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# Taking into account period variations and actuators saturation in sampled-data systems

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## Abstract

This paper deals with the problem of stability and stabilization of sampled-data systems under asynchronous samplings and actuators saturation. The method is based, on the first hand, on the use of a novel class of Lyapunov functionals whose derivative is negative along the trajectories of the continuous-time model of the sampled data system. It is shown that this fact guarantees that a quadratic Lyapunov function is strictly decreasing for the discrete-time asynchronous system. On the other side, the control saturation is taken into account from the use of a modified sector condition. These ingredients lead to the formulation of improved LMI conditions that can be cast in optimization problems aiming at enlarging estimates of the region of attraction of the closed-loop system or maximizing the bounds on the sampling period jitter for which stability and stabilization are ensured.

**Keywords:** Linear systems, saturation, sampled-data systems, aperiodic sampling.

## 1 Introduction

In the past decade, a large attention has been devoted to sampled-data systems [10]. This area addresses the problem of stability and stabilization of systems evolving in continuous-time whereas the controller delivers control inputs in discrete-time. Although the theory for linear sampled-data control systems is rather well established for the case of a constant sampling period [27], the problem of considering asynchronous sampling is an open domain of research. In particular, the interest to study sample-data systems has increased with the spreading of the so-called networked control systems [33]. In such applications several distributed plants are controlled over a communication network and the controllers can be implemented in a decentralized way (i.e. in local processors). In this case, communication delays, package losses and heavy temporary load of computation in a processor can lead to significant variations on the sampling time. These variations can dramatically affect the stability properties of the control system. It is then clear the motivation for developing robust stability and stabilization conditions which take into account the variations on the sampling period.

The sampling in a control loop can be modeled by a time-varying delay on the plant control input while considering a continuous-time dynamics. In this case, the delay variation rate between two sampling instants is equal to one. Thus, a fundamental problem regards the determination of the bounds on the delay variation (corresponding to the sampling period jitter), for which stability of the closed-loop system can be kept. Considering this perspective, we can cite for example the approach proposed in [5], and the improvements made in [18] and [4].

On the other side, many techniques have been proposed in the literature to deal with the stabilization of time-delay systems subject to actuator saturation. In this context, we can cite, for instance: [19] and [21], where globally stabilizing control laws are proposed; and [30], [2] and [5], where the regional stabilization problem is considered. These works are mainly concerned by state delayed systems. The stabilization

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conditions consider delay independent approaches (which allow to address time-varying delays, but in a possible conservative way) and delay dependent conditions considering fixed delays. Recently, in [7] and [34], results considering systems presenting time-varying delays on the states have been proposed. On the other hand, we can note a lack of results considering input delays, and, in particular, time-varying ones. In [9] and [31], anti-windup techniques are proposed for systems with fixed input delays.

In this paper we are interested in the problem of stabilizing a sampled-data system taking into account the possible variations on the sample period (due to packet losses in the network) and also the fact that the signals provided by the actuators are bounded (i.e saturating plant inputs). The method is based on the use of a particular functional that, differently from the Lyaunov-Krasovskii based approaches adopted for instance in [5], [18] and [4], does not need to be positive definite. It is shown that if the time-derivative of this functional along the trajectories of the continuous-time model is strictly negative, then a quadratic Lyapunov function is strictly decreasing for the discrete-time asynchronous system. On the other hand, the control saturation is taken into account from the use of a generalized sector condition. These ingredients lead to the formulation of LMI conditions that can be cast in optimization problems aiming at enlarging estimates of the region of attraction of the closed-loop system or maximizing the bounds on sampling period jitter for which the stability is kept.

The paper is organized as follows. The next section describes the problem formulation. Section 3 presents some preliminary lemmas on a LMI manipulation, on the generalized sector condition. Section 5 provides a generic result to ensure the stability of sample-data systems with constrained control signals. Based on this result, LMI-based stabilization conditions allowing to compute a state feedback gain in order to guarantee the regional (local) or the global stability of the closed-loop sample data system in the presence of actuator saturation is provided in section 5. In section 6, some optimization problems are proposed. Some numerical examples illustrating the potentialities of the method are provided in section 7.

**Notation.** Throughout the article, the sets  $\mathbb{N}$ ,  $\mathbb{R}^+$ ,  $\mathbb{R}^n$ ,  $\mathbb{R}^{n \times n}$  and  $\mathbb{S}^n$  denote respectively the set of positive integers, positive scalars,  $n$ -dimensional vectors,  $n \times n$  matrices and symmetric matrices of  $\mathbb{R}^{n \times n}$ . For a given positive scalar,  $\mathcal{T}_2$ , define  $\mathbb{K}$  as the set of continuous functions from an interval  $[0, T]$  to  $\mathbb{R}^n$ , where  $T$  is a positive scalar less than  $\mathcal{T}_2$ . The notations  $|\cdot|$  and  $\|\cdot\|$  stand for the absolute value of a scalar and for the Euclidean norm of a vector, respectively. The superscript ‘ $T$ ’ stands for matrix transposition. The notation  $P > 0$  for  $P \in \mathbb{S}^n$  means that  $P$  is positive definite. For any positive integer  $j \leq n$  any vector  $x \in \mathbb{R}^n$  and any matrix  $A \in \mathbb{R}^{n \times n}$ , the notations  $A_j$ ,  $x_j$  and  $\text{He}\{A\} > 0$  refer to the  $j^{\text{th}}$  line of matrix  $A$ , the  $j^{\text{th}}$  component of vector  $x$  and  $A + A^T > 0$ , respectively. The symbols  $I$  and  $0$  represent the identity and the zero matrices of appropriate dimension.  $\text{Co}\{\cdot\}$  denotes a convex hull.

## 2 Problem formulation

Let  $\{t_k\}_{k \in \mathbb{N}}$  be an increasing sequence of positive scalars such that  $\bigcup_{k \in \mathbb{N}} [t_k, t_{k+1}[ = [0, +\infty[$ . Assume that there exist two positive scalars  $\mathcal{T}_1 \leq \mathcal{T}_2$  such that the difference between two successive sampling instants  $T_k = t_{k+1} - t_k$  satisfies

$$\forall k \in \mathbb{N}, \quad 0 \leq \mathcal{T}_1 \leq T_k \leq \mathcal{T}_2. \quad (1)$$

Consider the linear system with a sampled-data input

$$\forall t \in [t_k, t_{k+1}[, \quad \dot{x}(t) = Ax(t) + Bu(t_k), \quad (2)$$

where  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$  represent the state variable and the input vector. The matrices  $A$  and  $B$  are assumed to be constant, known and of appropriate dimension. We suppose that the input vector  $u$  is subject to amplitude limitations defined by:

$$|u_i| \leq u_{0i}, \quad u_{0i} > 0, \quad i = 1, \dots, m. \quad (3)$$

Consider a linear state feedback control law  $u(t) = Kx(t)$ , where  $K \in \mathbb{R}^{m \times n}$ . Due to the control bounds defined in (3), the effective control signal to be applied to the system is given by

$$u(t) = \text{sat}(Kx(t_k)), \quad t_k \leq t < t_{k+1}, \quad (4)$$

with  $u_i(t) = \text{sat}(K_i x(t_k)) = \text{sign}(K_i x(t_k)) \min\{u_{0i}, |K_i x(t_k)|\}$ ,  $i = 1, \dots, m$ . Hence, the closed-loop system reads

$$\dot{x}(t) = Ax(t) + B\text{sat}(Kx(t_k)). \quad (5)$$

As noticed in [32], integrating the previous differential equation over a sampling period, the dynamics of the system satisfy

$$\forall \tau \in [0, T_k], \quad \begin{aligned} x(t_k + \tau) &= \Gamma(\tau)x(t_k) + \Upsilon(\tau)\text{sat}(Kx(t_k)), \\ \Gamma(\tau) &= e^{A\tau}, \Upsilon(\tau) = \int_0^\tau e^{A(\tau-\theta)}d\theta B. \end{aligned} \quad (6)$$

In particular, if the sampling period is constant, i.e.  $T_k = T \forall k \in \mathbb{N}$ , the dynamics become

$$x(t_{k+1}) = \Gamma(T)x(t_k) + \Upsilon(T)\text{sat}(Kx(t_k)),$$

and stability of the closed-loop system can be analyzed by the techniques proposed, for instance in [8]. Note that in the absence of the control saturation, it suffices that matrix  $(\Gamma(T) + \Upsilon(T)K)$  has all eigenvalues inside the unit circle to ensure the asymptotic stability of the closed-loop system. On the other hand, if the sampling period is time-varying, this does not hold anymore. Relevant stability analysis based on an uncertain representation of  $(\Gamma(T_k) + \Upsilon(T_k)K)$  (without saturations) have been already investigated for instance in [11, 20, 28]. However, extensions to systems with saturated control input lead to additional difficulties due to the definition of  $\Gamma(T_k)$  and  $\Upsilon(T_k)$ .

Concerning the analysis based directly on the continuous-time model (2), sufficient stability conditions were firstly designed using the Lyapunov-Kravovskii theorem for time-delay systems in [5] and refined in [18, 17, 4, 25]. This approach consists in modeling the sampling effect as an input varying delay and to apply results issued from the time-delay systems theory to derive stability criteria expressed with respect to the continuous-time model. However, this approach is still conservative in comparison to discrete-time approaches.

Although the system dynamics are considered to be linear, due to the control saturation, the closed-loop system is nonlinear. Hence, the determination of a global stabilizing controller is possible only when some stability assumptions are verified by the open-loop system ( $u(t) = 0$ ) (see [15, 29]). When this hypothesis is not verified, it is only possible to achieve semi-global or local/regional stabilization. In this case, given a stabilizing matrix  $K$ , we associate a *region of attraction* to the equilibrium point  $x_e(t) \equiv 0$  of the system (5). The region of attraction corresponds to all initial conditions  $x_0 \in \mathbb{R}$  such that the corresponding trajectories of the system (5) converge asymptotically to the origin [13]. Since the determination of the exact region of attraction is practically impossible, a problem of interest is to ensure asymptotic stability for a set of admissible initial conditions  $x_0$ . Hence, from the considerations above, in this paper, we are interested in studying the stabilization problems stated as follows.

- P1.** Given  $\mathcal{T}_1, \mathcal{T}_2$ , find  $K$  and a set of admissible initial conditions, as large as possible, for which asymptotic stability of the closed-loop system (5) is ensured.
- P2.** Maximize the bound on the delay  $\mathcal{T}_2$ , for which asymptotic stability of the closed-loop system (5) can be ensured for some set of admissible initial conditions.

Regarding a networked control system, problem **P1** can be seen as a controller design problem, which is defined from the network schedule constraints. On the other hand, problem **P2** can be seen as a network design problem. In this case, the network designer will consider the maximum allowable jitter (for which stability of the system can be guaranteed) as a constraint in the network scheduling. Of course, when it is possible, the objective will be global stabilization of the closed-loop system. Otherwise, a set of admissible initial conditions, included in the region of attraction of the closed-loop system, will be defined. This set can be seen as an estimate of the actual region of attraction and defines a region of “safe initialization” for the system.

In order to develop conditions to solve problems **P1** and **P2**, in the sequel a particular notation is adopted. For all integers  $k \in \mathbb{N}$ , a function  $\chi_k \in \mathbb{K}$  can be defined such that equation (5) can be equivalently represented by:

$$\forall \tau \in [0, T_k], \quad \begin{cases} \chi_k(\tau) = x(t_k + \tau), \\ \dot{\chi}_k(\tau) = \frac{d}{d\tau}\chi_k(\tau) = A\chi_k(\tau) + B\text{sat}(K\chi_k(0)). \end{cases} \quad (7)$$

### 3 Preliminaries

The following lemmas will be used in the sequel.

**Lemma 1** Let  $P$  be a positive definite matrix and define  $\tilde{P} = M^T P M$ , with  $M$  being a nonsingular matrix. If

$$\begin{bmatrix} P_0 & I \\ I & M + M^T - \tilde{P} \end{bmatrix} > 0, \quad (8)$$

then  $P < P_0$ .

**Proof.** If  $P > 0$  it follows that

$$(P^{-1} - M)^T P (P^{-1} - M) \geq 0,$$

which implies that  $P^{-1} \geq (M^T + M - \tilde{P})$ , or equivalently,  $P \leq (M^T + M - \tilde{P})^{-1}$ . Hence, from Schur's complement, if (8) is verified it follows that  $P_0 > (M^T + M + \tilde{P})^{-1}$ , which implies that  $P < P_0$ . □

Using the notation  $\chi_k$  introduced in (7), the following dead-zone function is defined

$$\psi(K\chi_k(0)) = \psi_k = K\chi_k(0) - \text{sat}(K\chi_k(0)). \quad (9)$$

Note that,  $\psi(K\chi_k(0))$  corresponds to a decentralized dead-zone nonlinearity. Considering the function  $\psi(K\chi_k(0))$ , the closed-loop system can be re-written as

$$\dot{\chi}_k(\tau) = A\chi_k(\tau) + BK\chi_k(0) - B\psi(K\chi_k(0)).$$

Consider now a matrix  $G \in \mathbb{R}^{m \times n}$  and define the polyhedral set

$$\mathcal{S} = \{x \in \mathbb{R}^n ; |(K_i - G_i)x| \leq u_{0i}, i = 1, \dots, m\}.$$

The following Lemma from [9], concerning the nonlinearity  $\psi(K\chi_k(0))$  is recalled.

**Lemma 2** Consider the function  $\psi(K\chi_k(0))$  defined in (9). If  $\chi_k(0) \in \mathcal{S}$  then the relation

$$\psi^T(K\chi_k(0))U[\psi(K\chi_k(0)) - G\chi_k(0)] \leq 0, \quad (10)$$

is verified for any matrix  $U \in \mathbb{R}^{m \times m}$  diagonal and positive definite.

The result in Lemma 2 can be seen as a generalized sector condition. As will be seen in the sequel, differently from the classical sector condition (used for instance in [31]), this condition will allow to obtain stability conditions directly in an LMI form. For notation simplicity, in the sequel we denote  $\psi(K\chi_k(0))$  as  $\psi_k$ .

## 4 Asymptotic stability of saturated and sampled-data systems

This section is motivated by the difference between the discrete and continuous-time Lyapunov theorems. As the problem of sampled-data systems is at the boundary of the discrete and the continuous-time theories, it is important to focus on the connections between them. More specifically, the main idea of this section consists in developing novel stability conditions for sampled-data systems with control saturation, modeled in continuous-time, using the discrete-time Lyapunov theorem.

**Theorem 1** Consider given matrices  $K$  and  $G$  in  $\mathbb{R}^{m \times n}$  and any positive definite diagonal matrix  $U$  in  $\mathbb{R}^{m \times m}$ . Let  $\mathcal{T}_1$  and  $\mathcal{T}_2$ ,  $\mathcal{T}_1 < \mathcal{T}_2$ , be two positive scalars and  $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$  be a function for which there exist real numbers  $0 < \mu_1 < \mu_2$  and  $p > 0$  such that

$$\forall x \in \mathbb{R}^n, \quad \mu_1|x|^p \leq V(x) \leq \mu_2|x|^p. \quad (11)$$

and such that, for all  $i = 1, \dots, m$  and  $x \in \mathbb{R}^n$

$$x^T(K_i - G_i)^T(K_i - G_i)x \leq u_{0i}^2 V(x). \quad (12)$$

Then, the two following statements are equivalent.

(i) For all  $k \in \mathbb{N}$ ,  $T_k \in [\mathcal{T}_1, \mathcal{T}_2]$ , the increment of the Lyapunov function satisfies

$$\Delta V(k) - 2T_k \psi_k^T U [\psi_k - G\chi_k(0)] < 0;$$

where  $\Delta V(k) = V(\chi_k(T_k)) - V(\chi_k(0))$

(ii) There exists a continuous functional  $\mathcal{V}_0 : [0, \mathcal{T}_2] \times \mathbb{K} \rightarrow \mathbb{R}$  which satisfies for all  $z \in \mathbb{K}$

$$\forall T_k \in [\mathcal{T}_1, \mathcal{T}_2] \quad \mathcal{V}_0(T_k, z) = \mathcal{V}_0(0, z). \quad (13)$$

and such that, for all  $k \in \mathbb{N}$ ,  $T_k \in [\mathcal{T}_1, \mathcal{T}_2]$  and  $\tau \in [0, T_k[$  and

$$\dot{W}(\tau, \chi_k) = \frac{d}{d\tau} [V(\chi_k(\tau)) + \mathcal{V}_0(\tau, \chi_k) - 2\tau \psi_k^T U [\psi_k - G\chi_k(0)]] < 0, \quad (14)$$

Moreover, if one of these two statements is satisfied, then, for all initial conditions  $x(0) = \chi_0(0)$  in the set  $\mathcal{E}$  defined by

$$\mathcal{E} = \{x \in \mathbb{R}^n; \quad V(x) \leq 1\}, \quad (15)$$

the solutions of system (2) with the saturated and sampled control law (4) converge asymptotically to the origin.

**Proof.** Consider a positive integer  $k$  and  $\tau \in [0, T_k[$ . Assume (ii) is satisfied. Integrating  $\dot{W}$  over the interval  $[0, T_k[$  and assuming that (13) and (14) hold, this directly implies  $\Delta V(k) < 2T_k \psi_k^T U [\psi_k - G\chi_k(0)]$  and then (i) holds.

Assume now that (i) is satisfied. Inspired by Lemma 2 in [22], consider the functional  $\mathcal{V}_0(\tau, \chi_k) = -V(\chi_k(\tau)) + \tau/T_k \Delta V(k)$ . Indeed,  $\mathcal{V}_0$  is a functional since it is expressed with respect to  $\Delta V(k)$  which depends on the function  $\chi_k(0)$ ,  $\chi_k(T_k)$  and  $\chi_k(\tau)$  for all  $\tau \in [0, T_k]$ . By simple computations, it is easy to obtain that this functional satisfies (13) and that

$$\dot{W}(\tau, \chi_k) = 1/T_k (\Delta V(k) - 2T_k \psi_k^T U [\psi_k - G\chi_k(0)]).$$

Then  $\dot{W}$  is negative if (i) holds. This proves the equivalence between (i) and (ii).

Since  $x(0)$  belongs to  $\mathcal{E}$ , inequality (12) directly implies that  $|(K_i - G_i)\chi_0(0)| \leq u_{i0}$ . Consequently  $\chi_0(0)$  also belongs to  $\mathcal{S}$  and, according to Lemma 2, the quantity  $\psi_0^T U [\psi_0 - G\chi_0(0)]$  is negative. Hence the increment of the Lyapunov function  $\Delta V(0)$  is negative and  $\chi_0(T_0) = \chi_1(0)$  also belongs to  $\mathcal{E}$ . Repeating the reasoning for  $k = 1, 2, \dots$ , we conclude that  $\chi_k(0) \in \mathcal{E}$  and  $\psi_k^T U [\psi_k - G\chi_k(0)] \leq 0$ , for all  $k \in \mathbb{N}$ . From the discrete-time Lyapunov theorem, the equilibrium of the discrete-time system is asymptotically stable and  $\chi_k(0)$  tends to zero as  $k$  tends to infinity.

The end of the proof consists in ensuring that the solutions of the continuous-time system are not diverging within a sampling period. From (6) and (11), the inequality

$$\begin{aligned} V(\chi_k(\tau)) &\leq \mu_2 |\Gamma(\tau)\chi_k(0) + \Upsilon(\tau) \text{sat}(K\chi_k(0))|^p, \\ &\leq \mu_2 (|\Gamma(\tau)|^p + |\Upsilon(\tau)K|^p) |\chi_k(0)|^p, \end{aligned}$$

is ensured, for all integer  $k$ , and  $\tau \in [0, T_k]$ . Noting that the functions  $\Gamma(\cdot) : [0, T_k] \rightarrow \mathbb{R}^{n \times n}$  and  $\Upsilon(\cdot) : [0, T_k] \rightarrow \mathbb{R}^{n \times m}$  are continuous and consequently bounded over  $[0, T_k]$ , there exists a positive scalar  $\mu_m$  such that,

$$|\Gamma(\tau)|^p + |\Upsilon(\tau)K|^p \leq \mu_m.$$

This ensures  $V(\chi_k(\tau)) \leq \mu_2 \mu_m |\chi_k(0)|^p$ , for all  $\tau$  in  $[0, T_k]$ . This proves that the continuous Lyapunov function uniformly and asymptotically tends to zero. □

In the literature several articles have introduced functionals satisfying the requirements of Theorem 1 (see for instance [18, 4]). However, those results use the Lyapunov-Krasovskii theorem which requires necessarily the positive definiteness of the functional  $\mathcal{V}_0$ . Theorem 1 relaxes the constraint on the positivity of the functional. The only requirement to ensure stability is on the discrete-time Lyapunov function  $V$ , and (13).

A graphical illustration of Theorem 1 is shown in Figure 1. The main idea consists in showing the equivalence between the conditions on the decreasing increment  $\Delta V(k) = V(\chi_k(T_k)) - V(\chi_k(0)) < 0$  and

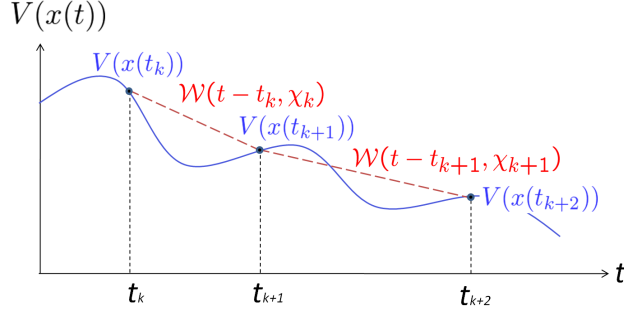


Figure 1: Illustration of the proof of Theorem 1

the existence of a continuous functional  $\mathcal{W}$  which coincides with the Lyapunov function  $V$  at the sampling instants and which is strictly decreasing within all sampling intervals. The main contribution of Theorem 1 is that the introduction of the functional  $\mathcal{W}$  allows the Lyapunov function  $V$  to be locally increasing.

Under the conditions of Theorem 1, it follows that the set  $\mathcal{E}$  is included in the region of attraction of the closed-loop system (5). Note this set is not necessarily positively invariant for the continuous-time system, but it is indeed for the discrete-time system, i.e. if  $\chi_0(0) \in \mathcal{E}$  it follows that  $\chi_k(0) \in \mathcal{E}$ , for all  $k > 0$ . Nonetheless, the convergence of the continuous-time system trajectories to the origin is guaranteed for any initial condition belonging to  $\mathcal{E}$ , i.e. the regional asymptotic stability is ensured. On the other hand, when the open-loop system is asymptotically stable, we may attempt to compute a control law that globally stabilizes the origin of the closed-loop system. The following corollary gives support to this case.

**Corollary 1** Consider a given matrix  $K$  in  $\mathbb{R}^{m \times n}$  and any positive definite diagonal matrix  $U$  in  $\mathbb{R}^{m \times m}$ . Let  $\mathcal{T}_1$  and  $\mathcal{T}_2$ ,  $\mathcal{T}_1 < \mathcal{T}_2$ , be two positive scalars and  $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$  be a function for which there exist real numbers  $0 < \mu_1 < \mu_2$  and  $p > 0$  such that condition (11) holds. Then the two following statements are equivalent.

(i) For all  $k \in \mathbb{N}$ ,  $T_k \in [\mathcal{T}_1, \mathcal{T}_2]$ , the increment of the Lyapunov function satisfies

$$\Delta V(k) - 2T_k \psi_k^T U [\psi_k - K \chi_k(0)] < 0;$$

where  $\Delta V(k) = V(\chi_k(T_k)) - V(\chi_k(0))$

(ii) There exists a continuous functional  $\mathcal{V}_0 : [0, \mathcal{T}_2] \times \mathbb{K} \rightarrow \mathbb{R}$  which satisfies for all  $z \in \mathbb{K}$

$$\forall T_k \in [\mathcal{T}_1, \mathcal{T}_2] \quad \mathcal{V}_0(T_k, z) = \mathcal{V}_0(0, z). \quad (16)$$

and such that, for all  $k \in \mathbb{N}$ ,  $T_k \in [\mathcal{T}_1, \mathcal{T}_2]$  and  $\tau \in [0, T_k]$  and

$$\dot{\mathcal{V}}(\tau, \chi_k) = \frac{d}{d\tau} [V(\chi_k(\tau)) + \mathcal{V}_0(\tau, \chi_k) - 2\tau \psi_k^T U [\psi_k - K \chi_k(0)]] < 0. \quad (17)$$

Moreover, if one of these two statements is satisfied, then the origin of system (2) with the saturated and sampled control law (4) is globally asymptotically stable.

**Proof.** The proof follows the same lines of the one of Theorem 1. The difference is that now, from the definition of the dead-zone function  $\psi_k$ , the relation  $\psi_k^T U [\psi_k - K \chi_k(0)] \leq 0$  holds for any  $\chi_k(0)$ . Then, it is ensured that  $\Delta V(k) < 0$ , for any  $\chi_k(0) \in \mathbb{R}^n$ , and global stability follows.  $\square$

Based on an appropriate choice of the functional  $\mathcal{V}_0$  and the application of Theorem 1 and Corollary 1, we derive LMI-based conditions for the stabilization of sampled data systems (1) under the saturated control law defined in (4), both in regional and global contexts.



**Remark 1** *The technique to handle saturation applied here follows the concepts used in the problem of absolute stability [13]. Basically, the idea consists in relaxing the conditions for the satisfaction of the decreasing of the Lyapunov function just for nonlinearities satisfying a sector bound condition. It is however important to emphasize that, as pointed in [9], considering deadzone nonlinearities, the relation (10) can be viewed as a generalized sector condition which encompasses the classical one used, for instance in [12] and [14]. This fact leads to a clear reduction of the conservatism in stability and stabilization conditions for systems with saturation nonlinearities. Moreover, as it will be clear in the sequel, the synthesis conditions can be obtained in an LMI form, while the classical approach leads to BMI conditions (see [23], Section 1.7.2, for a comprehensive discussion about this matter)*

## 5 Stabilization of sampled-data systems under input saturation

In this section, asymptotic stabilization conditions (both in regional as well global contexts) of the sampled-data system (2) with the saturating control law given by (4) are derived from the results of Theorem 1. These conditions allow the computation of a gain  $K$  that ensures the asymptotic stability of the closed-loop system.

**Theorem 2** *For given positive scalars  $\mathcal{T}_1 < \mathcal{T}_2$ , assume that there exist positive definite matrices  $\tilde{P}, \tilde{R} \in \mathbb{S}^n$ , a matrix  $\tilde{S}_1 \in \mathbb{R}^{n \times n}$ , a matrix  $\tilde{X} \in \mathbb{S}^{n+m}$ , a positive definite diagonal matrix  $\tilde{U} \in \mathbb{S}^m$ , matrices  $\tilde{Y}, \tilde{S}_2 \in \mathbb{R}^{n \times n}$  and  $\tilde{N} \in \mathbb{R}^{(3n+m) \times n}$ , two matrices  $\tilde{K}$  and  $\tilde{G} \in \mathbb{R}^{m \times n}$  and a positive scalar  $\epsilon$  that satisfy, for  $i = 1, 2$*

$$\Psi_1(\mathcal{T}_i) = \tilde{\Pi}_1 + \mathcal{T}_i \tilde{\Pi}_2 + \mathcal{T}_i \tilde{\Pi}_3 < 0, \quad \Psi_2(\mathcal{T}_i) = \begin{bmatrix} \tilde{\Pi}_1 - \mathcal{T}_i \tilde{\Pi}_3 & \mathcal{T}_i \tilde{N} \\ * & -\mathcal{T}_i \tilde{R} \end{bmatrix} < 0, \quad (18)$$

$$\Psi_{3j}(u_{0j}) = \begin{bmatrix} \tilde{P} & (\tilde{K} - \tilde{G})_j^T \\ * & u_{0j}^2 \end{bmatrix} \geq 0, \quad \forall j = 1, \dots, m, \quad (19)$$

with

$$\begin{aligned} \tilde{\Pi}_1 = & \text{He} \left\{ M_1^T \tilde{P} M_3 - M_{12}^T \tilde{S}_2 M_2 - \tilde{N} M_{12} \right\} - M_{12}^T \tilde{S}_1 M_{12} \\ & + \text{He} \left\{ (\epsilon M_1^T + M_3^T)(A \tilde{Y} M_1 + B \tilde{K} M_2 - \tilde{Y} M_3 - B \tilde{U} M_4) \right\} \\ & - 2M_4^T \tilde{U} M_4 + \text{He} \{ M_4^T \tilde{G} M_2 \}, \end{aligned} \quad (20)$$

$$\tilde{\Pi}_2 = M_3^T \tilde{R} M_3 + \text{He} \left\{ M_3^T (\tilde{S}_1 M_{12} + \tilde{S}_2 M_2) \right\}$$

$$\tilde{\Pi}_3 = M_{24}^T \tilde{X} M_{24},$$

where <sup>1</sup>

$$\begin{aligned} M_1 = & \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 & I & 0 \end{bmatrix}, \\ M_4 = & \begin{bmatrix} 0 & 0 & 0 & I \end{bmatrix}, \quad M_{12} = M_1 - M_2, \quad M_{24} = [M_2^T \ M_4^T]^T. \end{aligned}$$

Then, for all initial conditions  $x_0 = x(0)$  belonging to the ellipsoidal set

$$\mathcal{E}_P = \{x \in \mathbb{R}^n; x^T P x \leq 1\},$$

where  $P = \tilde{Y}^{-T} \tilde{P} \tilde{Y}^{-1}$ , the corresponding trajectories of the system (2) under the saturated control law defined in (4) with  $K = \tilde{K} \tilde{Y}^{-1}$  converge asymptotically to the origin for any asynchronous sampling satisfying (1).

**Proof.** Introduce a quadratic Lyapunov function candidate defined, for any  $x$  in  $\mathbb{R}^n$ , by  $V(x) = x^T P x$ , where  $P$  is a symmetric positive definite matrix from  $\mathbb{S}^n$ . Thus, the function  $V$  satisfies (11) since it has a quadratic form.

Considering now the result of Theorem 1, the idea is to prove that  $\Delta V_k = V(\chi_k(T_k)) - V(\chi_k(0)) - 2T_k \psi_k^T U [\psi_k - G \chi_k(0)] < 0$ , for all  $k \in \mathbb{N}$ . With this aim an appropriate functional  $\mathcal{V}_0$  satisfying (13) and (14) must be chosen. A candidate of such type of functionals is defined for all  $\tau \in [0 \ T_k]$ , as follows:

$$\begin{aligned} \mathcal{V}_0(\tau, \chi_k) = & (T_k - \tau)(\chi_k(\tau) - \chi_k(0))^T [S_1(\chi_k(\tau) - \chi_k(0)) + 2S_2 \chi_k(0)] \\ & + (T_k - \tau) \tau \begin{bmatrix} \chi_k(0) \\ \psi_k \end{bmatrix}^T X \begin{bmatrix} \chi_k(0) \\ \psi_k \end{bmatrix} + (T_k - \tau) \int_0^\tau \dot{\chi}_k^T(\theta) R \dot{\chi}_k(\theta) d\theta, \end{aligned}$$

<sup>1</sup>The matrices  $M_i$  are not of the same dimension. The notation 0 and  $I$  correspond to the zero and identity matrices of appropriate dimension.



with  $S_1 \in \mathbb{R}^{n \times n}$ ,  $S_2 \in \mathbb{R}^{n \times n}$ ,  $X \in \mathbb{S}^{n+m}$ ,  $R \in \mathbb{S}^n$ ,  $R > 0$ .

The first step of the proof focuses on the fact that the functional  $\mathcal{V}_0$  satisfies the condition (13). Since  $(T_k - \tau) = 0$  when  $\tau = T_k$ ,  $\chi_k(\tau) - \chi_k(0) = 0$  when  $\tau = 0$  and the integral term are equal to zero when  $\tau = 0$ , it follows that the functional  $\mathcal{V}_0(\tau, \chi_k)$  is equal to zero at  $\tau = 0$  and  $\tau = T_k$ . Thus, the functional  $\mathcal{V}_0$  satisfies condition (13) and, moreover, it is continuous at all sampling instants and differentiable over  $[0, T_k[$ . The end of the proof consists in showing that inequality (14) hold. The following equality is then obtained

$$\begin{aligned} \dot{\mathcal{W}}(\tau, \chi_k) &= 2\chi_k^T(\tau)P\dot{\chi}_k(\tau) - 2\psi_k^T U[\psi_k - G\chi_k(0)] \\ &\quad + (T_k - \tau)\dot{\chi}_k^T(\tau)[R\dot{\chi}_k(\tau) + 2S_1(\chi_k(\tau) - \chi_k(0)) + 2S_2\chi_k(0)] \\ &\quad - (\chi_k(\tau) - \chi_k(0))^T[S_1(\chi_k(\tau) - \chi_k(0)) + 2S_2\chi_k(0)] \\ &\quad + (T_k - 2\tau) \begin{bmatrix} \chi_k(0) \\ \psi_k \end{bmatrix}^T X \begin{bmatrix} \chi_k(0) \\ \psi_k \end{bmatrix} - \int_0^\tau \dot{\chi}_k^T(\theta)R\dot{\chi}_k(\theta)d\theta. \end{aligned} \quad (21)$$

Consider the extended vector  $\xi_k(\tau) = [\chi_k^T(\tau) \chi_k^T(0) \dot{\chi}_k^T(\tau) \psi_k^T]^T$ , for all  $\tau \in [0, T_k[$ , and a matrix  $N \in \mathbb{R}^{(3n+m) \times n}$ . Since  $R$  is assumed to be positive definite and thus non singular, the product  $(\dot{\chi}_k^T(\theta) - R^{-1}N^T\xi_k(\tau))^T R(\dot{\chi}_k(\theta) - R^{-1}N^T\xi_k(\tau))$  is positive for all  $\tau \in [0, T_k[$  and all  $\theta \in [0, \tau[$ . Integrating its development over  $[0, \tau]$ , the following inequality is obtained

$$\int_0^\tau \dot{\chi}_k^T(\theta)R\dot{\chi}_k(\theta)d\theta - 2\xi_k^T(\tau)N(\chi_k(\tau) - \chi_k(0)) + \tau\xi_k^T(\tau)NR^{-1}N^T\xi_k(\tau) \geq 0. \quad (22)$$

On the other hand, there exists a coupling relation between the components of the augmented vector  $\xi_k$ , which is given by

$$2(\chi_k^T(\tau)Y_1^T + \dot{\chi}_k^T(\tau)Y_2^T)(A\chi_k(\tau) - \dot{\chi}_k(\tau) + BK\chi_k(0) - B\psi_k) = 0,$$

for any square matrices  $Y_1$  and  $Y_2 \in \mathbb{R}^{n \times n}$ . The following null term can be added to inequality (21). This can be interpreted as the use of the descriptor approach introduced in [6]. Combining this inequality to equations (21) and (22), one obtains:

$$\dot{\mathcal{W}}(\tau, \chi_k) \leq \xi_k^T(\tau)[\Pi_1 + (T_k - \tau)\Pi_2 + \tau NR^{-1}N^T + (T_k - 2\tau)\Pi_3]\xi_k(\tau). \quad (23)$$

where

$$\begin{aligned} \Pi_1 &= \text{He}\{M_1^T P M_3 - M_{12}^T S_2 M_2 - N M_{12}\} - M_{12}^T S_1 M_{12} \\ &\quad + \text{He}\{(M_1^T Y_1^T + M_3^T Y_2^T)M_0 + M_4^T U G M_2\} - 2M_4^T U M_4 \\ \Pi_2 &= M_3^T R M_3 + \text{He}\{M_3^T (S_1 M_{12} + S_2 M_2)\} \\ \Pi_3 &= M_{24}^T X M_{24}, \end{aligned} \quad (24)$$

where  $M_0 = [A \quad BK \quad -I \quad -B]$ , Since the matrices  $K$  and  $G$  are decision variables, the previous inequality is not an LMI. Indeed the matrix  $\Pi_1$  contains products of matrices variables ( $Y_1$  with  $K$  and  $Y_2$  with  $K$ ). There exists several transformations to derive an LMI condition that allows to compute a gain  $K$ , while guaranteeing that  $\dot{\mathcal{W}}(\tau, \chi_k) < 0$ . A review of existing methods is given in [24]. In this paper, a method inspired from [3] is applied. Consider the contributions corresponding to terms of the form “ $\dot{\chi}_k^T(\tau)(\cdot)\dot{\chi}_k(\tau)$ ” in  $\dot{\mathcal{W}}$ . This is given by

$$M_3 [\Pi_1 + (T_k - \tau)\Pi_2 + \tau NR^{-1}N^T + (T_k - 2\tau)\Pi_3] M_3^T.$$

This leads to  $-Y_2 - Y_2^T + (T_k - \tau)R + \tau N_3 R^{-1} N_3^T$ , where  $N_3 \in \mathbb{R}^{n \times n}$  is a component of  $N$ . Since we are looking for a negative contribution of this term, a necessary conditions is that  $Y_2$  is non singular. Additionally,  $U$  is assumed to be a diagonal positive definite matrix. Thus it is possible to define the matrices  $\tilde{Y} = Y_2^{-1}$ ,  $\tilde{U} = U^{-1}$  and  $\tilde{\Xi} = \text{diag}\{\tilde{Y}, \tilde{Y}, \tilde{Y}, \tilde{U}\}$ . Consider now the vector  $\tilde{\xi} = \tilde{\Xi}^{-1}\xi$ . Rewriting (23), using the new variable  $\tilde{\xi}$  leads to

$$\dot{\mathcal{W}}(\tau, \chi_k) \leq \tilde{\xi}^T [\tilde{\Xi}^T \Pi_1 \tilde{\Xi} + (T_k - \tau)\tilde{\Xi}^T \Pi_2 \tilde{\Xi} + \tau \tilde{\Xi}^T N R^{-1} N^T \tilde{\Xi} + (T_k - 2\tau)\tilde{\Xi}^T \Pi_3 \tilde{\Xi}] \tilde{\xi}.$$

From the definition of the matrices  $M_i$  for  $i = 1, \dots, 4$ , one has

$$\begin{aligned} M_1\Xi &= \tilde{Y}M_1, \quad M_2\Xi = \tilde{Y}M_2, \quad M_3\Xi = \tilde{Y}M_3, \\ M_4\Xi &= \tilde{U}M_4, \quad M_{12}\Xi = \tilde{Y}M_{12}, \quad M_{24}\Xi = \begin{bmatrix} \tilde{Y} & 0 \\ 0 & \tilde{U} \end{bmatrix} M_{24}, \\ M_0\Xi &= A\tilde{Y}M_1 + BK\tilde{Y}M_2 - \tilde{Y}M_3 - B\tilde{U}M_4. \end{aligned}$$

Setting now  $Y_1 = \epsilon Y_2$  and considering the following change of variables  $\tilde{P} = \tilde{Y}^T P \tilde{Y}$ ,  $\tilde{S}_1 = \tilde{Y}^T S_1 \tilde{Y}$ ,  $\tilde{S}_2 = \tilde{Y}^T S_2 \tilde{Y}$ ,  $\tilde{X} = \begin{bmatrix} \tilde{Y} & 0 \\ 0 & \tilde{U} \end{bmatrix}^T X \begin{bmatrix} \tilde{Y} & 0 \\ 0 & \tilde{U} \end{bmatrix}$ ,  $\tilde{R} = \tilde{Y}^T R \tilde{Y}$ ,  $\tilde{N} = \Xi^T N \tilde{Y}$ ,  $\tilde{K} = K \tilde{Y}$  and  $\tilde{G} = G \tilde{Y}$ , the following inequality is obtained:

$$\dot{W}(\tau, \chi_k) \leq \tilde{\xi}^T [\tilde{\Pi}_1 + (T_k - \tau)\tilde{\Pi}_2 + \tau\tilde{N}\tilde{R}^{-1}\tilde{N}^T + (T_k - 2\tau)\tilde{\Pi}_3]\tilde{\xi}. \quad (25)$$

where  $\tilde{\Pi}_1$ ,  $\tilde{\Pi}_2$  and  $\tilde{\Pi}_3$  are defined in (20). To prove that  $\dot{W}$  is negative definite for all  $\tau$ , note that the right hand side of equation (25) is affine with respect to the variable  $\tau$  in  $[0, T_k]$ . Then, by convexity, it suffices to ensure that the right hand side of (25) is negative for  $\tau = 0$  and  $\tau = T_k$  (see [18] for more detail). This fact leads to the inequalities

$$\tilde{\Pi}_1 + T_k(\tilde{\Pi}_2 + \tilde{\Pi}_3) < 0, \quad \text{and} \quad \tilde{\Pi}_1 - T_k\tilde{\Pi}_3 + T_k\tilde{N}\tilde{R}^{-1}\tilde{N}^T < 0.$$

Applying the same argument on  $T_k$  in the interval  $[T_1, T_2]$  and by using the Schur's complement, conditions  $\Psi_1(T_i) < 0$  and  $\Psi_2(T_i) < 0$  given in (18) are obtained for  $i = 1, 2$ .

Right and left-multiplying (19) by  $\text{diag}\{Y_2, I\}$  and next applying Schur's complement, we conclude that (19) is equivalent to (12). Hence, it follows that  $\mathcal{E}_P \subset \mathcal{S}$  and the condition (10) is verified  $\forall \chi_k(0) \in \mathcal{E}_P$ .

Hence, by virtue of Theorem 1 the satisfaction of conditions (18)-(19) ensures the asymptotic convergence of the trajectories to the origin, provided that  $x(0) \in \mathcal{E}_P$ . □

Concerning the global stabilization, the following result based on the Corollary 1 can be stated.

**Corollary 2** *For given positive scalars  $T_1 < T_2$ , assume that there exist positive definite matrices  $\tilde{P}, \tilde{R} \in \mathbb{S}^n$ , a matrix  $\tilde{S}_1 \in \mathbb{R}^{n \times n}$ , a matrix  $\tilde{X} \in \mathbb{S}^{n+m}$ , a positive definite diagonal matrix  $\tilde{U} \in \mathbb{S}^m$ , matrices  $\tilde{Y}, \tilde{S}_2 \in \mathbb{R}^{n \times n}$  and  $\tilde{N} \in \mathbb{R}^{2n \times n}$  and a positive scalar  $\epsilon$  that satisfy, for  $i = 1, 2$*

$$\Psi_1(T_i) < 0 \quad \text{and} \quad \Psi_2(T_i) < 0 \quad (26)$$

with  $\tilde{\Pi}_1, \tilde{\Pi}_2$  and  $\tilde{\Pi}_3$  as in Theorem 2 but with  $\tilde{G} = \tilde{K}$ .

Then, the origin of system (2) under the saturated control law defined in (4) with  $K = \tilde{K}\tilde{Y}^{-1}$  is globally asymptotically stable for any asynchronous sampling satisfying (1).

**Proof.** It suffices to replace  $G$  by  $K$  in the proof of Theorem 2. □

**Remark 2** *The strategy to handle the saturation effects in the derived stabilization conditions is based on the use of a generalized sector condition. Nonetheless, it should be pointed out that the results presented can be straightforwardly re-derived considering a polytopic modeling for the saturation term (see [23], Section 1.7). For instance, if we consider the model used in [1] and [35], with  $B^j$ , denoting the  $j^{\text{th}}$  column of matrix  $B$ , and the sets  $\mathbb{S}_r$  and  $\mathbb{S}_r^c$ ,  $r = 1, \dots, 2^m$  of indices  $j = 1, \dots, m$ , with  $\mathbb{S}_1 = \emptyset$ , defined as in [1], it follows that [23]:*

$$B\text{sat}(Kx) \in \text{Co}\left\{ \sum_{j \in \mathbb{S}_r^c} B^j K_j x + \sum_{j \in \mathbb{S}_r} B^j H_{r,j} x, \quad r = 1, \dots, 2^m \right\} \quad (27)$$

provided that the state belongs to the set  $\tilde{\mathcal{S}} = x \in \cap_{r=2}^{2^m} S(H_r, u_0)$ , with

$$S(H_r, u_0) = \{x \in \mathbb{R}^n; |H_{r,j}x| \leq u_{0j}, \quad \forall j \in \mathbb{S}_r\}$$

Hence, following the same steps used in the proof of Theorem 2, new stabilization conditions can be basically obtained by:

1. eliminating the term  $\psi_k^T U[\psi_k - K\chi_k(0)] < 0$  from (14), this will imply the elimination of the terms in variables  $\tilde{U}$  and  $\tilde{G}$  appearing in  $\Psi_1(\mathcal{T}_i)$  and  $\Psi_2(\mathcal{T}_i)$ .
2. replacing the term  $B\tilde{K}M_2$  in  $\tilde{\Pi}_1$  by  $(\sum_{j \in \mathbb{S}_c^c} B^j \tilde{K}_j + \sum_{j \in \mathbb{S}_r} B^j \tilde{H}_{r,j})M_2$ , with  $\tilde{H}_{r,j} = H_{r,j}\tilde{Y}$  - in this case, the conditions  $\Psi_1(\mathcal{T}_i) < 0$  and  $\Psi_2(\mathcal{T}_i) < 0$  would be replaced, respectively, by  $\Psi_{1,r}(\mathcal{T}_i) < 0$  and  $\Psi_{2,r}(\mathcal{T}_i) < 0$ , for  $r = 1, \dots, 2^m$  - this corresponds to ensure  $\dot{W} < 0$  on the vertices of the convex hull (27).
3. replacing the inclusion condition  $\mathcal{E}_P \subset \mathcal{S}$  by  $\mathcal{E}_P \subset \tilde{\mathcal{S}}$  - this former inclusion can be verified by a set of inequalities similar to (19), just by replacing  $(\tilde{K}_j - \tilde{G}_j)$  by  $\tilde{H}_{r,j}$ .

In some cases, a small improvement (when the objective is the maximization of the set  $\mathcal{E}_P$ ) can be obtained with this kind of modeling at the expense of a higher numerical complexity (see a nice analysis in this sense in [35] and also in [23], Section 3.2.3). Basically, this comes from the fact that LMIs  $\Psi_1(\mathcal{T}_i) < 0$  and  $\Psi_2(\mathcal{T}_i) < 0$  would be replaced, each one, by  $2^m$  LMIs.

On the other hand, it should be mentioned that the sector approach allows to cope with the global stabilization and the results for the anti-windup synthesis (which can be seen as a future extension of the present work) would be convex, which is not the case with the polytopic representations of the saturation function (see Remark 6.11 in [23]).

The facts above justify our choice by the use of a generalized sector condition approach, instead of the polytopic one, to model the control saturation effects.

**Remark 3** The LMI conditions which are provided in Theorem 2 and Corollary 2 are linear with respect to the matrices  $A$  and  $B$  which characterizes the dynamics of the system. Then direct extensions of these two results are provided in the case of systems with polytopic uncertainties defined by some positive scalars  $\lambda_i$ 's such that  $\sum_{j=1}^M \lambda_j = 1$  and

$$[A \ B] = \sum_{l=1}^M \lambda_l [A_l \ B_l]. \quad (28)$$

Note that the  $\lambda_l$ 's could either be constant parameters or time-varying but in both cases they are considered unknown.

## 6 Optimization problems

From the theoretical result given by Theorem 2 and Corollary 2, this section provides the formulation of some optimization problems to address the stabilization problems **P1** and **P2** defined in Section 2.

### 6.1 Maximization of the estimate of the region of attraction

For a given  $u_0$  and an asynchronous sampling satisfying (1) with given  $\mathcal{T}_1$  and  $\mathcal{T}_2$ , the objective is to determine  $K$  that leads to the largest set of initial conditions for which the conditions of Theorem 2 are satisfied. In other words, we should find  $K$  such that  $\mathcal{E}_P$  is maximized considering some size criterion. For instance, we can maximize the minimal axis of  $\mathcal{E}_P$ , which corresponds to minimize the maximal eigenvalue of  $P$ . This can be accomplished from the following optimization problem

$$\begin{aligned} & \min \delta \\ & \text{subject to} \\ & (18), (19) \\ & \begin{bmatrix} \delta I & I \\ I & \tilde{Y} + \tilde{Y}^T - \tilde{P} \end{bmatrix} > 0. \end{aligned} \quad (29)$$

Note that, from Lemma 1, the last inequality above ensures that  $P < \delta I$ , which ensures that  $\lambda_{\max}(P) < \delta$ .

## 6.2 Maximization of the sampling period

Define a region of admissible initial states

$$\mathcal{E}_0 = \{x \in \mathbb{R}^n, \quad x^T P_0 x \leq 1\}.$$

Hence, given  $\mathcal{T}_1$  the idea is to find the maximal  $\mathcal{T}_2$  for which it is possible to compute  $K$  such that asymptotic stability is ensured for all initial conditions belonging to  $\mathcal{E}_0$ . An upper-bound on the maximal  $\mathcal{T}_2$  can therefore be obtained from the following optimization problem

$$\begin{aligned} & \max \mathcal{T}_2 \\ & \text{subject to} \\ & (18), (19) \\ & \begin{bmatrix} P_0 & I \\ I & \tilde{Y} + \tilde{Y}^T - \tilde{P} \end{bmatrix} > 0. \end{aligned} \tag{30}$$

Note that, from Lemma 1, the last inequality above ensures that  $P < P_0$ , which ensures that  $\mathcal{E}_0 \subset \mathcal{E}_P$ .

On the other hand, when the open-loop is asymptotically stable, a global stabilizing gain  $K$  leading to a maximized bound the sampling period jitter  $\mathcal{T}_2$  can be obtained by solving the following optimization problem:

$$\begin{aligned} & \max \mathcal{T}_2 \\ & \text{subject to} \\ & (26) \end{aligned} \tag{31}$$

## 6.3 Optimization of the actuator size

Given bounds  $\mathcal{T}_1$  and  $\mathcal{T}_2$  for the sampling period, the idea in this case is to design the control law in order to achieve a guaranteed stability for a given set  $\mathcal{E}_0$  of admissible initial conditions with less costly actuators. In general the cost of the  $j^{\text{th}}$  actuator is directly related with its size, which can be measured by the bound  $u_{0j}$ .

Hence, setting  $\bar{u}_j = u_{0j}^2$ ,  $j = 1, \dots, m$ , this problem can be addressed from the solution of the following optimization problem.

$$\begin{aligned} & \min \sum_{j=1}^m c_j \bar{u}_j \\ & \text{subject to} \\ & (18), (19) \\ & \begin{bmatrix} P_0 & I \\ I & \tilde{Y} + \tilde{Y}^T - \tilde{P} \end{bmatrix} > 0. \end{aligned} \tag{32}$$

with the coefficients  $c_j$  being the weights of the size/cost of each actuator in the composition of the cost function.

**Remark 4** For a fixed  $\epsilon$ , the constraints in (29), (30), (31) and (32) are LMIs. Then the optimal solution of these problems can be easily approached by solving LMI-based problems on a grid in  $\epsilon$ .

**Remark 5** The result of Theorem 2 concerns the synthesis of a stabilizing gain  $K$ . Nonetheless, the conditions (18), and (19) can also be used for analysis purposes, when the gain  $K$  is given. It suffices to replace  $\tilde{K}$  by  $K\tilde{Y}$ . In this case, problems (29) and (30) can be solved, to determine an estimate of the region of attraction of the closed-loop system or to determine a bound on the admissible sampling period jitter, respectively. Note that the conditions are still LMIs for a fixed  $\epsilon$ .

## 7 Illustrative Examples

### 7.1 Example 1

We consider (2) with the following matrices, taken from [16] :

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

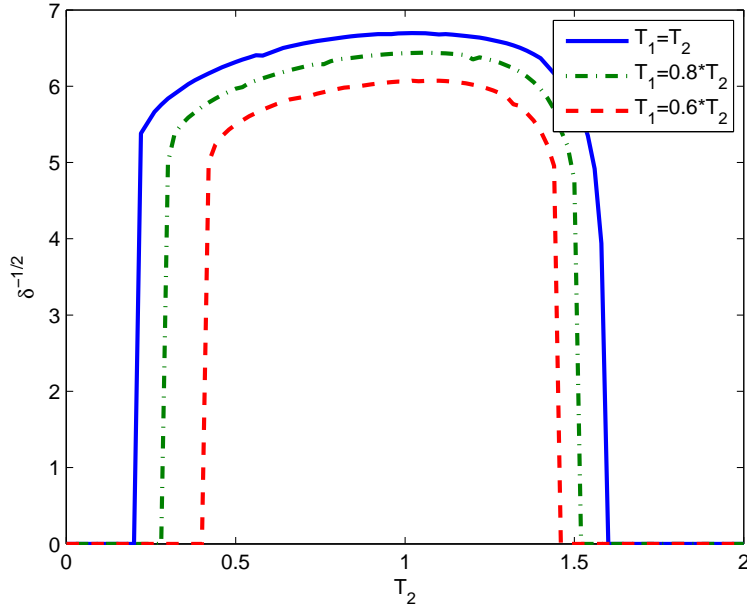


Figure 2: Evolution of  $1/\sqrt{\delta}$  with respect to the maximum available sampling period  $\mathcal{T}_2$  for different cases of  $\mathcal{T}_1$ .

and where  $u_0 = 5$ . The control gain  $K$  is chosen as  $[1 \ 0]$ . This system is unstable with a continuous-time state feedback control  $u(t) = Kx(t)$ . However, it was proven in [26] that the closed-loop system using a sampled version of the same control law becomes stable if the sampling period becomes sufficiently large. We will show now that the same behavior appears even if the system is subject to input saturation.

In order to assess the stability of the closed-loop system with the given gain  $K$ , we can consider the optimization problem (29) with the variable  $\tilde{K}$  replaced by  $K\tilde{Y}$  (see Remark 5). In this case, Figure 2 shows the influence of the maximum allowable sampling period on the size (measured in terms of  $\delta$ ) of the set of admissible initial states for which the stability can be ensured (i.e. the estimate of the region of attraction). As mentioned above, the set of admissible initial conditions is empty for small sampling periods. This comes from the fact that this particular system is not stable for small sampling periods. Moreover, it should be noticed that the case of constant sampling periods (i.e.  $\mathcal{T}_1 = \mathcal{T}_2$ ) leads to larger sets when compared to the case of asynchronous sampling.

## 7.2 Stabilization example

We consider (2) with the following matrices (taken from [2], where  $h = 0$ ):

$$A = \begin{bmatrix} 1.1 & -0.6 \\ 0.5 & -1.0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and where  $u_0 = 5$ . From [5], the gain  $K = [-1.696 \ 0.523]$  ensures local stability of the closed loop for constant sampling periods ( $T = T_k = \mathcal{T}_1 = \mathcal{T}_2, \forall k$ ), such that  $T \leq 0.75$ . The obtained set of admissible initial conditions is defined by

$$\mathcal{E}_1 = \{x \in \mathbb{R}^2 ; x^T \begin{bmatrix} 5.6933 \cdot 10^4 & -1.8807 \cdot 10^4 \\ -1.8807 \cdot 10^4 & 0.8653 \cdot 10^4 \end{bmatrix} x \leq 1\}.$$

Applying Theorem 2 with  $\epsilon = 1.3$ , it follows that the asymptotic stability of the system is ensured with the controller gain  $K = [-1.7491 \ 0.5417]$  for any asynchronous sampling period satisfying (1) with  $\mathcal{T}_1 = 0$  and  $\mathcal{T}_2 \leq 0.75$ . The obtained set of admissible initial conditions in this case is given by

$$\mathcal{E}_2 = \{x \in \mathbb{R}^2 ; x^T \begin{bmatrix} 0.4450 & 0.2307 \\ 0.2307 & 21.0091 \end{bmatrix} x \leq 1\}.$$

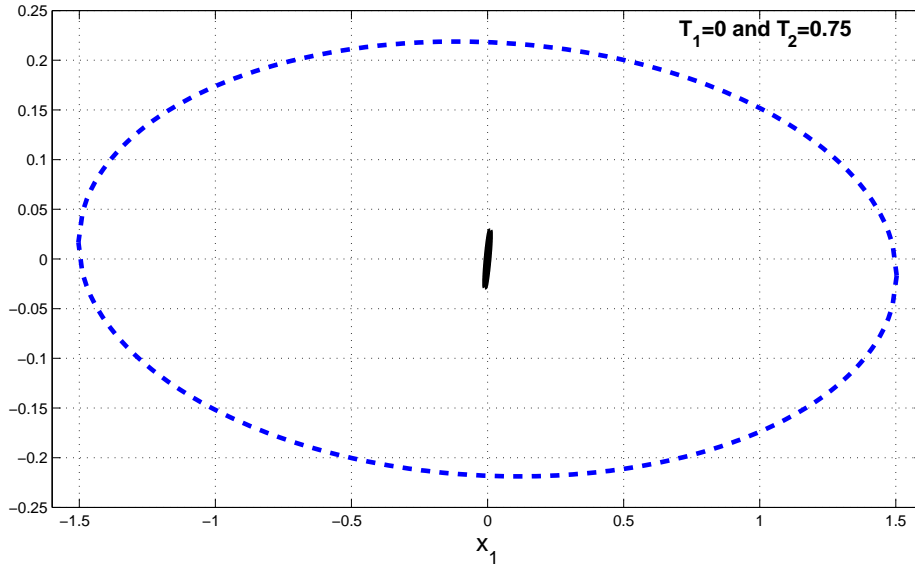


Figure 3: Set of admissible initial conditions using Theorem 3.3 from [5] (continuous-line) and using Theorem 2 (dashed-line).

The two obtained sets of admissible initial states (estimates of the region of attraction) are depicted in Figure 3. It can be noticed that the set obtained using Theorem 2 is significantly larger than the one obtained from the result in [5]. This proves the efficiency of the method. It should be pointed out that, although a different modeling (i.e. a convex hull approach) for the saturation is considered in [5], the reduction of conservatism is mainly due to the approach of dealing with stability, i.e. without using the Lyapunov-Krasovskii theorem. Indeed, similar improvements would be achieved if we consider the polytopic (convex hull) modeling for the saturated term adopted in [5], at the expense of a larger computational burden (see Remark 2).

Furthermore, solving problem (30) it is possible to design a stabilizing state control feedback for all asynchronous samplings satisfying with  $\mathcal{T}_1 = 0$  and  $\mathcal{T}_2 = 1.99$  (obtained with  $\epsilon = 0.1$ ).

## 8 Conclusion

This article proposes a novel constructive stabilization criterion for sampled and saturated controlled systems based on the discrete-time Lyapunov theorem. Another important feature of the proposed approach regards the fact of taking explicitly into account control limitations (saturation) in the stabilization problem. Moreover, as illustrated in the example the method provides less conservative results than existing approaches reported in the literature.

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