X} \times \mathbb{R}^n.$$

Proof. On the compact set $L = \{(x,\xi) \in \overline{X} \times \mathbb{R}^n; \; |\xi| = 1\}$ the simple-characteristic property (4.3) with (4.2) yields, for ν sufficiently large,

$$\nu|p(x,\xi)| + |\langle p'_\xi(x,\xi), \psi'(x)\rangle|^2 \geq C > 0.$$

For α sufficiently large we find (see the definition of $\theta_{p,\psi}(x,\xi)$ in (3.3))

$$\nu\alpha|p(x,\xi)| + \alpha|\langle p'_\xi(x,\xi), \psi'(x)\rangle|^2 + \theta_{p,\psi}(x,\xi) \geq \alpha C' > 0, \quad (x,\xi) \in L.$$

By homogeneity, recalling that $\theta_{p,\psi}(x,\xi)$ is homogeneous of degree $2m - 2$ in ξ, we then obtain

$$\nu\alpha|p(x,\xi)||\xi|^{m-2} + \alpha|\langle p'_\xi(x,\xi), \psi'(x)\rangle|^2 + \theta_{p,\psi}(x,\xi) \gtrsim \alpha|\xi|^{2m-2}, \quad x \in \overline{X}, \; \xi \in \mathbb{R}^n.$$

We first consider the case $\tilde{\tau}(x) \ll |\xi|$. We then obtain

$$\nu\alpha|p(x,\zeta)||\zeta|^{m-2} + \alpha|\langle p'_\xi(x,\zeta), \psi'(x)\rangle|^2 + \sum_{j,k} \partial^2_{x_j x_k}\psi(x)\, \partial_{\xi_k}\overline{p}(x,\overline{\zeta})\, \partial_{\xi_j} p(x,\zeta)$$
$$+ \operatorname{Re}\sum_j \partial_{x_j}\psi(x)\,\{\overline{p}, \partial_{\xi_j}p\}(x,\xi) \gtrsim \alpha|\zeta|^{2m-2}.$$

Since $p(x,\xi)$ is principally normal, Lemma 28.2.5 in [Hör85a], with $N = \psi'(x)$ and $\tilde{\tau}(x)$ in place of τ, yields

$$\Big|\tilde{\tau}(x)^{-1}\operatorname{Im}\sum_j \partial_{x_j}p(x,\zeta)\partial_{\xi_j}\overline{p}(x,\overline{\zeta}) - \operatorname{Re}\sum_j \partial_{x_j}\psi(x)\,\{\overline{p}, \partial_{\xi_j}p\}(x,\xi)\Big|$$
$$\leq C\Big(\tilde{\tau}(x)|\psi'(x)|\,|\zeta|^{2m-3} + \tilde{\tau}(x)^{-1}|p(x,\zeta)|\,|\zeta|^{m-1} + \langle p'_\xi(x,\zeta), \psi'(x)\rangle\,|\zeta|^{m-1}\Big).$$

We thus find

$$\nu\alpha|p(x,\zeta)||\zeta|^{m-2} + \alpha|\langle p'_\xi(x,\zeta), \psi'(x)\rangle|^2$$
$$+ \sum_{j,k} \partial^2_{x_j x_k}\psi(x)\, \partial_{\xi_k}\overline{p}(x,\overline{\zeta})\, \partial_{\xi_j} p(x,\zeta) + \tilde{\tau}(x)^{-1}\operatorname{Im}\sum_j \partial_{x_j}p(x,\zeta)\partial_{\xi_j}\overline{p}(x,\overline{\zeta})$$
$$+ C\Big(\tilde{\tau}(x)|\psi'(x)|\,|\zeta|^{2m-3} + \tilde{\tau}(x)^{-1}|p(x,\zeta)|\,|\zeta|^{m-1} + \langle p'_\xi(x,\zeta), \psi'(x)\rangle\,|\zeta|^{m-1}\Big) \geq C'\alpha|\zeta|^{2m-2}.$$

After a multiplication by $\tau\varphi(x)\tilde{\tau}(x)$ we thus obtain, with (3.4),

$$\nu\tilde{\tau}(x)^2|p(x,\zeta)||\zeta|^{m-2} + \tilde{\tau}(x)^2|\langle p'_\xi(x,\zeta), \psi'(x)\rangle|^2 + \tau\varphi(x)\Theta_{p,\psi}(x,\xi,\tilde{\tau}(x)) + C\Big(\tau\varphi(x)\tilde{\tau}(x)^2|\psi'(x)|\,|\zeta|^{2m-3}$$
$$+ \tau\varphi(x)|p(x,\zeta)|\,|\zeta|^{m-1} + \tau\varphi(x)\tilde{\tau}(x)\langle p'_\xi(x,\zeta), \psi'(x)\rangle\,|\zeta|^{m-1}\Big) \gtrsim C'\tilde{\tau}(x)^2|\zeta|^{2m-2}.$$

With the Young inequality we observe that we have

$$\nu \tilde{\tau}(x)^2 |p(x,\zeta)| |\zeta|^{m-2} \lesssim \nu |p(x,\zeta)|^2 + \nu \underbrace{\tilde{\tau}(x)^4 |\zeta|^{2m-4}}_{\ll \tilde{\tau}^2 |\zeta|^{2m-2}},$$

$$\tau \varphi(x) \tilde{\tau}(x)^2 |\psi'(x)| |\zeta|^{2m-3} \lesssim \tilde{\tau}(x)^3 |\zeta|^{2m-3} \ll \tilde{\tau}^2 |\zeta|^{2m-2}, \quad \text{as } \alpha \geq 1,$$

$$\tau \varphi(x) |p(x,\zeta)| |\zeta|^{m-1} \lesssim \nu |p(x,\zeta)|^2 + \underbrace{\nu^{-1}(\tau \varphi(x))^2 |\zeta|^{2m-2}}_{=\nu^{-1}\alpha^{-2}\tilde{\tau}^2 |\zeta|^{2m-2}},$$

$$\tau \varphi(x) \tilde{\tau}(x) \langle p'_\xi(x,\zeta), \psi'(x) \rangle |\zeta|^{m-1} \lesssim \tilde{\tau}(x)^2 |\langle p'_\xi(x,\zeta), \psi'(x) \rangle|^2 + \alpha^{-2} \tilde{\tau}(x)^2 |\zeta|^{2m-2}.$$

With α large these estimates yield

$$\nu |p(x,\zeta)|^2 + \tilde{\tau}(x)^2 |\langle p'_\xi(x,\zeta), \psi'(x) \rangle|^2 + \tau \varphi(x) \Theta_{p,\psi}(x,\xi,\tilde{\tau}(x)) \gtrsim \tilde{\tau}(x)^2 |\zeta|^{2m-2},$$

which by (3.8) reads

$$\nu |p(x,\zeta)|^2 + \frac{\tau \varphi(x)}{2i} \{\overline{p}(x,\overline{\zeta}), p(x,\zeta)\} \gtrsim \tilde{\tau}(x)^2 |\zeta|^{2m-2}.$$

We now treat the case $|\xi| \leq \delta \tilde{\tau}(x)$, for some fixed δ. Let $\hat{\tau} \geq 0$ and consider

$$f(x,\xi,\hat{\tau}) = \nu |p(x,\xi + i\hat{\tau}\psi'(x))|^2 + \hat{\tau}^2 |\langle p'_\xi(x,\xi + i\hat{\tau}\psi'(x)), \psi'(x) \rangle|^2.$$

We place ourselves on the compact set

$$\mathscr{C} = \{(x,\xi,\hat{\tau}); \ \hat{\tau}^2 + |\xi|^2 = 1, \ x \in \overline{X}, \ \xi \in \mathbb{R}^n, \ \hat{\tau} > 0, \ |\xi| \leq \delta\hat{\tau}\}.$$

By (4.3)-(4.2) we have

$$\forall (x,\xi,\hat{\tau}) \in \mathscr{C}, \quad p(x,\xi + i\hat{\tau}\psi'(x)) = 0 \ \Rightarrow \ \hat{\tau}\langle p'_\xi(x,\xi + i\hat{\tau}\psi'(x)), \psi'(x) \rangle \neq 0.$$

For ν sufficiently large, we then obtain $f \geq C > 0$ on $\mathscr{C}$. By homogeneity we deduce

(4.5) $$f(x,\xi,\hat{\tau}) \gtrsim (\hat{\tau}^2 + |\xi|^2)^m, \quad x \in \overline{X}, \ \xi \in \mathbb{R}^n, \ \hat{\tau} \in \mathbb{R}^+, \quad |\xi| \leq \delta \tilde{\tau}(x)$$

With (3.8) we have

$$\nu |p(x,\zeta)|^2 + \frac{\tau \varphi(x)}{2i} \{\overline{p(x,\zeta)}, p(x,\zeta)\} = \nu |p(x,\zeta)|^2 + \tau \alpha \varphi(x) \tilde{\tau}(x) |\langle p'_\xi(x,\zeta), \psi'(x) \rangle|^2 + \tau \varphi(x) \Theta_{p,\psi}(x,\xi,\tilde{\tau}(x))$$

$$= f(x,\xi,\tilde{\tau}(x)) + \tau \varphi(x) \Theta_{p,\psi}(x,\xi,\tilde{\tau}(x))$$

$$\geq C \mu^{2m} + \tau \varphi(x) \Theta_{p,\psi}(x,\xi,\tilde{\tau}(x)),$$

with $C > 0$. Observing that $\tau \varphi(x) |\Theta_{p,\psi}(x,\xi,\tilde{\tau}(x))| \leq \tau \varphi(x) \mu^{2m-1}(x,\xi,\tau)$, by choosing α sufficiently large we also obtain (4.4) in the case $|\xi| \leq \delta \tilde{\tau}(x)$. ∎

With Proposition 4.5 we obtain the following Carleman estimate.

Theorem 4.6. *Let $P = P(x, D_x)$ be a principally normal operator in Ω of order m and $\psi \in \mathscr{C}^\infty(\Omega)$, $\psi \geq C > 0$ on Ω, a function such that the simple-characteristic property (4.3) is fulfilled. We set $\varphi = e^{\alpha \psi}$. If X is an open subset, $X \Subset \Omega$, there exist $C > 0$, $\alpha_1 > 0$, and $\tau_1 \geq 1$ such that*

$$\alpha \sum_{|\beta| < m} (\tau \alpha)^{2(m-|\beta|)-1} \|\varphi^{m-|\beta|-\frac{1}{2}} e^{\tau \varphi} D_x^\beta u\|_{L^2}^2 \leq C \|e^{\tau \varphi} P u\|_{L^2}^2,$$

for all $u \in \mathscr{C}_c^\infty(X)$, $\alpha \geq \alpha_1$, and $\tau \geq \tau_1$.

We observe that we have gained a factor α on the l.h.s. in contrast to the Carleman estimate of Theorem 3.8.

Remark 4.7. If $\psi' \neq 0$ in Ω and $P = -\Delta$ then the simple characteristic property is fulfilled. With Proposition 1.3 we can conclude that the power of α just obtain the l.h.s. of the estimate is optimal. We shall refine this consideration at the end of Section 5.

Proof. Let Y be an open subset such that $X \Subset Y \Subset \Omega$. Proposition 4.5 applies in Y. As in the proof of Theorem 3.8 we restrict ourselves to a small neighborhood $V \subset W \Subset Y$ of a point $x_0 \in X$, where W is an open subset. The function ψ then satisfies Assumption 2.1. We assume $u \in \mathscr{C}_c^\infty(V)$.

We denote by $p = p(x,\xi)$ the principal symbol of P. We set $P_\varphi = e^{\tau\varphi}Pe^{-\tau\varphi}$ and $v = e^{\tau\varphi}u$. We have $P_\varphi = P(x, D_x + i\tau\varphi'(x)) \in \Psi(\mu^m, g)$. Its principal symbol of given by

$$p_\varphi = p_\varphi(x,\xi,\tau) = p(x,\xi + i\tau\varphi'(x)).$$

We first consider only the principal part and set $Q = \mathrm{Op}^w(p_\varphi)$ and introduce

$$Q_2 = \mathrm{Op}^w(q_2) \in \Psi(\mu^m, g), \quad \text{with } q_2 = \mathrm{Re}\, p_\varphi,$$
$$Q_1 = \mathrm{Op}^w(q_1) \in \Psi(\mu^m, g), \quad \text{with } q_1 = \mathrm{Im}\, p_\varphi.$$

We have $Q = Q_2 + iQ_1$ with both Q_2 and Q_1 selfadjoint.

We have $\|Qv\|_{L^2}^2 = \|Q_2v\|_{L^2}^2 + \|Q_1v\|_{L^2}^2 + i([Q_2, Q_1]v, v)_{L^2}$. With ν such that $\nu(\tau\varphi(x))^{-1} \leq 1$ we then write

$$\tau\|Qv\|_{L^2}^2 \geq \nu\|\varphi^{-\frac{1}{2}}Q_2v\|_{L^2}^2 + \nu\|\varphi^{-\frac{1}{2}}Q_1v\|_{L^2}^2 + i\tau([Q_2,Q_1]v,v)_{L^2}$$
$$= \Big(\big(\underbrace{\nu(Q_2\varphi^{-1}Q_2 + Q_1\varphi^{-1}Q_1) + i\tau[Q_2,Q_1]}_{=B \in \Psi(\varphi^{-1}\mu^{2m}, g)}\big)v, v\Big)_{L^2}.$$

With (A.3) and (A.4) we have

$$q_j \sharp \varphi^{-1} \sharp q_j = \varphi^{-1}q_j^2 \mod S(\alpha^2\varphi^{-1}\mu^{2m-2}, g), \quad j = 1, 2,$$
$$i(q_2 \sharp q_1 - q_1 \sharp q_2) = \{q_2, q_1\} \mod S(\alpha^3\mu^{2m-3}, g).$$

We thus find $B = B_0 + B_1$ with $B_0 \in \Psi(\varphi^{-1}\mu^{2m}, g)$ and $B_1 \in \Psi(\alpha^2\varphi^{-1}\mu^{2m-2}, g)$, and

$$B_0 = \mathrm{Op}^w(\nu\varphi^{-1}(q_2^2 + q_1^2) + \tau\{q_2, q_1\}) = \mathrm{Op}^w\Big(\nu\varphi^{-1}|p_\varphi|^2 + \frac{\tau}{2i}\{\overline{p_\varphi}, p_\varphi\}\Big).$$

By Lemma 2.4, there exists $w \in L^2(\mathbb{R}^n)$ such that $v = \mathrm{Op}^w(\varphi^{\frac{1}{2}}\mu^{1-m})w$. We obtain

$$\tau\|Qv\|_{L^2}^2 \geq (\tilde{B}_0 w, w) + (\tilde{B}_1 w, w),$$

with $\tilde{B}_1 \in \Psi(\alpha^2, g)$ and $\tilde{B}_0 \in \Psi(\mu^2, g)$. By (A.3) the Weyl symbol of $\tilde{B}_0$ is given by

$$\tilde{b}_0 = \mu^{2-2m}\Big(\nu|p_\varphi|^2 + \frac{\tau\varphi}{2i}\{\overline{p_\varphi}, p_\varphi\}\Big). \mod S(\alpha^2, g),$$

By Proposition 4.5 we have $\tilde{b}_0 - \tilde{\tau}(x)^2 \geq 0$ in W for ν, α and τ sufficiently large. Arguing as in the proof of Theorem 3.8 with the Fefferman-Phong inequality (see [FP78] and Theorem A.4 below) we obtain

$$\tau\|Qv\|_{L^2}^2 \geq C\|\tilde{\tau}w\|_{L^2}^2 + (\mathrm{Op}^w(r)w, w)_{L^2}$$

with $\|\mathrm{Op}^w(r)w\|_{L^2} \leq C_N\tau^{-N}\|v\|_{L^2}$. For τ sufficiently large, with Lemma 2.6 we find $\tau\|Qv\|_{L^2}^2 \gtrsim \|\mathrm{Op}^w(\tilde{\tau}\varphi^{-\frac{1}{2}}\mu^{m-1})v\|_{L^2}^2$, which gives

$$\|\mathrm{Op}^w(p_\varphi)v\|_{L^2}^2 = \|Qv\|_{L^2}^2 \gtrsim \alpha\|\mathrm{Op}^w(\tilde{\tau}^{\frac{1}{2}}\mu^{m-1})v\|_{L^2}^2 = \alpha\|v\|_{m-1,\frac{1}{2}}^2.$$

We conclude as in the proof of Theorem 3.8. ■

4.2. **Elliptic operators.** For elliptic operators stronger results can also be achieved. First, we shall consider elliptic operators and weight functions under pseudo-convexity conditions and, second, we shall consider elliptic operators along with weight functions such that the simple-characteristic property holds.

4.2.1. *Elliptic operators under strong pseudoconvexity condition.* Let P be an elliptic operator of order m. We first note that if ψ is a smooth function such that $|\psi'| > 0$ on Ω, then it is a pseudo-convex function on Ω for the operator P.

Proposition 4.8. *Let P be elliptic on Ω and let ψ satisfy* (s-Ψc) *(point 2 in Definition 3.3) for all $x \in \Omega$. We set $\varphi = e^{\alpha\psi}$ and*

$$\zeta = \zeta(x, \xi, \tau) = \xi + i\tau\varphi'(x) = \xi + i\tilde{\tau}(x)\psi'(x), \qquad \tilde{\tau}(x) = \tau\alpha\varphi(x).$$

Let X be an open subset such that $X \Subset \Omega$. There exist $C > 0$, $\tau_0 \geq 1$, $\alpha_0 > 0$ such that we have

$$(4.6) \qquad (\tau\alpha\varphi)^{-1}\mu^{2m} \leq C\Big(|p(x,\zeta)|^2 + \frac{1}{2i}\big\{\overline{p(x,\zeta)}, p(x,\zeta)\big\}\Big), \quad \tau \geq \tau_0, \ \alpha \geq \alpha_0, \ (x,\xi) \in \overline{X} \times \mathbb{R}^n.$$

Proof. We have $|p(x,\xi)|^2 \gtrsim |\xi|^{2m}$. We first consider the case $\tilde{\tau}(x) \ll |\xi|$. We then obtain

$$|p(x,\zeta)|^2 \gtrsim |\zeta|^{2m}.$$

With (3.8), by homogeneity we have $\big|\frac{1}{2i}\big\{\overline{p}(x,\overline{\zeta}), p(x,\zeta)\big\}\big| \lesssim \alpha|\zeta|^{2m-1}$. For τ sufficiently large we obtain

$$|p(x,\zeta)|^2 + \frac{1}{2i}\big\{\overline{p}(x,\overline{\zeta}), p(x,\zeta)\big\} \gtrsim |\zeta|^{2m}.$$

We now treat the case $|\xi| \leq \delta\tilde{\tau}(x)$, for some fixed δ. We introduce $\hat{\tau} > 0$ and place ourselves on the compact set

$$\mathscr{C} = \{(x, \xi, \hat{\tau}); \ \hat{\tau}^2 + |\xi|^2 = 1, \ |\xi| \leq \delta\hat{\tau}, \ x \in \overline{X}\}.$$

As ψ is strongly pseudo-convex, we have

$$\alpha\Big(|p(x, \xi + i\hat{\tau}\psi'(x))|^2 + \hat{\tau}^2|\langle p'_\xi(x, \xi + i\hat{\tau}\psi'(x)), \psi'(x)\rangle|^2\Big) + \hat{\tau}\Theta_{p,\psi}(x,\xi,\hat{\tau}) \geq C, \quad (x,\xi,\hat{\tau}) \in \mathscr{C},$$

for α sufficiently large. By homogeneity we find

$$\alpha\Big(|p(x, \xi + i\hat{\tau}\psi'(x))|^2 + \hat{\tau}^2|\langle p'_\xi(x, \xi + i\hat{\tau}\psi'(x)), \psi'(x)\rangle|^2\Big) + \hat{\tau}\Theta_{p,\psi}(x,\xi,\hat{\tau}) \gtrsim (|\hat{\tau}| + |\xi|)^{2m},$$

for all $x \in \overline{X}$, $\hat{\tau} > 0$ and ξ such that $|\xi| \leq \delta\hat{\tau}$. We apply this last inequality to $\hat{\tau} = \tilde{\tau}(x)$ and find

$$\alpha\Big(\tilde{\tau}(x)^{-1}|p(x,\zeta)|^2 + \tilde{\tau}(x)|\langle p'_\xi(x,\zeta), \psi'(x)\rangle|^2\Big) + \Theta_{p,\psi}(x,\xi,\tilde{\tau}(x)) \gtrsim \tilde{\tau}(x)^{-1}(|\tilde{\tau}| + |\xi|)^{2m},$$

in the case $|\xi| \leq \delta\tilde{\tau}(x)$. As $\alpha/\tilde{\tau}(x) = 1/(\tau\varphi(x)) \leq 1$, this yields

$$|p(x,\zeta)|^2 + \tilde{\tau}(x)\alpha|\langle p'_\xi(x,\zeta), \psi'(x)\rangle|^2 + \Theta_{p,\psi}(x,\xi,\tilde{\tau}(x)) \gtrsim \tilde{\tau}(x)^{-1}|\zeta|^{2m} \gtrsim \tilde{\tau}(x)^{-1}\mu^{2m}.$$

Hence, recalling (3.8) we also obtain (4.6) in this second case. ∎

With Proposition 4.8 we then obtain the following Carleman estimate.

Theorem 4.9. *Let $P = P(x, D_x)$ be an elliptic operator in Ω of order m and $\psi \in \mathscr{C}^\infty(\Omega)$, $\psi \geq C > 0$ on Ω, be a strongly pseudo-convex function w.r.t. Ω and P. We set $\varphi = e^{\alpha\psi}$. If X is an open subset, $X \Subset \Omega$, there exist $C > 0$, $\alpha_1 > 0$, and $\tau_1 \geq 1$ such that*

$$(4.7) \qquad \sum_{|\beta| \leq m} (\tau\alpha)^{2(m-|\beta|)-1}\|\varphi^{m-|\beta|-\frac{1}{2}}e^{\tau\varphi}D_x^\beta u\|_{L^2}^2 \leq C\|e^{\tau\varphi}Pu\|_{L^2}^2,$$

for all $u \in \mathscr{C}_c^\infty(X)$, $\alpha \geq \alpha_1$, and $\tau \geq \tau_1$.

We observe that we have gained an additional term, for $|\beta| = m$, in the sum on the l.h.s. in contrast to the Carleman estimate of Theorem 3.8.

Remark 4.10. From an estimate of the form of (4.7) by fixing the values of τ and α we obtain

$$\|u\|_{H^m(\mathbb{R}^n)} = \sum_{|\beta| \le m} \|D_x^\beta u\|_{L^2} \le C\|Pu\|_{L^2},$$

which implies that P is elliptic. The additional term we have obtained in the previous theorem is thus a privilege of elliptic operators.

Remark 4.11. Note that ellipticity is not sufficient for a Carleman estimate of the form (4.7) to hold. Pseudo-convexity and strong pseudo-convexity are also needed (see Section 5 for the necessity of these conditions). The bi-Laplace operator is a typical example of elliptic operators for which this type of estimate cannot be achieved. See (1.7), Proposition 1.1, (1.13) and Proposition 1.4, in the introductory section for the weaker estimate that can be obtained for the bi-Laplace operator.

Proof. The proof goes along that of Theorem 3.8. We only point out differences. For the symbol ρ obtained in (3.11) we have $\rho - \mu^2/(\tau\alpha\varphi) \ge 0$ by Proposition 4.8. The counterpart to (3.12) is then

$$\|Qw\|_{L^2}^2 \ge (\mu^2/(\tau\alpha\varphi)w, w)_{L^2} - C\|\alpha w\|_{L^2}^2 + (\mathrm{Op}^{\mathrm{w}}(r)w, w)_{L^2}.$$

By Proposition 2.8 we have $(\mu^2/(\tau\alpha\varphi)w, w)_{L^2} \gtrsim \|w\|_{1,-\frac{1}{2}}^2$. Since $\|w\|_{1,-\frac{1}{2}} \gtrsim \|w\|_{0,\frac{1}{2}} \gtrsim \|\varphi^{\frac{1}{2}}\alpha^{\frac{1}{2}}w\|_{L^2}$, we obtain

$$\|Qw\|_{L^2}^2 \ge C\|w\|_{1,-\frac{1}{2}}^2 + (\mathrm{Op}^{\mathrm{w}}(r)w, w)_{L^2}.$$

for α sufficiently large. The conclusion of the proof is then as in the proof of Theorem 3.8. ∎

Remark 4.12. Readers may not be at ease with the use of the Fefferman-Phong inequality as in the proof of Theorem 3.8 for the derivation of a Carleman estimate for an elliptic operators. Indeed, the original proof of L. Hörmander in [Hör63, Section 8.3], for Carleman estimates with one large paramter for elliptic operators, only relies on integrations by parts. We thus provide in Appendix B.12 an alternative proof of Theorem 4.9 that only relies on the *classical* Gårding inequality for homogeneous differential operators. This remark originates from the work [BL13] where similar estimates are derived for elliptic operators at a boundary, typically in a half-space, where strong estimates such as the Fefferman-Phong inequality or the sharp Gårding inequality are either not available or require severe additional assumptions on the symbol of the operator.

Remark 4.13. In Section 4.2.1 we have considered elliptic operators under the strong pseudoconvexity condition. In the next section we shall obtain an improved result in the case of elliptic operators satisfying the simple-characteristic property. It is natural to question the existence of elliptic operators (and weight functions) not fulfilling the simple-characteristic property and yet satisfying the strong pseudo-convexity condition. An example is provided in the following proposition whose proof can be found in Appendix B.13.

Proposition 4.14. *Let* $P = (D_{x_1}^2 + D_{x_2}^2 + D_{x_3}^2)^2 + D_{x_1}D_{x_2}^3 + iD_{x_1}^3 D_{x_2}$ *in* $\mathbb{R}^3$. *The operator* P *is elliptic and, with the weight function* $\psi(x) = x_3 + (x_1^2 + x_2^2)/2$, *it fulfills the strong pseudo-convexity condition at* $x = (0,0,0)$ *and does not satisfy the simple-characteristic property there.*

4.2.2. *Elliptic operators under the simple-characteristic property.* Combining the simple-characteristic and elliptic properties we obtain the following estimate.

Proposition 4.15. *Let* $p(x,\xi)$ *and* $\psi(x)$ *be such that* (4.3) *holds. Assume moreover that* $p(x,\xi)$ *is elliptic. We set* $\varphi = e^{\alpha\psi}$ *and*

$$\zeta = \zeta(x,\xi,\tau) = \xi + i\tau\varphi'(x) = \xi + i\tau\alpha\psi'(x)\varphi(x).$$

Let X *be an open subset such that* $X \Subset \Omega$. *There exist* $C > 0$, $\tau_0 \ge 1$, $\alpha_0 > 0$, $\nu_0 > 0$, *such that we have*

$$(4.8) \qquad \mu^{2m} \le C\Big(\nu|p(x,\zeta)|^2 + \frac{\tau\varphi(x)}{2i}\big\{\overline{p(x,\zeta)}, p(x,\zeta)\big\}\Big), \quad \tau \ge \tau_0, \ \alpha \ge \alpha_0, \ \nu \ge \nu_0, \ (x,\xi) \in \overline{X} \times \mathbb{R}^n.$$

Proof. Let $\hat{\tau} \geq 0$ and consider

$$f(x, \xi, \hat{\tau}) = \nu |p(x, \xi + i\hat{\tau}\psi'(x))|^2 + \hat{\tau}^2 |\langle p'_\xi(x, \xi + i\hat{\tau}\psi'(x)), \psi'(x) \rangle|^2.$$

We place ourselves on the compact set

$$\mathscr{C} = \left\{ (x, \xi, \hat{\tau}); \ \hat{\tau}^2 + |\xi|^2 = 1, \ x \in \overline{X}, \ \xi \in \mathbb{R}^n, \ \hat{\tau} \in \mathbb{R}^+ \right\}.$$

By (4.3) and the ellipticity of p we find

$$\forall (x, \xi, \hat{\tau}) \in \mathscr{C}, \quad p(x, \xi + i\hat{\tau}\psi'(x)) = 0 \ \Rightarrow \ \hat{\tau} \langle p'_\xi(x, \xi + i\hat{\tau}\psi'(x)), \psi'(x) \rangle \neq 0.$$

For ν sufficiently large, we then obtain $f \geq C > 0$ on $\mathscr{C}$. By homogeneity we deduce

(4.9) $$f(x, \xi, \hat{\tau}) \gtrsim \left(\hat{\tau}^2 + |\xi|^2 \right)^m, \quad x \in \overline{X}, \ \xi \in \mathbb{R}^n, \ \hat{\tau} \in \mathbb{R}^+.$$

With (3.8) we have

$$\nu |p(x, \zeta)|^2 + \frac{\tau \varphi(x)}{2i} \left\{ \overline{p(x, \zeta)}, p(x, \zeta) \right\} = \nu |p(x, \zeta)|^2 + \tau \alpha \varphi(x) \tilde{\tau}(x) |\langle p'_\xi(x, \zeta), \psi'(x) \rangle|^2 + \tau \varphi(x) \Theta_{p,\psi}(x, \xi, \tilde{\tau}(x))$$

$$= f(x, \xi, \tilde{\tau}(x)) + \tau \varphi(x) \Theta_{p,\psi}(x, \xi, \tilde{\tau}(x))$$

$$\geq C\mu^{2m} + \tau \varphi(x) \Theta_{p,\psi}(x, \xi, \tilde{\tau}(x)),$$

with $C > 0$. Observing that $\tau \varphi(x) |\Theta_{p,\psi}(x, \xi, \tilde{\tau}(x))| \leq \tau \varphi(x) \mu^{2m-1}(x, \xi, \tau)$ we conclude by choosing α sufficiently large. ∎

With Proposition 4.15 we then obtain the following Carleman estimate.

Theorem 4.16. *Let $P = P(x, D_x)$ be an elliptic differential operator in Ω of order m and $\psi \in \mathscr{C}^\infty(\Omega)$, be such that the simple-characteristic property (4.3) holds. We set $\varphi = e^{\alpha\psi}$. If X is an open subset, $X \Subset \Omega$, there exist $C > 0$, $\alpha_1 > 0$, and $\tau_1 \geq 1$ such that*

$$\alpha \sum_{|\beta| \leq m} (\tau\alpha)^{2(m-|\beta|)-1} \|\varphi^{m-|\beta|-\frac{1}{2}} e^{\tau\varphi} D_x^\beta u\|_{L^2}^2 \leq C \|e^{\tau\varphi} Pu\|_{L^2}^2,$$

for all $u \in \mathscr{C}_c^\infty(X)$, $\alpha \geq \alpha_1$, and $\tau \geq \tau_1$.

We observe that we have gained a factor α and an additional term in the sum on the l.h.s., for $|\beta| = m$, in contrast to the Carleman estimate of Theorem 3.8. Note that elliptic operators of order two with real coefficients fit the framework of this result if ψ' does not vanish in Ω. This was used in the introduction for the Laplace operator (see (1.12)).

Proof. Let Y be an open subset of Ω such that $X \Subset Y \Subset \Omega$. Proposition 4.15 applies in Y.

As in the proof of Theorem 3.8 we restrict ourselves to a small neighborhood V of a point $x_0 \in X$ with $V \Subset Y$. We also choose a small neighborhood W such that $V \Subset W \Subset Y$. The function ψ then satisfies Assumption 2.1. We assume $u \in \mathscr{C}_c^\infty(V)$.

We set $P_\varphi = e^{\tau\varphi} P e^{-\tau\varphi}$ and $v = e^{\tau\varphi} u$. We have $P_\varphi = P(x, D_x + i\tau\varphi'(x)) \in \Psi(\mu^m, g)$. Its principal symbol of given by

$$p_\varphi = p_\varphi(x, \xi, \tau) = p(x, \xi + i\tau\varphi'(x)).$$

We introduce the selfadjoint operators

$$Q_2 = \frac{P_\varphi + P_\varphi^*}{2} \quad \text{and} \quad Q_1 = \frac{P_\varphi - P_\varphi^*}{2i},$$

with principal symbols $q_2 = \operatorname{Re} p_\varphi$ and $q_1 = \operatorname{Im} p_\varphi$.

We have $\|P_\varphi v\|_{L^2}^2 = \|Q_2 v\|_{L^2}^2 + \|Q_1 v\|_{L^2}^2 + i\big([Q_2, Q_1]v, v\big)_{L^2}$. With ν such that $\nu(\tau\varphi(x))^{-1} \le 1$ we then write

$$\tau\|P_\varphi v\|_{L^2}^2 \ge \nu\|\varphi^{-\frac{1}{2}} Q_2 v\|_{L^2}^2 + \nu\|\varphi^{-\frac{1}{2}} Q_1 v\|_{L^2}^2 + i\tau\big([Q_2, Q_1]v, v\big)_{L^2}$$

$$= \Big(\underbrace{\big(\nu(Q_2\varphi^{-1}Q_2 + Q_1\varphi^{-1}Q_1) + i\tau[Q_2, Q_1]\big)}_{=B \in \Psi(\varphi^{-1}\mu^{2m}, g)} v, v \Big)_{L^2}.$$

By Lemma 2.4, there exists $w \in L^2(\mathbb{R}^n)$ such that $v = \mathrm{Op}^{\mathrm{w}}(\varphi^{\frac{1}{2}}\mu^{\frac{1}{2}-m})w$. We then have

$$\tau\|P_\varphi v\|_{L^2}^2 \ge \Big(\underbrace{\mathrm{Op}^{\mathrm{w}}(\varphi^{\frac{1}{2}}\mu^{\frac{1}{2}-m}) \, B \, \mathrm{Op}^{\mathrm{w}}(\varphi^{\frac{1}{2}}\mu^{\frac{1}{2}-m})}_{=\tilde{B} \in \Psi(\mu, g)} w, w \Big)_{L^2}.$$

From pseudo-differential calculus (see (A.2)) the principal symbol of $\tilde{B}$ is

$$\tilde{b} = \mu^{1-2m}\big(\nu|q_2|^2 + \nu|q_1|^2 + \tau\varphi\{q_2, q_1\}\big) = \mu^{1-2m}\big(\nu|p_\varphi|^2 + \frac{\tau\varphi}{2i}\{\overline{p_\varphi}, p_\varphi\}\big).$$

We thus have $\tilde{b} - C\mu \ge 0$ in W by Proposition 4.15. Let $\chi \in \mathscr{C}_c^\infty(W)$ such that, $0 \le \chi \le 1$ and $\chi = 1$ in a neighborhood of V. We then write

$$\tilde{b} = b + r, \qquad b = \tilde{b}\chi + \mu(1 - \chi), \quad r = (\tilde{b} - \mu)(1 - \chi).$$

Then $b \in S(\mu, g)$ and $b - C\mu \ge 0$ in the whole phase-space. Recalling that $\lambda_g = \mu/\alpha$, the sharp Gårding inequality (see Theorem A.3) then yields,

$$\tau\|P_\varphi v\|_{L^2}^2 \ge (\mathrm{Op}^{\mathrm{w}}(\mu)w, w)_{L^2} - C\|\alpha^{\frac{1}{2}}w\|_{L^2}^2 + (\mathrm{Op}^{\mathrm{w}}(r)w, w)_{L^2}.$$

By Proposition 2.8 we then have

$$\tau\|P_\varphi v\|_{L^2}^2 \ge C\|w\|_{\frac{1}{2},0}^2 - C'\|\alpha^{\frac{1}{2}}w\|_{L^2}^2 + (\mathrm{Op}^{\mathrm{w}}(r)w, w)_{L^2}.$$

and, as $\varphi = e^{\alpha\psi}$ with $\psi \ge C > 0$ on W, for α sufficiently large we obtain

$$\tau\|P_\varphi v\|_{L^2}^2 \ge C\|w\|_{\frac{1}{2},0}^2 + (\mathrm{Op}^{\mathrm{w}}(r)w, w)_{L^2}.$$

With Lemmata 2.6 and 3.9 we then find

$$\|P_\varphi v\|_{L^2}^2 \gtrsim \|(\tau\varphi)^{-\frac{1}{2}}v\|_{m,0}^2.$$

The conclusion of the proof is then as in the proof of Theorem 3.8.   ■

Remark 4.17. In connection to Remark 4.12, an alternative proof of Theorem 4.16 can be written without relying on the sharp Gårding inequality but rather the *classical* Gårding inequality for homogeneous differential operators. The argument resembles that of the proof of Theorem 4.9 given in Appendix B.12. The adaptation to the case of operators satisfying the simple characteristic property is left to the reader (see also [BL13]).

5. Necessary conditions on the weight function and optimality

Starting from Carleman estimates L. Hörmander derived necessary conditions on the weight function [Hör63, Hör85a]. We apply the same approach in the case of Carleman estimates with two large parameters and weight functions of the form $\varphi = e^{\alpha\psi}$.

Lemma 5.1. *Let P be a differential operator of order m with smooth principal symbol $p(x, \xi)$ and let $\psi \in \mathscr{C}^\infty(\Omega)$. Let X be an open subset of Ω.*

(1) *If the following estimate holds,*

$$(5.1) \qquad \sum_{|\beta|<m} \left(\tau\alpha\right)^{2(m-|\beta|)-1} \|\varphi^{m-|\beta|-\frac{1}{2}} e^{\tau\varphi} D_x^\beta u\|_{L^2}^2 \leq K \|e^{\tau\varphi} Pu\|_{L^2}^2, \quad \varphi = e^{\alpha\psi},$$

for $\tau \geq \tau_0 > 0$ and $\alpha \geq \alpha_0 > 0$ and $u \in \mathscr{C}_c^\infty(X)$, we then have

$$(5.2) \qquad \sum_{|\beta|<m} \left(\tau\alpha\varphi(x)\right)^{2(m-|\beta|)-1} |\zeta^\beta|^2 \leq \frac{K}{i}\{\overline{p}(x,\overline{\zeta}), p(x,\zeta)\}, \quad \zeta = \xi + i\tau\varphi'(x) = \xi + i\tau\alpha\psi'(x)\varphi(x),$$

for $\tau > 0$ and $\alpha \geq \alpha_0$ and $(x,\xi) \in T^*(X)$, if $p(x,\zeta) = 0$. If $m \geq 2$ we have $\psi' \neq 0$ in X. Moreover $p(x,\xi)$ does not vanish at second order at any point of $T^*(X) \setminus 0$.

(2) *If the sum in (5.1) is replaced by $\sum_{|\beta|\leq m}$ (resp. $\alpha^\gamma \sum_{|\beta|<m}$ or $\alpha^\gamma \sum_{|\beta|\leq m}$ with $\gamma > 0$) then the same is true for (5.2).*

This lemma is the counterpart with two large parameters of Theorem 28.2.1 in [Hör85a]. The proof is along the same lines and we refer to Appendix B.14 for it.

In Lemma 5.1 the case of first-order operators stands out. Then, in fact, the Carleman estimate (5.1) does not imply $\psi' \neq 0$ in Ω. This illustrated by the following result.

Proposition 5.2. *Let $P = D_{x_1}$ and let $\psi \in \mathscr{C}^\infty(\Omega)$ be such that $\partial_{x_1}^2 \psi \geq C_0 > 0$ in Ω. Let X be an open subset such that $X \Subset \Omega$. We then have*

$$\|e^{\tau\varphi} Pu\|_{L^2}^2 \geq 2\tau\alpha C_0 \|\varphi^{\frac{1}{2}} e^{\tau\varphi} u\|_{L^2}^2, \quad \varphi = e^{\alpha\psi}, \quad u \in \mathscr{C}_c^\infty(X), \quad \tau > 0, \quad \alpha > 0.$$

A possible choice for the function ψ is $\psi = x_1^2$ whose gradient vanishes at $x_1 = 0$.

Proof. With $v = ue^{\tau\varphi}$ we write

$$\|e^{\tau\varphi} Pu\|_{L^2}^2 = \|D_{x_1}v + i\tau v \partial_{x_1}\varphi\|_{L^2}^2 = \|D_{x_1}v\|_{L^2}^2 + \|\tau v \partial_{x_1}\varphi\|_{L^2}^2 - 2\tau \operatorname{Re} \int_{\mathbb{R}^n} (\partial_{x_1}\varphi)(\partial_{x_1}v)\overline{v}dx$$

$$= \|D_{x_1}v - i\tau v \partial_{x_1}\varphi\|_{L^2}^2 + 2\tau \int_{\mathbb{R}^n} \partial_{x_1}^2 \varphi |v|^2 dx \geq 2\tau \int_{\mathbb{R}^n} \partial_{x_1}^2 \varphi |v|^2 dx$$

with an integration by parts. We have $\partial_{x_1}^2 \varphi = \alpha^2(\partial_{x_1}\psi)^2\varphi + \alpha\partial_{x_1}^2\psi\varphi \geq \alpha C_0\varphi$, which yields the result. ∎

Remark 5.3. In the light of Section 4.1 and Theorem 4.6 one however hopes to have stronger estimates than (5.1) for first-order operators. We shall prove below that such stronger estimates then imply that $\psi' \neq 0$, regardless of the operator order.

Note in particular that the proof of Proposition 5.2 yields an estimate of the form given in Theorem 4.6, i.e., with an additional factor α if we further assume that $\partial_{x_1}\psi \neq 0$ in Ω.

Lemma 5.4. *Let P be a principally normal differential operator of order m with smooth principal symbol $p(x,\xi)$ and let $\psi \in \mathscr{C}^\infty(\Omega)$ be such that $\psi'(x) \neq 0$. We set $\varphi = e^{\alpha\psi}$. If (5.1) holds then*

$$(5.3) \qquad \sum_{|\beta|=m-1} \alpha\varphi(x)|\xi^\beta|^2 \leq 2K \operatorname{Re}\{\overline{p}, \{p, \varphi\}\}(x,\xi),$$

for $\alpha \geq \alpha_0$ if $(x,\xi) \in T^*(X)$, $p(x,\xi) = 0$ and $\{p,\varphi\}(x,\xi) = 0$. If the sum in (5.1) is replaced by $\alpha \sum_{|\beta|<m}$ then the sum in (5.3) is replaced by $\alpha \sum_{|\beta|=m-1}$.

The proof of Lemma 5.4 can be adapted from that of Theorem 28.2.1' in [Hör85a].

We next prove that the strong pseudo-convexity condition on the function ψ is in fact *necessary* for the Carleman estimate (5.1) to hold.

Theorem 5.5. *Assume that estimate (5.1) holds for all open subsets $X \Subset \Omega$ and assume further that P is principally normal. In the case of a first-order operator, we also suppose that $\psi' \neq 0$ in Ω. Then the function ψ is strongly pseudo-convex w.r.t. P and Ω.*

Along with Theorem 3.8 we then obtain the following result.

Corollary 5.6. *Let P be a principally normal operator. The strong pseudo-convexity of ψ in Ω is* necessary *and* sufficient *for the Carleman estimate* (5.1) *to hold for every open subset $X \Subset \Omega$ (with the additional assumption that $\psi' \neq 0$ in Ω in the case of a first-order operator).*

Remark 5.7. Note that this result is in contrast with L. Hörmander's work where there is a gap between the necessary and the sufficient conditions on the weight function φ to have a Carleman estimate (compare Theorem 28.2.1 and Theorem 28.2.3 and the connection that is made with pseudo-convexity in Section 28.3 in [Hör85a]). Here, of course we impose a particular structure on the weight function φ, viz. $\varphi = e^{\alpha\psi}$.

Proof of Theorem 5.5. We have $\psi' \neq 0$ in Ω (this is assumed if $m = 1$ or this follows from Lemma 5.1 if $m > 1$). We first prove that (Ψc) holds everywhere. Let $x \in \Omega$ and $\xi \in \mathbb{R}^n \setminus 0$ such that $p(x, \xi) = 0$ and $\{p, \psi\}(x, \xi) = \langle p'_\xi(x, \xi), \psi'(x) \rangle = 0$. Then $\{p, \varphi\}(x, \xi) = 0$. Lemma 5.4 then yields $\theta_{p,\varphi}(x, \xi) = \mathrm{Re}\{\overline{p}, \{p, \varphi\}\} > 0$. By (3.7) we have here

$$\mathrm{Re}\{\overline{p}, \{p, \psi\}\} = \theta_{p,\psi}(x, \xi) = (\alpha\varphi)^{-1}\theta_{p,\varphi}(x, \xi) > 0.$$

We next prove that (s-Ψc) holds everywhere. Let $x \in \Omega$, $\xi \in \mathbb{R}^n$ and $\hat{\tau} > 0$ such that

$$p(x, \xi + i\hat{\tau}\psi'(x)) = \langle p'_\xi(x, \xi + i\hat{\tau}\psi'(x)), \psi'(x) \rangle = 0.$$

We then choose $\tau > 0$ and $\alpha > \alpha_0$ such that $\hat{\tau} = \tau\alpha\varphi(x) = \tilde{\tau}(x)$. We set $\zeta = \xi + i\hat{\tau}\psi'(x) = \xi + i\tau\varphi'(x)$, and we have $p(x, \xi + i\tau\varphi'(x)) = 0$. With (5.2) and (3.4)–(3.5) we then have

$$\Theta_{p,\varphi}(x, \xi, \tau) = \hat{\tau}\sum_{j,k} \partial^2_{x_j x_k}\psi(x)\,\partial_{\xi_j}p(x, \zeta)\,\partial_{\xi_k}\overline{p}(x, \overline{\zeta}) + \mathrm{Im}\sum_j \partial_{x_j}p(x, \zeta)\,\partial_{\xi_j}\overline{p}(x, \overline{\zeta}) > 0,$$

i.e., $\Theta_{p,\psi}(x, \xi, \hat{\tau}) = \frac{1}{2i}\{\overline{p}(x, \xi - i\tau\psi'(x)), p(x, \xi + i\tau\psi'(x))\} > 0$ by (3.8) as $\langle p'_\xi(x, \zeta), \psi'(x) \rangle = 0$. ∎

We shall now obtain necessary conditions for a stronger Carleman estimate as in Theorem 4.6 to hold.

Theorem 5.8. *Let P be a principally normal differential operator of order m with smooth principal symbol $p(x, \xi)$ and let $\psi \in \mathscr{C}^\infty(\Omega)$ be such that the following estimate holds, for all open subsets $X \Subset \Omega$,*

$$(5.4) \qquad \alpha^\gamma \sum_{|\beta| < m} (\tau\alpha)^{2(m-|\beta|)-1}\|\varphi^{m-|\beta|-\frac{1}{2}}e^{\tau\varphi}D_x^\beta u\|_{L^2}^2 \leq K\|e^{\tau\varphi}Pu\|_{L^2}^2, \quad \varphi = e^{\alpha\psi},$$

for $\tau \geq \tau_0 > 0$ and $\alpha \geq \alpha_0 > 0$ and $u \in \mathscr{C}_c^\infty(X)$ and $\gamma > 0$. Then the function ψ is such that $\psi'(x) \neq 0$ in Ω and the simple-characteristic property (4.3) *holds in Ω.*

Note that in the case of a first-order operator, the stronger estimate (5.4) implies $\psi' \neq 0$ as opposed to the "regular" Carleman estimate with two large parameters (5.1). See also Proposition 5.2 and Remark 5.3.

With Theorem 4.6 we have thus obtained the following result.

Corollary 5.9. *Let P be a principally normal operator and ψ be a weight function. The simple-characteristic property* (4.3) *is* necessary *and* sufficient *for the Carleman estimate of Theorem 4.6 to hold for every open subset $X \Subset \Omega$.*

Proof of Theorem 5.8. Assume that $\psi'(x) = 0$ with $x \in \Omega$. Choosing $\xi = 0$, i.e., $\zeta = 0$ we have $p(x, \zeta) = 0$ and thus Lemma 5.1 yields

$$\alpha^\gamma \tilde{\tau}(x)^{2m-1} \lesssim \Theta_{p,\psi}(x, 0, \tilde{\tau}(x)) \lesssim \tilde{\tau}(x)^{2m-1}, \qquad \tilde{\tau}(x) = \tau\alpha\varphi(x)$$

by (3.5) and (3.8) for $\alpha \geq \alpha_0$ and $\tau > 0$, which is clearly impossible if $\gamma > 0$.

First, we prove that property (4.3) holds for $\hat{\tau} > 0$. Let $x \in \Omega$ and $\xi \in \mathbb{R}^n$ and $\hat{\tau} > 0$ be such that $\rho_{x,\xi}(\hat{\tau}) = p(x, \xi + i\hat{\tau}\psi'(x)) = 0$. For $\alpha \geq \alpha_0$ we set $\tau = \tau(\alpha) = \hat{\tau}/(\alpha\varphi(x))$. We then have $\hat{\tau} = \tilde{\tau}(x)$. Note that τ goes to zero as α goes to ∞. Yet $\hat{\tau}$ remains fixed. By Lemma 5.1 and (3.5) and (3.8) we have

$$\alpha^\gamma \hat{\tau}^{2m-1} \lesssim \Theta_{p,\psi}(x, \xi, \hat{\tau}) + \alpha\hat{\tau}|\langle p'_\xi(x, \xi + i\hat{\tau}\psi'(x)), \psi'(x) \rangle|^2.$$

If $\rho'_{x,\xi}(\hat{\tau}) = i\langle p'_\xi(x, \xi + i\hat{\tau}\psi'(x)), \psi'(x)\rangle = 0$ we reach the same contradiction as above,

$$\alpha^\gamma \hat{\tau}^{2m-1} \lesssim \Theta_{p,\psi}(x, \xi, \hat{\tau}),$$

since here the value of $\Theta_{p,\psi}(x, \xi, \hat{\tau})$ is kept fixed and α is free to increase. The root $\hat{\tau}$ is thus simple.

Second, we consider the case $\hat{\tau} = 0$. Let $x \in \Omega$ and $\xi \in \mathbb{R}^n$ be such that $p(x, \xi) = 0$ and let us assume that $\rho'_{x,\xi}(0) = i\langle p'_\xi(x, \xi), \psi'(x)\rangle = 0$. Then $\{p, \varphi(x)\} = 0$ and with Lemma 5.4 we obtain

$$\alpha^{1+\gamma}\varphi(x) \sum_{|\beta|=m-1} |\xi^\beta|^2 \le 2K \operatorname{Re}\{\overline{p}, \{p, \varphi\}\}(x, \xi).$$

With (3.2) and (3.7) we get $\alpha^{1+\gamma}\varphi(x) \sum_{|\beta|=m-1} |\xi^\beta|^2 \lesssim \alpha\varphi\theta_{p,\psi}(x, \xi)$, yielding as above a contradiction if we let α go to ∞, if $\xi \ne 0$. This concludes the proof. ∎

We finish this section with some consideration regarding the optimality of the powers in α. As a direct consequence of Theorem 5.8, we have the following proposition.

Corollary 5.10. *If the simple-characteristic property does not hold one can only hope for an estimate of the form (5.4) with $\gamma = 0$. In such case the powers of α in the Carleman estimate of Theorem 3.8 is optimal.*

The following proposition gives an optimality results when the simple-characteristics property holds.

Proposition 5.11. *Let P be a differential operator of order m with smooth principal symbol $p(x, \xi)$ and let $\psi \in \mathscr{C}^\infty(\Omega)$ be such that the following estimate holds,*

$$(5.5) \qquad \alpha^\gamma \sum_{|\beta|<m} (\tau\alpha)^{2(m-|\beta|)-1} \|\varphi^{m-|\beta|-\frac{1}{2}}e^{\tau\varphi}D_x^\beta u\|_{L^2}^2 \le K\|e^{\tau\varphi}Pu\|_{L^2}^2, \quad \varphi = e^{\alpha\psi},$$

for $\tau \ge \tau_0 > 0$ and $\alpha \ge \alpha_0 > 0$ and $u \in \mathscr{C}_c^\infty(\Omega)$ and $\gamma \ge 1$. If there exists $(x, \xi, \tau) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^+$ such that $p(x, \xi + i\hat{\tau}\psi'(x)) = 0$ with $\hat{\tau} \ne 0$, then we have $\gamma = 1$.

Note that the Laplace operator, in dimension greater than or equal to two, fits the assumption of this proposition with ψ such that ψ' does not vanish in Ω. This (further) justifies the result of Proposition 1.3 in the introductory section. If $P = -\Delta$ then $p(x, \xi + i\hat{\tau}\psi'(x)) = 0$ if $\xi \perp \psi'(x)$ and $|\xi| = |\hat{\tau}\psi'(x)|$.

Proof. The proof is contained in that of Theorem 5.8. If $\rho_{x,\xi}(\hat{\tau}) = p(x, \xi + i\hat{\tau}\psi'(x)) = 0$ we have

$$\alpha^\gamma \hat{\tau}^{2m-1} \lesssim \Theta_{p,\psi}(x, \xi, \hat{\tau}) + \alpha\hat{\tau}|\langle p'_\xi(x, \xi + i\hat{\tau}\psi'(x)), \psi'(x)\rangle|^2,$$

where $\alpha > \alpha_0$. This implies $\gamma \le 1$. ∎

Remark 5.12. Note that the proofs of Theorem 5.8 and Proposition 5.11 show that the powers of τ are also optimal under the assumptions of these two results.

In Proposition 5.11 the condition on the existence of a non-zero root for $\rho_{x,\xi}(\hat{\tau}) = p(x, \xi + i\hat{\tau}\psi'(x)) = 0$ cannot be avoided, as explained by the following examples.

Example 5.13. We place ourselves in $\mathbb{R}^n$ with $x = (x', x_n)$. For the operator $P = D_{x_n}$ and $\varphi(x) = e^{\alpha\psi(x)}$ with $\psi(x) = x_n + f(x')$, we have $p(x, \xi + i\tau\psi'(x)) = \xi_n + i\tau$ that vanishes if and only if $\xi_n = 0$ and $\tau = 0$. Here $\xi = (\xi', \xi_n)$ and ξ' can take any value. Note that the simple-characteristic property (4.3) is fulfilled. However we have a stronger estimate than that of Theorem 4.6:

$$(5.6) \qquad (\alpha\tau)^2 \|\varphi e^{\tau\varphi}u\|_{L^2}^2 \le \|e^{\tau\varphi}Pu\|_{L^2}^2,$$

for $\tau > 0$ and $\alpha > 0$ and $u \in \mathscr{C}_c^\infty(\mathbb{R}^n)$. More generally, if $j \ge 0$ we have

$$(\alpha\tau)^2 \|\varphi^{(j+1)/2}e^{\tau\varphi}u\|_{L^2}^2 \le \|\varphi^{(j-1)/2}e^{\tau\varphi}Pu\|_{L^2}^2.$$

Note that this is not an ellipticity result as D_{x_n} is not elliptic in $\mathbb{R}^n$. In fact an estimate of the form (5.5) with $\gamma = 1$ would be

$$(5.7) \qquad \tau\alpha^2\|\varphi^{\frac{1}{2}}e^{\tau\varphi}u\|_{L^2}^2 \lesssim \|e^{\tau\varphi}Pu\|_{L^2}^2.$$

As we have $\alpha^{2N} \lesssim \varphi$, for any $N \in \mathbb{N}$, since $\psi \geq C > 0$, from (5.6) we deduce

$$\tau^2\alpha^N\|\varphi^{\frac{1}{2}}e^{\tau\varphi}u\|_{L^2}^2 \lesssim \|e^{\tau\varphi}Pu\|_{L^2}^2,$$

for any $N \in \mathbb{N}$, which is stronger than (5.7) with respect to the power of α (and τ).

Proof. Here the conjugated operator is $P_\varphi = e^{\tau\varphi}Pe^{-\tau\varphi} = D_{x_n} + i\tau\varphi'_{x_n}(x)$. For $v = e^{\tau\varphi}u$ we compute

$$\operatorname{Re}(P_\varphi v, i\varphi^j v)_{L^2} = i([D_{x_n}, \varphi^j]v, v)_{L^2} + \tau\operatorname{Re}(i\varphi'_{x_n}v, i\varphi^j v)$$
$$= \alpha j\|\varphi^{j/2}v\|_{L^2}^2 + \alpha\tau\|\varphi^{(j+1)/2}v\|_{L^2}^2$$
$$\geq \alpha\tau\|\varphi^{(j+1)/2}v\|_{L^2}^2.$$

We thus find

$$\alpha\tau\|\varphi^{(j+1)/2}v\|_{L^2}^2 + (\alpha\tau)^{-1}\|^{(j-1)/2)}P_\varphi v\|_{L^2}^2 \geq 2\alpha\tau\|\varphi^{(j+1)/2}v\|_{L^2}^2.$$

yielding the result. ∎

Such strong estimates are not limited to the order one. In $\mathbb{R}^2$ we have the following example.

Example 5.14. For the operator $P = D_{x_1}D_{x_2}$ of order two in $\mathbb{R}^2$ and $\varphi(x) = e^{\alpha\psi(x)}$ with $\psi(x) = x_1 + x_2$ the simple-characteristic property is fulfilled (see Examples 4.2 and Appendix B.11).

However, we have the following strong estimate:

$$(\alpha\tau)^4\|\varphi^2e^{\tau\varphi}u\|_{L^2}^2 + \sum_{k=1,2}(\alpha\tau)^2\|\varphi e^{\tau\varphi}D_{x_k}u\|_{L^2}^2 \leq 3\|e^{\tau\varphi}Pu\|_{L^2}^2,$$

for $\tau > 0$ and $\alpha > 0$ and $u \in \mathscr{C}_c^\infty(\mathbb{R}^n)$.

This operator $D_{x_1}D_{x_2}$ is nothing but the wave operator in the particular coordinates $t = x_1 + x_2$ and $y = x_1 - x_2$. The level sets of the weight function correspond to space-like surfaces.

Proof. The result of Example 5.13 gives, for $j \geq 0$ and $k = 1, 2$,

$$(\alpha\tau)^2\|\varphi^{(j+1)/2}e^{\tau\varphi}w\|_{L^2}^2 \leq \|\varphi^{(j-1)/2}e^{\tau\varphi}D_{x_k}w\|_{L^2}^2,$$

for $\tau > 0$ and $\alpha > 0$ and $w \in \mathscr{C}_c^\infty(\mathbb{R}^n)$. If set $w = D_{x_1}u$ or $w = D_{x_2}u$ we thus obtain (for $j = 1$)

$$(\alpha\tau)^2\|\varphi e^{\tau\varphi}D_{x_k}u\|_{L^2}^2 \leq \|e^{\tau\varphi}Pu\|_{L^2}^2, \qquad k = 1, 2.$$

If we set $w = u$ with $j = 3$ and $k = 1$ we have

$$(\alpha\tau)^2\|\varphi^2e^{\tau\varphi}u\|_{L^2}^2 \leq \|\varphi e^{\tau\varphi}D_{x_1}u\|_{L^2}^2.$$

A linear combination of these inequality yields the result. ∎

Remark 5.15. (1) The cascade proof used for the previous example does not yield a proper estimate for the operator $P = D_{x_1}D_{x_2}D_{x_3}$ in $\mathbb{R}^3$ and the weight function $\psi(x) = x_1 + x_2 + x_3$ as one could naively think: one cannot obtain estimates for the weighted norm of $D_{x_j}^2u$, $j = 1, 2, 3$, as should be in a Carleman estimate for a third-order operator. In fact, note that the weight function ψ is not pseudo-convex. If $p(x, \xi) = \{p, \psi\}(x, \xi) = 0$ then $\{p, \{p, \psi\}\} = 0$. However for the wave operator $D_t^2 + \Delta_x$ in $\mathbb{R}^{n+1}$ such a strong estimate can be derived with the weight function $\psi(t, x) = t$.

(2) Note that in *one-dimension* the operator $P = D_x^k$ with $\psi(x) = x$ fulfills the simple characteristic property (4.3) and arguing (by induction) as in the proof of Example 5.14 we find

$$\sum_{j<k} (\alpha\tau)^{2(k-j)} \|\varphi^{k-j} e^{\tau\varphi} D_x^j u\|_{L^2}^2 \le k \|e^{\tau\varphi} Pu\|_{L^2}^2,$$

for $\tau > 0$ and $\alpha > 0$ and $u \in \mathscr{C}_c^\infty(\mathbb{R}^n)$.

APPENDIX A. SOME FACTS ON PSEUDO-DIFFERENTIAL OPERATORS

A.1. **Symbols.** Here, we consider the following metric on phase-space $W = \mathbb{R}^n \times \mathbb{R}^n$:

$$g = \alpha^2 |dx|^2 + \frac{|d\xi|^2}{\mu^2}, \qquad \text{with } \mu^2 = \mu^2(x, \xi, \tau) = \big(\tau\alpha\varphi(x)\big)^2 + |\xi|^2, \text{ and } \tau \ge 1, \ \alpha \ge 1,$$

and τ and α are two parameters.

A positive function $m(X; \tau, \alpha)$, with $X = (x, \xi) \in W$, is called an admissible order function if it is slowing varying and temperate, meaning that we have:

(1) There exist $C > 0$ and $r > 0$ such that

$$\forall X, Y \in W, \quad g_X(X - Y) \le r^2 \quad \Rightarrow \quad C^{-1} \le \frac{m(X; \tau, \alpha)}{m(Y; \tau, \alpha)} \le C.$$

(2) There exist $C > 0$ and $N > 0$ such that

$$\forall X, Y \in W, \quad \frac{m(X; \tau, \alpha)}{m(Y; \tau, \alpha)} \le C\big(1 + g_X^\sigma(X - Y)\big)^N,$$

where the dual metric is given by $g^\sigma = \mu^2 |dx|^2 + \frac{|d\xi|^2}{\alpha^2}$.

We also introduce $h_g^2(X) = \sup g_X/g_X^\sigma = \alpha^2 \mu^{-2}$. Here, the uncertainty principle is fulfilled as we have $0 < h_g \le 1$.

With an admissible order function m we may then define the following symbol class.

Definition A.1. Let $a(x, \xi; \tau, \alpha) \in \mathscr{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{+,*} \times \mathbb{R}^{+,*}; \mathbb{C})$. We say that $a \in S(m, g)$ if

$$\forall \beta, \gamma \in \mathbb{N}^n, \quad \exists C_{\beta,\gamma} > 0, \quad \big|\partial_x^\beta \partial_\xi^\gamma a(x, \xi; \tau, \alpha)\big| \le C_{\beta,\gamma} \alpha^{|\beta|} \mu^{-|\gamma|} m(x, \xi; \tau, \alpha), \quad (x, \xi) \in W, \quad \tau > 1, \ \alpha > 1.$$

We define the principal symbol of a as its equivalent class in $S(m, g)/S(h_g m, g)$.

A.2. **Standard quantization.** With $a \in S(m, g)$ we can define the following pseudo-differential operator

$$\mathrm{Op}(a)u(x) = a(x, D_x)u(x) = (2\pi)^{-n} \iint e^{i\langle x-y, \xi\rangle} a(x, \xi; \tau, \alpha)u(y) \, dy d\xi, \qquad u \in \mathscr{S}(\mathbb{R}^n),$$

in the sense of oscillatory integrals (see e.g. Section 7.8 in [Hör90]).

We have the following composition formula: for m_1 and m_2 two admissible order functions then $m_1 m_2$ is also an admissible order function and if $a \in S(m_1, g)$ and $b \in S(m_2, g)$ we have $\mathrm{Op}(a)\,\mathrm{Op}(b) = \mathrm{Op}(c)$ with $c = a \circ b \in S(m_1 m_2, g)$ with moreover

$$a \circ b(x, \xi; \tau, \alpha) - \sum_{0 \le j < N} \frac{1}{j!} \big(i\langle D_y, D_\eta\rangle\big)^j a(x, \eta)b(y, \xi) \,\big|_{\substack{y=x \\ \eta=\xi}} \in S(\alpha^N \mu^{-N} m_1 m_2, g),$$

for all $N \in \mathbb{N}$.

A.3. **Weyl quantization.** With $a \in S(m, g)$ we can define the following pseudo-differential operator

$$\mathrm{Op}^{\mathrm{w}}(a)u(x) = (2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle} a\big(\frac{x+y}{2}, \xi; \tau, \alpha\big) u(y) \, dy d\xi, \qquad u \in \mathscr{S}(\mathbb{R}^n),$$

in the sense of oscillatory integrals. Connection with the standard quantization is as follows: $\mathrm{Op}^{\mathrm{w}}(a) = \mathrm{Op}(\tilde{a})$ with

$$\tilde{a}(x, \xi) = e^{i\langle D_x, D_\xi\rangle/2} a(x, \xi), \quad \text{or equivalently} \quad a(x, \xi) = e^{-i\langle D_x, D_\xi\rangle/2} \tilde{a}(x, \xi),$$

yielding in particular

$$\tilde{a}(x, \xi) - \sum_{0 \le j < N} \frac{1}{2^j j!} \big(i\langle D_x, D_\xi\rangle\big)^j a(x, \xi) \in S(\alpha^N \mu^{-N} m, g),$$

and

$$a(x, \xi) - \sum_{0 \le j < N} \frac{1}{2^j j!} \big(-i\langle D_x, D_\xi\rangle\big)^j \tilde{a}(x, \xi) \in S(\alpha^N \mu^{-N} m, g).$$

In particular, the principal symbols coincide in both quantizations.

We have the following composition formula: for m_1 and m_2 two admissible order functions, with $a \in S(m_1, g)$ and $b \in S(m_2, g)$, by Theorem 18.5.4 in [Hör85b] we have $\mathrm{Op}^{\mathrm{w}}(a)\mathrm{Op}^{\mathrm{w}}(b) = \mathrm{Op}^{\mathrm{w}}(c)$ with $c = a \sharp b \in S(m_1 m_2, g)$ and

$$(A.1) \qquad a \sharp b(x, \xi) - \sum_{0 \le j < N} \frac{1}{j!} \Big(i\sigma\big(D_x, D_\xi; D_y, D_\eta\big)/2\Big)^j a(x, \xi) b(y, \eta)\big|_{\substack{y=x \\ \eta=\xi}} \in S(\alpha^N \mu^{-N} \rho\rho', g), \quad N \in \mathbb{N},$$

where σ is the symplectic form: $\sigma(x, \xi; y, \eta) = \langle\xi, y\rangle - \langle x, \eta\rangle$.

With this formula we find

$$(A.2) \qquad a \sharp b(x, \xi) = ab(x, \xi) - \frac{i}{2}\{a, b\}(x, \xi) \mod S(\alpha^2 \mu^{-2} \rho\rho', g),$$

and it follows that

$$(A.3) \qquad b \sharp a \sharp b(x, \xi) = ab^2(x, \xi) \mod S(\alpha^2 \mu^{-2} \rho\rho'^2, g),$$

From (A.1) we also find

$$(A.4) \qquad \big(a \sharp b - b \sharp a\big)(x, \xi) = -i\{a, b\}(x, \xi) \mod S(\alpha^3 \mu^{-3} \rho\rho', g).$$

Concerning the boundedness of the pseudo-differential operators we have just defined we have the following result (See Theorem 18.6.3 in [Hör85b]).

Theorem A.2. *Let $a \in S(1, g)$. Then $\mathrm{Op}^{\mathrm{w}}(a)$ maps $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ continuously.*

In the main text, we shall invoke two important inequalities, viz., the sharp Gårding inequality and the Fefferman-Phong inequality. With $h_g^2 = \sup g_X/g^\sigma(X) = \alpha\mu^{-1}$ here, the statements read as follows.

Theorem A.3 (Sharp Gårding inequality)**.** *Let $a \in S(h_g^{-1}, g)$ be such that $a \ge 0$. Then*

$$(\mathrm{Op}^{\mathrm{w}}(a)u, u) \ge -C\|u\|_{L^2}^2, \qquad u \in \mathscr{S}(\mathbb{R}^n).$$

Theorem A.4 (Fefferman-Phong inequality)**.** *Let $a \in S(h_g^{-2}, g)$ be such that $a \ge 0$. Then*

$$(\mathrm{Op}^{\mathrm{w}}(a)u, u) \ge -C\|u\|_{L^2}^2, \qquad u \in \mathscr{S}(\mathbb{R}^n).$$

We refer to Theorems 18.6.7 and 18.6.8 in [Hör85b] for proofs. The Fefferman-Phong inequality is obviously stronger than the sharp Gårding inequality. Yet, as can be easily seen, for the sharp Gårding inequality only the principal symbol needs to be nonnegative. This is not the case for the Fefferman-Phong inequality. Moreover the sharp Gårding inequality extends to the case of systems, which does not hold for the second inequality [Bru90]. We also refer to [Bon99] for refined statements of these inequalities (see also [Ler10].

For a presentation of the Weyl-Hörmander calculus we refer to [Ler10], [Hör85b, Sections 18.4–6] and [Hör79].

APPENDIX B. PROOFS OF SOME INTERMEDIATE TECHNICAL RESULTS

B.1. **Useful computations for the proofs of necessary conditions and optimality.** Let $\psi \in \mathscr{C}^\infty(\Omega)$ and $\varphi(x) = e^{\alpha\psi(x)}$. Let $x_0 \in \Omega$ and $\xi_0 \in \mathbb{R}^n$. Without any loss of generality we shall assume that $x_0 = 0$. We set $\zeta_0 = \xi_0 + i\varphi'(x_0)$ and $\phi(x) = \varphi(x) - \varphi(x_0)$.

We then introduce $w(x) = \langle x, \zeta_0 \rangle$. We note that

(B.1)
$$\phi(x) - \operatorname{Im} w(x) = G(x) + |x|^3 \mathscr{O}_\alpha(1),$$

with

(B.2)
$$G(x) = \frac{1}{2} \sum_{j,k} \partial^2_{x_j x_k} \varphi(x_0) x_j x_k.$$

We then pick $f \in \mathscr{C}_c^\infty(\mathbb{R}^n)$, $f \not\equiv 0$, and set

(B.3)
$$u_\tau(x) = e^{i\tau w(x)} f\big((\lambda\tau)^{\frac{1}{2}} x\big),$$

For applications we shall make various choices for λ. It will either be constant or a function of α and x_0.

For $p \in \mathbb{R}$, we then write

$$\|\varphi^p e^{\tau\phi} u_\tau\|_{L^2}^2 = \int_{\mathbb{R}^n} e^{2\tau\big(G(x)+|x|^3\mathscr{O}_\alpha(1)\big)} \varphi(x)^{2p} \big|f\big((\lambda\tau)^{\frac{1}{2}}x\big)\big|^2 dx$$

$$= (\lambda\tau)^{-n/2} \int_{\mathbb{R}^n} e^{2\lambda^{-1}G(y)+\tau^{-\frac{1}{2}}\lambda^{-3/2}|y|^3\mathscr{O}_\alpha(1))} \varphi\big((\lambda\tau)^{-1/2}y\big)^{2p} |f(y)|^2 dy$$

(B.4)
$$\underset{\tau\to\infty}{\sim} \tau^{-n/2} I_p,$$

with the change of variables $y = (\lambda\tau)^{\frac{1}{2}} x$ and the dominated convergence theorem, and where

$$I_p = \lambda^{-n/2} \varphi(x_0)^{2p} \int_{\mathbb{R}^n} e^{2\lambda^{-1}G(y)} \big|f(y)\big|^2 dy.$$

We note that

$$e^{-i\tau w(x)} D_x^\beta u_\tau = (D_x + \tau\zeta_0)^\beta f\big((\lambda\tau)^{\frac{1}{2}}x\big) = \tau^{|\beta|}\zeta_0^\beta f\big((\lambda\tau)^{\frac{1}{2}}x\big) + \tau^{|\beta|-\frac{1}{2}} \mathscr{O}_\lambda(1),$$

which gives, arguing as above,

(B.5)
$$\|\varphi^p e^{\tau\phi} D_x^\beta u_\tau\|_{L^2}^2 \underset{\tau\to\infty}{\sim} \tau^{2|\beta|-n/2} I_{p,\beta},$$

with

$$I_{p,\beta} = \lambda^{-n/2} \varphi(x_0)^{2p} |\zeta_0^\beta|^2 \int_{\mathbb{R}^n} e^{2\lambda^{-1}G(y)} \big|f(y)\big|^2 dy.$$

B.2. **Proof of Proposition 1.1.** We pick $x_0 \in X$ and $\xi_0 \in \mathbb{R}^n$ such that $|\xi_0| = |\varphi'(x_0)|$ and $\langle\xi_0, \varphi'(x_0)\rangle = 0$, which is possible as $n \geq 2$, and we set $\zeta_0 = \xi_0 + i\varphi'(x_0)$ and $w(x) = \langle x, \zeta_0 \rangle$. We set $\phi(x) = \varphi(x) - \varphi(x_0)$. Observe then that $\nabla_x w = \zeta_0$ is constant and that we have

(B.6)
$$\sum_j (\partial_{x_j} w)^2 = 0.$$

We may assume that $x_0 = 0$ without any loss of generality. We then pick a test function u_τ as in (B.3). For τ sufficiently large $\operatorname{supp}(u_\tau)$ is sufficiently close to x_0 to apply estimate (1.9).

With (B.4) in Appendix B.1 (in the case $\lambda = 1$) we find

(B.7)
$$\tau^{\gamma_0} \|e^{\tau\varphi} u_\tau\|_{L^2}^2 \underset{\tau\to\infty}{\sim} \tau^{\gamma_0-\frac{n}{2}} \int_{\mathbb{R}^n} e^{2G(y)} |\phi(y)|^2 dy.$$

With (B.6) we obtain $\Delta u_\tau = \left(\tau\Delta\phi\big(\tau^{\frac{1}{2}}x\big) + 2i\tau^{\frac{3}{2}}\langle\nabla_x w, \nabla_x\phi\big(\tau^{\frac{1}{2}}x\big)\rangle\right)e^{i\tau w(x)}$ and

$$\Delta^2 u_\tau = \left(\tau^2\Delta^2\phi\big(\tau^{\frac{1}{2}}x\big) + 4i\tau^{\frac{5}{2}}\langle\nabla_x w, \nabla_x\Delta\phi\big(\tau^{\frac{1}{2}}x\big)\rangle - 4\tau^3\phi''\big(\tau^{\frac{1}{2}}x\big)(\nabla_x w, \nabla_x w)\right)e^{i\tau w(x)},$$

With (B.1) we obtain

$$\|e^{\tau\varphi}Bu_\tau\|_{L^2}^2 = \int_{\mathbb{R}^n} e^{2\tau(G(x)+|x|^3\mathscr{O}_\alpha(1))}\left|\tau^2\Delta^2\phi\big(\tau^{\frac{1}{2}}x\big) + 4i\tau^{\frac{5}{2}}\langle\nabla_x w, \nabla_x\Delta\phi\big(\tau^{\frac{1}{2}}x\big)\rangle - 4\tau^3\phi''\big(\tau^{\frac{1}{2}}x\big)(\nabla_x w, \nabla_x w)\right|^2 dx.$$

As $\gamma_j \geq 6$, with estimate (1.9), (B.7), the change of variables $y = \tau^{\frac{1}{2}}x$, and the dominated convergence theorem, we find that $\phi''(\nabla_x w, \nabla_x w)$ does not vanish identically and

$$(B.8)\qquad \|e^{\tau\varphi}\Delta^2 u_\tau\|_{L^2}^2 \underset{\tau\to\infty}{\sim} 16\tau^{6-\frac{n}{2}}\int_{\mathbb{R}^n} e^{2G(y)}\big|\phi''(y)(\nabla_x w, \nabla_x w)\big|^2 dy.$$

This implies that $\gamma_0 = 6$ and $\nabla_x w \neq 0$. In particular $\varphi'(x_0) \neq 0$ and $\xi_0 \neq 0$.

With (B.5) we find

$$(B.9)\qquad \sum_{j=0}^4 \sum_{|\beta|=j} \tau^{\gamma_j-2j}\|e^{\tau\varphi}D_x^\beta u_\tau\|_{L^2}^2 \underset{\tau\to\infty}{\sim} \sum_{j=0}^4 \sum_{|\beta|=j} \tau^{\gamma_j-\frac{n}{2}}|\zeta_0^\beta| \int_{\mathbb{R}^n} e^{2G(y)}|\phi(y)|^2 dy,$$

which by estimate (1.9) implies $\gamma_j = 6$, $j = 1, 2, 3, 4$.   ∎

B.3. **Proofs of Propositions 1.3 and 1.4.** We start with the proof of Proposition 1.3. We choose $x_0 \in X$ and $\xi_0 \in \mathbb{R}^n$ and define $w(x)$ as in the proof of Proposition 1.1. We may assume that $x_0 = 0$ without any loss of generality. We then pick $f \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ and set

$$u_\tau(x) = e^{i\tau w(x)}f\big((\lambda\tau)^{\frac{1}{2}}x\big), \quad \lambda = \alpha^2\varphi(x_0) = \alpha^2 e^{\alpha\psi(x_0)},$$

For τ sufficiently large $\text{supp}(u_\tau)$ is sufficiently close to x_0 to apply estimate (1.15). This test function differs from that in proof of Proposition 1.1 with $\lambda\tau$ replacing τ.

With (B.4) we find

$$\tau^3\alpha^{3+\gamma}\|\varphi^{\frac{3}{2}}e^{\tau\varphi}u_\tau\|_{L^2}^2 \underset{\tau\to\infty}{\sim} \tau^{3-n/2}I_\alpha, \qquad \text{with} \qquad I_\alpha = \alpha^{3+\gamma}\varphi(x_0)^3\lambda^{-n/2}\int_{\mathbb{R}^n} e^{2\lambda^{-1}G(y)}\big|\phi(y)\big|^2 dy.$$

We then find

$$(B.10)\qquad I_\alpha \underset{\alpha\to\infty}{\sim} \alpha^{3+\gamma-n}\varphi(x_0)^{3-n/2}\|e^{\langle y,\psi'(x_0)\rangle^2}\phi(y)\|_{L^2}^2 \neq 0,$$

as

$$G(x) = \frac{1}{2}\sum_{j,k}\partial^2_{x_j x_k}\varphi(x_0)x_j x_k = \frac{1}{2}\sum_{j,k}\big(\alpha^2\partial_{x_j}\psi(x_0)\partial_{x_k}\psi(x_0) + \alpha\partial^2_{x_j x_k}\psi(x_0)\big)\varphi(x_0)x_j x_k.$$

With (B.6) we obtain $\Delta u_\tau = \left(\lambda\Delta\phi\big(\lambda^{\frac{1}{2}}x\big) + 2i\tau\lambda^{\frac{1}{2}}\langle\nabla_x w, \nabla_x\phi\big(\lambda^{\frac{1}{2}}x\big)\rangle\right)e^{i\tau w(x)}$, yielding arguing as above

$$\|e^{\tau\varphi}\Delta u_\tau\|_{L^2}^2 \underset{\tau\to\infty}{\sim} \tau^{3-n/2}J_\alpha, \qquad \text{with} \qquad J_\alpha = 4\alpha^{2-n}\varphi(x_0)^{1-n/2}\int_{\mathbb{R}^n} e^{2\lambda^{-1}G(y)}|\langle\nabla_x w, \nabla_x\phi(y)\rangle|^2 dy.$$

Since $\nabla_x w \sim i\alpha\varphi(x_0)\psi'(x_0)$ as $\alpha \to \infty$, we find

$$(B.11)\qquad J_\alpha \underset{\alpha\to\infty}{\sim} 4\alpha^{4-n}\varphi(x_0)^{3-n/2}\|e^{\langle y,\psi'(x_0)\rangle^2}\langle\psi'(x_0), \nabla_x\phi(y)\rangle\|_{L^2}^2,$$

From (1.14) we have $I_\alpha \leq CJ_\alpha$. With (B.10) and (B.11), we find that $\gamma \leq 1$, thus concluding the proof of Proposition 1.3.

We now prove the result of Proposition 1.4. Arguing as above we have

$$\tau^6\alpha^{6+\gamma}\|\varphi^{\frac{6}{2}}e^{\tau\varphi}u_\tau\|_{L^2}^2 \underset{\tau\to\infty}{\sim} \tau^{6-n/2}K_\alpha, \qquad K_\alpha = \alpha^{6+\gamma-n}\varphi(x_0)^{6-n/2}\int_{\mathbb{R}^n} e^{2\lambda^{-1}G(y)}\big|\phi(y)\big|^2 dy,$$

and

$$K_\alpha \underset{\alpha\to\infty}{\sim} \alpha^{6+\gamma-n} \varphi(x_0)^{6-n/2} \|e^{\langle y,\psi'(x_0)\rangle^2}\phi(y)\|_{L^2}^2 \neq 0. \tag{B.12}$$

We have

$$\Delta^2 u_\tau = \left(\lambda^2\Delta^2\phi(\lambda^{\frac12}x) + 4i\tau\lambda^{\frac32}\langle\nabla_x w,\nabla_x\Delta\phi(\lambda^{\frac12}x)\rangle - 4\tau^2\lambda\phi''(\lambda^{\frac12}x)(\nabla_x w,\nabla_x w)\right)e^{i\tau w(x)},$$

yielding

$$\|e^{\tau\varphi}\Delta^2 u_\tau\|_{L^2}^2 \underset{\tau\to\infty}{\sim} \tau^{6-n/2}L_\alpha, \qquad \text{with} \qquad L_\alpha = 16\alpha^{4-n}\varphi(x_0)^{2-n/2}\int_{\mathbb{R}^n} e^{2\lambda^{-1}G(y)}\big|\phi''(y)(\nabla_x w,\nabla_x w)\big|^2 dy.$$

We have

$$L_\alpha \underset{\alpha\to\infty}{\sim} 16\alpha^{8-n}\varphi(x_0)^{6-n/2}\|e^{\langle y,\psi'(x_0)\rangle^2}\phi''(y)(\psi',\psi')\|_{L^2}^2. \tag{B.13}$$

From (1.15) we have $K_\alpha \leq CL_\alpha$. With (B.12) and (B.13), we find that $\gamma \leq 2$, thus concluding the proof of Proposition 1.4. ∎

B.4. **Proof of Lemma 2.3.** From the symbolic calculus we find $H = \mathrm{Id} + R_1$ with $R_1 \in \Psi(h_g, g)$. As $h_g = \alpha/\mu$ we observe that $S_1 = \tau R_1 \in \Psi(1,g)$, uniformly in τ. From L^2 boundedness (see Theorem A.2) we have $S_1 : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ continuously and $\|S_1\|_{(L^2,L^2)} \leq C$, uniformly in τ. Hence, for τ sufficiently large, $\mathrm{Id} + \tau^{-1}S_1$ is invertible in $L(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$. ∎

B.5. **Proof of Lemma 2.4.** Let $r, r' \in \mathbb{R}$. We first prove that $\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{r'}\mu^r) : L^2(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n)$ is injective. Let $w \in L^2$ be such that $\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{r'}\mu^r)w = 0$. By Lemma 2.3 $\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{-r'}\mu^{-r})\,\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{r'}\mu^r)$ is invertible in $L(L^2, L^2)$. It follows that $w = 0$.

Let $u \in \mathscr{S}(\mathbb{R}^n)$ and set $H = \mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^s\mu^k)\,\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{-s}\mu^{-k})$. By Lemma 2.3 and its proof, the operator $H = \mathrm{Id} + R_1$, with $R_1 \in \Psi(\alpha/\mu, g)$, is invertible in $L(L^2, L^2)$ for τ sufficiently large. We set $S_1 = \tau R_1 \in \Psi(\tau\alpha/\mu, g) \subset \Psi(\varphi^{-1}, g) \subset \Psi(1,g)$. We set $v = H^{-1}\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^s\mu^k)u$ that is in $L^2(\mathbb{R}^n)$, since $w = \mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^s\mu^k)u \in \mathscr{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$. Moreover, for some $C > 0$,

$$1/C\|w\|_{L^2} \leq \|v\|_{L^2} \leq C\|w\|_{L^2}.$$

We have $v = H^{-1}w = \lim_{n\to\infty}\sum_{j=0}^n \tau^{-j}S_1^j w$ in $L^2(\mathbb{R}^n)$. Let $s', k' \in \mathbb{R}$ and $a \in S(\tilde{\tau}^{s'}\mu^{k'}, g)$. Since $\mathrm{Op}^{\mathrm{w}}(a)$ is continuous from $L^2(\mathbb{R}^n)$ into $\mathscr{D}'(\mathbb{R}^n)$ we see that

$$\mathrm{Op}^{\mathrm{w}}(a)v = \lim_{n\to\infty}\sum_{j=0}^n \tau^{-j}\mathrm{Op}^{\mathrm{w}}(a)S_1^j w \quad \text{in } \mathscr{D}'(\mathbb{R}^n). \tag{B.14}$$

Set $\tilde{k} = \max(\lfloor k'\rfloor + 1, 0)$ and $\tilde{s} = \max(\lfloor s'\rfloor + 1, 0)$. For $j \geq \tilde{k} + \tilde{s} + 1$ we write

$$\tau^{-j}\mathrm{Op}^{\mathrm{w}}(a)S_1^j = \tau^{-j+\tilde{k}+\tilde{s}}\underbrace{\mathrm{Op}^{\mathrm{w}}(a)(\tau\alpha)^{-\tilde{k}-\tilde{s}}S_1^{\tilde{k}+\tilde{s}}}_{\in\Psi(1,g)}\underbrace{\alpha^{\tilde{k}+\tilde{s}}S_1}_{\in\Psi(1,g)}\underbrace{S_1^{j-1-\tilde{k}-\tilde{s}}}_{\in\Psi(1,g)},$$

as $\tau\alpha^{\tilde{k}+\tilde{s}+1}/\mu$ is bounded. We thus have

$$\|\tau^{-j}\mathrm{Op}^{\mathrm{w}}(a)S_1^j w\|_{L^2} \leq \tau^{-j+\tilde{k}+\tilde{s}}C_1 C_2 C_3^{j-1-\tilde{k}-\tilde{s}}\|w\|_{L^2}, \tag{B.15}$$

with

$$C_1 = \|\mathrm{Op}^{\mathrm{w}}(a)(\tau\alpha)^{-\tilde{k}-\tilde{s}}S_1^{\tilde{k}+\tilde{s}}\|_{(L^2,L^2)}, \quad C_2 = \|\alpha^{\tilde{k}+\tilde{s}}S_1\|_{(L^2,L^2)}, \quad C_3 = \|S_1\|_{(L^2,L^2)}.$$

For τ sufficiently large, from (B.15) it follows that the series in (B.14) actually converges in $L^2(\mathbb{R}^n)$ and thus for any $k', s' \in \mathbb{R}$ we have $\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{s'}\mu^{k'})v \in L^2(\mathbb{R}^n)$. In particular, $\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{-s}\mu^{-k})v \in L^2(\mathbb{R}^n)$.

Observe now that $\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^s\mu^k)\left(\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{-s}\mu^{-k})v\right) = \mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^s\mu^k)u$. We conclude that $u = \mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{-s}\mu^{-k})v$ with the injectivity of $\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^s\mu^k)$ from $L^2(\mathbb{R}^n)$ into $\mathscr{D}'(\mathbb{R}^n)$. ∎

B.6. **Proof of Proposition 2.5. Point (1).** Setting $(u, u')_{\mathscr{H}_{k,s}} = (\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^s \mu^k)u, \mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^s \mu^k)u')$ we obtain an inner product for $\mathscr{H}_{k,s}(\mathbb{R}^n)$.

Let us now consider $(u_n)_n$ a Cauchy sequence in $(\mathscr{H}_{k,s}(\mathbb{R}^n), \|.\|_{k,s})$. We have $u_n = \mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{-s}\mu^{-k})v_n$ with $v_n \in L^2(\mathbb{R}^n)$. Define $H = \mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^s\mu^k)\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{-s}\mu^{-k})$. By Lemma 2.3, the sequence $v_n = H^{-1}\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^s\mu^k)u_n$, $n \in \mathbb{N}$, is a Cauchy sequence in $L^2(\mathbb{R}^n)$ that converges to $v \in L^2(\mathbb{R}^n)$. Introducing $u = \mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{-s}\mu^{-k})v \in \mathscr{H}_{k,s}(\mathbb{R}^n)$, since $\|u_n - u\|_{k,s} = \|H(v_n - v)\|_{L^2}$, we see that $(u_n)_n$ converges to u in $(\mathscr{H}_{k,s}(\mathbb{R}^n), \|.\|_{k,s})$.

Density and point (2). Let now $u \in \mathscr{H}_{k,s}(\mathbb{R}^n)$ be such that $u = \mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{-s}\mu^{-k})v$ with $v \in L^2(\mathbb{R}^n)$ and let $(v_n)_n \subset \mathscr{S}(\mathbb{R}^n)$ be convergent to v in $L^2(\mathbb{R}^n)$. We set $u_n = \mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{-s}\mu^{-k})v_n \in \mathscr{S}(\mathbb{R}^n)$. Then $\|u_n - u\|_{k,s} = \|H(v_n - v)\|_{L^2}$ and we see that the sequence $(u_n)_n \subset \mathscr{S}(\mathbb{R}^n)$ converges to u in $(\mathscr{H}_{k,s}(\mathbb{R}^n), \|.\|_{k,s})$. Moreover the inequality

$$1/C\|u_n\|_{k,s} \leq \|v_n\|_{L^2} \leq C\|u_n\|_{k,s}, \qquad \tau \geq \tau_1(k,s).$$

given by Lemma 2.4 yields, $1/C\|u\|_{k,s} \leq \|v\|_{L^2} \leq C\|u\|_{k,s}$, by passing to the limit.

Point (3). We first prove that $\mathrm{Op}^{\mathrm{w}}(a)$ maps $\mathscr{H}_{k,s}$ into L^2 continuously if $a \in S(\tilde{\tau}^s \mu^k, g)$. We choose $\tau > \tau_1(k,s)$ so as to apply Proposition 2.5. Let $u \in \mathscr{H}_{k,s}(\mathbb{R}^n)$. There exists $v \in L^2(\mathbb{R}^n)$ such that $u = \mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{-s}\mu^{-k})v$ and $\|v\|_{L^2} \lesssim \|\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^s\mu^k)u\|_{L^2}$. Then

$$(\mathrm{B.16}) \qquad \|\mathrm{Op}^{\mathrm{w}}(a)u\|_{L^2} = \|\mathrm{Op}^{\mathrm{w}}(a)\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{-s}\mu^{-k})v\|_{L^2} \lesssim \|v\|_{L^2} \lesssim \|\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^s\mu^k)u\|_{L^2},$$

by the L^2 boundedness of Theorem A.2 since $\mathrm{Op}^{\mathrm{w}}(a)\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{-s}\mu^{-k}) \in \Psi(1, g)$.

Next, we let $u \in \mathscr{S}(\mathbb{R}^n)$. We have $\mathrm{Op}^{\mathrm{w}}(a)u \in \mathscr{S}(\mathbb{R}^n)$ and

$$\|\mathrm{Op}^{\mathrm{w}}(a)u\|_{k',s'} = \|\underbrace{\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{s'}\mu^{k'})\mathrm{Op}^{\mathrm{w}}(a)}_{\in \Psi(\tilde{\tau}^{s+s'}\mu^{k+k'},g)}u\|_{L^2} \leq \|u\|_{k+k',s+s'},$$

by (B.16). We then conclude by density. $\blacksquare$

B.7. **Proof of Lemma 2.6.** Let $u \in \mathscr{S}(\mathbb{R}^n)$ and $v \in L^2(\mathbb{R}^n)$ such that $u = \mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{-s}\mu^{-k})v$. We saw in the proof of Lemma 2.4 that $\mathrm{Op}^{\mathrm{w}}(a)v \in L^2(\mathbb{R}^n)$ for all $a \in S(\tilde{\tau}^{s'}\mu^{k'}, g)$, if $\tau \geq \tau_1(k', s')$. In particular $v \in \cap_{l,r\in\mathbb{R}}\mathscr{H}_{l,r}(\mathbb{R}^n)$.

We set $H = \mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{-s}\mu^{-k})\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^s\mu^k) \in \Psi(1, g)$. For any $\ell, r \in \mathbb{R}$, with Proposition 2.5, the operator H maps $\mathscr{H}_{\ell,r}(\mathbb{R}^n)$ into itself continuously and is invertible, for τ sufficiently large, arguing as in the proof of Lemma 2.3. We thus set $\tilde{u} = H^{-1}u \in \mathscr{H}_{\ell,r}(\mathbb{R}^n)$. We have

$$(\mathrm{B.17}) \qquad C_{\ell,r}^{-1}\|\tilde{u}\|_{\ell,r} \leq \|u\|_{\ell,r} \leq C_{\ell,r}\|\tilde{u}\|_{\ell,r}.$$

In fact, recall from the proof of Lemma 2.3 that H^{-1} can be expressed as a Neumann series. Here, we obtain the convergence of the series in any space $\mathscr{H}_{\ell,r}(\mathbb{R}^n)$.

We then observe that

$$\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{-s}\mu^{-k})\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^s\mu^k)\tilde{u} = H\tilde{u} = u = \mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{-s}\mu^{-k})v.$$

As $\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^s\mu^k)\tilde{u} \in L^2(\mathbb{R}^n)$ the injectivity of $\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{-s}\mu^{-k})$ from L^2 into $\mathscr{H}_{k,s}(\mathbb{R}^n)$ (see Lemma 2.3) we obtain $\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^s\mu^k)\tilde{u} = v$. In particular, as $\ell, r \in \mathbb{R}$ are chosen arbitrarily we find $v \in \mathscr{H}_{k',s'}(\mathbb{R}^n)$ for any $k', s' \in \mathbb{R}$, if τ is chosen sufficiently large and we have by (B.17)

$$\|v\|_{k',s'} = \|\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{s'}\mu^{k'})\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^s\mu^k)\tilde{u}\|_{L^2} \lesssim \|\tilde{u}\|_{k+k',s+s'} \lesssim \|u\|_{k+k',s+s'}.$$

As $v \in \cap_{l,r\in\mathbb{R}}\mathscr{H}_{l,r}(\mathbb{R}^n)$, we also have

$$\|u\|_{k+k',s+s'} = \|\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{s+s'}\mu^{k+k'})u\|_{L^2} = \|\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{s+s'}\mu^{k+k'})\mathrm{Op}^{\mathrm{w}}(\tilde{\tau}^{-s}\mu^{-k})v\|_{L^2} \lesssim \|v\|_{k',s'}.$$

$\blacksquare$

B.8. Proof of Lemma 2.7. With Lemma 2.6, let $v \in L^2$ be such that $u = \mathrm{Op^w}(\tilde{\tau}^{\frac{-s}{2}} \mu^{\frac{-k}{2}})v$ with $\|v\|_{L^2} \lesssim \|u\|_{\frac{k}{2}, \frac{s}{2}}$. We then have

$$\left| (\mathrm{Op^w}(a)u, u) \right| = \left| (\mathrm{Op^w}(\tilde{\tau}^{\frac{-s}{2}} \mu^{\frac{-k}{2}}) \mathrm{Op^w}(a) \mathrm{Op^w}(\tilde{\tau}^{\frac{-s}{2}} \mu^{\frac{-k}{2}})v, v) \right| \lesssim \|v\|_{L^2}^2 \lesssim \|u\|_{\frac{k}{2}, \frac{s}{2}}^2,$$

by Theorem A.2, as $\mathrm{Op^w}(\tilde{\tau}^{\frac{-s}{2}} \mu^{\frac{-k}{2}}) \mathrm{Op^w}(a) \mathrm{Op^w}(\tilde{\tau}^{\frac{-s}{2}} \mu^{\frac{-k}{2}}) \in \Psi(1, g)$. ∎

B.9. Proof of Proposition 2.8. We write $\tilde{\tau}^s \mu^k = \tilde{\tau}^{\frac{s}{2}} \mu^{\frac{k}{2}} \sharp \tilde{\tau}^{\frac{s}{2}} \mu^{\frac{k}{2}} + r$ with $r \in S(\alpha \tilde{\tau}^s \mu^{k-1}, g)$. We thus have

$$(\mathrm{Op^w}(\tilde{\tau}^s \mu^k)u, u) = \|u\|_{\frac{k}{2}, \frac{s}{2}}^2 + (\mathrm{Op^w}(r)u, u) \geq \|u\|_{\frac{k}{2}, \frac{s}{2}}^2 - C\alpha \|u\|_{\frac{k-1}{2}, \frac{s}{2}}^2.$$

We conclude by choosing τ sufficiently large. ∎

B.10. Proof of Lemma 3.9. Set $H = \mathrm{Op^w}(\mu^{m-1}) \mathrm{Op^w}(\mu^{1-m})$. By Lemma 2.4, for τ sufficiently large, we have $w = H^{-1} \mathrm{Op^w}(\mu^{m-1})v$, with $H^{-1} : L^2 \to L^2$ continuously. We then write

$$\mathrm{Op^w}(r)w = H^{-1} H \mathrm{Op^w}(r)w = y_1 + z_1, \quad y_1 = H^{-1}[H, \mathrm{Op^w}(r)]w, \quad z_1 = H^{-1} \mathrm{Op^w}(r)Hw.$$

We have

$$z_1 = H^{-1} \mathrm{Op^w}(r) \mathrm{Op^w}(\mu^{m-1})v = H^{-1} \mathrm{Op^w}(r) \mathrm{Op^w}(\mu^{m-1})\hat{\chi}v.$$

Where $\hat{\chi} \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ with $\hat{\chi} = 1$ in a neighborhood of V and such that $\chi = 1$ in a neighborhood of $\mathrm{supp}(\hat{\chi})$. As $\mathrm{supp}(r) \cap \mathrm{supp}(\hat{\chi}) = \emptyset$ we find $\mathrm{Op^w}(r) \mathrm{Op^w}(\mu^{m-1})\hat{\chi} \in \Psi\big((\mu/\alpha)^{-N}, g\big)$ for any $N \in \mathbb{N}$. We thus obtain

(B.18) $$\|z_1\|_{L^2} \lesssim \tau^{-N} \|v\|_{L^2}.$$

We have $[H, \mathrm{Op^w}(r)] = R_1 \in \Psi\big((\mu/\alpha)^{-1}, g\big)$ and $y_1 = H^{-2} H R_1 w = y_2 + z_2$ with

$$y_2 = H^{-2}[H, R_1]w, \quad z_2 = H^{-2} R_1 \mathrm{Op^w}(\mu^{m-1})v,$$

Similarly we have an estimate for $|z_2|_{L^2}$ of the same form as that of $|z_1|_{L^2}$ in (B.18) and by induction for any $k \geq 2$ we find $\mathrm{Op^w}(r)w = y_k + z_1 + \cdots + z_k$ with $z_3, \ldots, z_k$ also satisfying such an estimate and

$$y_k = H^{-k} R_k w,$$

with $R_k \in \Psi\big((\mu/\alpha)^{-k}, g\big)$. With Lemma 2.6 we obtain

$$\|y_k\|_{L^2} \lesssim \|R_k w\|_{L^2} \lesssim \alpha^k \|w\|_{-k,0} \lesssim \alpha^k \|v\|_{m-1-k,0},$$

with k arbitrary. Hence,

$$\tau^N \|y_k\|_{L^2} \lesssim \alpha^k \tau^N \|v\|_{m-1-k,0} \lesssim \|v\|_{L^2},$$

for k sufficiently large, as $\alpha^k \tau^N \mu^{m-1-k} \in S(\mu^{m-1+N-k}, g)$ since $\alpha^k \lesssim (\alpha\varphi)^N$, for $\alpha \geq 1$. ∎

B.11. Details on Examples 4.2 and Remark 4.3. Consider the bi-Laplace operator with symbol $p(\xi) = |\xi|^4$ in an open subset of $\mathbb{R}^n$, $n \geq 2$. Then

$$\rho_{x,\xi}(\hat{\tau}) = \left(\sum_j (\xi_j + \hat{\tau}\psi_j')^2 \right)^2 = \left(|\xi|^2 - |\hat{\tau}\psi'|^2 + 2i\hat{\tau}\langle \xi, \psi' \rangle \right)^2$$

We then have

$$\rho_{x,\xi}'(\hat{\tau}) = 2\big(-2\hat{\tau}|\psi'|^2 + 2i\langle \xi, \psi' \rangle \big)\big(|\xi|^2 - |\hat{\tau}\psi'|^2 + 2i\hat{\tau}\langle \xi, \psi' \rangle \big).$$

There exist roots since $\rho_{x,\xi}(\hat{\tau}) = 0$ is equivalent to $\xi \perp \psi'$ and $|\xi| = |\hat{\tau}\psi'|$. Then $\rho_{x,\xi}(\hat{\tau}) = 0$ implies $\rho_{x,\xi}'(\hat{\tau}) = 0$. Roots are always (at least) double.

We now consider the elliptic symbol $p(\xi) = \xi_1^4 + \xi_2^4$ in $\mathbb{R}^2$. We have

$$\rho_{x,\xi}(\hat{\tau}) = (\xi_1 + i\hat{\tau}\psi_{x_1}')^4 + (\xi_2 + i\hat{\tau}\psi_{x_2}')^4, \qquad \rho_{x,\xi}'(\hat{\tau}) = 4i\Big(\psi_{x_1}'(\xi_1 + i\hat{\tau}\psi_{x_1}')^3 + \psi_{x_2}'(\xi_2 + i\hat{\tau}\psi_{x_2}')^3 \Big).$$

and we assume that $\rho_{x,\xi}(\hat{\tau}) = 0$. Observe that $\hat{\tau} = 0$ implies $\xi = 0$. We may thus assume $\hat{\tau} > 0$. Observe also that $\xi_2 + i\hat{\tau}\psi'_{x_2} = 0$ is excluded as it implies $\xi_1 + i\hat{\tau}\psi'_{x_1} = 0$, and we then have $\psi' = 0$. We may thus write

$$\frac{(\xi_1 + i\hat{\tau}\psi'_{x_1})^4}{(\xi_2 + i\hat{\tau}\psi'_{x_2})^4} = -1,$$

or equivalently

$$\frac{\xi_1 + i\hat{\tau}\psi'_{x_1}}{\xi_2 + i\hat{\tau}\psi'_{x_2}} = e^{i(2k+1)\pi/4}, \quad k = 0, 1, 2, 3.$$

We then obtain

$$\rho'_{x,\xi}(\hat{\tau}) = 4i(\xi_2 + i\hat{\tau}\psi'_{x_2})^3\big(\psi'_{x_1}e^{i3(2k+1)\pi/4} + \psi'_{x_2}\big)$$

that cannot vanish if $\psi' \neq 0$.

We now consider the operator with symbol $p = \frac{1}{2}\xi^2 + \xi_1\xi_2$ along with $\psi(x) = \frac{1}{2}(x_1 - a)^2$, in the case $\Omega \subset \{x_1 > a\}$. We then have

$$\rho_{x,\xi}(\hat{\tau}) = \frac{1}{2}(\xi_1 + i\hat{\tau}(x_1 - a))^2 + (\xi_1 + i\hat{\tau}(x_1 - a))\xi_2, \qquad \rho'_{x,\xi}(\hat{\tau}) = i(x_1 - a)\big(\xi_1 + \xi_2 + i\hat{\tau}(x_1 - a)\big),$$

Let us assume that $\rho_{x,\xi}(\hat{\tau}) = \rho'_{x,\xi}(\hat{\tau}) = 0$. From $\rho'_{x,\xi}(\hat{\tau}) = 0$ we have $\hat{\tau} = 0$ and $\xi_1 + \xi_2 = 0$. From $\rho_{x,\xi}(\hat{\tau}) = 0$ we find $0 = -\frac{1}{2}\xi_1^2$. Hence $\xi = 0$. Property (4.3) is thus fulfilled.

We now consider the operator with symbol $p = \xi_1\xi_2$ along with $\psi(x) = x_1 + x_2$, in the case $\Omega \subset \mathbb{R}^2$. We then have

$$\rho_{x,\xi}(\hat{\tau}) = (\xi_1 + i\hat{\tau})(\xi_2 + i\hat{\tau}), \qquad \rho'_{x,\xi}(\hat{\tau}) = i(\xi_1 + i\hat{\tau}) + i(\xi_2 + i\hat{\tau}).$$

We have $\rho_{x,\xi}(\hat{\tau}) = 0$ if and only if $\hat{\tau} = 0$ and $\xi_1\xi_2 = 0$. Assume that $\xi_1 = 0$ we then have $\rho'_{x,\xi}|_{\hat{\tau}=0} = i\xi_2$. If the root is double we then have $\xi = 0$ and $\hat{\tau} = 0$. The simple characteristic property (4.3) is thus satisfied.

We finally prove the statement of Remark 4.3. Let $\xi \in \mathbb{R}^n \setminus 0$. By (4.2) Property 4.3 precisely prevents to have $p(x,\xi) = 0$ and $\{p,\psi\}(x,\xi) = 0$ simultaneously. The implication in the definition of pseudo-convexity (Definition 3.2) is thus clearly fulfilled.

Similarly let $\xi \in \mathbb{R}^n$ and let $\tau > 0$. Property (4.3) precisely prevents to have $p(x, \xi + i\tau\psi'(x)) = 0$ and $\{p,\psi\}(x, \xi + i\tau\psi'(x)) = 0$ simultaneously. The implication in the definition of strong pseudo-convexity (Definition 3.3) is thus clearly fulfilled.

Consider now the operator $P(D_x) = D_{x_1}$ in $\Omega \subset \mathbb{R}^2$ and the weight function $\psi(x) = x_1^2/2 + x_2$. The gradient and the Jacobian matrix of the weight function are given by

$$\nabla\psi = \begin{pmatrix} x_1 \\ 1 \end{pmatrix}, \quad J_\psi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and we have

$$p(\xi) = \xi_1, \quad \{p,\psi\} = x_1, \quad \{p, \{p,\psi\}\} = 1, \quad \frac{1}{2i}\big\{p(\xi - i\tau\psi'), p(\xi + i\tau\psi')\big\} = \Theta_{p,\psi} = \tau > 0.$$

We thus see that strong pseudo-convexity is fulfilled. Set $\rho_{x,\xi}(\tau) = p(\xi + i\tau\psi') = \xi_1 + i\tau x_1$. We have $\rho'_{x,\xi}(\tau) = ix_1$. We thus see that $\tau \neq 0$ is a double root if we have $x_1 = \xi_1 = 0$.

B.12. **Alternative proof of Theorem 4.9.** Here we restrict ourselves to using the standard Gårding inequality for homogeneous symbols to derive the Carleman estimate as can be done for elliptic operators.

We start with the following symbol inequality.

Proposition B.1. *Let P be elliptic on Ω and let ψ satisfy* (s-Ψc) *(point 2 in Definition 3.3) for all $x \in \Omega$. We set $\varphi = e^{\alpha\psi}$ and*

$$\zeta = \zeta(x, \xi, \tau) = \xi + i\tau\varphi'(x) = \xi + i\tilde{\tau}(x)\psi'(x), \qquad \tilde{\tau}(x) = \tau\alpha\varphi(x).$$

Let X be an open subset such that $X \Subset \Omega$. There exist $C > 0$, $\tau_0 \geq 1$, $\alpha_0 > 0$, ν_0 such that we have

(B.19) $$C\mu^{2m} \leq \nu|p(x, \zeta)|^2 + \tilde{\tau}(x)\Theta_{p,\psi}(x, \xi, \tilde{\tau}(x)) + \nu\tilde{\tau}(x)^2|\langle p'_\xi(x, \zeta), \psi'(x)\rangle|^2,$$

for $\tau \geq \tau_0$, $\alpha \geq \alpha_0$, $\nu \geq \nu_0$, and $(x, \xi) \in \overline{X} \times \mathbb{R}^n$.

Proof. With the ellipticity of p we have $|p(x, \xi)|^2 \gtrsim |\xi|^{2m}$. We first consider the case $\tilde{\tau}(x) \ll |\xi|$. For $\nu_1 > 0$, we then obtain

$$\nu\big(|p(x, \zeta)|^2 + \tilde{\tau}(x)^2|\langle p'_\xi(x, \zeta), \psi'(x)\rangle|^2\big) + \tilde{\tau}(x)\Theta_{p,\psi}(x, \xi, \tilde{\tau}(x)) \gtrsim |\zeta|^{2m} \gtrsim |\mu|^{2m}.$$

for $\nu \geq \nu_1$, and $x \in \overline{X}$.

We now treat the case $|\xi| \leq \delta\tilde{\tau}(x)$, for some fixed δ. We introduce $\hat{\tau} > 0$ and place ourselves on the compact set

$$\mathscr{C} = \{(x, \xi, \hat{\tau}); \ \hat{\tau}^2 + |\xi|^2 = 1, \ |\xi| \leq \delta\hat{\tau}, \ x \in \overline{X}\}.$$

As ψ is strongly pseudo-convex, we have

$$\nu\big(|p(x, \xi + i\hat{\tau}\psi'(x))|^2 + \hat{\tau}^2|\langle p'_\xi(x, \xi + i\hat{\tau}\psi'(x)), \psi'(x)\rangle|^2\big) + \hat{\tau}\Theta_{p,\psi}(x, \xi, \hat{\tau}) \geq C, \quad (x, \xi, \hat{\tau}) \in \mathscr{C},$$

for $\nu \geq \nu_0$ with $\nu_0 \geq \nu_1$ sufficiently large. Since the symbol is homogeneous of degree $2m$ in $(\tilde{\tau}, \xi)$ we have

$$\nu\big(|p(x, \xi + i\hat{\tau}\psi'(x))|^2 + \hat{\tau}^2|\langle p'_\xi(x, \xi + i\hat{\tau}\psi'(x)), \psi'(x)\rangle|^2\big) + \hat{\tau}\Theta_{p,\psi}(x, \xi, \hat{\tau}) \gtrsim |(\hat{\tau}, \xi)|^{2m},$$

for $\nu \geq \nu_0$, $x \in \overline{X}$ and $|\xi| \leq \delta\hat{\tau}$. We apply this last inequality to $\hat{\tau} = \tilde{\tau}(x)$ in the case $|\xi| \leq \delta\tilde{\tau}(x)$, which gives the sought inequality in this second case. ∎

Alternative proof of Theorem 4.9. Let Y be an open subset such that $X \Subset Y \Subset \Omega$. Proposition B.1 applies in Y. As in the proof of Theorem 3.8 we restrict ourselves to a small neighborhood $V \subset W \Subset Y$ of a point $x_0 \in X$, where W is an open subset. The function ψ then satisfies Assumption 2.1. We assume $u \in \mathscr{C}_c^\infty(V)$.

We denote by $p = p(x, \xi)$ the principal symbol of P. We set $P_\varphi = e^{\tau\varphi}Pe^{-\tau\varphi}$ and $v = e^{\tau\varphi}u$. We have $P_\varphi = P(x, D_x + i\tau\varphi'(x)) \in \Psi(\mu^m, g)$. Its principal symbol of given by

$$p_\varphi = p_\varphi(x, \xi, \tau) = p(x, \xi + i\tau\varphi'(x)).$$

We first consider only the principal part and set $Q = \mathrm{Op}^w(p_\varphi)$ and introduce

$$Q_2 = \mathrm{Op}^w(q_2) \in \Psi(\mu^m, g), \quad \text{with } q_2 = \mathrm{Re}\, p_\varphi,$$
$$Q_1 = \mathrm{Op}^w(q_1) \in \Psi(\mu^m, g), \quad \text{with } q_1 = \mathrm{Im}\, p_\varphi.$$

We have $Q = Q_2 + iQ_1$ with both Q_2 and Q_1 selfadjoint.

We have $\|Qv\|_{L^2}^2 = \|Q_2v\|_{L^2}^2 + \|Q_1v\|_{L^2}^2 + i\big([Q_2, Q_1]v, v\big)_{L^2}$. With ν such that $\nu(\tau\varphi(x))^{-1} \leq 1$ we then write

$$\|Qv\|_{L^2}^2 \geq \nu\|\tilde{\tau}^{-\frac{1}{2}}Q_2v\|_{L^2}^2 + \nu\|\tilde{\tau}^{-\frac{1}{2}}Q_1v\|_{L^2}^2 + i\big([Q_2, Q_1]v, v\big)_{L^2}$$
$$= \Big(\underbrace{\big(\nu(Q_2\tilde{\tau}^{-1}Q_2 + Q_1\tilde{\tau}^{-1}Q_1) + i[Q_2, Q_1]\big)}_{=B \in \Psi(\alpha\mu^{2m-1}, g)}v, v\Big)_{L^2}.$$

We then set $e = \tilde{\tau}^{\frac{1}{2}}\alpha^{\frac{1}{2}}\langle p'_\xi(x, \zeta), \psi'(x)\rangle \in S(\tilde{\tau}^{\frac{1}{2}}\alpha^{\frac{1}{2}}\mu^{m-1}, g)$ and consider $E = \mathrm{Op}^w(e)$. We write, with $\nu\alpha^{-1} \leq 1$,

$$0 = \|Ev\|_{L^2}^2 - \|Ev\|_{L^2}^2 \geq \nu\alpha^{-1}\|Ev\|_{L^2}^2 - \|Ev\|_{L^2}^2 = (\nu\alpha^{-1} - 1)(E^*Ev, v)_{L^2}.$$

And we find

$$\|Qv\|_{L^2}^2 \geq \Big(\big(\nu(Q_2\tilde{\tau}^{-1}Q_2 + Q_1\tau^{-1}Q_1) + i[Q_2, Q_1] + (\nu\alpha^{-1} - 1)E^*E\big)v, v\Big)_{L^2}.$$

The symbol ρ of $\nu(Q_2\tilde{\tau}^{-1}Q_2 + Q_1\tau^{-1}Q_1) + i[Q_2, Q_1] + (\nu\alpha^{-1} - 1)E^*E$ is

$$\rho = \nu\tilde{\tau}^{-1}(q_2^2 + q_1^2) + \{q_2, q_1\} + (\nu\alpha^{-1} - 1)\tilde{\tau}(x)\alpha|\langle p'_\xi(x, \zeta), \psi'(x)\rangle|^2$$
$$\mod S(\alpha\tilde{\tau}^{-1}\mu^{2m-1}, g) + S(\alpha^3\mu^{2m-3}, g) + S(\alpha^2\tilde{\tau}\mu^{2m-3}, g).$$

As we have

$$\{q_2, q_1\} = \{\operatorname{Re} p_\varphi(x, \xi, \tau), \operatorname{Im} p_\varphi(x, \xi, \tau)\} = \frac{1}{2i}\{\overline{p}(x, \overline{\zeta}), p(x, \zeta)\},$$

with (3.8) we obtain

$$\rho = \rho_0 + r, \qquad \rho_0 = \nu\tilde{\tau}^{-1}(q_2^2 + q_1^2) + \Theta_{p,\psi}(x, \xi, \tilde{\tau}(x)) + \nu\tilde{\tau}(x)|\langle p'_\xi(x, \zeta), \psi'(x)\rangle|^2,$$

where $r \in S(\alpha\tilde{\tau}^{-1}\mu^{2m-1}, g) + S(\alpha^2\tilde{\tau}\mu^{2m-3}, g) \subset S(\alpha^2\tilde{\tau}^{-1}\mu^{2m-1}, g)$. By Proposition B.1 we have

$$\rho_0 \gtrsim \tilde{\tau}^{-1}\mu^{2m}, \quad x \in W, \xi \in \mathbb{R}^n, \tau \geq \tau_0,$$

for ν chosen sufficiently large. Let $\chi \in \mathscr{C}_c^\infty(W)$ be such that, $0 \leq \chi \leq 1$ and $\chi = 1$ in a neighborhood of V. We then write

$$\rho = \hat{\rho} + \hat{r}, \qquad \hat{\rho} = \rho\chi + \tilde{\tau}^{-1}\mu^{2m}(1 - \chi), \quad \hat{r} = (\rho - \tilde{\tau}^{-1}\mu^{2m})(1 - \chi).$$

We have

(B.20) $$|(\operatorname{Op}^{\mathrm{w}}(r)v, v)| \lesssim \alpha^2\|v\|_{m-\frac{1}{2}, -\frac{1}{2}}^2,$$

With $\hat{\chi} \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ such that $\hat{\chi} = 1$ in a neighborhood of V and such that $\chi = 1$ in a neighborhood of $\operatorname{supp}(\hat{\chi})$ we have, for any $N \in \mathbb{N}$,

(B.21) $$|(\operatorname{Op}^{\mathrm{w}}(\hat{r})v, v)| = |(\operatorname{Op}^{\mathrm{w}}(\hat{r})\hat{\chi}v, v)| \lesssim \|v\|_{-N, 0}^2,$$

since $\operatorname{Op}^{\mathrm{w}}(\hat{r})\hat{\chi} \in \cap_{N \in \mathbb{N}} S(\mu^{-N}, g)$.

Since $\hat{\rho}_0 \in S(\tilde{\tau}^{-1}\mu^{2m})$ is homogeneous of degree $2m - 1$ in $(\xi, \tilde{\tau})$ and $\hat{\rho}_0 \gtrsim \tilde{\tau}^{-1}\mu^{2m}$, with the *classical* Gårding inequality we thus find

$$(\operatorname{Op}^{\mathrm{w}}(\hat{\rho}_0)v, v) \gtrsim \|v\|_{m, -\frac{1}{2}}^2,$$

for τ chosen sufficiently large.

As $\mu \geq \tilde{\tau} = \alpha\varphi\tau$ with $\varphi = e^{\alpha\psi}$ with $\psi > 0$, from the remainder estimates (B.20)–(B.21) we obtain, taking α sufficiently large,

$$\|Qv\|_{L^2}^2 \gtrsim \|v\|_{m, -\frac{1}{2}}^2$$

We then conclude the proof as in the end of the proof of Theorem 3.8. ∎

B.13. **Proof of Proposition 4.14.** First note that $p(\xi) = |\xi|^4 + \xi_1\xi_2^3 + i\xi_1^3\xi_2$ being elliptic the pseudo-convexity condition of Definition 3.2 is trivially fulfilled. We now wish to solve for $\hat{\tau} > 0$

(B.22) $$p(\xi + i\hat{\tau}\psi'(x_0)) = 0 \quad \text{and} \quad \{p, \psi\}(x_0, \xi + i\hat{\tau}\psi'(x_0)) = 0,$$

where we have set $x_0 = (0, 0, 0)$.

We set $a(\xi) = |\xi|^2$. We have $\{p, \psi\}(x_0, \xi) = 4\xi_3 a(\xi)$. Hence $\{p, \psi\}(x_0, \xi + i\hat{\tau}\psi'(x_0)) = 0$ reads $4(\xi_3 + i\hat{\tau})a(\xi + i\hat{\tau}\psi'(x_0))$. As $\hat{\tau} > 0$ this implies that $a(\xi + i\hat{\tau}\psi'(x_0)) = 0$, meaning that $\xi_3 = 0$ and $\xi_1^2 + \xi_2^2 = \hat{\tau}^2$. If we consider the additional equation $p(\xi + i\hat{\tau}\psi'(x_0)) = 0$ we then have $\xi_1\xi_2^3 + i\xi_1^3\xi_2 = 0$, that is, either $\xi_1 = 0$ or $\xi_2 = 0$. We have thus found that (B.22) is equivalent to

(B.23) $$|\xi_1| = \hat{\tau}, \ \xi_2 = 0, \ \xi_3 = 0 \qquad \text{or} \qquad \xi_1 = 0, \ |\xi_2| = \hat{\tau}, \ \xi_3 = 0.$$

We thus have non trivial double roots for the map $\hat{\tau} \mapsto \rho_{x_0,\xi}(\hat{\tau})$ introduced in Section 4.1. The simple characteristic is not fulfilled in any neighborhood of x_0.

Let us now check that the strong pseudo-convexity condition is fulfilled at $x = x_0$, meaning that $\Theta_{p,\psi}(x_0, \xi, \hat{\tau}) > 0$ if (B.22) holds, with $\Theta_{p,\psi}$ as introduced in (3.5). In the present case the form of $\Theta_{p,\psi}$ reduces to

$$\Theta_{p,\psi}(x_0, \xi, \hat{\tau}) = \hat{\tau}\big(|\partial_{\xi_1} p(x, \xi + i\hat{\tau}\psi'(x_0))|^2 + |\partial_{\xi_2} p(x, \xi + i\hat{\tau}\psi'(x_0))|^2\big)$$

$$= \hat{\tau}\big(|4\xi_1 a(\xi + i\hat{\tau}\psi'(x_0)) + \xi_2^3 + 3i\xi_1^2\xi_2|^2 + |4\xi_2 a(\xi + i\hat{\tau}\psi'(x_0)) + 3\xi_1\xi_2^2 + i\xi_1^3|^2\big)$$

If (B.22) holds, that is, if we have (B.23), we then find $\Theta_{p,\psi}(x_0, \xi, \hat{\tau}) = \hat{\tau}^7 > 0$, which concludes the proof. ∎

B.14. **Proof of Lemma 5.1.** Let $x_0 \in X$ and $\xi_0 \in \mathbb{R}^n$ such that $p(x_0, \xi_0 + i\varphi'(x_0)) = 0$. If $\varphi'(x_0) = 0$ we can simply choose $\xi_0 = 0$. We shall come back to this case bellow and prove that $\varphi'(x_0) = 0$ does not occur if $m \geq 2$. Without any loss of generality we may assume that $x_0 = 0$. We set $\zeta_0 = \xi_0 + i\varphi'(x_0)$. For all $\tau > 0$ we have $p(x_0, \tau\zeta_0) = 0$. We set $\phi(x) = \varphi(x) - \varphi(0)$. We have

(B.24)
$$\sum_{|\beta| < m} (\tau\alpha)^{2(m-|\beta|)-1} \|\varphi^{m-|\beta|-\frac{1}{2}} e^{\tau\phi} D_x^\beta u\|_{L^2}^2 \leq K \|e^{\tau\phi} Pu\|_{L^2}^2,$$

for $\tau \geq \tau_0 > 0$ and $\alpha \geq \alpha_0 > 0$ and $u \in \mathscr{C}_c^\infty(X)$.

We use the computations carried out in Section B.1 with the test function u_τ as introduced in (B.3) with $\lambda = 1$ here. We choose τ sufficiently large so that $\text{supp}(u_\tau)$ is close enough to x_0 to apply (B.24).

With (B.4) and (B.5) we find

(B.25)
$$\sum_{|\beta| \leq m-1} (\tau\alpha)^{2(m-|\beta|)-1} \|\varphi^{m-|\beta|-\frac{1}{2}} e^{\tau\phi} D_x^\beta u\|_{L^2}^2 \underset{\tau \to \infty}{\sim} \tau^{2m-1-\frac{n}{2}} \sum_{|\beta| \leq m-1} (\alpha\varphi(0))^{2(m-|\beta|)-1} |\zeta_0^\beta|^2 \int_{\mathbb{R}^n} e^{2G(y)} |f(y)|^2 dy.$$

With the Taylor formula we observe that

$$e^{-i\tau w(x)} p(x, D_x) e^{i\tau w(x)} = p(x, D_x + \tau\zeta_0) = p(x, \tau\zeta_0) + \langle p'_\xi(x, \tau\zeta_0), D_x \rangle + \sum_{k=2}^m p_\xi^{(k)}(x, \tau\zeta_0) \underbrace{(D_x, \dots, D_x)}_{k \text{ times}}.$$

As $D_x^\beta f\big(\tau^{\frac{1}{2}} x\big) = \tau^{\frac{|\beta|}{2}} \mathscr{O}(1)$ we have $p_\xi^{(k)}(x, \tau\zeta_0)(D_x, \dots, D_x) f\big(\tau^{\frac{1}{2}} x\big) = \tau^{m-\frac{k}{2}} \mathscr{O}(1)$, we then find

$$e^{-i\tau w(x)} Pu_\tau = p(x, \tau\zeta_0) f\big(\tau^{\frac{1}{2}} x\big) + \langle p'_\xi(x, \tau\zeta_0), D_x \rangle f\big(\tau^{\frac{1}{2}} x\big) + \tau^{m-1} \mathscr{O}(1)$$

$$= \tau^{m-\frac{1}{2}} \Big(\tau^{\frac{1}{2}} p(x, \zeta_0) f\big(\tau^{\frac{1}{2}} x\big) + \langle p'_\xi(x, \zeta_0), (D_x f)\big(\tau^{\frac{1}{2}} x\big) \rangle + \tau^{-\frac{1}{2}} \mathscr{O}(1)\Big).$$

Next, we write

$$p(x, \zeta_0) = \underbrace{p(0, \zeta_0)}_{=0} + \langle x, p'_x(0, \zeta_0) \rangle + \frac{1}{2} \int_0^1 (1-\sigma) p_x^{(2)}(\sigma x, \zeta_0)(x, x) d\sigma,$$

which gives

$$\tau^{\frac{1}{2}} p(x, \zeta_0) f\big(\tau^{\frac{1}{2}} x\big) = \langle x, p'_x(0, \zeta_0) \rangle\big) f\big(\tau^{\frac{1}{2}} x\big) + \frac{1}{2} \tau^{-\frac{1}{2}} \Big(\int_0^1 (1-\sigma) p_x^{(2)}(\sigma x, \zeta_0)\big(\tau^{\frac{1}{2}} x, \tau^{\frac{1}{2}} x\big) d\sigma\Big) f\big(\tau^{\frac{1}{2}} x\big)$$

$$= \langle x, p'_x(0, \zeta_0) \rangle\big) f\big(\tau^{\frac{1}{2}} x\big) + \tau^{-\frac{1}{2}} \mathscr{O}(1),$$

as f has compact support. We have thus obtained

$$e^{-i\tau w(x)} Pu_\tau = \tau^{m-\frac{1}{2}} \Big(\langle \tau^{\frac{1}{2}} x, p'_x(0, \zeta_0) \rangle f\big(\tau^{\frac{1}{2}} x\big) + \langle p'_\xi(x, \zeta_0), (D_x f)\big(\tau^{\frac{1}{2}} x\big) \rangle + \tau^{-\frac{1}{2}} \mathscr{O}(1)\Big).$$

We then obtain

$$\|e^{\tau\phi} Pu_\tau\|_{L^2}^2 = \tau^{2m-1} \int_{\mathbb{R}^n} e^{2\tau\big(G(x) + |x|^3 \mathscr{O}_\alpha(1)\big)} \big|\langle \tau^{\frac{1}{2}} x, p'_x(0, \zeta_0) \rangle f\big(\tau^{\frac{1}{2}} x\big) + \langle p'_\xi(x, \zeta_0), (D_x f)\big(\tau^{\frac{1}{2}} x\big) \rangle + \tau^{-\frac{1}{2}} \mathscr{O}(1)\big|^2 dx$$

$$= \tau^{2m-1-\frac{n}{2}} \int_{\mathbb{R}^n} e^{2\big(G(y) + \tau^{-\frac{1}{2}} |y|^3 \mathscr{O}_\alpha(1)\big)} \big|\langle y, p'_x(0, \zeta_0) \rangle f(y) + \langle p'_\xi(\tau^{-\frac{1}{2}} y, \zeta_0), (D_x f)(y) \rangle + \tau^{-\frac{1}{2}} \mathscr{O}(1)\big|^2 dx,$$

with the change of variable $y = \tau^{\frac{1}{2}} x$. The dominated convergence theorem yields

$$(B.26) \qquad \|e^{\tau\phi} P u_\tau\|_{L^2}^2 \underset{\tau\to\infty}{\sim} \tau^{2m-1-\frac{n}{2}} \int_{\mathbb{R}^n} e^{2G(y)} \big| \langle y, p_x'(0, \zeta_0) \rangle f(y) + \langle p_\xi'(0, \zeta_0), (D_x f)(y) \rangle \big|^2 dy,$$

with (B.25) and estimate (B.24), where the integral *does not* vanish. In particular, if $m \geq 2$ this yields $\zeta_0 \neq 0$ that rules out the possibility of having $\varphi'(x_0) = 0$, i.e., $\psi'(x_0) = 0$ (see the first observations at the beginning of the proof).

With (B.24) we obtain

$$\sum_{|\beta|<m} \big(\alpha\varphi(0)\big)^{2(m-|\beta|)-1} |\zeta_0^\beta|^2 \int_{\mathbb{R}^n} e^{2G(y)} |f(y)|^2 dy \leq K \int_{\mathbb{R}^n} e^{2G(y)} \big| \langle y, p_x'(0, \zeta_0) \rangle f(y) + \langle p_\xi'(0, \zeta_0), (D_x f)(y) \rangle \big|^2 dy.$$

Changing f into $e^{-G} f$ yields

$$\sum_{|\beta|<m} \big(\alpha\varphi(0)\big)^{2(m-|\beta|)-1} |\zeta_0^\beta|^2 \|f\|_{L^2}^2 \leq K \|L(x, D_x) f\|_{L^2}^2,$$

with $L(x, D_x) = \langle x, p_x'(0, \zeta_0) \rangle + \langle p_\xi'(0, \zeta_0), D_x - (D_x G) \rangle$. We compute

$$\frac{1}{2i} \{\overline{L}, L\} = \sum_{j,k} \partial^2_{x_j x_k} \varphi(0) \, \partial_{\xi_j} p(0, \zeta_0) \, \partial_{\xi_k} \overline{p}(0, \overline{\zeta_0}) + \mathrm{Im} \sum_j \partial_{x_j} p(0, \zeta_0) \, \partial_{\xi_j} \overline{p}(0, \overline{\zeta_0}).$$

Lemma 28.2.2 in [Hör85a] gives

$$(B.27) \qquad \sum_{|\beta|<m} \big(\alpha\varphi(0)\big)^{2(m-|\beta|)-1} |\big(\xi_0 + i\varphi'(0)\big)^\beta|^2 \leq 2K \Big(\sum_{j,k} \partial^2_{x_j x_k} \varphi(0) \, \partial_{\xi_j} p(0, \xi_0 + i\varphi'(0)) \, \partial_{\xi_k} \overline{p}(0, \xi_0 - i\varphi'(0))$$
$$+ \mathrm{Im} \sum_j \partial_{x_j} p(0, \xi_0 + i\varphi'(0)) \, \partial_{\xi_j} \overline{p}(0, \xi_0 - i\varphi'(0)) \Big).$$

Let now $\tau > 0$, $x \in X$, and $\xi \in \mathbb{R}^n$ be such that $p(x, \xi + \tau\varphi'(x)) = 0$. Setting $x_0 = x$, $\xi_0 = \xi/\tau$ we can use (B.27). By homogeneity we then have

$$(B.28)$$
$$\sum_{|\beta|<m} \big(\tau\alpha\varphi(x)\big)^{2(m-|\beta|)-1} |\big(\xi + i\tau\varphi'(x)\big)^\beta|^2 \leq 2K \Big(\tau \sum_{j,k} \partial^2_{x_j x_k} \varphi(x) \, \partial_{\xi_j} p(x, \xi + i\tau\varphi'(x)) \, \partial_{\xi_k} \overline{p}(x, \xi - i\tau\varphi'(x))$$
$$+ \mathrm{Im} \sum_j \partial_{x_j} p(x, \xi + \tau i\varphi'(x)) \, \partial_{\xi_j} \overline{p}(x, \xi - i\tau\varphi'(x)) \Big).$$

We conclude to (5.2) thanks to (3.4)-(3.5).

The last statement of of the first part of Lemma 5.1 is proven in [Hör85a, Theorem 28.2.1]. The different statements in the second point of the lemma follow as above. ∎

References

[Ali83]   S. Alinhac, *Non-unicité du problème de Cauchy*, Ann. of Math. (2) **117** (1983), no. 1, 77–108.

[Bar00]   V. Barbu, *Exact controllability of the superlinear heat equation*, Appl. Math. Optim. **42** (2000), 73–89.

[BC94]   J.-M. Bony and J.-Y. Chemin, *Espaces fonctionnels associés au calcul de Weyl-Hörmander*, Bull. Soc. Math. France **122** (1994), 77–118.

[BK81]   A. L. Bukhgeim and M. V. Klibanov, *Global uniqueness of class of multidimensional inverse problems*, Soviet Math. Dokl. **24** (1981), 244–247.

[BL13]   M. Bellassoued and J. Le Rousseau, *Carleman estimates for elliptic boundary value problems*, Preprint (2013), 50 pages.

[BLR92]   C. Bardos, G. Lebeau, and J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim. **30** (1992), 1024–1065.

[Bon99]   J.-M. Bony, *Sur l'inégalité de Fefferman-Phong*, Sémin. Equ. Dériv. Partielles, Ecole Polytech., Palaiseau, France (1999).

[Bru90]   R. Brummelhuis, *A counterexample to the Fefferman-Phong inequality for systems*, C. R. Acad. Sci. Paris Sér. I Math. **310** (1990), no. 3, 95–98.

[BY12] M. Bellassoued and M. Yamamoto, *Carleman estimate with second large parameter for second order hyperbolic operators in a Riemannian manifold and applications in thermoelasticity cases*, Appl. Anal. **91** (2012), no. 1, 35–67.

[Cal58] A.-P. Calderón, *Uniqueness in the Cauchy problem for partial differential equations.*, Amer. J. Math. **80** (1958), 16–36.

[Car39] T. Carleman, *Sur une problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes*, Ark. Mat. Astr. Fys. **26B** (1939), no. 17, 1–9.

[CDS95] F. Colombini and D. Del Santo, *Condition (P) is not sufficient for uniqueness in the Cauchy problem*, Comm. Partial Differential Equations **20** (1995), 2113–2128.

[CDSZ96a] F. Colombini, D. Del Santo, and C. Zuily, *The Fefferman-Phong inequality in the locally temperate Weyl calculus*, Osaka J. Math. **33** (1996), 847–861.

[CDSZ96b] ———, *Singular principal normality in the Cauchy problem*, J. Math. Kyoto Univ. **36** (1996), 365–384.

[Deh84] B. Dehman, *Unicité du problème de cauchy pour une classe d'opérateurs quasi-homogènes*, J. Math. Kyoto Univ. **24** (1984), 453–471.

[DSFKSU09] D. Dos Santos Ferreira, C. E. Kenig, M. Salo, and G. Uhlmann, *Limiting carleman weights and anisotropic inverse problems*, Invent. Math. **178** (2009), 119–171.

[EI00] M. Eller and V. Isakov, *Carleman estimates with two large parameters and applications*, Contemporary Math. **268** (2000), 117–136.

[Ell00] M. Eller, *Carleman estimates with a second large parameter*, Journal of Mathematical Analysis and Applications **249** (2000), 491–514.

[FCZ00] E. Fernández-Cara and E. Zuazua, *Null and approximate controllability for weakly blowing up semilinear heat equations*, Ann. Inst. H. Poincaré, Analyse non lin. **17** (2000), 583–616.

[FI96] A. Fursikov and O. Yu. Imanuvilov, *Controllability of evolution equations*, vol. 34, Seoul National University, Korea, 1996, Lecture notes.

[FP78] C. Fefferman and D. H. Phong, *On positivity of pseudo-differential operators*, Proc. Nat. Acad. Sci. **75** (1978), 4673–4674.

[Hör58] L. Hörmander, *On the uniqueness of the Cauchy problem*, Math. Scand. **6** (1958), 213–225.

[Hör63] ———, *Linear Partial Differential Operators*, Springer-Verlag, Berlin, 1963.

[Hör79] ———, *The Weyl calculus of pseudo-differential operators*, Comm. Pure Appl. Math. **32** (1979), 359–443.

[Hör83] Lars Hörmander, *Uniqueness theorems for second order elliptic differential equations*, Comm. Partial Differential Equations **8** (1983), 21–64.

[Hör85a] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, vol. IV, Springer-Verlag, 1985.

[Hör85b] ———, *The Analysis of Linear Partial Differential Operators*, vol. III, Springer-Verlag, 1985, Second printing 1994.

[Hör90] ———, *The Analysis of Linear Partial Differential Operators*, second ed., vol. I, Springer-Verlag, 1990.

[IIY03] O Yu. Imanuvilov, V. Isakov, and M. Yamamoto, *An inverse problem for the dynamical Lamé system with two sets of boundary data*, Comm. Pure Appl. Math. **56** (2003), 1366–1382.

[IK08] V. Isakov and N. Kim, *Carleman estimates with second large parameter and applications to elasticity with residual stress*, Applicationes Mathematicae **35** (2008), 447–465.

[Isa93] V. Isakov, *Carleman type estimates in an anisotropic case and applications*, J. Differential Equations **105** (1993), 217–238.

[Isa98] ———, *Inverse problems for partial differential equations*, Springer-Verlag, Berlin, 1998.

[JK85] D. Jerison and C. E. Kenig, *Unique continuation and absence of positive eigenvalues for Schrödinger operators*, Ann. of Math. (2) **121** (1985), no. 3, 463–494, With an appendix by E. M. Stein.

[KSU07] C. E. Kenig, J. Sjöstrand, and G. Uhlmann, *The Calderón problem with partial data*, Ann. of Math. **165** (2007), 567–591.

[KT01] H. Koch and D. Tataru, *Carleman estimates and unique continuation for second-order elliptic equations with non-smooth coefficients*, Comm. Pure Appl. Math. **54** (2001), no. 3, 339–360.

[KT02] ———, *Sharp counterexamples in unique continuation for second order elliptic equations*, J. Reine Angew. Math. **542** (2002), 133–146.

[Ler85] N. Lerner, *Unicité de Cauchy pour des opérateurs différentiels faiblement principalement normaux*, J. Math. Pures Appl. **64** (1985), 1–11.

[Ler10] ———, *Metrics on the Phase Space and Non-Selfadjoint Pseudo-Differential Operators*, Pseudo-Differential Operators, Vol. 3, Birkhäuser, Basel, 2010.

[LL] J. Le Rousseau and N. Lerner, *Carleman estimates for anisotropic elliptic operators with jumps at an interface*, Anal. PDE, to appear.

[LL12] J. Le Rousseau and G. Lebeau, *On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations*, ESAIM: Control, Optimisation and Calculus of Variations **18** (2012), 712–747.

[LM68] J.-L. Lions and E. Magenes, *Problèmes aux Limites Non Homogènes*, vol. 1, Dunod, 1968.

[LR95]   G. Lebeau and L. Robbiano, *Contrôle exact de l'équation de la chaleur*, Comm. Partial Differential Equations **20** (1995), 335–356.

[LR10]   J. Le Rousseau and L. Robbiano, *Carleman estimate for elliptic operators with coefficents with jumps at an interface in arbitrary dimension and application to the null controllability of linear parabolic equations*, Arch. Rational Mech. Anal. **105** (2010), 953–990.

[LR11]   _____ , *Local and global Carleman estimates for parabolic operators with coefficients with jumps at interfaces*, Invent. Math. **183** (2011), 245–336.

[LR85]   N. Lerner and L. Robbiano, *Unicité de Cauchy pour des opérateurs de type principal*, J. Analyse Math. **44** (1984/85), 32–66.

[Sog89]   C. D. Sogge, *Oscillatory integrals and unique continuation for second order elliptic differential equations*, J. Amer. Math. Soc. **2** (1989), no. 3, 491–515.

[Zui83]   C. Zuily, *Uniqueness and Non Uniqueness in the Cauchy Problem*, Birkhauser, Progress in mathematics, 1983.

Jérôme Le Rousseau.

Laboratoire de Mathématiques - Analyse, Probabilités, Modélisation - Orléans, CNRS UMR 7349, Fédération Denis-Poisson, FR CNRS 2964, Université d'Orléans, B.P. 6759, 45067 Orléans cedex 2, France.

Institut Universitaire de France.

E-mail address: `jlr@univ-orleans.fr`