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► To cite this version:

Vincent Cossart, Olivier Piltant. Resolution of singularities of threefolds in mixed characteristic: case of small multiplicity. 2012. hal-00735929

HAL Id: hal-00735929

<https://hal.science/hal-00735929>

Preprint submitted on 28 Sep 2012

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RESOLUTION OF SINGULARITIES OF THREEFOLDS IN MIXED CHARACTERISTIC: CASE OF SMALL MULTIPLICITY

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INTRODUCTION

This article is part of the authors' program whose purpose is to prove the following conjecture on Resolution of Singularities of threefolds in mixed characteristic. The conjecture is a special case of Grothendieck's Resolution conjecture for quasi-excellent schemes.

CONJECTURE 0.1 *Let C be an integral regular excellent curve with function field F . Let S/F be a reduced algebraic projective surface and $\mathcal{X}$ be a flat projective C -scheme with generic fiber $\mathcal{X}_F = S$. There exists a birational projective C -morphism $\pi : \mathcal{Y} \rightarrow \mathcal{X}$ such that*

- (i) $\mathcal{Y}$ is everywhere regular.
- (ii) $\pi^{-1}(\text{Reg}\mathcal{X}) \rightarrow \text{Reg}\mathcal{X}$ is an isomorphism.

Let us point out that the equicharacteristic techniques designed in [CP1] extend to the situation described in the above conjecture. In particular, [CP1] theorem 3.3 extends and reduces conjecture **0.1** to the following variant:

CONJECTURE 0.2 *Let A be an excellent DVR with quotient field F and residue characteristic $p > 0$. Let (S, M, k) be a regular local ring of dimension three dominating A , essentially of finite type over A with $K := QF(S)$ of transcendence degree two over F . Let finally L/K be a finite field extension and v be a valuation of L . Assume:*

- (i) L/K is cyclic Galois or purely inseparable of degree p .
- (ii) v has rank one and is centered in S .

Then there exists a regular local ring T essentially of finite type over A with $QF(T) = L$ such that v is centered in T .

Applying embedded resolution techniques for surfaces, it can be assumed that such a v is centered in a local model of L of the form $B = (S[X]/(h))_{(M, X)}$ with h monic of degree p ; more precisely, $h = X^p - g^{p-1}X - f$, $f, g \in M$ and ($g = 0$ if $\text{char}A = 0$). In particular the Local Uniformization statement of conjecture **0.2** only involves certain hypersurface singularities $(\text{Spec}B, x)$, of multiplicity $m(x) \leq p = \text{char}k$, and embedded in an excellent fourfold $(Z = \text{Spec}S[X], x)$. We prove here:

MAIN THEOREM 0.3 *Let $(R, \mathfrak{M}, k = k(x) := \frac{R}{\mathfrak{M}})$ be an excellent regular local ring of dimension four, $(Z, x) := (\text{Spec}R, \mathfrak{M})$ and $(X, x) := (\text{Spec}R/(h), x)$ be a reduced hypersurface. Assume that the multiplicity $m(x)$ of (X, x) satisfies $m(x) < p := \text{char}k(x)$. Let v be a valuation of $K(X)$ centered at x . Then there exists a finite sequence of local blowing ups*

$$(X, x) =: (X_0, x_0) \longleftarrow (X_1, x_1) \longleftarrow \cdots \longleftarrow (X_n, x_n),$$

where $x_i \in X_i$, $0 \leq i \leq n$ is the center of v , each blowing up center $Y_i \subset X_i$ is permissible at x_i (in Hironaka's sense), such that x_n is regular.

The proof of theorem **0.3** builds upon classical Resolution of Singularities techniques. We use systematically the Hironaka characteristic polyhedron and Hironaka's invariants: the multiplicity $m(x)$ and τ -number $\tau(x)$ for the hypersurface singularity $(X, x) := (\text{Spec} R/(h), x)$.

Since we are working without any ground field (at least when R is not equicharacteristic), the Tschirnhausen trick (killing the degree $(m(x) - 1)$ -term in the equation) cannot be directly applied even though $m(x) < p$. Rather, we use it for the initial face of the Hironaka polyhedron (theorem **II.5**) to define well behaved invariants. The pair $(m(x), \tau(x))$ is then further on completed to a 6-tuple $\iota(x)$ defined in **IV.2**.

The main technical part is concentrated in section VI. We consider projections to a two dimensional space to define a refined invariant in sections **VI.3** and **VI.4.1**. Controlling the transformation law for this refined invariant under blowing up is much harder but leads essentially to the same formulæ as for the characteristic polygon of a surface singularity. The proof of the Main Theorem follows rather easily from these computations (section **VI.4.5**).

It is worth pointing out that these techniques are global in nature and it is to be expected that theorem **0.3** can be extended to a global version, i.e. without referring to a given valuation v and using global blowing up centers. We use the valuation only at a few specific places (mostly in section V) to make the argument quicker.

This article is organized as follows: section I states the reduction of the Main Theorem to the case $\tau(x) = 1$, immediate from [CJS]. This means that the initial form $\text{in}_x(h)$ can be written

$$\text{in}_x(h) = \lambda Y^{m(x)}, \lambda \in k(x), \lambda \neq 0, y \text{ a regular parameter of } R, Y := \text{in}_x(y).$$

Section II first recollects known material from [H1][H2] about characteristic polyhedra and associated invariants (definition **II.2**). Special coordinates $(z, u) := (z, u_1, u_2, u_3)$ on R are said to be *fully prepared* if they compute the Hironaka characteristic polyhedron $\Delta(h; u)$ and if the δ -initial $\text{in}_\delta(h)$ of h is Tschirnhausen transformed, i.e. has no term of degree $m(x) - 1$ (theorem **II.5** and definition **II.5.3**). The form $\text{in}_\delta(h)$ is defined in **II.2(iii)** and is the sum of the initial forms of all those terms in h contributing to the face of minimal order at x of $\Delta(h; u)$. In such special coordinates, invariants $d_1, d_2, d_3, \epsilon(x)$ (each of them is a nonnegative rational number) can be computed from the polyhedron $\Delta(h; u)$ and the δ -initial $\text{in}_\delta(h)$ (definition **II.7**). Theorem **II.9** proves the Main Theorem when $\epsilon(x) = 0$, in which case only combinatorial blowing ups are used.

When $\epsilon(x) > 0$, some preparations are required in order to get the locus

$$\Sigma := \{y \in X : m(y) = m(x), \tau(y) = \tau(x), \epsilon(y) > 0\}$$

Zariski closed and of dimension at most one (theorems **II.10** and **II.11**). Section III prepares $\text{Spec} R$ and constructs an equicharacteristic p normal crossings divisor

$$E \subseteq \text{div}(u_1 u_2 u_3) \subset \text{Spec} R$$

which contains Σ ($E = E_n$ in proposition **III.1**).

Section IV then provides some further invariants build up from the ideal of coefficients of $\text{in}_\delta(h)$ once this preparation is achieved: a refined directrix $\mathcal{V} \subseteq \langle U_1, U_2, U_3 \rangle$ (definition **IV.1**) and associated refined numerical invariant $\iota(x)$ (definition **IV.2**):

$$\iota(x) := (m(x), -\tau(x), \epsilon(x), -\rho(x), -t(x), -e(x)).$$

Section V introduces the notion of ϵ -permissible blowing up centers (definition **V.1**). For curves, being ϵ -permissible is stronger than being Hironaka permissible (proposition **V.2**); blowing up along an ϵ -permissible center does not increase the invariant $\iota(x)$ (proposition **V.3**). Furthermore, $\iota(x)$ can be decreased by blowing up along ϵ -permissible centers except possibly when $\mathcal{V} = \langle U_3 \rangle$ and (either $\text{div}(u_3) \subseteq E$ or $E \subseteq \text{div}(u_1 u_2)$) (propositions **V.4** and **V.5**).

Section VI proves the same result in these remaining cases (theorem **VI.1**), thus concluding the proof of the Main Theorem. We now project to the (u_1, u_2) -space and define *well prepared* coordinates by minimizing the induced image of $\Delta(h; u)$ by this projection (this requires choosing special coordinates (z, u_3)). There are further associated invariants $\beta(u, z), C(u, z), \gamma(u, z)$ defined in **VI.4.1**. The behaviour of these invariants by blowing up ϵ -permissible curves and closed points are studied respectively in propositions **VI.4.2** and **VI.4.3**. Section **VI.4.5** contains the proof of theorem **VI.1** and is basically a consequence of the former computations.

The notation and assumption in the Main Theorem will be kept all along this article. The proof will be made by induction on the multiplicity $m(x) = \text{ord}_x(h)$ of $x \in X$. Since it is assumed that $m(x) < p$, (X, x) is already regular if $p = 2$, so we assume $p \geq 3$ from now on. The formal completion of R with respect to $\mathfrak{M}$ is denoted by $\widehat{R}$.

I BASIC INVARIANTS

Two basic invariants are attached to the hypersurface singularity $(X, x) = (\text{Spec} R/(h), x)$. The first invariant is its multiplicity $m(x)$ (or m for short) of (X, x) . The second invariant is $\tau(x)$ (or τ for short), which is the dimension of the smallest $k(x)$ -vector subspace $\mathcal{T}$ of $\frac{\mathfrak{M}}{\mathfrak{M}^2}$ such that $\text{in}_x(h) \in k(x)[\mathcal{T}]$ [H1, Ch.2, Lemma 10]. This vector space is called the *directrix* of $\text{in}_x(h)$.

Proving the Main Theorem in the cases $\dim(Z) - \tau(x) \in \{0, 1, 2\}$, i.e. $\tau(x) \in \{2, 3, 4\}$ is done in [CJS]. So from now on, we assume that $\tau(x) = 1$. Equivalently:

$$\text{in}_x(h) = \lambda Y^m, \lambda \in k(x), \lambda \neq 0, y \text{ a regular parameter of } R, Y := \text{in}_x(y).$$

II CHARACTERISTIC POLYHEDRON

DEFINITION II.1

- (i) An F -subset $\Delta \subset \mathbb{R}_+^d$ is a closed complex subset of $\mathbb{R}_+^d$ such that $v \in \Delta$ implies $v + \mathbb{R}_+^d \subset \Delta$.
- (ii) A point $v \in \Delta$ is called a *vertex* if there is a positive linear form L on $\mathbb{R}^d$ (i.e. has strictly positive coefficients) such that

$$\{v\} = \Delta \cap \{A \in \mathbb{R}^d \mid L(A) = 1\}.$$

- (iii) The *essential boundary* $\partial\Delta$ of an F -subset Δ is the subset of Δ consisting of those $v \in \Delta$ such that $v \notin v' + \mathbb{R}_+^d$ with $v' \in \Delta$ unless $v' = v$. We write $\Delta^+ = \Delta - \partial\Delta$.

For the next definition and proposition, we will forget the hypothesis $\dim(R) = 4$: we will have to use the notions defined there for different regular rings of dimension at most three. Given a r.s.p. $(y, u_1, u_2, \dots, u_d) =: (y, u)$ of a regular local ring R and $f \in R$, there exists a *finite* sum expansion

$$f = \sum_{A, b} C_{A, b} y^b u^A, \quad b \in \mathbb{N}, \quad A \in \mathbb{N}^d. \quad (1)$$

where each $C_{A,b}$ is a unit in R . This follows easily from the facts that R is Noetherian and the map $R \subseteq \widehat{R}$ faithfully flat. We regard u as “fixed” parameters and y as “varying”, which is reflected in the indexing below. Assume furthermore that

$$h \in \mathfrak{M}, \quad h \notin (u_1, \dots, u_d). \quad (2)$$

We let $\overline{R} := R/(u_1, \dots, u_d)$, $\overline{h} \in \overline{R}$ be the image of h and “ord” be the valuation of the discrete valuation ring $\overline{R}$. We extend our conventions by letting now

$$m := \text{ord} \overline{h} \geq 1. \quad (3)$$

Assumption (2) and notation (3) are maintained all along this article. Our original concern is for $\tau(x) = 1$, say $\text{in}_x(h) = \lambda Y^m$, $0 \neq \lambda \in k(x)$ which fits into these conventions provided $Y = \text{in}_x(y)$.

DEFINITION II.2

(i) The polyhedron $\Delta(h; u; y) \subset \mathbb{R}_{\geq 0}^d$ is defined as the smallest F -subset containing all points of

$$S(h) := \left\{ v = \frac{A}{m-b} \mid 0 \leq b < m \right\}.$$

The characteristic polyhedron $\Delta(h; u) \subset \mathbb{R}_{\geq 0}^d$ is defined by the formula

$$\Delta(h; u) := \bigcap_{(\widehat{y}, u_1, \dots, u_d)} \Delta(h; u; \widehat{y}), \quad (4)$$

where the intersection runs over all r.s.p's of $\widehat{R}$ of the form $(\widehat{y}, u_1, \dots, u_d)$.

(ii) For $v \in \partial \Delta(h; u; y)$, the v -initial of h is defined as

$$\text{in}_v(h) := \sum_{A,b} \overline{C_{A,b}} Y^b U^A \in k[U, Y] = k[U_1, U_2, \dots, U_d, Y],$$

where $\overline{C_{A,b}} \in k$ is the residue of $C_{A,b}$ and the sum ranges over such (A, b) that

$$C_{A,b} \neq 0, \quad (b \leq m, A = 0) \text{ or } (b < m \text{ and } v = \frac{A}{m-b}).$$

(iii) For $A \in \mathbb{N}^d$, let $|A| := a_1 + \dots + a_d$. We put

$$\delta(h, u, y) := \min \left\{ \frac{|A|}{m-b} : C_{A,b} \neq 0, b < m \right\}.$$

This is in fact an invariant of the polyhedron $\Delta(h; u; y)$ since

$$\delta(h, u, y) = \min \{ |v| : v \in \Delta(h; u; y) \}.$$

The δ -initial of h is defined as

$$\text{in}_{\delta, u, y}(h) := \sum_{A,b} \overline{C_{A,b}} Y^b U^A \in k(x)[U, Y] = k(x)[U_1, U_2, \dots, U_d, Y],$$

where the sum ranges over such (A, b) that

$$C_{A,b} \neq 0, \quad (b \leq m, A = 0) \text{ or } (b < m \text{ and } \frac{|A|}{m-b} = \delta(h, u, y)).$$

(iv) More generally, let

$$L : (x_1, x_2, \dots, x_d) \mapsto L(x_1, x_2, \dots, x_d) = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_d x_d, \quad \lambda_1, \lambda_2, \dots, \lambda_d \in \mathbb{Q}_{\geq 0},$$

be a nonzero nonnegative linear form on $\mathbb{R}^d$. We define

$$l(h, u, y) := \min\{L(A) \mid A \in \Delta(h; u, y)\} \geq 0.$$

We define a monomial valuation $v_{L,h,u,y}$ on R by setting

$$I_\lambda := (\{y^b u^A \mid l(h, u, y)b + L(A) \geq \lambda\}) \subseteq R,$$

for $\lambda \geq 0$ and $v_{L,h,u,y}(g) := \min\{\lambda \in \mathbb{Q} \mid g \in I_\lambda\}$ for any nonzero $g \in R$.

PROPOSITION II.3 Let L be a nonzero nonnegative linear form as above, and let

$$I := \{i \mid \lambda_i > 0\}, \quad I' := \{i \mid \lambda_i = 0\} = \{1, \dots, d\} \setminus I.$$

The graded algebra $\text{gr}_{v_{L,h,u,y}}(R)$ of R w.r.t. $v_{L,h,u,y}$ is given by

(i) if $l(h, u, y) \neq 0$, then

$$\text{gr}_{v_{L,h,u,y}}(R) = \frac{R}{(y, \{u_i\}_{i \in I})} [Y, \{U_i\}_{i \in I}];$$

(ii) if $l(h, u, y) = 0$, then

$$\text{gr}_{v_{L,h,u,y}}(R) = \frac{R}{(\{u_i\}_{i \in I})} [\{U_i\}_{i \in I}].$$

In particular, we have $\text{gr}_{v_{L,h,u,y}}(R) \simeq k[Y, U_1, U_2, \dots, U_d]$ whenever L is positive.

The above proposition is obvious. One also checks easily the following:

REMARK II.3.1 Let v be a vertex of $\Delta(h, u, y)$. We have:

(i) $\text{in}_v(h)$ is independent of the presentation **II.1** (1),

(ii) $\text{in}_v(h) \neq \text{in}_x(h)$,

(iii) L being a positive linear form L on $\mathbb{R}^d$ such that $\{v\} = \Delta \cap \{A \in \mathbb{R}^d \mid L(A) = 1\}$, (cf. **II.1** (ii)), then

$$\text{in}_v(h) = \text{in}_{v_{L,h,u,y}}(h) \in \text{gr}_{v_{L,h,u,y}}(R) = k(x)[U_1, U_2, \dots, U_d, Y].$$

When there is no ambiguity, we will write $\text{gr}_\delta(R)$ and $\text{in}_\delta(h) \in \text{gr}_\delta(R)$ instead of respectively $\text{gr}_{v_{L,h,u,y}}(R)$ and $\text{in}_{v_{L,h,u,y}}(h)$, where $L(x_1, x_2, \dots, x_d) = x_1 + x_2 + \dots + x_d$.

REMARK II.4 With notations as above, we have:

$\delta(h, u, y) \in \frac{1}{m!}\mathbb{N}$ and $\delta(h, u, y) > 1$ if $(m = \text{ord}_x(h) \text{ and } \langle \text{in}_x(h) \rangle = \langle \text{in}_x(y) \rangle^m)$.

ASSUMPTIONS II.4.1 We now apply these constructions to the case $R := \mathcal{O}_{Z,x}$, $\dim(R) = 4$; the element $h \in R$ verifies assumptions **II.1**(2)(3) with $m = \text{ord}_x(h) < p = \text{char}_k(x)$ and $\langle \text{in}_x(h) \rangle = \langle Y^m \rangle$. In addition, $X = \text{Spec}(R/(h))$ is reduced.

THEOREM II.5 *Given $(y, u_1, u_2, u_3) =: (y, u)$ as above, there exists $z \in R$, $z \equiv y \pmod{(u_1, u_2, u_3)}$ such that*

$$\Delta(h, u, z) = \Delta(h, u) \neq \emptyset, \quad (1)$$

$$\text{in}_{\delta, u, z}(h) = \sum_{A, b, b \neq m(x)-1} \overline{C_{A,b}} z^b U^A. \quad (2)$$

Proof. Suppose $\Delta(h, u) = \emptyset$, then, in $\widehat{R}$, we should have $h = \gamma z^m$, γ invertible in $\widehat{R}$ and $z \in \widehat{R}$ a local parameter: h should be nonreduced in $\widehat{R}$. By excellence, h should be nonreduced in R , in contradiction with the hypothesis X reduced.

Since $\Delta(h, u) \neq \emptyset$, $\Delta(h, u)$ may be defined by a finite number n of inequalities $L_i(x_1, x_2, x_3) \geq 1$, $1 \leq i \leq n$, with $L_i(x_1, x_2, x_3) = a_{i,1}x_1 + a_{i,2}x_2 + a_{i,3}x_3$, $a_{i,1}, a_{i,2}, a_{i,3} \in \mathbb{Q}_{\geq 0}$, $L_i \neq 0$. In a few words:

$$\Delta(h, u) = \{(x_1, x_2, x_3) | L_i(x_1, x_2, x_3) \geq 1, 1 \leq i \leq n\}.$$

We choose $L_1(x_1, x_2, x_3) = \frac{1}{\delta(h, u)}(x_1 + x_2 + x_3)$, with $\delta(h, u) := \min\{|v| : v \in \Delta(h, u)\}$.

Suppose (1) does not hold for (y, u) . Then, with notations as in **II.2**(iv), some L_i , $1 \leq i \leq n$ satisfies

$$L_i(\Delta(h, u, y)) = [l_i(h, u, y), \infty[\not\subset [1, +\infty[\Leftrightarrow l_i(h, u, y) < 1.$$

We skip the index i of L_i and of $l_i(h, u, y)$ to simplify the notations. Following **II.2**(iv), we define the initial form of h with respect to L, u, y : in the case $l(h, u, y) > 0$

$$\text{in}_{L, u, y}(h) := \sum_{A, b} \overline{C_{A,b}} Y^b U^A \in \text{gr}_{L, u, y}(R), \quad (3)$$

with $bl(h, u, y) + L(A) = ml(h, u, y)$, $\overline{C_{A,b}} \in \frac{R}{(u_i)_{a_i > 0 + (y)}}$. In the case $l(h, u, y) = 0$, we have

$$\text{in}_{L, u, y}(h) := \sum_{A, b} \overline{C_{A,b}} y^b U^A \in \text{gr}_{L, u, y}(R), L(A) = 0, \overline{C_{A,b}} y^b \in \frac{R}{(u_i)_{a_i > 0}}. \quad (4)$$

Claim II.5.1 In (3) (resp. (4)), there exists A with $\overline{C_{A, m-1}} \neq 0 \in \frac{R}{(u_i)_{a_i > 0}}$ (resp. $\overline{C_{A, m-1}} y^{m-1} \neq 0 \in \frac{R}{(u_i)_{a_i > 0}}$).

Indeed, in the face with equation $L(x_1, x_2, x_3) = l(h, u, y)$ of $\Delta(h, u, y)$ there is at least a vertex v which is solvable [H2, (3.8)]. Then $\text{in}_v(h)$ is collinear to an m^{th} -power: $C_{v, m-1} \notin \mathfrak{M}$ since $m < p$ and the claim is proved. Take $A = v$ and let

$$y_1 = y + \frac{1}{m} C_{0, m}^{-1} \sum_A C_{A, m-1} u^A \in R. \quad (5)$$

Note that, for any A with $C_{A, m-1} \neq 0$, $A \in \Delta(h, u, y)$. So, for any i , $1 \leq i \leq n$, and any A with $C_{A, m-1} \neq 0$, $L(A) \geq l(h, u, y)$. So if in the expansion of **II.2**(1) we set

$$y = y_1 - \frac{1}{m} C_{0, m}^{-1} \sum_A C_{A, m-1} u^A,$$

we get a new expansion

$$h = \sum_{A,b} D_{A,b} y_1^b u^A, \quad D_{A,b} \in R^\times \cup \{0\}, \quad b \in \mathbb{N}, \quad A \in \mathbb{N}^3, \quad (6)$$

and $D_{A,b} \neq 0 \Rightarrow L(A) + bl_i(h, u, y) \geq m$. So

$$l(h, u, y_1) \geq l(h, u, y), \quad 1 \leq i \leq n.$$

Suppose $l(h, u, y_1) = l(h, u, y)$. Then $v_{L,u,y_1}(y) = v_{L,u,y_1}(y_1) = l(h, u, y)$,

$$Y := \text{in}_{L,u,y_1}(y) = Y_1 - \text{in}_{L,u,y_1}\left(\frac{1}{m} C_{0,m}^{-1} \sum_A C_{A,m-1} u^A\right), \quad Y_1 := \text{in}_{L,u,y_1}(y_1),$$

$$\text{in}_{L,u,y_1}(h) = \sum_{A,b, L(A)+l(h,u,y)b=l(h,u,y)m} \text{in}_{L,u,y_1}(D_{A,b}) Y_1^b (\text{in}_{L,u,y_1}(u^A)), \quad (7)$$

$$\text{in}_{L,u,y_1}(h) = \sum_{A,b, L(A)+l(h,u,y)b=l(h,u,y)m} \text{in}_{L,u,y_1}(C_{A,b}) (Y_1 - \text{in}_{L,u,y_1}\left(\frac{1}{m} C_{0,m}^{-1} \sum_A C_{A,m-1} u^A\right))^b, \quad (8)$$

where (7) is the expansion of (8). In (7), the terms with $b = m - 1$ are all zero; in (6), $D_{A,m-1} \neq 0$ implies $L(A) + l(h, u, y_1)b > ml(h, u, y_1)$. By the claim, $l(h, u, y_1) = 1$: a contradiction, hence

$$l(h, u, y_1) > l(h, u, y).$$

Note that (5) is independent of the linear form L , so

$$l_i(h, u, y_1) > l_i(h, u, y), \quad \text{for all } i \quad 1 \leq i \leq n.$$

By induction on the $l_i(h, u, y)$'s, we get (1). If (1) holds but not (2) for h, u, y , we make the change of variable (3) and get (2).

PROPOSITION II.5.2 *With notations as above, assume furthermore that (z, u) and (z', u') satisfy (1) of the previous theorem. Then $\delta(h, u, z) = \delta(h, u', z')$.*

Proof. This is obvious if $u_i = u'_i$, $1 \leq i \leq 3$. On the other hand, the condition

$$\text{in}_\delta(h) \neq \lambda(Z - \Phi(U_1, U_2, U_3))^m \text{ for every } \Phi \in k(x)[U_1, U_2, U_3] \quad (9)$$

is preserved if $z = z'$ since $\delta(h, u, z) > 1$ (remark II.4). In particular we have

$$\delta(h, u, z) = \delta(h, u', z) \leq \delta(h, u', z')$$

and we conclude by symmetry that $\delta(h, u, z) = \delta(h, u', z')$.

DEFINITION II.5.3 *When $z \in R$ is such that (1) (resp. (2) holds), we say that (z, u) is prepared (resp. δ -prepared). If both of (1) and (2) hold for $z \in R$, we say that (z, u) is fully prepared. If there is no ambiguity on $u = (u_1, u_2, u_3)$, we simply say that z is prepared, δ -prepared, or fully-prepared.*

If (z, u) is prepared, the invariant $\delta(h, u, z)$ will be henceforth denoted by $\delta(x)$.

THEOREM II.6 *Let $(y, u_1, u_2, u_3) =: (y, u)$ be as before and E be a normal crossings divisor $E \subseteq \text{div}(u_1 u_2 u_3) \subset \text{Spec}(R)$. For a component $\text{div}(u_i)$ of E , we define*

$$d_i(u, y) := \inf\{x_i | (x_1, x_2, x_3) \in \Delta(h, u, y)\},$$

$$d_i(u) := \inf\{x_i | (x_1, x_2, x_3) \in \Delta(h, u)\}.$$

(i) Let $z \in R$ be such that $(z, u) = (z, u_1, u_2, u_3)$ is prepared. Then

$$d_i(u) \geq 1 \Leftrightarrow V(z, u_i) \text{ is a permissible blowing up center of } \text{Spec}(R/(h)). \quad (1)$$

(ii) We have $d_i(u) > 0$ if and only if $Y_i := V(h, u_i)_{\text{red}}$ is a regular surface.

(iii) Assume that $d_i(u) > 0$. Then $d_i(u) = \delta(\eta_i)$, where $\eta_i \in Z$ is the generic point of $Y_i = V(h, u_i)_{\text{red}}$ as above; in particular, $d_i(u)$ is independent of the choice of a prepared $(z, u) = (z, u_1, u_2, u_3)$ containing u_i .

Proof of (ii). Take $z \in R$ such that $\Delta(h, u) = \Delta(h, u, z)$, so $h = z^m \text{mod}(u_i)$ iff $d_i(u) > 0$ iff $Y := V(z, u_i) = V(h, u_i)_{\text{red}}$. We get

$$d_i(u) > 0 \Leftrightarrow (V(h, u_i))_{\text{red}} \text{ is regular at } x.$$

Proof of (i). In that hypersurface case, $Y := V(z, u_i) \in \text{Spec} R$ permissible means $h \in (z, u_i)^m$ which is equivalent to $d_i(u) \geq 1$.

Proof of (iii). Let η be the generic point of $Y = V(h, u_i)_{\text{red}} = V(z, u_i)$. The following equivalence is straightforward

$$d_i(u) = 1 \Leftrightarrow V(z, u_i) \text{ is a permissible center of } \text{Spec}(R/(h)) \text{ and } \tau(\eta) = 2.$$

If $0 < d_i(u) < 1$, we obviously have $d_i(u) = d_i(u, y)$ for every r.s.p. (y, u_1, u_2, u_3) (with $\langle \text{in}_x(h) \rangle = \langle Y^m \rangle$).

We turn to the case:

$$d_i(u) > 1 \Leftrightarrow V(z, u_i) \text{ is a permissible center of } \text{Spec}(R/(h)) \text{ and } \tau(\eta_i) = 1.$$

We claim that $d_i(u) = \delta(\eta_i)$. As $V(z, u_i) = V(h, u_i)_{\text{red}}$, this will prove the invariance of $d_i(u)$. We take $i = 1$ and write η for η_1 in the following lemma.

LEMMA II.6.1 *Let (y, u_1, u_2, u_3) be as above and assume that $d_1(u, y) > 1$. Then*

$$d_1(u, y) = \delta(h, u_1, y), \quad (1)$$

where the right hand side is computed w.r.t. the datum $(h) \subset R_{(y, u_1)}$. If furthermore (y, u) is prepared, then

$$d_1(u) = \delta(\eta). \quad (2)$$

Proof. By II.1 (1), we have a finite expansion

$$h = \sum_{A, b} C_{A, b} y^b u^A, \quad C_{A, b} \in R^\times, \quad b \in \mathbb{N}, \quad A \in \mathbb{N}^3$$

that we rewrite as:

$$h = \sum_{a_1, b} \left(\sum_{a_2, a_3} C_{A, b} u_2^{a_2} u_3^{a_3} \right) y^b u_1^{a_1}, \quad A = (a_1, a_2, a_3) \in \mathbb{N}^3.$$

As $d_1(u, y) > 1$, we have $b + a_1 > m$ for all $b < m$. Note that every $\sum_{a_2, a_3} C_{A, b} u_2^{a_2} u_3^{a_3}$ is invertible in $R_{(y, u_1)} = \mathcal{O}_{Z, \eta}$. Then $\delta(h, u_1, y) = \inf\{\frac{a_1}{m-b} | b < m\} = d_1(u, y)$: this is (1).

Now suppose that y, u_1, u_2, u_3 is prepared. We claim that y, u_1 is prepared w.r.t. the datum $(h) \subset R_{(y, u_1)}$. If not, then

$$in_\delta(h) = \sum_{a_1 + b d_1 = d_1 m} \left(\sum_{a_2, a_3} \overline{C_{A, b} u_2^{a_2} u_3^{a_3}} \right) Y^b u_1^{a_1} \in \text{gr}_\delta(R_{(y, u_1)}),$$

is proportional to an m^{th} -power. So there exists some a_1 with

$$a_1 + (m-1)d_1 = d_1 m, \text{ and } \sum_{a_2, a_3} C_{A, b} u_2^{a_2} u_3^{a_3} \neq 0,$$

so there exists some A with $a_1 + (m-1)d_1 = d_1 m$ and $C_{A, m-1} \neq 0$. Then, as in **II.5(5)**, we change the variable y by $y_1 = y + \frac{1}{m} C_{0, m}^{-1} \sum_A C_{A, m-1} u^A \in R$, we get a new expansion

$$h = \sum_{A, b} D_{A, b} y_1^b u^A, \quad D_{A, b} \in R^\times \cup \{0\}, \quad b \in \mathbb{N}, \quad A \in \mathbb{N}^3.$$

Now $D_{A, b} \neq 0$ implies $\frac{L_i(A)}{l_i(h, u, y)} + b \geq m$ for each linear form L_i such that

$$\Delta(h, u) = \{(x_1, x_2, x_3) | L_i(x_1, x_2, x_3) \geq 1, \quad 1 \leq i \leq n\}.$$

This holds in particular for the linear form $L(x_1, x_2, x_3) = \frac{1}{d_1} x_1$. Since y is supposed to be nonprepared for u_1 , the unique vertex $d_1(u_1, y)$ of $\Delta(h, u_1, y) \subset \mathbb{R}^+$ does not belong to $\Delta(h, u_1, y_1)$. We get $d_1(u_1, y_1) > d_1(u, y)$, a contradiction with the fact that $\Delta(h, u, y)$ was minimal.

DEFINITION and NOTATION II.7 *Given $(y, u_1, u_2, u_3) =: (y, u)$, h reduced, with assumptions **II.1(2)(3)** and a normal crossings divisor $E \subset \text{div}(u_1 u_2 u_3) \subset \text{Spec}(R)$, we let $d_i := d_i(u)$ for each irreducible component $\text{div}(u_i)$ of E . We let $d_i := 0$ whenever $\text{div}(u_i)$ is not an irreducible component of E .*

We define $\epsilon(x, E) \in \mathbb{Q}_{\geq 0}$ (or $\epsilon(x)$ for short) by:

$$\epsilon(x, E) = \delta(x) - \sum_{\text{div}(u_i) \subset E} d_i.$$

These invariants appear in [CP2] Ch.1 (II.3.3) in an equal characteristic context. The following remarks are obvious from the definitions.

REMARK II.8 *We have*

- (i) $\epsilon(x, E) \in \frac{1}{m!} \mathbb{N}$,
- (ii) if $\epsilon(x, E) = 0$, $\Delta(h, u)$ has only one vertex: the point $v = (d_1, d_2, d_3)$.

THEOREM II.9 *Given $(y, u_1, u_2, u_3) =: (y, u)$ and a divisor $E \subset \text{div}(u_1 u_2 u_3) \subset \text{Spec}(R)$ as above, assume that $\epsilon(x, E) = 0$. There exists a finite sequence of local blowing ups*

$$(X, x) =: (X_0, x_0) \longleftarrow (X_1, x_1) \longleftarrow \cdots \longleftarrow (X_n, x_n),$$

where $x_0 = x$, $x_i \in X_i$, $0 \leq i \leq n$ is the center of v , each blowing up center $Y_i \subset X_i$ is permissible in Hironaka's sense, such that $m(x_n) < m(x)$.

Proof. See the connection with [CP2] Ch.1 (II.4.6). Let $z \in R$ be such that $\Delta(h, u, z) = \Delta(h, u)$. Then $\delta(x) = \sum_{1 \leq i \leq 3} d_i$. Let $\mathcal{I} \subset \{1, 2, 3\}$ satisfy the two following conditions:

P (for permissibility): $\sum_{i \in \mathcal{I}} d_i \geq 1$,

M (for maximality): $|\mathcal{I}|$ minimal for P, i.e. the dimension of $V(z, < u_i, i \in \mathcal{I} >)$ is maximal for P.

Note that $\mathcal{I} \subset \{1, 2, 3\}$ is not unique in general. Then we choose $\mathcal{I}$ with PM and we blow up Z along $V(z, < u_i, i \in \mathcal{I} >)$. Let $e : Z' \rightarrow Z$ denote this blowing up, X' be the strict transform of X , $x' \in X'$ be a point above x , $E' \subset Z'$ be the reduced inverse image of E .

We claim that for either $(m(x'), -\tau(x')) <_{lex} (m(x), -\tau(x))$ or

$$((m(x'), \tau(x')) = (m(x), \tau(x)) = (m(x), 1) \text{ and } \epsilon(x') = 0 \text{ and } \delta(x') < \delta(x)).$$

Since $\delta(h, u, y) \in \frac{1}{m!}\mathbb{N}$, a descending induction on $\delta(x)$ reduces to $m(x') < m(x)$ or $(m(x') = m(x), \tau(x') \geq 2)$. As stated in section I, this completes the proof.

Proof of the claim. We only treat the case $\mathcal{I} = \{1, 2, 3\}$, the other cases being similar, if somewhat simpler. By PM, $\mathcal{I} = \{1, 2, 3\}$ means

$$d_i > 0, \quad d_i + d_j < 1 \text{ when } i \neq j, \quad 1 \leq i, j \leq 3. \quad (1)$$

By [H3, thm.3, p.331], if $m(x') = m(x)$, then x' lies on the strict transform of $z = 0$. The variables u_1, u_2, u_3 play symmetric roles; so after reordering, it can be assumed that x' belongs to the affine chart $\text{Spec} R[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$. Let

$$(z', u'_1, u'_2, u'_3) := (z/u_1, u_1, u_2/u_1, u_3/u_1).$$

Let

$$h = C_{0,m} z^m + \sum_{\substack{|A| \\ \delta(x) + b \geq m}} C_{A,b} z^b u_1^{a_1} u_2^{a_2} u_3^{a_3},$$

be an expansion **II.1**(1) of h with (z, u) fully prepared (theorem **II.5** and definition **II.5.3**),

$$h' := h/u_1^m = C_{0,m} z'^m + \sum_{\substack{|A| \\ \delta(x) + b = m}} C_{A,b} z'^b u_1^{(m-b)(\delta(x)-1)} u_2'^{a_2} u_3'^{a_3} + h'_1 \quad (2)$$

where $h'_1 \in I'_{\delta(x)}{}^+ := (z'^{m+1}, z'^b u_1^{(m-b)(\delta(x)-1)+1}, 0 \leq b \leq m)$.

Since $\epsilon(x) = 0$, $C_{A,b}$ invertible in (2) implies $a_2 = (m-b)d_2$, $a_3 = (m-b)d_3$. Note also that $\delta(x) - 1 = d_1 + d_2 + d_3 - 1 < d_1$ by (1). As $d_1 + d_2 < 1$ and $d_1 + d_3 < 1$, $m(x') = m(x)$ implies $x' = (z', u'_1, u'_2, u'_3)$. The coordinate change $(z, u_1, u_2, u_3) \mapsto (z', u'_1, u'_2, u'_3)$ is a monomial substitution, so $\Delta(h', u', z')$ is again minimal. With natural notations, we get $(d'_1, d'_2, d'_3) = (\delta(x) - 1, d_2, d_3)$, $\epsilon(x') = 0$ and

$$\delta(x') = d'_1 + d'_2 + d'_3 = \delta(x) - 1 + d_2 + d_3 < \delta(x)$$

provided $< \text{in}_{x'}(h') > = < Z'^2 >$, i.e. $\tau(x') = \tau(x) = 1$.

THEOREM II.10 *Given $(y, u_1, u_2, u_3) =: (y, u)$ and a divisor $E \subset \text{div}(u_1 u_2 u_3) \subset \text{Spec}(R)$ as above, assume that E is equicharacteristic $p = \text{chark}(x)$ and*

$$\text{Sing}_{m(x)}(X) \subset E.$$

Then the set $\{y \in X \mid m(y) = m(x), \epsilon(y) > 0 \text{ and } \tau(y) = 1\}$ is locally closed.

Proof. It is well known that the set

$$\{y \in X | m(y) = m(x), \tau(y) \geq 2\} \subseteq E$$

is locally closed. Suppose $\epsilon(x) = 0$ for some closed point $x \in E$. We choose a r.s.p. (z, u_1, u_2, u_3) of R at x which is fully prepared. There is a finite expansion

$$h = C_{0,m} z^m + \sum_{\frac{|A|}{\delta(x)} \geq b} C_{A,b} z^{m-b} u_1^{a_1} u_2^{a_2} u_3^{a_3} \quad (1)$$

with each $C_{A,b}$ invertible in R and $a_i \geq b d_i$, $i = 1, 2, 3$. Since $\epsilon(x) = 0$, there exists $C_{A,b}$ such that $a_i = b d_i$, $i = 1, 2, 3$, with $b \geq 2$ by full preparedness. Then the locus

$$\{y \in X | m(y) = m(x), \epsilon(y) = 0 \text{ or } \tau(y) \geq 2\}$$

contains the intersection of $\text{Sing}_{m(x)}(X)$ with the complement of the hypersurface $V(C_{A,b})$.

THEOREM II.11 *With assumptions as in II.10, assume furthermore $\epsilon(x) > 0$ and let F be an irreducible component of E with $x \in F$. Then*

$$\dim(\{y \in X | m(y) = m(x), \epsilon(y) > 0 \text{ and } \tau(y) = 1\} \cap F) \leq 1.$$

Proof. Say $\text{div}(u_1)$ is the given component. If $d_1 < 1$, then $\dim(\{y \in X | m(y) = m(x)\} \cap F) \leq 1$ and the result is clear.

Assume now $d_1 > 1$ and pick a fully prepared (y, u_1) w.r.t. to datum $h \in R_{y,u_1}$. There exists a nonempty Zariski open set $\Omega \subseteq F$ such that for $y \in \Omega$ there is an expansion

$$h = \gamma_0 z^m + \sum_{1 \leq i \leq m} \gamma_i z^{m-i} u_1^{a_i},$$

with $a_i \geq i d_1$, $\gamma_i \in \mathcal{O}_{X,y}$ for $i \geq 1$. By definition of d_1 , some $i \geq 2$ satisfies $(a_i = i d_1 \text{ and } \gamma_i \notin (u_1))$. By full preparedness, we have $\gamma_1 \in (u_1)$ if $a_1 = d_1$. Let Ω' be the intersection of Ω with complement of the proper closed subset $V(\gamma_i)$, so

$$\Omega' \subseteq \{y \in X | m(y) = m(x), \epsilon(y) = 0\}.$$

Assume finally $d_1 = 1$. The same construction now yields

$$\Omega' \subseteq \{y \in X | m(y) = m(x), \tau(y) \geq 2\}.$$

III CONSTRUCTION OF THE DIVISOR E .

In this section, we reach the assumptions of II.10. We show they are stable under a class of local permissible blowing ups which we will prove ahead are sufficient to prove theorem 0.3. We stick to assumptions II.4.1.

PROPOSITION III.1 *With assumptions as above, there exists a finite sequence of local blowing ups*

$$(Z, x) =: (Z_0, x_0) \longleftarrow (Z_1, x_1) \longleftarrow \cdots \longleftarrow (Z_n, x_n),$$

where $x_0 = x$, $x_i \in X_i$ (X_i denoting the strict transform of X), $0 \leq i \leq n$, is the center of v , each blowing up center $Y_i \subset X_i$ is permissible for X_i in Hironaka's sense, such that one the following properties holds:

- (i) $(m(x_n), -\tau(x_n)) <_{lex} (m(x), -\tau(x))$, or
- (ii) $(m(x_n), -\tau(x_n)) = (m(x), -\tau(x))$ and there exists a (reduced) normal crossings divisor $E_n \subset (Z_n, x_n)$ of equicharacteristic $p = \text{chark}(x_n) = \text{chark}(x)$ such that

$$C_{x_n}(E_n) \perp C_{x_n}(X_n)_{red}, \quad (1)$$

$$S_n := \text{Sing}_{m(x_n)}(X_n) \subset E_n, \quad (2)$$

where C_{x_n} denotes the tangent cone and $\text{Sing}_{m(x_n)}(X_n)$ is the stalk at x_n of the set of multiplicity $m(x)$.

Proof. We begin with the following lemma.

LEMMA III.2 *With assumptions as above, assume furthermore that there exists a normal crossings divisor $E \subset (Z, x)$ such that*

$$C_x(E) \perp C_x(X)_{red}. \quad (1)$$

Then for any local blowing up :

$$\pi : (Z', x') \longrightarrow (Z, x)$$

of center $Y \subset X$, permissible for (X, x) and at normal crossing with E , we have $(m(x'), \tau(x')) \leq (m(x), \tau(x))$, where $x' \in X'$ is the center of v ; if equality holds, then

$$C_{x'}(E') \perp C_{x'}(X')_{red},$$

where $E' := \pi^{-1}(E)_{red}$, X' the strict transform of X .

Proof. The normal crossing assumption implies that we can choose a r.s.p. (v_1, v_2, v_3, v_4) of $R := \mathcal{O}_{Z, x}$ such that $Y = V(v_1, \dots, v_e)$ and $E \subseteq \text{div}(v_1 \cdots v_4)$. By permissibility, we have $h \in (v_1, \dots, v_e)^{m(x)}$. Assumption (1) means that $\langle \text{in}_x(h) \rangle = \langle Z^d \rangle$, where $Z \notin \langle \text{in}_x(v_j), \text{div}(u_j) \subseteq E \rangle$. Changing generators of the ideal of Y , we relabel parameters as (z, u_1, u_2, u_3) with

$$E = \text{div}(u_1 \cdots u_d) \subset \text{div}(u_1 u_2 u_3), \quad I(Y) = (z, \{u_i, i \in A\}) \text{ for some } A \subseteq \{1, 2, 3\}. \quad (2)$$

If $m(x') = m(x)$, x' belongs to the strict transform of $\text{div}(z)$ by (1). Let $i \in A$ such that u_i generates the ideal of the exceptional divisor of π in a neighbourhood of x' and let $z' = z/u_i$. A local equation for (X', x') is $h' := h/u_i^{m(x)}$, where

$$h' \equiv \gamma z'^{m(x)} \pmod{(u_i, \{u_k, k \notin A\})},$$

and

$$E' \subset \text{div}(u_i \times \prod_{j \neq i, j \in A} \left(\frac{u_j}{u_i} \right) \prod_{k \notin A} u_k),$$

which proves the lemma.

Proof of III.1. If $\text{Sing}_{m(x)}(X) = \{x\}$, take $E = \text{div}(u_1 u_2 u_3)$ with coordinates as in (2) above.

If $\dim(\text{Sing}_{m(x)}(X)) \geq 1$, then any regular closed set $Y \subset S := S_0$ is permissible for X . In any case, we have $(m(x_n), -\tau(x_n)) \leq (m(x), -\tau(x))$ since centers are permissible.

Let (C, x) be any curve contained in S . Since it is assumed that v has rank one, x_n does not belong to the strict transform C_n of C in X_n for $n \gg 0$ if we take $Y_i = \{x_i\}$, the center of v in

X_i for $i \geq 0$. In particular, it can be assumed that the strict transform S_n of S in X_n has pure dimension two. Take $n = 0$ in what follows.

We now apply classical embedded resolution theorems for S with $\dim(S) = 2$ ([CJS] for suitable generality). This involves blowing up closed points or regular curves on the successive strict transforms of S . By blowing up finitely closed many points as before, it can be assumed that every blown up curve is equicharacteristic $p = \text{char}(k(x))$. We reach the following situation: the strict transform S_n of S at x_n is empty or an irreducible surface with normal crossings with the (equicharacteristic) reduced exceptional divisor E_n of $(Z_n, x_n) \rightarrow (Z, x)$. If S itself is equicharacteristic, enlarge E_n to $E_n \cup S$. Otherwise, we blow up finitely many times irreducible components of $S \cap E_n$ (i.e. equicharacteristic curves) to get $x_n \notin S_n$. This is possible again because v has rank one.

IV REFINED DIRECTRIX, TRANSVERSENESS, ENCOMBREMENT

Assume that the conclusion of proposition **III.1** (ii) holds. We will perform local blowing ups which are permissible in Hironaka's sense, with center Y_n having normal crossing with E_n . Take $n = 0$ in what follows, $E = E_0$, and consider a local blowing up :

$$\pi : (Z', x') \longrightarrow (Z, x)$$

of center $Y \subset X$, permissible for (X, x) and at normal crossing with E . We assume that

$$(m(x'), \tau(x')) = (m(x), \tau(x)),$$

where $x' \in X'$ is the center of v . By lemma **III.2**, we have

$$C_{x'}(E') \perp C_{x'}(X')_{red},$$

where $E' := \pi^{-1}(E)_{red}$, X' the strict transform of X in Z' .

DEFINITION and NOTATION IV.1 *Let (z, u) be fully prepared with $E \subset \text{div}(u_1 u_2 u_3) \subset \text{Spec}(R)$ as above. Let*

$$F := \text{in}_\delta(h) = Z^m + \sum_{2 \leq j \leq m} Z^{m-j} F_j(U_1, U_2, U_3) \in \text{gr}_\delta(R) = k(x)[Z, U_1, U_2, U_3],$$

where $Z = \text{in}_\delta(z)$, $U_i = \text{in}_\delta(u_i)$, $1 \leq i \leq 3$ (notations of **II.3.1**). Each F_j is zero or homogeneous of degree $j\delta(x)$; we have $F_j = 0$ if $j\delta(x) \notin \mathbb{N}$.

We define the refined tangent ideal of X at x as the ideal

$$I_x := (Z, \prod_{\text{div}(u_i) \subset E} U_i^{-m!d_i} F_j^{\frac{m!}{j}}, 1 \leq j \leq m) \subset k(x)[Z, U_1, U_2, U_3].$$

We define the refined directrix of X at x as the smallest vector subspace $\mathcal{V} \subseteq \langle U_1, U_2, U_3 \rangle$ such that

$$\{U_i^{-m!d_i} F_j^{\frac{m!}{j}} \mid 1 \leq j \leq m, \text{div}(u_i) \subset E\} \subseteq k(x)[\mathcal{V}].$$

Let $\rho(x) := \dim(\mathcal{V})$.

REMARK IV.1.1 *The following holds:*

- (i) $U_i^{[jd_i]}$ divides F_j , $2 \leq j \leq m$, for $1 \leq i \leq 3$ such that $\text{div}(u_i) \subset E$.
- (ii) I_x and $\mathcal{V}$ do not depend upon choices of z, u satisfying the assumptions.
- (iii) the polynomials

$$U_i^{-m!d_i} F_j^{\frac{m!}{j}}, \quad 1 \leq j \leq m, \quad 1 \leq i \leq 3$$

are zero or homogeneous of degree $m!e(x)$.

Statement (i) is a consequence of the definition of d_i . For (ii), suppose (z', u') is fully prepared, where $E \subset \text{div}(u'_1 u'_2 u'_3)$. Let

$$u_j = a_{1,j} u'_1 + a_{2,j} u'_2 + a_{3,j} u'_3 + b_j z, \quad a_{i,j}, b_j \in R, \quad 1 \leq i, j \leq 3,$$

for some matrix $(a_{i,j}) \in \text{GL}(3, R)$. Since $\tau(x) = 1$, we have

$$\deg U_j = \deg U'_j = \frac{1}{\delta(x)} < \deg Z = 1$$

in gr_δ . Computing w.r.t. the r.s.p. (z, u') , we get

$$\text{in}_\delta(h) = Z^m + \sum_{2 \leq j \leq m} Z^{m-j} F'_j(U'_1, U'_2, U'_3) \in \text{gr}_\delta(R) = k(x)[Z, U'_1, U'_2, U'_3],$$

with $F'_j(U'_1, U'_2, U'_3) = F_j(\overline{\mathcal{M}}.(U'_1, U'_2, U'_3))$, $\overline{\mathcal{M}}$ being the residue of $\mathcal{M}$ in $\text{GL}(3, k(x))$. Since (z', u') is fully prepared, no term in Z^{m-1} occurs in $\text{in}_\delta(h) \in k(x)[Z', U'_1, U'_2, U'_3]$ and this implies that $< \text{in}_\delta(Z) > = < \text{in}_\delta(Z') >$.

Statement (iii) immediately follows from definition **II.7**.

DEFINITION IV.2 Let E be a fixed normal crossings divisor and (z, u) be fully prepared (always with the condition $E \subset \text{div}(u_1 u_2 u_3) \subset \text{Spec}(R)$) as above.

We call “transverseness” index of x , denoted by $t(x)$, the maximal dimension of a subspace of $\mathcal{V}$ which is transverse to $< U_i, \text{div}(u_i) \subset E >$. This is independent of the choice of a fully prepared r.s.p. (z, u) by remark **IV.1.1(ii)**.

We call “encombrement” of x , denoted by $e(x)$, the minimum number of U_i ’s among all possible fully prepared (z, u) necessary to write a basis of $\mathcal{V}$.

We define an invariant

$$\iota(x) := (m(x), -\tau(x), e(x), -\rho(x), -t(x), -e(x)) \in \mathbb{N} \times \{-4, -3, -2, -1\} \times \mathbb{Q}_{\geq 0} \times \{-3, -2, -1, 0\}^3.$$

For convenience, we extend the definition when $\tau(x) \geq 2$ by letting $\iota(x) := (m(x), -\tau(x), 0, 0, 0)$, theorem **0.3** being already proved in this special case (section I). Note that $e(x) \in \frac{1}{m(x)!} \mathbb{N}$, so any decreasing sequence of values (for the lexicographical ordering) taken by ι is finite.

EXAMPLES IV.2.1 Assume $p \geq 5$.

(i) $h = z^3 + u_1^2(u_1 + u_2 + u_3)^2$, $E = \text{div}(u_1)$.

Then $\delta(x) = \frac{4}{3}$, $d_1(x) = \frac{2}{3}$, $\mathcal{V} = < U_1 + U_2 + U_3 >$, $\rho(x) = 1$, $t(x) = 1$, $e(x) = 1$: take the r.s.p. $(z, u_1, u_1 + u_2 + u_3, u_3)$.

(ii) $h = z^3 + u_1^2(u_1 + u_2 + u_3)^2$, $E = \text{div}(u_1 u_2 u_3)$.

Then $\delta(x) = \frac{4}{3}$, $d_1(x) = \frac{2}{3}$, $d_2(x) = d_3(x) = 0$, $\mathcal{V} = < U_1 + U_2 + U_3 >$, $\rho(x) = 1$, $t(x) = 0$, $e(x) = 3$: the only choice allowed upon (u_1, u_2, u_3) is permuting or multiplying by a unit in R .

(iii) $h = z^3 + u_1^2(u_1^2 + u_2^2)$, $E = \text{div}(u_1)$.

Then $\delta(x) = \frac{4}{3}$, $d_1(x) = \frac{2}{3}$, $\mathcal{V} = \langle U_1, U_2 \rangle$, $\rho(x) = 2$, $t(x) = 1$, $e(x) = 2$.
(iv) $h = z^3 + u_1^2(u_1^2 + u_2^2)$, $E = \text{div}(u_1 u_2)$.
Then $\delta(x) = \frac{4}{3}$, $d_1(x) = \frac{2}{3}$, $d_2(x) = 0$, $\mathcal{V} = \langle U_1, U_2 \rangle$, $\rho(x) = 2$, $t(x) = 0$, $e(x) = 2$.

REMARK IV.2.2 The French “encombrement” was proposed by J. Giraud twenty years ago (English: “cumbersomeness index” roughly).

V PERMISSIBLE BLOWING UPS, BEHAVIOUR OF THE INVARIANTS

We stick to the assumptions of the previous section and assume furthermore that $\epsilon(x) > 0$.

DEFINITION V.1 An ϵ -permissible center (permissible center for short) Y at x is one of the following:

- (i) either $Y := \{x\} = V(z, u_1, u_2, u_3)$,
- (ii) or $Y := V(z, u_1, u_2)$ with (z, u) fully prepared,

$$d_1 + d_2 + \epsilon(x) \geq 1 \quad (1)$$

and

$$l(h, u, z) = m, \quad (2)$$

where L denotes the linear form $L(x_1, x_2, x_3) = \frac{x_1 + x_2}{d_1 + d_2 + \epsilon(x)}$ (definition **II.2**).

PROPOSITION V.2 An ϵ -permissible center at x is permissible in Hironaka’s sense.

Proof. Indeed, we have just to look at the case of a curve $V(z, u_1, u_2)$. In that latter case, as $d_1 + d_2 + \epsilon(x) \geq 1$, we have

$$\text{ord}_\eta(h) \geq v_{L, h, u, z}(h) = m,$$

where η is the generic point of $V(z, u_1, u_2)$, so

$$\text{ord}_\eta(h) = m,$$

which means exactly that $V(z, u_1, u_2)$ is permissible in Hironaka’s sense.

PROPOSITION V.3 Let $\pi : Z' \rightarrow Z$ be the blowing up along an ϵ -permissible center Y at x , X' be the strict transform of X (with transformed equation h' at the center $x' \in X'$ of v). We have:

- (i) $\iota(x') \leq \iota(x)$ (definition **IV.2**). If equality holds (in which case we say that x' is “very near” x), then $E' := \pi^{-1}(E)_{\text{red}}$ is transverse to the directrix T' of X' at x' .
- (ii) if $Y = \{x\}$ and $(m(x'), -\tau(x'), \epsilon(x')) = (m(x), -\tau(x), \epsilon(x))$, then x' lies on

$$\text{Proj}_k(x)[Z, U_1, U_2, U_3]/(Z, \mathcal{V}) \subset \text{Proj}(\text{gr}_{\mathfrak{M}}(R)) = \mathbb{P}_{k(x)}^3$$

with notations as in **IV.1**. The refined directrix $\mathcal{V}'$ at x' satisfies $\mathcal{V}' \equiv U^{-1}\mathcal{V} \pmod{\langle U \rangle}$ where $U = \text{in}_{x'}(u)$, u an equation of the exceptional divisor of π .

Proof. First assume that $Y = \{x\}$. By [H3, thm.3, p.331], if $m(x') = m(x)$, then x' lies on the strict transform of $z = 0$. The variables u_1, u_2, u_3 play symmetric roles; so after reordering, it can be assumed that x' belongs to the affine chart $\text{Spec} R[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$. Let

$$(z', u'_1, u'_2, u'_3) := (z/u_1, u_2/u_1, u_3/u_1).$$

We can choose a r.s.p. at x' in the following way: if x' is the origin, take (z', u'_1, u'_2, u'_3) ; if x' belongs to the strict transform of, say $\text{div}(u_2)$, we can take (z', u'_1, u'_2, v_3) with $v_3 = \sum_a \lambda_a u'_3{}^a$, $\lambda_a \in R$ a unit or zero (the sum is finite) whose residue $\sum_a \overline{\lambda_a} U_3^a \in k(x)[U_3]$ is an irreducible polynomial; in the general case $u'_2(x')u'_3(x') \neq 0$, we take (z', u'_1, v_2, v_3) where $v_c = \sum_{a,b} \lambda_{a,b,c} u'_2{}^a u'_3{}^b$, $c = 2, 3$ (sums are finite), $\lambda_{a,b,c} \in R$ a unit or zero, and

$$< \sum_{a,b} \overline{\lambda_{a,b,c}} U_2^a U_3^b, c = 1, 2 > \subset k(x)[U_2, U_3]$$

is a maximal ideal. Let

$$F := Z^m + \sum_{2 \leq j \leq m} Z^{m-j} U_1^{a(1,j)} U_2^{a(2,j)} U_3^{a(3,j)} G_j(U_1, U_2, U_3) :=$$

$$Z^m + \sum_{2 \leq j \leq m} Z^{m-j} F_j(U_1, U_2, U_3) := \text{in}_\delta(h) \in \text{gr}_\delta(R) = k(x)[Z, U_1, U_2, U_3], \quad (1)$$

with $a(i, j) \geq j d_i$, $2 \leq j \leq m$, $1 \leq i \leq 3$, $G_j \in k(x)[U_1, U_2, U_3]$ homogeneous, $G_j = 0$ or $\deg(G_j) = j\delta(x) - (a(1, j) + a(2, j) + a(3, j))$ and G_j not divisible by U_i , $1 \leq i \leq 3$.

Let $h' := h/u_1^m$ define the strict transform of h . We define the linear form

$$L'(x'_1, x'_2, x'_3) := \frac{x'_1}{\delta(x) - 1}$$

with associated valuation $v' := v_{L', h', u', z'}$ (definition **II.2**). We have

$$\text{in}_{v'}(h') = Z'^m + \sum_{2 \leq j \leq m} Z'^{m-j} U_1^{j(\delta(x)-1)} u'_2{}^{a(2,j)} u'_3{}^{a(3,j)} G_j(1, u'_2, u'_3) \in \text{gr}_{v'}(R'), \quad (2)$$

where $\text{gr}_{v'}(R') = R'/(z', u'_1)[Z', U_1]$. Here, the meaning of $G_j(1, u'_2, u'_3)$ is given by the inclusion

$$k(x) = R/\mathfrak{M} \rightarrow R'/(u'_1) \rightarrow R'/(z', u'_1).$$

By (2), $x'_1 = \delta(x) - 1$ is the minimum value of the first coordinate of points in $\Delta(h', u'_1, v_2, v_3, z')$. Since z is δ -prepared, no vertex of $\Delta(h', u'_1, v_2, v_3, z')$ with first coordinate equal to $x'_1 = \delta(x) - 1$ is solvable. We get

$$d_1(x') = \delta(x) - 1,$$

and for at least one vertex $(x'_1 = \delta(x) - 1, x'_2, x'_3)$, we have

$$x'_2 + x'_3 \leq \min\left\{\frac{\text{ord}_{x'}(u'_2{}^{a(2,j)} u'_3{}^{a(3,j)} G_j(1, u'_2, u'_3))}{j}, 2 \leq j \leq m\right\}.$$

In case x' belongs to the strict transform of some $\text{div}(u_i)$, $i = 2, 3$, we have $d_i(x') = d_i(x)$ for $u'_i(x') = 0$ by **II.6**(iii). This leads to:

$$\epsilon(x') \leq \min\left\{\frac{\text{ord}_{x'}(u'_2{}^{a(2,j)-jd_2} u'_3{}^{a(3,j)-jd_3} G_j(1, u'_2, u'_3))}{j}, 2 \leq j \leq m\right\} \leq \epsilon(x) \quad (3)$$

with the convention: $\text{ord}_{x'}(u_i'^a) = 0$ when $a \in \mathbb{Q}_+$ and $u'_i(x') \neq 0$, $\text{ord}_{x'}(u_i'^a) = a$ when $u'_i(x') = 0$. This proves $(-\tau(x'), \epsilon(x')) \leq (-\tau(x), \epsilon(x))$.

Assume that $(-\tau(x'), \epsilon(x')) = (-\tau(x), \epsilon(x))$. Then

$$\text{ord}_{x'} \left(\prod_{\text{div}(u_i) \subset E} u_i'^{-m!(a(i,j)-jd_i)} F_j(1, u_2', u_3')^{\frac{m!}{j}} \right) = \deg \left(\prod_{\text{div}(u_i) \subset E} U_i^{-m!(a(i,j)-jd_i)} F_j^{\frac{m!}{j}} \right)$$

for each j with $F_j \neq 0$. By [H3, Theorems 2 and 3], this means that x' lies on

$$\text{Proj}k(x)[Z, U_1, U_2, U_3]/(Z, \mathcal{V}) \subset \text{Proj}k(x)[Z, U_1, U_2, U_3] = \mathbb{P}_{k(x)}^3. \quad (4)$$

This proves the first assertion of (ii) in this case. All other assertions are easy consequences of (3) and of its explicitation (4).

Assume now that $u_2'(x')u_3'(x') \neq 0$. If x' is rational over x , i.e. $u_2'(x') = \lambda \in k(x)$, $u_3'(x') = \mu \in k(x)$, we have

$$I_{x'} \equiv (Z', G_j(1, V_2 - \lambda, V_3 - \mu)^{\frac{m!}{j}}, 2 \leq j \leq m) \text{ mod } (U_1'), \quad (5)$$

where $I_{x'}$ is the refined tangent ideal of x' (cf. **IV.1(ii)**). This proves the last assertion of (ii) in this case. Finally, if x' is not rational over x , then $\dim(\mathcal{V}) = 1$. We get

$$\mathcal{V} = \langle aU_1 + bU_2 + cU_3 \rangle, \quad a, b, c \in k(x), \quad (b, c) \neq (0, 0).$$

If $b \neq 0$, we take $v_2 := a + bu_2' + cu_3' \text{ mod } (u_1)$ and we get by **IV.1.1**

$$I_{x'} = (Z', V_2^{m!\epsilon(x)}) \text{ mod } (U_1'), \quad (6)$$

which proves the last assertion of (ii) in this case. All other assertions are easy as in the previous case.

We now consider blowing up along a curve $Y = V(z, u_1, u_2)$.

By [H3, thm.3, p.331], if $m(x') = m(x)$, then x' lies on the strict transform of $z = 0$. The variables u_1, u_2 play symmetric roles; so after reordering, it can be assumed that x' belongs to the affine chart $\text{Spec}R[z/u_1, u_2/u_1, u_3] \subset Z'$. Let

$$(z', u_1', u_2', u_3') := (z/u_1, u_1, u_2/u_1, u_3).$$

We can choose a r.s.p. at x' in the following way: if x' is the origin, take (z', u_1', u_2', u_3') ; otherwise take (z', u_1', v_2, u_3') where $v_2 = \sum_a \lambda_a u_3'^a$, $\lambda_a \in R$ a unit or zero (the sum is finite) whose residue $\sum_a \bar{\lambda}_a U_3^a \in k(x)[U_3]$ is an irreducible polynomial.

With notations (1), since $V(z, u_1, u_2)$ is ϵ -permissible, we have

$$a(3, j) = jd_3, \quad G_j \in k(x)[U_1, U_2].$$

Let $h' := h/u_1^m$ define the strict transform of h . Equation (2) gets replaced by

$$\text{in}_{v'}(h') = \Gamma_0 Z'^m + \sum_{2 \leq j \leq m} Z'^{m-j} U_1'^{j(\delta(x)-1-d_3)} u_2'^{a_2} u_3'^{jd_3} \Gamma_j \in \text{gr}_{v'}(R') = R'/(z', u_1')[Z', U_1'], \quad (2*)$$

with $\Gamma_j \in R'/(z', u_1')$ whose residue in $R'/(z', u_1', u_3')$ is $G_j(1, u_2')$ for $2 \leq j \leq m$, Γ_0 a unit.

By (2*), $\delta(x) - 1 - d_3$ is the minimum value of the first coordinate of points in $\Delta(h', u'_1, v_2, u'_3, z')$. As in (1) there is no Z'^{m-1} , each vertex of $\Delta(h', u'_1, v_2, u'_3, z')$ with first coordinate equal to $\delta(x) - 1 - d_3$ is not solvable. Then

$$d_1(x') = \delta(x) - 1 - d_3,$$

and for at least one vertex $(x'_1 = \delta(x) - 1 - d_3, x'_2, x'_3)$, we have

$$x'_2 + x'_3 \leq \min\left\{\frac{\text{ord}_{x'}(u_2'^{a_2} u_3'^{a_3} G_j(1, u'_2))}{j}, 2 \leq j \leq m\right\}.$$

By theorem **II.6**(iii), we have $d_3(x') = d_3$. If $u'_2(x') \neq 0$, this gives

$$\epsilon(x') \leq \min\left\{\frac{\text{ord}_{x'}(u_3'^{a_3-d_3} G_j(1, u'_2))}{j}, 2 \leq j \leq m\right\} \leq \epsilon(x).$$

If $u'_2(x') = 0$, we also have $d_2(x') = d_2$ by **II.6**(iii). This leads to:

$$\epsilon(x') \leq \min\left\{\frac{\text{ord}_{x'}(u_2'^{a_2-d_2} u_3'^{a_3-d_3} G_j(1, u'_2))}{j}, 2 \leq j \leq m\right\} \leq \epsilon(x)$$

with the convention: $\text{ord}_{x'}(u_i'^a) = 0$ when $a \in \mathbb{Q}^+$ and $u_i'(x') \neq 0$, $\text{ord}_{x'}(u_i'^a) = a$ when $u_i'(x') = 0$.

If $\epsilon(x') = \epsilon(x)$, then

$$\text{ord}_{x'} \left(\prod_{\text{div}(u_i) \subset E} u_i'^{-m!(a(i,j)-jd_i)} G_j(1, u'_2)^{\frac{m!}{j}} \right) = \deg \left(\prod_{\text{div}(u_i) \subset E} U_i^{-m!(a(i,j)-jd_i)} G_j^{\frac{m!}{j}} \right)$$

for each j with $G_j \neq 0$. By [H3, Theorems 2 and 3], this means that x' lies on

$$\text{Proj}k(x)[Z, U_1, U_2]/(Z, \mathcal{V}) \subset \text{Proj}k(x)[Z, U_1, U_2], \quad (4*)$$

where the latter is identified with $\pi^{-1}(x) \subset Z'$. The proof now runs parallel to the case $Y = \{x\}$.

PROPOSITION V.4 *With assumption as in V.3, assume $e(x) = 3$ and $Y = \{x\}$. Then x' is not very near x , i.e. $\iota(x') < \iota(x)$ (definition **IV.2**).*

Proof. If $\rho(x) = 3$, this follows from **V.3**(ii), since $\text{Proj}k(x)[Z, U_1, U_2, U_3]/(Z, \mathcal{V}) = \emptyset$.

When $\rho(x) = 2$, we have $t(x) < 2$ necessarily: otherwise we should have

$$E \subseteq \text{div}(u_1), \mathcal{V} = \langle U_2, U_3 \rangle \pmod{(U_1)}.$$

By a linear change on the free variables (u_2, u_3) , we would get

$$\mathcal{V} = \langle U_2, U_3 \rangle,$$

i.e. $e(x) = 2$, a contradiction.

When $\rho(x) = 2$, $t(x) = 1$, we can choose parameters such that

$$E = \text{div}(u_1 u_2), \mathcal{V} = \langle U_3, \alpha U_1 + U_2 \rangle, \alpha \in k(x)^\times.$$

By proposition **V.3**(ii), we have

$$\mathcal{V}' \equiv \langle U'_3, V'_2 \rangle \pmod{(U'_1)},$$

with $E' = \text{div}(u'_1)$, $V'_2 = \text{in}_{x'}(a + u_2/u_1)$, where $a \in R$ is a preimage of α . Then $\rho(x') \geq t(x') \geq 2$.

When $\rho(x) = 2$, $t(x) = 0$, then, up to a permutation on u_1, u_2, u_3 , we have $E = \text{div}(u_1 u_2 u_3)$ and

$$\mathcal{V} = \langle U_1 + \alpha U_2, \beta U_2 + U_3, \rangle, \quad \alpha \in k(x)^\times, \quad \beta \in k(x).$$

By proposition **V.3(ii)**, we can take $\pi^{-1}(x) = \text{div}(u_2)$ locally at x' , and r.s.p.

$$(z', v'_1, u'_2, v'_3) := (z/u_2, u_1/u_2 + a, u_2, u_3/u_2 + b),$$

where $a, b \in R$ are preimages of α, β . In particular we get $E \subseteq \text{div}(u'_2 u'_3)$. On the other hand, we have

$$\mathcal{V}' \equiv \langle V'_1, V'_3 \rangle \pmod{(U'_2)},$$

and this proves that $t(x') \geq 1$.

When $\rho(x) = 1$, then $e(x) = 3$ implies $E = \text{div}(u_1 u_2 u_3)$ (so $t(x) = 0$) and

$$\mathcal{V} = \langle \alpha U_1 + \beta U_2 + U_3, \rangle, \quad \alpha, \beta \in k(x)^\times$$

up to renumbering parameters. By proposition **V.3(ii)**, we can choose $E' \subseteq \text{div}(u'_1 u'_2)$, say $\pi^{-1}(x) = \text{div}(u_1)$ locally at x' and r.s.p.

$$(z', u'_1, v'_2, v'_3) := (z/u_1, u_1, v'_2, a + b u_2/u_1 + u_3/u_1),$$

at x' , where $a, b \in R$ are preimages of α, β . Since $V'_3 \in \mathcal{V}' \oplus \langle U_1 \rangle$, we get $t(x') \geq 1$.

PROPOSITION V.5 *Let x satisfy the conclusion of proposition **III.1(ii)** and assume $e(x) = 2$. There exists a finite sequence of local blowing ups*

$$(Z, x) =: (Z_0, x_0) \longleftarrow (Z_1, x_1) \longleftarrow \cdots \longleftarrow (Z_n, x_n), \quad (1)$$

where $x_0 = x$, $x_i \in X_i$ (X_i denoting the strict transform of X), $0 \leq i \leq n$, is the center of v , each blowing up center is $Y_i = \{x_i\}$ such that $\iota(x') < \iota(x)$.

Proof. First assume that $\rho(x) = 1$. Then $t(x) = 0$, $\text{div}(u_2 u_3) \subseteq E$ and we have $\mathcal{V} = \langle \alpha U_2 + U_3, \rangle$, $\alpha \in k(x)^\times$ after possibly renumbering parameters. If x' does not belong to the strict transform of $\text{div}(u_2)$, we can take $\pi^{-1}(x) = \text{div}(u_2)$ locally at x' , and r.s.p.

$$(z', v'_1, u'_2, v'_3) := (z/u_2, v'_1, u_2, a + u_3/u_2),$$

where $a \in R$ is a preimage of α . In particular we get $E' \subseteq \text{div}(u'_1 u'_2)$, with $u'_1 = u_1/u_2$. On the other hand, we have

$$\mathcal{V}' \equiv \langle V'_3 \rangle \pmod{(U'_2)},$$

whence $t(x') \geq 1$, so $\iota(x') < \iota(x)$. Assume now that x' belongs to the strict transform of $\text{div}(u_2)$. We can take $\pi^{-1}(x) = \text{div}(u_1)$ locally at x' , and r.s.p.

$$(z', u'_1, u'_2, u'_3) := (z/u_1, u_1, u_2/u_1, u_3/u_1).$$

We get $E = \text{div}(u'_1 u'_2 u'_3)$ and

$$\mathcal{V}' \equiv \langle \alpha U'_2 + U'_3 \rangle \pmod{(U'_1)}.$$

If $\iota(x') = \iota(x)$, then $\mathcal{V}' = \langle \alpha U'_2 + U'_3 \rangle$ and iterate. Since the valuation v has rank one, say $v(u_2) < n v(u_1)$ for some $n > 0$, the process stops after iterating n times.

Assume that $\rho(x) = 2$. Then $\mathcal{V} = \langle U_2, U_3 \rangle$ after possibly renumbering parameters. We can take $\pi^{-1}(x) = \text{div}(u_1)$ locally at x' and r.s.p.

$$(z', u'_1, u'_2, u'_3) := (z/u_1, u_1, u_2/u_1, u_3/u_1).$$

We get $\text{div}(u'_1) \subseteq E'$ and

$$\mathcal{V}' \equiv \langle U'_2, U'_3 \rangle \pmod{(U'_1)},$$

hence $\langle U'_2 + \alpha_2 U'_1, U'_3 + \alpha_3 U'_1 \rangle \subseteq \mathcal{V}'$ for some $\alpha_2, \alpha_3 \in k(x)$. If $\iota(x') = \iota(x)$, then equality holds; moreover $\alpha_i = 0$ whenever $\text{div}(u_i) \subseteq E$, $i = 2$ or $i = 3$.

Iterating, there exists a *regular* formal curve $\hat{Y} \subset X$ passing through all points $x, x_1 := x', \dots, x_n$, taking $Y_i = \{x_i\}$ for each $i \geq 0$. By standard arguments, $\hat{Y} \subseteq \text{Sing}_{m(x)}(X)$. Our assumptions (beginning of section IV) force $\hat{Y} \subset E$, say $\hat{Y} \subset \text{div}(u_2)$. One concludes as in the case $\rho(x) = 1$.

VI PROOF OF THE MAIN THEOREM.

By theorem **II.9**, a reduction in $m = m(x)$ can be achieved when $\epsilon(x, E) = 0$ for some normal crossings divisor $E \subseteq \text{div}(u_1 u_2 u_3)$. The previous section (propositions **V.4** and **V.5**) reduces theorem **0.3** to the only case $\epsilon(x) > 0$, $e(x) = 1$. There remains to prove the following:

THEOREM VI.1 *Let x satisfy the conclusion of proposition **III.1(ii)** (w.r.t. $E \subseteq \text{div}(u_1 u_2 u_3)$) and assume $\epsilon(x) > 0$, $e(x) = 1$. There exists a finite sequence of local blowing ups*

$$(Z, x) =: (Z_0, x_0) \longleftarrow (Z_1, x_1) \longleftarrow \cdots \longleftarrow (Z_n, x_n), \quad (1)$$

where $x_0 = x$, $x_i \in X_i$ (X_i denoting the strict transform of X), $0 \leq i \leq n$, is the center of v , each blowing up center $Y_i \subset X_i$ is permissible in Hironaka's sense, such that

- (i) $\iota(x_n) < \iota(x)$, and
- (ii) x_n satisfies the conclusion of proposition **III.1(ii)** (w.r.t. the strict transform E_n of E in Z_n) if $((m(x_n), \tau(x_n)) = (m(x), \tau(x))$ and $\epsilon(x_n) > 0$).

The proof is long, needing new invariants and the control of their behavior under permissible blowing ups. There are two different cases:

- (i) $t(x) = 0$, $e(x) = 1$,
- (ii) $t(x) = e(x) = 1$.

In both cases, we choose the indices so that $\mathcal{V} = \langle U_3 \rangle$. We assume that

- (P1) (z, u) is fully prepared, and
- (P2) $E \subseteq \text{div}(u_1 u_2 u_3)$.

VI.2 A new invariant B , preparation of the free variable (case (ii)).

Let us remind the convention $d_i(x) = 0$ for $\text{div}(u_i) \not\subset E$, $1 \leq i \leq 3$. In particular, $d_3(x) = 0$ in case (ii). For $B \in \mathbb{Q}_+$, define the monomial valuation v_B by

$$v_B(z) = 1, \quad v_B(u_3) := \frac{1}{\epsilon(x) + d_3 + \frac{d_1 + d_2}{B}} =: Bv_B(u_1) = Bv_B(u_2).$$

We choose $B \in \mathbb{N} \cup \{+\infty\}$ maximal such that, up to the multiplication by an element of $k(x)^\times$, $\text{in}_{v_B}(h)$ takes the following form:

$$\text{in}_{v_B}(h) = Z^m + \sum_{1 \leq j \leq m} Z^{m-j} \Phi_j(U_1, U_2, U_3),$$

with

$$\deg_{U_3}(\Phi_j) \leq j(d_3 + \epsilon(x)), \quad 1 \leq j \leq m,$$

$$\Phi_j(U_1, U_2, U_3) = U_1^{jd_1} U_2^{jd_2} U_3^{jd_3} (\lambda_j U_3^{j\epsilon(x)} + P_j(U_1, U_2, U_3)) \quad (1)$$

with $\lambda_j \in k(x)$ and $(jd_1, jd_2, j\epsilon(x)) \in \mathbb{N}^3$ whenever equality holds; furthermore, equality holds for some j , $2 \leq j \leq m$.

Note that we necessarily have $\Phi_j(U_1, U_2, U_3) \neq \lambda_j U_3^{j\epsilon(x)}$ for some j if $B < +\infty$, since B is taken to be maximal. Moreover, since (z, u) is fully prepared and $\mathcal{V} = \langle U_3 \rangle$, we necessarily have $B \geq 1$ and $\deg_{U_3}(\Phi_1) < d_3 + \epsilon(x)$.

This construction builds up a face of $\Delta(h, u, z)$ with equation

$$\frac{x_1 + x_2}{B(\epsilon(x) + d_3) + d_1 + d_2} + \frac{x_3}{(\epsilon(x) + d_3 + \frac{d_1 + d_2}{B})} = 1,$$

for some B which contains the point $\mathbf{x} := (d_1, d_2, \epsilon(x) + d_3)$ and at least another point.

Let p be the projection

$$p : \mathbb{R}^3 - \{\mathbf{x}\} \longrightarrow \{x_3 = 0\}.$$

For analytic computations, note that if $M = z^{m-j} u_1^{a_1} u_2^{a_2} u_3^{a_3}$ is a monomial appearing with nonzero coefficient in some expansion **II.1**(1) of h and $j \geq 1$, then M defines the point $\mathbf{x}_M$:

$$M = z^{m-j} u_1^{a_1} u_2^{a_2} u_3^{a_3} \leftrightarrow \mathbf{x}_M = \left(\frac{a_1}{j}, \frac{a_2}{j}, \frac{a_3}{j} \right) \in \Delta(h, u, z),$$

and

$$p(\mathbf{x}_M) = \left(d_1 + \frac{\frac{a_1}{j} - d_1}{d_3 + \epsilon(x) - \frac{a_3}{j}}, d_2 + \frac{\frac{a_2}{j} - d_2}{d_3 + \epsilon(x) - \frac{a_3}{j}} \right). \quad (2)$$

Then $B + d_1 + d_2$ is the minimum value $x_1 + x_2$ for points in $p(\Delta(h, u, z) \cap \{x_3 < \epsilon(x) + d_3\})$. We define $\Delta_2(h; u_1, u_2; u_3) \subseteq (\mathbb{R}_+)^2$ by the formula

$$(d_1, d_2) + \Delta_2(h; u_1, u_2; u_3) := p(\Delta(h, u, z) \cap \{x_3 < \epsilon(x) + d_3\}).$$

The main idea is that $\Delta_2(h; u_1, u_2; u_3)$ acts as the characteristic polyhedron of a surface singularity and in the following, we mimic [CJS],[2], all these following Hironaka.

In case (ii) ($\text{div}(u_3) \not\subseteq E$), we will require two extra conditions (to be achieved in **VI.3** below by possibly changing u_3):

(P3) *there is no homogeneous $P \in k(x)[U_1, U_2]$, $P \neq 0$, such that*

$$\text{in}_{v_B}(h) = Z^m + \sum_{2 \leq j \leq m} Z^{m-j} \lambda_j U_1^{jd_1} U_2^{jd_2} (U_3 + P(U_1, U_2))^{j\epsilon(x)}, \quad (3)$$

with the convention $\lambda_j = 0$ when $(jd_1, jd_2, j\epsilon(x)) \notin \mathbb{N}^3$;

(P4) *if $B < +\infty$, let $\mathbf{x}_2 = (d_1 + A(1), \beta + d_2)$ be the vertex of $\Delta_2(h; u_1, u_2; u_3)$ with minimal first coordinate. Then $\mathbf{x}_2$ does not vanish by changing u_3 to $u_3 + \gamma u_1^\alpha u_2^\beta$, $\gamma \in R$, γ invertible.*

PROPOSITION VI.3 *With assumptions as above, there exist (z, u_1, u_2, u_3) , $z, u_3 \in \widehat{R}$ such that (P1)(P2) and (P3)(P4) (in case (ii) with $B < \infty$) are satisfied.*

Proof. The conditions (P1)(P2) can be achieved easily. If (P3) or (P4) is not achieved, we make a translation on u_3 : we replace u_3 by $u_3 + \sum_{a_1, a_2} \gamma_{a_1, a_2} u_1^{a_1} u_2^{a_2}$, $\gamma_{a_1, a_2} \in R$, $(a_1, a_2) \in p(\Delta(h, u, z) \cap \{x_3 < \epsilon(x) + d_3\})$. To achieve (P3), we take

$$\sum_{a_1, a_2} \overline{\gamma_{a_1, a_2}} U_1^{a_1} U_2^{a_2} := P(U_1, U_2),$$

$P(U_1, U_2)$ as in (3), which makes B increase if (P3) is not achieved. To achieve (P4) we change u_3 to $u_3 + \gamma u_1^\alpha u_2^\beta$ as in (P4)), which makes $(A(1), \beta)$ strictly increase for the lexicographical ordering.

In both cases, this translation makes $\Delta_2(h; u_1, u_2; u_3)$ smaller. These translations may spoil (P1), so each must be followed by a translation on z to get again (P1). This translation makes $\Delta_2(h; u_1, u_2; u_3)$ not bigger. The process may be infinite, but since $\Delta_2(h; u_1, u_2; u_3)$ gets smaller at each step, this converges to some $z, u_3 \in \widehat{R}$.

DEFINITION VI.3.1 *With assumptions as above, a r.s.p. (z, u_1, u_2, u_3) , $z, u_3 \in \widehat{R}$ such that (P1)(P2) and (P3)(P4) (in case (ii) with $B < \infty$) are satisfied is said to be well prepared. For such (z, u_1, u_2, u_3) , the number B defined above is denoted by $B(z, u_1, u_2, u_3)$ or $B(x)$ for short, even if it may depend on the choice of (z, u_1, u_2, u_3) .*

VI.4. We begin the proof of Theorem VI.1 by the special case $B(x) = \infty$.

When $B(z, u_1, u_2, u_3) = \infty$, $\Delta(h, u, z)$ has only one vertex with coordinates $(d_1, d_2, \epsilon(x) + d_3)$. Since $\epsilon(x) > 0$, we have $\text{div}(u_3) \not\subseteq E$, hence $t(x) = 1$ (case (ii)) and $E \subseteq \text{div}(u_1 u_2)$, $d_3 = 0$. The proof is a variation of that of theorem II.9, checking carefully the algebraicity of the blowing up centers.

It has been assumed from section IV on that $\text{Sing}_m(X) \subseteq E$, so $\epsilon(x, E) < 1$ necessarily since $V(z, u_3) \subseteq \text{Sing}_m(X)$ otherwise.

By blowing up the surfaces $V(z, u_i)$, $\text{div}(u_i) \subseteq E$, it can be assumed w.l.o.g. that $d_i < 1$. Similarly, it can be assumed that $d_1 + d_2 < 1$ by blowing up $V(z, u_1, u_2)$.

Assume that $V(z, u_i, u_3) \subseteq \text{Sing}_m(X) \subseteq E$, i.e. $d_i + \epsilon(x) \geq 1$, $i = 1$ or $i = 2$. Then $C_i := V(z, u_i, u_3)$ is a formal irreducible component of $\text{Sing}_m(X)$. By excellence, its Zariski closure $\overline{C}_i$ is a curve on X . On the other hand, $\overline{C}_i$ is contained in $V(z, u_i)$, so C_i itself is a curve on X . By blowing up C_i , we may assume that $d_i + \epsilon(x) < 1$, $i = 1, 2$.

At this point, we have reached the situation of theorem II.9(1) and the proof therein extends without changes: we eventually get reduction in $(m(x), \tau(x))$ by blowing up closed points. We observe that theorem VI.1 can also be phrased as follows in this case: E_n can be enlarged to a new normal crossings divisor F_n such that $\epsilon(x_n, F_n) = 0$.

From now on, we assume that

$$e(x) = 1, \quad B(x) < \infty. \quad (\text{Hyp})$$

DEFINITION VI.4.1 (New invariants) *We define A_1, β by: $(d_1 + A_1, \beta + d_2)$ is the vertex of minimal first coordinate of*

$$p(\Delta(h, u, z) \cap \{x_3 < \epsilon(x) + d_3\}).$$

We define A_2 by: $d_2 + A_2$ is the minimal second coordinate of the points of

$$p(\Delta(h, u, z) \cap \{x_3 < \epsilon(x) + d_3\}).$$

We define $C(u, z)$ (or $C(x)$ for short) by;

$$C(u, z) = B(u, z) - A_1 - A_2.$$

Finally, we define $\gamma(u, z)$ (or $\gamma(x)$ for short) as follows:

- (i) $\gamma(u, z) := \lceil \beta(u, z) \rceil \geq 0$ if $(E \subseteq \text{div}(u_1) \text{ and } t(x) = 1)$;
- (ii) $\gamma(u, z) := \lceil \beta(u, z) \rceil \geq 0$ if $(E \subseteq \text{div}(u_1 u_3) \text{ and } t(x) = 0)$;
- (iii) $\gamma(u, z) := 1 + \lfloor C(u, z) \rfloor \geq 1$ otherwise, i.e. if $(E = \text{div}(u_1 u_2) \text{ and } t(x) = 1)$ or if $(E = \text{div}(u_1 u_2 u_3) \text{ and } t(x) = 0)$.

PROPOSITION VI.4.2 (Behaviour of the new invariants under blowing up along an ϵ -permissible curve). Assume that (Hyp) is true, (z, u) is well prepared and let

$$\mathcal{C}_i := V(z, u_i, u_3), \quad i = 1 \text{ or } i = 2.$$

Assume that $\mathcal{C}_i$ is ϵ -permissible in $\widehat{X} = \text{Spec}(\widehat{R}/(h))$, for some i , $i = 1, 2$, then:

- (i) $\epsilon(x) + d_3 + d_i \geq 1$,
- (ii) $\mathcal{C}_i$ is algebraic, i.e., if in achieving (P3)(P4), we get $z, u_3 \in \widehat{R}$, then there exists a curve in $\text{Spec } R$ whose formal completion is $V(z, u_i, u_3)$.
- (iii) let $\pi_i : (Z', x') \rightarrow (Z, x)$ be the blowing up along $\mathcal{C}_i$, $X' \subset Z'$ the strict transform of X and $x' \in X'$ the center of v , with $\iota(x') = \iota(x)$. Then:
- (iv) if $i = 1$ and x' is the point of Z' with parameters

$$(z', u'_1, u'_2, u'_3) := (z/u_1, u_1, u_2, u_3/u_1),$$

these are well-prepared parameters and

$$\beta(x') = \beta(x), \quad A_1(x') = A_1(x) - 1, \quad A_2(x') = A_2(x),$$

$$d_1(x') = d_1(x) + \epsilon(x) + d_3(x) - 1, \quad d_2(x') = d_2(x), \quad d_3(x') = d_3(x);$$

- (v) if $i = 2$ and x' is the point of Z' with parameters

$$(z', u'_1, u'_2, u'_3) := (z/u_2, u_1, u_2, u_3/u_2),$$

these are well-prepared parameters and

$$\beta(x') = \beta(x) - 1, \quad A_1(x') = A_1(x), \quad A_2(x') = A_2(x) - 1,$$

$$d_2(x') = d_1(x) + \epsilon(x) + d_3(x) - 1, \quad d_1(x') = d_1(x).$$

Proof of (i). Condition (i) is equivalent to $h \in (z, u_3, u_i)^m$.

Proof of (ii)(iii). Let us note that (ii) is clear when $\text{div}(u_3) \subset E$, because in that case, we do not make (P3)(P4), $z, u_1, u_3 \in R$. When $\text{div}(u_3) \not\subset E$, we will prove that

- (ii)' $\mathcal{C}_i$ is the only analytic branch in $\text{div}(u_i) \cap \text{Sing}_m(X) \cap \{y \in X : \epsilon(y) > 0\}$ not contained in $\text{div}(u_j)$, $j = 1, 2$, $j \neq i$.

By **II.10 II.11**, this will prove (ii). We compute $\pi_i : X' \subset Z' \rightarrow X' \subset Z'$. By symmetry, we suppose $i = 1$. Let us expand:

$$h = \sum_{A, m-j, 0 \leq j \leq m} C_{A, m-j} z^{m-j} u_1^{a_1} u_2^{a_2} u_3^{a_3},$$

$C_{A, m-j} \in R$, $C_{A, m-j}$ invertible or zero, $C_{A, m-j} \neq 0 \Rightarrow a_1 + a_2 + a_3 \geq j\delta(x)$, $a_i \geq jd_i$, $i = 1, 2, 3$, $C_{0,0}$ invertible.

Since $h \equiv \delta z^m \pmod{(u_1, u_2, u_3)}$, $\delta \in R$ a unit, $X' \cap \text{Spec} R[u_1/z, u_3/z] \subset Z'$ does not contain the point $(z, u_1/z, u_2, u_3/z)$. Assume now that x' belongs to the affine chart $\text{Spec} R[z/u_3, u_1/u_3] \subset Z'$. Let

$$(z', u'_1, u'_2, u'_3) := (z/u_3, u_1/u_3, u_2, u_3).$$

We have

$$h' := u_3^{-m} h = \sum C_{A, m-j} z'^{m-j} u_1^{a_1} u_2^{a_2} u_3'^{a_1+a_3-j}, \quad (1)$$

$$h' = C_{0, m} z'^m + \sum_{2 \leq j \leq m} \gamma_j u_1'^{j d_1} u_2'^{j d_2} u_3'^{j(d_1+d_3+\epsilon(x)-1)} \pmod{I^+(z', u'_1, u_2, u_3)}, \quad (2)$$

where, $\gamma_j \in R$, γ_j invertible or zero, $\gamma_j = 0$ when one exponent is not integer,

$$\gamma_j = C_{j d_1, j d_2, j(d_3+\epsilon(x)), m-j} \pmod{\mathfrak{M}}$$

when γ_j is invertible and $I^+(z', u'_1, u_2, u_3)$ is generated by

$$z'^{m+1}, z'^{m-j} u_1'^a u_2'^b u_3'^c,$$

with $1 \leq j \leq m, a \geq j d_1, b \geq j d_2, c \geq j(d_1+d_3+\epsilon(x)-1), a+b+c > j(d_1+d_2+d_1+d_3+\epsilon(x)-1)$.

Note that (2) implies $d_3(x') \geq d_1 + d_3 + \epsilon(x) - 1$, in fact there is equality. Otherwise, by [H2], there would exist $t = z' + \gamma u_3^e$, $e \geq d_1 + d_3 + \epsilon(x) - 1$, $\gamma \in R$, $\text{ord}_{u_i} \gamma \geq d_i$, $i = 1, 2$, with $h' = C_{0,0} t^m \pmod{I^+(t, u'_1, u_2, u_3)}$. As $I^+(z', u'_1, u_2, u_3) = I^+(t, u'_1, u_2, u_3)$ and, in (2), there is no term in z'^{m-1} , this is impossible. As $d_1(x') = d_1(x)$ and $d_2(x') = d_2(x)$, by (2), we get $\epsilon(x') = 0$: there is no x' very near x in this chart. This gives the first statement in (iv). This gives also (ii)', because if there was a curve in $\text{div}(u_2) \cap \text{Sing}_m(X) \cap \{y \in X : \epsilon(y) > 0\}$, the strict transform of this curve would have a non empty intersection with our affine chart and there would exist in this chart some x' with $\epsilon(x') \geq 1$.

Proof of (iv). Now $x' \in \text{Spec} R[z/u_1, u_3/u_1] \subset Z'$ is the point with parameters $(z', u'_1, u'_2, u'_3) := (z/u_1, u_1, u_2, u_3/u_1)$. Then, using the notations of (1),

$$h' := u_1^{-m} h = \sum C_{A, m-j} z'^{m-j} u_1^{a_1+a_3-j} u_2^{a_2} u_3'^{a_3},$$

$\Delta(h', u', z')$ is obtained as follows: take the convex hull of the set $\{(a+c-1, b, c) | (a, b, c) \in \Delta(h, u, z)\}$ and add $\mathbb{R}^{+3}$, then

$$\partial(\Delta(h', u', z')) \subset \{(a+c-1, b, c) | (a, b, c) \in \partial(\Delta(h, u, z))\},$$

$$\text{in}(h', \Delta')_{u', z'} = \overline{C_{0, m}} Z'^m + \sum_{2 \leq j \leq m, A/j \in \partial(\Delta(h', u', z'))} \lambda_{j, A} Z'^{m-j} U_1'^{a_1+a_3-j} U_2'^{a_2} U_3'^{a_3},$$

where the $\lambda_{j, A} \in k(x)$ are defined by:

$$\text{in}(h, \Delta)_{u, z} = \overline{C_{0, m}} Z^m + \sum_{2 \leq j \leq m, \frac{A}{j} \in \partial(\Delta(h, u, z))} \lambda_{j, A} Z^{m-j} U_1^{a_1+a_3-j} U_2^{a_2} U_3^{a_3}.$$

Let $\mathcal{M}'$ be the set of monomials $M' = z'^{m-j} u_1^{a_1+a_3-j} u_2^{a_2} u_3'^{a_3}$ which appear with a non zero coefficient in the expansion of h' , let $\mathcal{M}$ be the set of monomials $M = z^{m-j} u_1^{a_1} u_2^{a_2} u_3^{a_3}$ which appear with a non zero coefficient in the expansion of h :

$$d_1(x') = \text{inh}_{M' \in \mathcal{M}'} \left(\frac{a_1 + a_3 - j}{j} \right) = d_1 + d_3 + \epsilon(x) - 1, d_i(x') = \text{inh}_{M' \in \mathcal{M}'} \left(\frac{a_i}{j} \right) = d_i, \quad i = 2, 3.$$

As x' is very near x , $\epsilon(x) = \epsilon(x')$, $\delta(x') = d_1(x') + d_2(x') + d_3(x') + \epsilon(x)$. The only point on the first side of $\Delta(h', u', z')$ is

$$(d_1(x'), d_2(x), d_3(x) + \epsilon(x))$$

let p' be the projection on $x_3 = 0$ from this vertex. A monomial M' defines a point $(\frac{a_1+a_3}{j}-1, \frac{a_2}{j}, \frac{a_3}{j})$ that we call also M' , when $a_3 < d_3(x) + \epsilon(x)$,

$$p'(M') = (d_1(x') + \frac{\frac{a_1+a_3}{j}-1-d_1(x')}{d_3(x)+\epsilon(x)-\frac{a_3}{j}}, d_2(x) + \frac{\frac{a_2}{j}-d_2(x)}{d_3(x)+\epsilon(x)-\frac{a_3}{j}}),$$

as $\frac{\frac{a_1+a_3}{j}-1-d_1(x')}{d_3(x)+\epsilon(x)-\frac{a_3}{j}} = \frac{\frac{a_1}{j}-d_1(x)}{d_3(x)+\epsilon(x)-\frac{a_3}{j}} - 1$ and by **VI.3(2)**

$$p(M) = (d_1(x) + \frac{\frac{a_1}{j}-d_1(x)}{d_3(x)+\epsilon(x)-\frac{a_3}{j}}, d_2(x) + \frac{\frac{a_2}{j}-d_2(x)}{d_3(x)+\epsilon(x)-\frac{a_3}{j}}).$$

We get

$$p'(\Delta(h', u', z') \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x'), d_2(x')),$$

i.e. the polyhedron $p'(\Delta(h', u', z') \cap \{x_3 < d_3(x) + \epsilon(x)\})$ translated by the vector $-(d_1(x'), d_2(x'))$, from

$$p(\Delta(h, u, z) \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x), d_2(x))$$

by making an horizontal translation of -1 . This gives the other assertions of (iv). Mutatis mutandis, we get (v).

PROPOSITION VI.4.3 (Behaviour of the new invariants under blowing up a closed point). *Assume that (Hyp) is true and (z, u) is well prepared. Let $\pi_i : (Z', x') \rightarrow (Z, x)$ be the blowing up along x , $X' \subset Z'$ the strict transform of X and $x' \in X'$ the center of v , with $\iota(x') = \iota(x)$. Then*

- (i) x' belongs to the strict transform of $V(z, u_3)$,
- (ii) if $x' \in \text{Spec}R[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$ is the point

$$(z', u'_1, u'_2, u'_3) := (z/u_1, u_1, u_2/u_1, u_3/u_1),$$

these parameters are well-prepared and

$$\beta(x') \leq \beta(x), \quad A_1(x') = B(x) - 1, \quad A_2(x') = A_2(x), \quad C(u', z') \leq C(u, z),$$

$$d_1(x') = d_1(x) + d_2(x) + d_3(x) + \epsilon(x) - 1, \quad d_2(x') = d_2(x), \quad d_3(x') = d_3(x);$$

(iii) if $x' \in \text{Spec}R[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$ and $x' \neq (z', u'_1, u'_2, u'_3)$, then

$$\beta(x') \leq 1 + \left\lfloor \frac{C(u, z)}{2} \right\rfloor, \quad A_1(x') = d(x) - 1,$$

$$d_1(x') = d_1(x) + d_2(x) + d_3(x) + \epsilon(x) - 1, \quad d_2(x') = 0, \quad d_3(x') = d_3(x),$$

where $\lfloor \cdot \rfloor$ denotes lower integral part. If moreover $(E \subseteq \text{div}(u_1 u_3)$ and $0 < \beta(x)$), then $\beta(x') \leq \beta(x)$.

We have

$$\gamma(x') \leq \gamma(x).$$

More precisely: if $(x'$ is not rational over x and $\gamma(x) \geq 3$), then $\gamma(x') < \gamma(x)$; if $(\gamma(x') = \gamma(x) = 2$ and x' is not rational over x), then $\beta(x) = 2$, $\text{div}(u_2) \not\subset E$ and $\beta(x') < \beta(x) = 2$;
(iv) if $x' \in \text{Spec}R[z/u_2, u_1/u_2, u_3/u_2] \subset Z'$ is the point with parameters

$$(z', u'_1, u'_2, u'_3) := (z/u_2, u_1/u_2, u_2, u_3/u_2),$$

these are well prepared parameters and

$$\beta(x') = \beta(x) + A_1(x) - 1, \quad A_1(x') = A_1(x), \quad A_2(x') = B(x) - 1,$$

$$d_2(x') = d_1(x) + d_2(x) + d_3(x) + \epsilon(x) - 1, \quad d_1(x') = d_1(x),$$

$$\gamma(x') \leq \gamma(x), \quad C(u', z') \leq \frac{\beta(x)}{2}.$$

Proof. (i) is a consequence of **V.3(ii)** and **V.3(3)**.

Proof of (ii). Write

$$h = \sum C_{A, m-b} z^{m-b} u_1^{a_1} u_2^{a_2} u_3^{a_3}, \quad C_{A, m-b} \in R^\times \text{ or } C_{A, m-b} = 0,$$

where the sum runs along $b \leq m$, $A = 0$ when $b = 0$, and $A = (a_1, a_2, a_3) \in b\Delta(h, u, z)$. Then

$$h' := u_1^{-m} h = \sum C_{A, m-j} z'^{m-j} u_1^{a_1+a_2+a_3-j} u_2^{a_2} u_3^{a_3},$$

and $\Delta(h', u', z')$ is obtained as follows: take the convex hull of the set

$$\{(a+b+c-1, b, c) + \mathbb{R}_+^3 \mid (a, b, c) \in \Delta(h, u, z)\}.$$

Let $\mathcal{M}'$ be the set of monomials $M' = z'^{m-j} u_1^{a_1+a_2+a_3-j} u_2^{a_2} u_3^{a_3}$ which appear with a non zero coefficient in the expansion of h' , let $\mathcal{M}$ be the set of monomials $M = z^{m-j} u_1^{a_1} u_2^{a_2} u_3^{a_3}$ which appear with a non zero coefficient in the expansion of h :

$$d_1(x') = \inf_{M' \in \mathcal{M}'} \left(\frac{a_1 + a_2 + a_3 - j}{j} \right) = d_1 + d_2 + d_3 + \epsilon(x) - 1,$$

$$d_i(x') = \inf_{M' \in \mathcal{M}'} \left(\frac{a_i}{j} \right) = d_i(x), \quad i = 2, 3.$$

As x' is very near to x , $\epsilon(x) = \epsilon(x')$, $\delta(x') = d_1(x') + d_2(x') + d_3(x') + \epsilon(x)$. The only point on the first side of $\Delta(h', u', z')$ is

$$(d_1(x'), d_2(x), d_3(x) + \epsilon(x)).$$

Let p' be the projection on $x_3 = 0$ from this vertex. A monomial M' corresponds to a point $\mathbf{x}_{M'} \left(\frac{a_1+a_2+a_3}{j} - 1, \frac{a_2}{j}, \frac{a_3}{j} \right)$. When $a_3 < d_3(x) + \epsilon(x)$,

$$p'(M') = (d_1(x') + \frac{\frac{a_1+a_2+a_3}{j} - 1 - d_1(x')}{d_3(x) + \epsilon(x) - \frac{a_3}{j}}, d_2(x) + \frac{\frac{a_2}{j} - d_2(x)}{d_3(x) + \epsilon(x) - \frac{a_3}{j}}),$$

as $\frac{\frac{a_1+a_3}{j}-1-d_1(x')}{d_3(x)+\epsilon(x)-\frac{a_3}{j}} = \frac{\frac{a_1}{j}-d_1(x)}{d_3(x)+\epsilon(x)-\frac{a_3}{j}} - 1$ and by **VI.3**(2)

$$p(M) = (d_1(x) + \frac{\frac{a_1}{j} - d_1(x)}{d_3(x) + \epsilon(x) - \frac{a_3}{j}}, d_2(x) + \frac{\frac{a_2}{j} - d_2(x)}{d_3(x) + \epsilon(x) - \frac{a_3}{j}}).$$

So we get

$$p'(\Delta(h', u', z') \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x'), d_2(x'))$$

from

$$p(\Delta(h, u, z) \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x), d_2(x))$$

as follows: take the convex hull of the set

$$\{(a + b - 1, b) + \mathbb{R}_+^2 | (a, b) \in p(\Delta(h, u, z) \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x), d_2(x))\}.$$

These are the usual transformation laws of the characteristic polyhedra of surfaces see the appendix of H. Hironaka in [3]. To get the other assertions of (ii), the proof runs along the same lines as **VI.4.2**(1).

Proof of (iv). Mutatis mutandis, we get all assertions of (iv), except the last line that we prove now. In fact, we get

$$p'(\Delta(h', u', z') \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x'), d_2(x'))$$

from

$$p(\Delta(h, u, z) \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x), d_2(x))$$

as follows: take the convex hull of the set $\{(a, a + b - 1) + \mathbb{R}_+^2 | (a, b) \in p(\Delta(h, u, z) \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x), d_2(x))\}$. We get $A_1(x') = A_1(x)$, $\beta(x') = \beta(x) + A_1(x) - 1$ and $A_2(x) = d(x) - 1$.

Let us denote by (α_2, β_2) and (α_3, β_3) with $\alpha_2 \leq \alpha_3$, the coordinates of the (maybe equal) vertices of the first side of $p(\Delta(h, u, z) \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x), d_2(x))$.

Then $(\alpha_2, \alpha_2 + \beta_2 - 1) = (\alpha_2, B(x) - 1)$ is the vertex of smaller second coordinate of

$$p'(\Delta(h', u', z') \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x'), d_2(x')).$$

Note that $(A_1(x), A_1(x) + \beta(x) - 1)$ is the vertex of smaller first coordinate of

$$p'(\Delta(h', u', z') \cap \{x_3 < d_3(x) + \epsilon(x)\}) - (d_1(x'), d_2(x')).$$

All this leads to:

$$A_1(x') = A_1(x), \quad A_2(x') = B(x) - 1,$$

$$C(u', z') \leq \beta(x') - A_2(x') = A_1(x) + \beta(x) - 1 - (B(x) - 1) = \beta(x) - (B(x) - A_1(x)),$$

$$C(u', z') \leq \alpha_2 - A_1(x') = \alpha_2 - A_1(x) \leq \alpha_2 + \beta_2 - A_1(x) = B(x) - A_1(x).$$

Then either $B(x) - A_1(x) \leq \frac{\beta(x)}{2}$, then $C(u', z') \leq \frac{\beta(x)}{2}$ by the last inequality; or $B(x) - A_1(x) > \frac{\beta(x)}{2}$, then $C(u', z') < \frac{\beta(x)}{2}$ by the first of the two inequalities just above. The inequality $\gamma(x') \leq \gamma(x)$ is left to the reader.

Proof of (iii). Recall the notations and assumptions of **VI.2**(1). We write

$$\Phi_j(U_1, U_2, U_3) = U_1^{jd_1} U_2^{jd_2} U_3^{jd_3} (\lambda_j U_3^{j\epsilon(x)} + \sum_{i \in \mathbb{Q}^+} U_1^{a(i,j)} U_2^{b(i,j)} U_3^{j\epsilon(x)-i} Q_{i,j}(U_1, U_2)) \quad (1)$$

with $\lambda_j \in k(x)$, $\lambda_j = 0$ if $(jd_1, jd_2, j\epsilon(x)) \notin \mathbb{N}^3$. In this expansion, we take:

$$Q_{i,j} \in k(x)[U_1, U_2], \quad Q_{i,j} = 0 \text{ or } (U_1 \nmid Q_{i,j} \text{ and } U_2 \nmid Q_{i,j}),$$

$$Q_{i,j} = 0 \text{ when } (jd_1 + a(i,j), jd_2 + b(i,j), jd_3 + j\epsilon(x) - i) \notin \mathbb{N}^3.$$

Note that at least one $Q_{i,j}$, $2 \leq j \leq m$ is nonzero and at least one $\lambda_{j'}$, $2 \leq j' \leq m$ is nonzero.

By definition of $C(u, z)$, when $Q_{i,j} \neq 0$, $\deg(Q_{i,j}) \leq iC(u, z)$, where $\deg$ is the usual homogeneous degree. When $Q_{i,j} \neq 0$, let us denote $d(i, j) = \deg(Q_{i,j})$. Then we have, with natural notations, the relation:

$$v_B(u_3^{j\epsilon(x)}) = v_B(U_1^{a(i,j)} U_2^{b(i,j)} U_3^{j\epsilon(x)-i} Q_{i,j}(U_1, U_2))$$

$$j\epsilon(x)v_B(u_3) = (j\epsilon(x) - i)v_B(u_3) + (a(i, j) + b(i, j) + d(i, j))v_B(u_1)$$

$$j\epsilon(x)v_B(u_3) = (j\epsilon(x) - i)v_B(u_3) + (a(i, j) + b(i, j) + d(i, j))\frac{v_B(u_3)}{B},$$

which leads to:

$$a(i, j) + b(i, j) + d(i, j) - j(d_1 + d_2) = iB. \quad (2)$$

Then, in the expansion of $U_3^{j\epsilon(x)+jd_3-i} U_1^{a(i,j)} U_2^{b(i,j)} Q_{i,j}(U_1, U_2)$, the monomial with non zero coefficient and minimal exponent in U_1 is

$$U_1^{a(i,j)} U_2^{iB-a(i,j)} U_3^{j\epsilon(x)+jd_3-i}$$

which gives the point (cf. **VI.2(2)**)

$$(d_1 + \frac{\frac{a(i,j)}{j} - d_1}{d_3 + \epsilon(x) - \frac{jd_3+j\epsilon(x)-i}{j}}, d_2 + \frac{\frac{iB-a(i,j)}{j} - d_1}{d_3 + \epsilon(x) - \frac{jd_3+j\epsilon(x)-i}{j}})$$

in $p(\Delta(h, u, z) \cap \{x_3 < \epsilon(x) + d_3\})$. As

$$d_1 + \frac{\frac{a(i,j)}{j} - d_1}{d_3 + \epsilon(x) - \frac{jd_3+j\epsilon(x)-i}{j}} = d_1 + \frac{a(i, j) - jd_1}{i},$$

we deduce that

$$A_1(u, z) = \inf_i \left\{ \frac{a(i, j) - jd_1}{i} \mid 2 \leq j \leq m(x), 0 < i \leq j\epsilon(x), j\epsilon(x) + jd_3 - i \in \mathbb{N}, i \in \mathbb{Q}, Q_{i,j} \neq 0 \right\}. \quad (3)$$

Similarly,

$$A_2(u, z) = \inf_i \left\{ \frac{b(i, j) - jd_1}{i} \mid 2 \leq j \leq m(x), 0 < i \leq j\epsilon(x), j\epsilon(x) + jd_3 - i \in \mathbb{N}, i \in \mathbb{Q}, Q_{i,j} \neq 0 \right\} \quad (4)$$

and, finally, by (2), when $Q_{i,j} \neq 0$,

$$d(i, j) = i(B - \frac{a(i, j) - jd_1}{i} - \frac{b(i, j) - jd_1}{i}) \leq iC(u, z). \quad (5)$$

Since $x' \in \text{Spec}R[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$, x' is not the origin

$$(z', u'_1, u'_2, u'_3) = (z/u_1, u_1, u_2/u_1, u_3/u_1),$$

and x' belongs to the strict transform of $V(z, u_3)$ then $z'(x') = u'_1(x') = u'_3(x') = 0$. We complete (z', u'_1, u'_3) to a r.s.p. (z', u'_1, v', u'_3) at x' where

$$v' = u'_2{}^n + \sum_{0 \leq a \leq n-1} \mu_a u'_2{}^{n-a}, \quad \mu_a = 0 \text{ or } \mu_a \in R^\times,$$

for some irreducible polynomial

$$P := U_2^n + \sum_{0 \leq a \leq n-1} \bar{\mu}_a U_2^{n-a} \in k(x)[U_2].$$

The following lemma will end the proof of **VI.4.3**(iii).

LEMMA VI.4.4 *With hypotheses and notations as in **VI.4.3**(iii), let $d := [k(x') : k(x)]$. We have:*

- (i) $A_1(x') = B(u, z) - 1$;
- (ii) if $\text{div}(u_3) \subset E$, then

$$\beta(x') \leq \frac{C(u, z)}{d} \leq \frac{\beta(x)}{d}$$

(iii) in general,

$$\beta(x') < 1 + \left\lfloor \frac{C(u, z)}{d} \right\rfloor, \quad (1)$$

(iv) if $(E \subseteq \text{div}(u_1), 0 < \beta(x)$ and x' is rational over x), then

$$\beta(x') \leq \beta(x).$$

Proof. As x' is very near to x , we have $\epsilon(x) = \epsilon(x')$, $\delta(x') = d_1(x') + d_2(x') + d_3(x') + \epsilon(x)$. As x' is on the strict transform of $\text{div}(u_3)$ and not on the strict transform of $\text{div}(u_2)$, we get:

$$d_2(x') = 0, \quad d_3(x') = d_3(x).$$

With notations as in the proof of **VI.4.3**(ii):

$$h = \sum C_{A, m-b} z^{m-b} u_1^{a_1} u_2^{a_2} u_3^{a_3}, \quad C_{A, m-b} \in R^\times \text{ or } C_{A, m-b} = 0,$$

where the sum runs along $b \leq m$, $A = 0$ when $b = 0$, $A = (a_1, a_2, a_3) \in b\Delta(h, u, z)$,

$$h' := u_1^{-m} h = \sum C_{A, m-j} z'^{m-j} u_1^{a_1+a_2+a_3-j} u_2'^{a_2} u_3'^{a_3}.$$

Up to multiplying h by an unit, we may assume $\overline{C_{0,m}} = 1 \in k(x)$. Then, with the notations of **II.2**, we have

$$\begin{aligned} \delta(x) &= d_1 + d_2 + d_3 + \epsilon(x), \\ \text{in}_{\delta, u, z}(h) &= Z^m + \sum_{2 \leq j \leq m} \mu_j Z^{m-j} U_1^{jd_1} U_2^{jd_2} U_3^{jd_3 + j\epsilon(x)}, \quad \mu_j \in k(x), \end{aligned} \quad (1)$$

$\mu_j = 0$ whenever $(jd_1, jd_2, jd_3 + j\epsilon(x)) \notin \mathbb{N}^3$, $\mu_j = \overline{C_{jd_1, jd_2, jd_3 + j\epsilon(x), m-j}} \in k(x)$ otherwise. This leads to

$$h' = C_{0,m} z'^m + \sum_{2 \leq j \leq m} z'^{m-j} C_{jd_1, jd_2, jd_3 + j\epsilon(x), m-j} u_1'^{j(d_1 + d_2 + d_3 + \epsilon(x) - 1)} u_2'^{jd_2} u_3'^{jd_3 + j\epsilon(x)} + h'_1 \quad (2)$$

where $h'_1 \in \{z'^{m-j} u_1'^{a(j)}, j \in \mathbb{N}, a(j) > j(d_1 + d_2 + d_3 + \epsilon(x) - 1) = j(\delta(x) - 1)\}$. As a consequence,

$$(\delta(x) - 1, 0, d_3(x) + \epsilon(x))$$

is the vertex of smallest first coordinate of $\Delta(h', u'_1, v', u'_3, t)$ and is not solvable. In the preparation, we may replace z' by $t = z' + \lambda u_1'^a$ with $a \geq \delta(h) - 1$, but, this cannot erase the vertex $(\delta(x) - 1, 0, d_3(x) + \epsilon(x))$. We get

$$d_1(x') = \delta(x) - 1.$$

Let us study the projection of $\Delta(h', u'_1, v', u'_3, t) \cap \{x'_3 < d_3 + \epsilon(x')\}$ on $x'_3 = 0$, in particular we are interested in the vertex of smallest first coordinate of this projection. Let w be the monomial valuation on $R' := \mathcal{O}_{X', x'}$ defined by

$$\begin{aligned} w(z') &= 1, \quad w(u'_3) = \frac{1}{\epsilon(x) + d_3 + \frac{d_1 + d_2}{B(u, z) - 1}}, \\ w(u'_1) &= \frac{1}{(B(u, z) - 1)(\epsilon(x) + d_3) + d_1 + d_2} = \frac{1}{B(u, z) - 1} w(u'_3). \end{aligned}$$

There is an expansion

$$\text{in}_w(h') = Z'^m + \sum_{2 \leq j \leq m} Z'^{m-j} \Phi'_j(U'_1, U'_3) \in \text{gr}_w(R') = R'/(z', u'_1, u'_3)[Z', U'_1, U'_3],$$

where

$$\begin{aligned} \Phi'_j(U'_1, U'_3) &= \lambda_j U_1'^{j(\delta(x) - 1)} \overline{u_2'}^{jd_2} U_3'^{j\epsilon(x) + jd_3} + \\ &\sum_{0 < i \leq j\epsilon(x), j\epsilon(x) + jd_3 - i \in \mathbb{N}, i \in \mathbb{Q}} U_3'^{j\epsilon(x) + jd_3 - i} U_1'^{a(i, j) + b(i, j) + d(i, j) + j\epsilon(x) + jd_3 - i - j} \overline{u_2'}^{b(i, j)} Q_{i, j}(1, \overline{u_2'}), \end{aligned}$$

where $d(i, j) = \deg(Q_{i, j}) \overline{u_2'}$ is the image of u_2' in $\frac{R'}{(z', u_1, u_3)} = k(x)[\overline{u_2'}]_{(\overline{v'})}$, $\overline{v'}$ being the image of v' in $\frac{R'}{(z', u_1, u_3)}$. Let us recall that

$$\text{in}_w(h') \in \text{gr}_w(R') := \bigoplus_{r \in \mathbb{Q}_{\geq 0}} \frac{I_r}{I_r^+},$$

with $I_r = \{a \in R' | w(a) \geq r\}$, $I_r^+ = \{a \in R' | w(a) > r\}$. By **VI.4.3(2)**, $\text{in}_w(h') =$

$$Z'^m + \sum_{2 \leq j \leq m} Z'^{m-j} U_1'^{j(\delta(x) - 1)} [\lambda_j U_3'^{j\epsilon(x) + jd_3} + U_3'^{j\epsilon(x) + jd_3 - i} U_1'^{i(B(u, z) - 1)} \overline{u_2'}^{b(i, j)} Q_{i, j}(1, \overline{u_2'})]. \quad (3)$$

This means that

$$\frac{1}{(B(u, z) - 1)(\epsilon(x) + d_3) + d_1 + d_2} x'_1 + \frac{1}{\epsilon(x) + d_3 + \frac{d_1 + d_2}{B(u, z) - 1}} x'_3 = 1$$

is the defining equation of a face of $\Delta(h', u'_1, v', u'_3)$.

VI.4.4.1 When $\text{div}(u_3) \subset E$, then $\text{div}(u'_3) \subset E'$, we have just to make (P2) in the preparation, we may replace z' by $t = z' + r$, $r \in R'$ and, as Z'^{m-1} does not appear in (2), $w(r) > 1$, $w(t) = w(z') = 1$. This means that

$$\frac{1}{(B(u, z) - 1)(\epsilon(x) + d_3) + d_1 + d_2} x'_1 + \frac{1}{\epsilon(x) + d_3 + \frac{d_1 + d_2}{B(u, z) - 1}} x'_3 = 1$$

is the defining equation of a face of $\Delta(h', u'_1, v', u'_3, t)$. By **VI.2(2)**,

$$A_1(u'_1, v', u'_3, t) = B(u, z) - 1 \text{ and } \beta(x') = \inf\{\text{ord}_{x'}(Q_{i,j}(1, \overline{u'_2}))/id\}.$$

By **VI.4.3(5)**, $\beta(x') \leq C(u, z)/d$ and this gives **VI.4.4** in the case $\text{div}(u_3) \subset E$.

VI.4.4.2 From now on, $\text{div}(u_3) \not\subset E$, in particular $d_3(x) = 0$. Then, to get (P1),..., (P4), we may replace z' by $t = z' + r$, $r \in R'$ and, as $Z'^{m(x)-1}$ does not appear in (2), $w(r) > 1$, $w(t) = w(z') = 1$. We possibly have to make the projection of $\Delta(h', u'_1, v', u'_3, t) \cap \{x'_3 < d_3 + \epsilon(x')\}$ on $x'_3 = 0$ smaller by changing u'_3 to $v_3 = u'_3 + \lambda u'_1{}^a$ with $a \geq B(u, z) - 1$ and $\lambda \in R'$, λ not divisible by u'_1 .

Assume that $a > B(u, z) - 1$ (this is always the case when $B(u, z) \notin \mathbb{N}$). Then $\text{in}_w(v_3) = \text{in}_w(u'_3)$, we get $A_1(u'_1, v', u'_3, t) = B(u, z) - 1$ and

$$\beta(x') = \inf\{\text{ord}_{x'}(Q_{i,j}(1, \overline{u'_2}))/id\}.$$

By **VI.4.3(5)**,

$$\beta(x') \leq C(u, z)/d$$

which gives **VI.4.4** in this case.

VI.4.4.3 From now on,

$$B(u, z) \in \mathbb{N}, \quad a = B(u, z) - 1.$$

If there exists a couple (i, j_0) such that in (3) above

$$\lambda_{j_0} = 0 \text{ and } Q_{i,j_0} \neq 0,$$

then the translations $t = z' + r$ and $v_3 = u'_3 + \lambda u'_1{}^a$ will not modify the term

$$U_3'^{j_0 \epsilon(x) - i_0} U_1'^{i_0(B(u, z) - 1)} \overline{u_2'}^{b(i_0, j_0)} Q_{i_0, j_0}(1, \overline{u_2'})$$

with $i_0 := \min\{i : Q_{i,j_0} \neq 0\}$. More precisely, in the expansion

$$\text{in}_w(h') = T^m + \sum_{2 \leq j \leq m} T^{m-j} U_1'^{j(\delta(x)-1)} [\lambda_j V_3^{j\epsilon(x)} + \mu_{i,j} U_3'^{j\epsilon(x)-i} U_1'^{i(B(u, z)-1)} \times V'^{e(i,j)}],$$

$\mu_{i,j} \in \frac{R'}{(t, u'_1, u'_3)}$, $e(i, j) \in \mathbb{N}$, we will have

$$\overline{u_2'}^{b(i_0, j_0)} Q_{i_0, j_0}(1, \overline{u_2'}) = \mu_{i_0, j_0} \times V'^{e(i_0, j_0)}.$$

Then

$$\beta(x') \leq \text{ord}_{x'}(Q_{i_0, j_0}(1, \overline{u'_2}))/id,$$

which, by **VI.4.3**(5), gives

$$A_1(x') = B(u, z) - 1, \quad \beta(x') \leq C(u, z)/d$$

and implies **VI.4.4** in this case.

VI.4.4.4 From now on, we assume the implication:

$$Q_{i,j} \neq 0 \Rightarrow \lambda_j \neq 0.$$

In particular, we have $j\epsilon(x) \in \mathbb{N}$, $j\delta(x) \in \mathbb{N}$ and all the indices i in (2)(3) are integers. Let us define

$$F_j \in \text{gr}_{v_B}(R) = k(x)[U_1, U_2, U_3, Z]$$

by

$$F_j = \lambda_j U_3^{j\epsilon(x)} + \sum_{1 \leq i \leq j\epsilon(x)-1} U_3^{j\epsilon(x)-i} U_1^{a(i,j)} U_2^{b(i,j)} Q_{i,j}(U_1, U_2).$$

$$F'_j \in \text{gr}_w(R') = \frac{R'}{(u'_1, u'_3, z')} [U'_1, U'_3, Z'] = \frac{R'}{(u'_1, u'_3, z')} [U'_1, U'_3, T]$$

by

$$F'_j = \lambda_j U'_3^{j\epsilon(x)} + \sum_{1 \leq i \leq j\epsilon(x)-1} U'_3^{j\epsilon(x)-i} U'_1^{i(B(u,z)-1)} \overline{u'_2}^{b(i,j)} Q_{i,j}(1, \overline{u'_2}),$$

so (3) can be rewritten:

$$\text{in}_w(h') = T^{m(x)} + \sum_{2 \leq j \leq m(x)} T^{m(x)-j} U'_1^{j(\delta(x)-1)} F'_j. \quad (3')$$

The preceding remarks rewrite $j\epsilon(x) \notin \mathbb{N} \Rightarrow F_j = 0, F'_j = 0$. Let

$$G_j = F_j^{\frac{m!\epsilon(x)}{j\epsilon(x)}}, \quad G'_j = F'_j{}^{\frac{m!\epsilon(x)}{j\epsilon(x)}} \quad 2 \leq j \leq m, \quad j\epsilon(x) \in \mathbb{N},$$

$\deg_{U_3}(G_j) = m!\epsilon(x)$ or $G_j = 0$, and $\deg_{U'_3}(G'_j) = m!\epsilon(x)$ or $G'_j = 0$.

Let $\mu_1, \mu_2 \in k(x)$, j_1, j_2 , $2 \leq j_1, j_2 \leq m$, let

$$G = \mu_1 G_{j_1} + \mu_2 G_{j_2} = \mu_{m!\epsilon(x)} U_3^{m!\epsilon(x)} + \sum_{1 \leq i \leq m!\epsilon(x)-1} U_3^{m!\epsilon(x)-i} U_1^{a(i)} U_2^{b(i)} Q_i(U_1, U_2),$$

where $Q_i = 0$ or Q_i neither divisible by U_1 nor by U_2 . Let us denote $d(i) := \deg(Q_i)$. Assume that for some i , $Q_i \neq 0$, then, by **VI.4.3**(5),

$$d(i) = i(B - \frac{a(i)}{i} - \frac{b(i)}{i}) \leq iC(u, z). \quad (4)$$

Assume that not all G_j 's are collinear in the $k(x)$ -vector space $\text{gr}_{v_B}(R)$. Then there is some $G \neq 0$ as above with $\lambda = 0$. Let

$$G = \mu_1 G_{j_1} + \mu_2 G_{j_2} = \sum_{1 \leq i \leq m!\epsilon(x)-1} U_3^{m!\epsilon(x)-i} U_1^{a(i)} U_2^{b(i)} Q_i(U_1, U_2),$$

with some $Q_i \neq 0$. Let $i_0 := \min\{i : Q_{i_0} \neq 0\}$. Let

$$G' = \lambda G'_{j_1} + \mu G'_{j_2} = \sum_{1 \leq i \leq m! \epsilon(x) - 1} U_3^{m! \epsilon(x) - i} U_1^{i(B(u, z) - 1)} \overline{u_2}^{b(i)} Q_i(1, u'_2).$$

Replacing U'_3 by V_3 , we get

$$G' = \mu_1 G'_{j_1} + \mu_2 G'_{j_2} = V_3^{m! \epsilon(x) - i_0} U_1^{i_0(B(u, z) - 1)} \overline{u_2}^{b(i_0)} Q_{i_0}(1, u'_2) + H', \quad \deg_{V_3} H' < m! \epsilon(x) - i_0.$$

Then

$$A_1(x') = B(u, z) - 1, \quad \beta(x') \leq C(u, z)/d$$

which implies **VI.4.4** in this case.

VI.4.4.5 From now on, we assume that all G_j 's are collinear in the $k(x)$ -vector space $\text{gr}_{v_B}(R)$.

By (P3) for (z, u) , any $G_j \neq 0$ is not collinear to a $(m! \epsilon(x))^{\text{th}}$ -power, any $F_j \neq 0$ is not collinear to a $(j \epsilon(x))^{\text{th}}$ -power. Take some $F_j \neq 0$, and let

$$j \epsilon(x) = p^e q, \quad (p, q) = 1. \quad (5)$$

Let $v_3 = u'_3 + \lambda u_1'^a$, with $a \geq B(u, z) - 1$ and $\lambda \in R'$, λ not divisible by u'_1 . Let

$$\bar{\lambda} \in \frac{R'}{(z', u_1, u'_3)} = k(x)[\overline{u'_2}]_{\overline{v'}}, \quad b := \text{ord}_{v'}(\bar{\lambda}), \quad \beta_0 := \min_i \left(\frac{\text{ord}_{v'}(Q_{i,j}(1, u'_2))}{i} \right) \leq C(u, z)/d.$$

When $b < \beta_0$, we have

$$A_1(x') = B(u, z) - 1, \quad \beta(x') = b < C(u, z)/d.$$

When $b > \beta_0$, we have

$$A_1(x') = B(u, z) - 1, \quad \beta(x') = \beta_0 \leq C(u, z)/d.$$

When $b = \beta_0$ and there exists $i < p^e$ such that $Q_{i,j} \neq 0$, say i_0 is the smallest such i , we get

$$F'_j = \lambda_j V_3^{j \epsilon(x)} + V_3^{j \epsilon(x) - i_0} U_1^{i_0(B(u, z) - 1)} \overline{u'_2}^{b(i_0, j)} Q_{i_0, j}(1, \overline{u'_2}) + H'_j, \quad \deg_{V_3} H'_j < j \epsilon(x) - i_0,$$

$$A_1(x') = B(u, z) - 1, \quad \beta(x') \leq \frac{\text{ord}_{v'}(Q_{i_0, j}(1, u'_2))}{i_0} \leq C(u, z)/d.$$

When $b = \beta_0$ and for $i \leq p^e$ $Q_{i,j} = 0$, then

$$F'_j = \lambda_j V_3^{j \epsilon(x)} + V_3^{j \epsilon(x) - p^e} U_1^{p^e(B(u, z) - 1)} \lambda^q + H'_j, \quad \deg_{V_3} H'_j < j \epsilon(x) - p^e,$$

$$A_1(x') = B(u, z) - 1, \quad \beta(x') \leq b = \beta_0 \leq C(u, z)/d.$$

When $b = \beta_0$ and for $i < p^e$ $Q_{i,j} = 0$ and $\lambda_j^{-1} U_1^{a(p^e, j)} U_2^{b(p^e, j)} Q_{p^e, j}$ is a $(p^e)^{\text{th}}$ -power, then

$$\lambda_j^{-1} U_1^{a(p^e, j)} U_2^{b(p^e, j)} Q_{p^e, j} = U_1^{p^e a} U_2^{p^e b} Q_0^{p^e},$$

with $a \geq A_1(x)$, $b \geq A_2(x)$ and $B_0 := \deg(Q_0) \leq C(u, z)$. Let $W_3 := U_3 + U_1^a U_2^b Q_0$, we get

$$F_j = \lambda_j W_3^{j\epsilon(x)} + \sum_{1+p^e \leq i \leq j\epsilon(x)-1} W_3^{j\epsilon(x)-i} U_1^{a_0(i,j)} U_2^{b_0(i,j)} Q_{0,i,j}(U_1, U_2),$$

$$a_0(i, j) \geq iA_1(x), \quad b_0(i, j) \geq iA_2(x), \quad d_0(i, j) := \deg(Q_{0,i,j}) \leq iC(u, z), \quad \text{or } Q_{0,i,j} = 0.$$

Let $w_3 \in R$ such that $\text{in}_{v_B}(w_3) = W_3$, then, with $w'_3 = w_3/u_1$, $W'_3 = \text{in}_w(w'_3)$:

$$F'_j = \lambda_j W'_3{}^{j\epsilon(x)} + \deg_{W'_3} < j\epsilon(x) - p^e.$$

Let $v_3 = w'_3 + \lambda' u_1^{a'}$, with $a' \geq B(u, z) - 1$ and $\lambda' \in R'$, λ' not divisible by u'_1 . Then we conclude as above:

$$A_1(x') = B(u, z) - 1, \quad \beta(x') \leq b = \beta_0 \leq C(u, z)/d.$$

VI.4.4.6 From now on, we assume **VI.4.4.2**, **VI.4.4.3**, **VI.4.4.4**, **VI.4.4.5** and for $i < p^e$ $Q_{i,j} = 0$ and $\lambda_j^{-1} U_1^{a(p^e, j)} U_2^{b(p^e, j)} Q_{p^e, j}$ is NOT a $(p^e)^{\text{th}}$ -power (c.f.(5)). In particular $e \geq 1$. Let us recall the following elementary lemma [CP2, **II.5.3.2**].

Lemma VI.4.4.7 *Let $F(U_1, U_2) \in k(x)[U_1, U_2]$ be a homogeneous polynomial of degree $d_0 \geq 0$, and $a, b \in \mathbb{N}$ be such that $U_1^a U_2^b F(U_1, U_2) \notin (k(x)[U_1, U_2])^p$.*

Let $x' \in \text{Spec}k(x)[\frac{U_2}{U_1}]$ be a closed point with ideal $(v := P(1, \frac{U_2}{U_1}))$, $P \in k(x)[U_1, U_2]$ a nonzero homogeneous irreducible polynomial of degree $d := [k(x') : k(x)]$, unitary in U_2 .

Let $A \in T' := k(x)[U_1, \frac{U_2}{U_1}]_{(U_1, v)}$ be such that $U_1^{a+b+d_0}$ (resp. $U_1^{a+b+d_0} v^b$) divides A^p in T' if $P \neq U_2$ (resp. $P = U_2$). There exists an integer $c \geq 0$ such that

$$U_1^{a+b+d_0} (\frac{U_2}{U_1})^b F(1, \frac{U_2}{U_1}) + A^p \equiv U_1^{a+b+d_0} (\frac{U_2}{U_1})^b \gamma v^c \pmod{(U_1^{a+b+d_0+1} T')},$$

with γ invertible in T' . We have the following estimates for c :

- (i) *if $P \neq U_2$ (resp. $P = U_2$), then $c \leq 1 + \frac{d_0}{d}$ (resp. $c \leq d_0$);*
- (ii) *if $P \neq U_2$, then $c < p(1 + \lfloor \frac{d_0}{pd} \rfloor)$ (equivalently: for every $N \in \mathbb{N}$ such that $\frac{d_0}{pd} < N$, we have $c < Np$);*
- (iii) *if $d_0 \geq 1$ and $b = 0$, then $c \leq i$.*

Let $f < e$ be the integer defined by:

$$\lambda_j^{-1} U_1^{a(p^e, j)} U_2^{b(p^e, j)} Q_{p^e, j} = Q_0^{p^h}, \quad Q_0 \text{ is not a } p^{\text{th}} - \text{power}.$$

Let

$$Q_0 =: (U_1^a U_2^b F(U_1, U_2)),$$

with $p^h a = a(p^e, j)$, $p^h b = b(p^e, j)$, $p^h d_0 = d(p^e, j)$, where $d_0 := \deg(Q_0)$. In particular,

$$d_0 \leq p^{e-h} C(u, z).$$

Then,

$$F'_j = \lambda_j (V_3^{j\epsilon(x)} + V_3^{j\epsilon(x)-p^e} \lambda_j^{-1} (U_1^{p^e(B(u, z)-1)} \overline{u_2}^{b(i_0, j)} Q_{i_0, j}(1, \overline{u_2}) + \overline{\lambda}^{p^e}) + \deg_{V_3} < j\epsilon(x) - p^e).$$

By lemma **VI.4.4.7**, $(\overline{u'_2}^{b(i_0,j)} Q_{i_0,j}(1, \overline{u'_2}) + \overline{\lambda}^{p^e}) = (\gamma v'^c)^{(p^h)} \neq (0)$, so

$$A_1(x') = B(u, z) - 1, \quad p^e \beta(x') \leq p^h c.$$

Furthermore, by (i) above, $c \leq 1 + \frac{d_0}{d}$, so:

$$p^e \beta(x') \leq p^h \left(1 + \frac{d_0}{d}\right) \leq p^h + p^h \frac{p^{e-h} C(u, z)}{d} = p^h + \frac{p^e C(u, z)}{d},$$

$$\beta(x') \leq \frac{1}{p} + \frac{C(u, z)}{d}.$$

By (ii),

$$p^e \beta(x') < p^h p \left(1 + \left\lfloor \frac{d_0}{pd} \right\rfloor\right) \leq p^{f+1} \left(1 + \left\lfloor \frac{p^{e-h} C(u, z)}{pd} \right\rfloor\right), \quad \beta(x') < 1 + \left\lfloor \frac{C(u, z)}{d} \right\rfloor,$$

which is **VI.4.4(iii)**. Now **VI.4.4(iv)** is a consequence of (iii) above.

VI.4.5 Proof of Theorem VI.1: some cases with $\gamma(u, z) = 1$.

The strategy to make the proof is to make a list of different subcases covering this case, from the easiest to the most difficult and to prove them up to the former ones.

All cases ($\beta(u, z) < 1$ and $\text{div}(u_1 u_2) \not\subseteq E$) are covered by **VI.4.5.3** below. All cases with ($\gamma(u, z) < 1$ and $\text{div}(u_1 u_2) \subseteq E$) are dealt with in **VI.4.5.6**. This includes in particular all remaining cases with $\beta(u, z) < 1$ since $C(u, z) \leq \beta(u, z)$ for $\text{div}(u_1 u_2) \subseteq E$ (see definition **VI.4.1**).

LEMMA VI.4.5.1 *With assumptions as in VI.4.3, assume furthermore that*

$$A_1(u, z) < 1, \quad \beta(u, z) < 1. \tag{1}$$

There exist well prepared parameters (z', u') at x' such that

$$(A_1(u', z'), \beta(u', z')) <_{\text{lex}} (A_1(u, z), \beta(u, z)), \quad \text{and } \beta(u', z') < 1.$$

Proof. This is a direct consequence of lemma **VI.4.4**.

LEMMA VI.4.5.2 *With assumptions as in VI.4.3, assume furthermore that*

$$\beta(u, z) < 1, A_1(u, z) \geq 1, \quad (d_1(x) + d_3(x) + \epsilon(x) \geq 1 \text{ or } E \subseteq \text{div}(u_1 u_3)). \tag{2}$$

Then $\mathcal{C}_1 := V(z, u_1, u_3)$ is an ϵ -permissible algebraic curve on X .

Let $\pi : (Z', x') \rightarrow (Z, x)$ be the blowing up along $\mathcal{C}_1$, $X' \subset Z'$ the strict transform of X and $x' \in X'$ the center of v and assume $\iota(x') = \iota(x)$. Then $(z', u') = (z/u_1, u_1, u_2, u_3/u_1)$ are well prepared parameters at x' and we have

$$(A_1(u', z'), \beta(u', z')) = (A_1(u, z) - 1, \beta(u, z)).$$

Proof. This follows from proposition **VI.4.2**.

REMARK VI.4.5.3 Lemmas **VI.4.5.1** **VI.4.5.2** prove theorem **VI.1** when (Hyp) is true and

$$\beta(u, z) < 1, A_1(u, z) \geq 1, (d_3(x) + \epsilon(x) \geq 1 \text{ or } E \subset \text{div}(u_1 u_3)). \quad (3)$$

Indeed, $d_3(x) + \epsilon(x) = d_3(x') + \epsilon(x')$ if $\iota(x') = \iota(x)$ after blowing up. If $E \subseteq \text{div}(u_1 u_3)$, then $E' \subseteq \text{div}(u'_1 u'_3)$, so condition (3) remains stable after blowing up. A descending induction on $A_1(u, z)$ ends the proof.

VI.4.5.4 Proof of Theorem VI.1 in the case $C(u, z) = 0$.

In that special case, we have $\beta(u, z) = A_2(u, z)$. When $A_2(u, z) < 1$ and $A_1(u, z) < 1$, **VI.4.5.1** gives the result. Let us see the other cases:

$$A_1(u, z) \geq 1 \text{ or } A_2(u, z) \geq 1. \quad (1)$$

Case 1. $d_3(x) + \epsilon(x) \geq 1$. We may assume $A_2(u, z) \geq 1$ by symmetry on u_1, u_2 . Then $\mathcal{C} := (z, u_2, u_3)$ is an ϵ -permissible algebraic curve by proposition **VI.4.2** and we get

$$A_1(x') = A_1(x), \quad A_2(x') = A_2(x) - 1, \quad C(u', z') = 0$$

after blowing up along $\mathcal{C}$ if $\iota(x') = \iota(x)$. A descending induction on $A_2(x)$ and **VI.4.5.2** give the result. From now on, we assume

$$d_3(x) + \epsilon(x) < 1. \quad (2)$$

Case 2. $d_3(x) + d_i(x) + \epsilon(x) < 1$, $i = 1$ and $i = 2$. We blow up $\{x\}$ in this case. By proposition **VI.4.3**(ii) or (iv), we have

$$d_3(x') = d_3(x), \quad d_2(x') = d_2(x), \quad d_1(x') = d_1(x) + d_2(x) + d_3(x) + \epsilon(x) - 1 < d_1(x), \quad C(u', z') = 0,$$

$$\delta(x') = d_1(x') + d_2(x') + d_3(x') + \epsilon(x') < \delta(x)$$

if $\iota(x') = \iota(x)$ and x' is the origin of a chart. Otherwise, lemma **VI.4.4**(ii)(iii) gives **VI.4.5.2**(2) at x' for some well prepared r.s.p. (z', u') .

Case 3. $d_3(x) + \epsilon(x) < 1$, $d_3(x) + d_i(x) + \epsilon(x) \geq 1$ for some $i = 1$ or $i = 2$, $A_j(x) \geq 1$, $j = 1, 2$. We choose an ϵ -permissible blowing up center Y as follows:

if $V(z, u_i, u_3)$, for $i = 1, 2$ are ϵ -permissible, then $Y := (z, u_i, u_3)$ with

$$(A_i(x), d_i(x)) \geq (A_{i'}(x), d_{i'}(x)), \quad \{i, i'\} = \{1, 2\};$$

if $V(z, u_i, u_3)$ is ϵ -permissible for a unique $i \in \{1, 2\}$, then $Y := (z, u_i, u_3)$;

if $V(z, u_i, u_3)$ is not ϵ -permissible for $i \in \{1, 2\}$, then $Y := \{x\}$.

Let $n(x) := 2$ if $(A_1(x), d_1(x)) = (A_2(x), d_2(x))$, $n(x) := 1$ otherwise. If $\iota(x') = \iota(x)$, we claim that x' satisfies **VI.4.6.3**(2) or falls into cases 1,2 above, or there is a well prepared r.s.p. (z', u') at x' with $C(u', z') = 0$ and

$$(\max_{i=1,2} \{A_i(x)\}, \max_{i=1,2} \{d_i(x)\}, n(x)) <_{\text{lex}} (\max_{i=1,2} \{A_i(x')\}, \max_{i=1,2} \{d_i(x')\}, n(x')). \quad (3)$$

Note that this ends the proof of the case $C(u, z) = 0$, since (3) can repeat but finitely many times. To prove the claim, first assume that $Y = (z, u_1, u_3)$. By proposition **VI.4.2**, we have

$$A_1(x') = A_1(x) - 1, \quad A_2(x') = A_2(x), \quad d_1(x') = d_1(x) + d_3(x) + \epsilon(x) - 1 < d_1(x), \quad d_2(x') = d_2(x)$$

and the result is clear. The case $Y = (z, u_2, u_3)$ is similar.

Assume now that $Y = \{x\}$. By symmetry on u_1, u_2 , we assume $A_2(x) \geq 1$. If $x' \in \text{Spec}R[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$ and x' is the point with parameters $(z/u_1, u_1, u_2/u_1, u_3/u_1)$ (origin of the first chart), we get

$$A_1(x') = A_1(x) + A_2(x) - 1, \quad A_2(x') = A_2(x), \quad d_1(x') = d_1(x) + d_2(x) + d_3(x) + \epsilon(x) - 1, \quad d_2(x') = d_2(x).$$

Since $A_2(x) \geq 1$ and $V(z, u_2, u_3)$ is not ϵ -permissible, we have

$$d_2(x) + d_3(x) + \epsilon(x) < 1, \quad d_1(x) + d_3(x) + \epsilon(x) \geq 1 \quad \text{and} \quad A_1(x) < 1.$$

We get $d_1(x) > d_2(x)$ and $d_1(x') = d_1(x) + d_2(x) + d_3(x)\epsilon(x) - 1 < d_1(x)$, $A_1(x') < A_1(x) \leq A_2(x') = A_2(x)$ which proves the claim.

If $x' \in \text{Spec}R[z/u_1, u_2/u_1, u_3/u_1] \subset Z'$ and x' is not the above point, we have **VI.4.5.2(2)** at x' for some well prepared r.s.p. (z', u') at x' by lemma **VI.4.4(ii)(iii)**.

If $x' \in \text{Spec}R[z/u_2, u_1/u_2, u_3/u_2] \subset Z'$ and x' is the point with parameters $(z/u_2, u_1/u_2, u_2, u_3/u_2)$ (origin of the second chart), we get

$$A_2(x') = A_1(x) + A_2(x) - 1, \quad A_1(x') = A_1(x), \quad d_2(x') = d_1(x) + d_2(x) + d_3(x) + \epsilon(x) - 1, \quad d_1(x') = d_1(x).$$

We have $A_1(x) < 1$: otherwise, as $V(z, u_1, u_3)$ is not ϵ -permissible, this would imply $d_1(x) + d_3(x) + \epsilon(x) < 1$, $d_2(x) + d_3(x) + \epsilon(x) \geq 1$, hence $V(z, u_2, u_3)$ ϵ -permissible since $A_2 \geq 1$: a contradiction. We now get $A_1(x) < 1 \leq A_2(x)$ and $A_2(x') = A_1(x) + A_2(x) - 1 < A_2(x)$ which completes the proof of the claim.

VI.4.5.5 Proof of Theorem **VI.1** in the case $C(u, z) < 1$, $\text{div}(u_1 u_2) \subseteq E$.

We perform the sequence of local blowing ups

$$(Z, x) =: (Z_0, x_0) \longleftarrow (Z_1, x_1) \longleftarrow \cdots \longleftarrow (Z_n, x_n) \longleftarrow \cdots,$$

where $x_0 = x$, $x_i \in X_i$ (X_i denoting the strict transform of X), $0 \leq i \leq n$, is the center of v , each blowing up center is $Y_i = \{x_i\}$.

If $\iota(x_1) = \iota(x)$, and is not the origin of a chart (*viz.* case 3 in **VI.4.5.4**), then x_1 verifies the assumptions of **VI.4.5.1** by lemma **VI.4.4(iii)**.

Assume now that $\iota(x_i) = \iota(x)$ and x_i is the origin of a chart for all $i \geq 0$. By **VI.4.3(ii)(iv)**, x_i verifies the assumptions of **VI.4.5.5** and $C(x_{i+1}) \leq C(x_i)$ for all $i \geq 0$. It is then a very well known fact that $C(x_i) = 0$ for $i \gg 0$, i.e. the assumptions of **VI.4.5.4** are satisfied.

VI.4.5.6 End of the proof of Theorem **VI.1**. As our invariants $C(u, z)$, $\beta(u, z)$ are discrete, the next lemma shows that we will reach one of the cases (ii) $\beta(u, z) < 1$ or (iii) $C(u, z) < 1$. This ends the proof of theorem **VI.1** (see comments right after **VI.4.5.**).

LEMMA VI.4.5.7. *With assumptions as in **VI.4.3**, consider the sequence of local blowing ups*

$$(Z, x) =: (Z_0, x_0) \longleftarrow (Z_1, x_1) \longleftarrow \cdots \longleftarrow (Z_n, x_n) \longleftarrow \cdots,$$

where $x_0 = x$, $x_i \in X_i$ (X_i denoting the strict transform of X), $0 \leq i \leq n$, is the center of v , each blowing up center is $Y_i = \{x_i\}$.

Assume that $\iota(x_i) = \iota(x)$ for all $i \geq 0$. There exists some $i \geq 0$ and a well prepared r.s.p. $(z_i, u_{1,i}, u_{2,i}, u_{3,i})$ at x_i (w.r.t. the reduced inverse image of E in Z_i) such that one of the following holds:

- (i) $\gamma(u_i, z_i) < \gamma(u, z)$;
- (ii) $\beta(u_i, z_i) < 1$;
- (iii) $C(u_i, z_i) < 1$;

Proof. This breaks up in three cases:

Case 1. for all $i \geq 0$, the point x_i is the origin of one of the two charts of **VI.4.3**, i.e. we are always in one of the cases **VI.4.3(ii)(iv)**. Then $C(u_i, z_i) = 0$ for $i \gg 0$ (see **VI.4.5.5** above).

Case 2. for all $i \geq 0$, (x_i is rational over x and belongs to the first chart), i.e. x_i is a rational point not on the strict transform of $\text{div}(u_1)$. By **VI.4.3(ii)(iii)**, x_1 has a r.s.p. of the form $(z/u_1, u_1, u_2/u_1 + \mu_1, u_3/u_1)$ for some $\mu_1 \in R$. A well prepared r.s.p. is of the form

$$z/u_1 + \lambda_1 u_1, u_1, u_2/u_1 + \mu_1, u_3/u_1 + \mu_2 u_1, \lambda_1, \mu_1, \mu_2 \in R,$$

with $\mu_2 = 0$ if $\text{div}(u_3) \subseteq E$. Then there exists a regular *formal* curve $\mathcal{C}$ of the form $\mathcal{C} = V(\widehat{z}, \widehat{u_2}, \widehat{u_3})$ on $\text{Spec}(\widehat{R}/(h))$, transverse to E_i for all $i \geq 0$, $\widehat{u_3} = u_3$ if $\text{div}(u_3) \subseteq E$, whose strict transform goes through all points x_i , $i \geq 0$. Necessarily $\mathcal{C} \subseteq \text{Sing}_m(X)$, so we may assume that $\mathcal{C} \subset \text{div}(u_j) \subseteq E$ for $j = 2$ or $j = 3$. In particular, we may take $\widehat{u_j} = u_j$ for $j = 2$ or $j = 3$. This implies that $v(u_j) > v(u_1^n) = nv(u_1)$ for all $n \geq 1$: a contradiction, since our given valuation v has rank one.

Case 3. $E \subseteq \text{div}(u_1 u_3)$ and we are not in case 2, i.e. there exists $i_0 \geq 0$ such that either x_{i_0+1} is not rational over x_{i_0} or E_{i_0+1} has one more component than E_{i_0} , i_0 minimal. Suppose $\beta(x_{i_0}) \geq 1$.

If x_{i_0+1} is not rational over x_{i_0} , we get

$$\beta(x) \geq \beta(x_{i_0}) > \beta(x_{i_0+1})$$

by **VI.4.3(ii)(iii)** and **VI.4.4(iii)**: note that $C(u, z) \leq \beta(u, z)$ since $\text{div}(u_2) \not\subseteq E$ and

$$1 + \left\lceil \frac{x}{2} \right\rceil \leq x \text{ for every } x \geq 1.$$

If E_{i_0+1} has one more component than E_{i_0} , we have

$$C(u_{i_0+1}, z_{i_0+1}) \leq \frac{\beta(x_{i_0})}{2} \leq \frac{\beta(x)}{2}$$

by **VI.4.3(ii)(iii)(iv)**. This gives **VI.4.5.7(iii)** if $1 \leq \beta(x) < 2$.

Now, $\gamma(x_{i_0+1}) = 1 + \lfloor C(u_{i_0+1}, z_{i_0+1}) \rfloor$, $\gamma(x) = \lceil \beta(x) \rceil$, so we get **VI.4.5.7(i)** if $\beta(x) > 2$.

Assume that

$$\beta(x_{i_0}) = \beta(x) = 2. \tag{1}$$

Since $\gamma(x) = 2$, we get **VI.4.5.7(i)** unless $\gamma(x_i) = 2$ for $i \geq 0$ by proposition **VI.4.3**.

Let $i_1 > i_0$ be the largest index such that E_i has as many components as E_{i_0} for $i_0 \leq i \leq i_1$. We may assume $i_1 < +\infty$ by case 1 and we have

$$\gamma(x_{i_1}) = 2 = 1 + \lfloor C(u_{i_1}, z_{i_1}) \rfloor.$$

By **VI.4.4(iii)**, we get $\beta(x_{i_1+1}) < 2$. Now the point x_{i_1+1} falls into case 2 above or into case 3 with (1) not satisfied. This concludes the proof in case 3.

The end of the proof of **VI.4.5.7** is just a logical game: we reach the assumption $E \subseteq \text{div}(u_1 u_3)$ for some point x_i , $i \geq 0$ provided we are not in case 1.

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