Optimal liquidation with additional information

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Noname manuscript No.
(will be inserted by the editor)

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Received: date / Accepted: date

Abstract We consider the problem of how to optimally close a large asset position in a market with a linear temporary price impact. We take the perspective of an agent who obtains a signal about the future price evolvement. By means of classical stochastic control we derive explicit formulas for the closing strategy that minimizes the expected execution costs. We compare agents observing the signal with agents who do not see it. We compute explicitly the expected additional gain due to the signal, and perform a comparative statics analysis.

Introduction

For many companies it is part of day-to-day business to build up and close large asset positions on financial markets. For example, whenever a fund modifies its investment strategy, it will reduce the position of some of its assets, while enlarging the holdings of other ones. Energy companies have to unwind

The first author gratefully acknowledges the financial support by the *Ecole Centrale de Lyon* during his visit in March 2012.

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long positions of power and buy the commodities they need for the power generation.

Selling or buying a large amount of an asset in short time usually entails a price impact. This is why in practice financial institutions, from now on referred to as agents, frequently unwind large positions by splitting them into smaller parts and closing them successively. Spreading orders over time implies the price impact to be smaller.

Agents closing a large asset position sometimes have additional information about the future price. The extra information can be based for example on assessments of market analysts. Trading houses have teams of analysts constantly observing markets. The analysts provide market assessments or even forecasts that are incorporated in the company's trading decisions.

The main aim of the paper is to study the value of a market assessment *before* it is revealed. To put it differently, we look at the *expected additional value* of market expertise. To this end we introduce into our model an expert who obtains a signal about the asset price at time T, the time up to which the position has to be closed. If the expert passes the knowledge on to the agent having to close the position, then we say that the agent is *informed*; else she is *non-informed*. We use the technique of *filtration enlargements* for modeling the information flow of the informed agent.

By comparing the optimal execution strategy of a non-informed agent with the one of an informed agent, one can derive the additional gain due to the signal. In order to obtain a closed form formula for the additional gain, we work in a stylized model. We assume that the price signal is the asset price at T disturbed by an independent centered Gaussian noise. Morever, we suppose that any transaction has a linear absolute temporary (abbreviated by LAT in the pioneering paper [7]) impact on the asset's price. The fundamental (i.e. non-influenced) price process is assumed to be a Brownian motion, complemented by a *drift*. We suppose that the agent aims at maximizing the expected proceeds (resp. minimizing expected costs) from closing a position.

We characterize optimal closing strategies by using standard stochastic control methods. The simple model set-up allows to obtain explicit formulas for the value functions and the optimal controls, and hence for the expected additional gain due to the signal.

We find that the expected additional gain due to the signal does not dependent on the initial price and on the agent's initial position size. The gain from the signal is *only* determined by three factors, namely the signal's noise, the volatility and the liquidity of the asset. We perform a comparative statics analysis of the additional gain to distill the precise dependence on these three factors.

We consider also the case where the signal reveals the asset's *exact* fundamental price at T, i.e. where the signal is not distorted by noise. We show that in this case the additional gain is finite. The market would admit arbitrage if there were no market frictions. The price impact entailed by any trading implies that the gain from exactly knowing the fundamental price at a future date is only finite.

The value of a price signal has so far been studied mainly within utility maximization models. In [13] the authors calculate, also by employing filtration enlargements, the expected additional *logarithmic utility* of an investor possessing inside information. They do *not* consider market frictions and hence obtain that the additional utility is infinite if the exact asset's price at T is known to the investor. The model of [13] has been put forward in many succeeding papers, e.g. in [2], [4], [5].

As an auxiliary step in the calculation of the additional gain we compute first the signal's a posteriori value. After receiving the signal, the agent perceives the price dynamics with an additional linear drift. Therefore, we shortly discuss also the liquidation problem under directional views. In Section 1 we summarize findings on optimal liquidation in a Bachelier model with a linear drift.

So far the liquidation literature has only briefly analyzed the impact of market opinions on trading strategies. Almgren & Chriss [1] calculate optimal *deterministic* liquidation strategies, allowing for directional views. They assume that the agent's objective is to minimize a weighted sum of the mean and the variance of the proceeds. It is remarkable that the optimal strategy from [1] maximizes CARA utility - not only among all deterministic, but even among all *predictable* trajectories. This is shown in [16] for a time continuous version of the Almgren & Chriss model. The paper [15] also studies the influence of price drifts on optimal liquidation strategies. A general semimartingale perspective is taken, leading to a more abstract representation of liquidation strategies in terms of conditional expectations.

The paper is organized as follows: In Section 1 we set up the model framework and provide optimal position strategies within a Bachelier-type model with a linear price drift. In Section 2, we estimate the value of additional information from an agent's perspective before the information is revealed. Section 3 performs a comparative statics analysis.

1 Closing positions in a Bachelier model with drift

1.1 The model set-up

Consider an agent who has to unwind a position of $X_0 \in \mathbb{R}$ shares of an asset until a time horizon $T > 0$. We assume that the *fundamental asset price* is a drifted Brownian motion satisfying the SDE

$$dS_t = a(t, S_t)dt + \sigma dW_t,$$

where $\sigma > 0$ is a constant volatility, $a : [0, T] \times \mathbb{R} \to \mathbb{R}$ a measurable drift function and W a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{H}_t)_{t\in[0,T]}, \mathbb{P})$. We assume that there exists $C \in \mathbb{R}_+$ such that $|a(t, s)| \leq C(1 + s)$ for all

$t \in [0,T]$ and $s \in \mathbb{R}_+$. We interpret the drift as the agent's directional view or extra information about the future price evolution. Moreover, we assume that all prices are *forward prices* so that no discounting is needed. The filtration $\mathcal{H}$ represents the agent's information.

A *closing strategy* (or simply strategy) of a position $x \in \mathbb{R}$ at time $t \in [0,T)$ is a $(\mathcal{H}_t)$-predictable strategy $\xi = (\xi_u)$ satisfying $\int_t^T \xi_u du = x$. We interpret ξ_t as the *selling rate* at time $t \in [0,T]$. Given ξ, the total position at time $t \in [0,T]$ is given by

$$X_t = X_0 - \int_0^t \xi_s ds.$$

Notice that $X_T = 0$, i.e. the position is closed at T.

For technical reasons we impose the following integrability condition on the closing strategies: a strategy (ξ_u), resp. its associated position process (X_t), is called *admissible* if

(A1) the process ξ is L^2-integrable, i.e $E(\int_0^T \xi_u^2 du) < \infty$,

(A2) the family $\left(\left(\frac{X_t^*}{T-t}\right)^2\right)_{0\le t\le T}$ is uniformly integrable,

and $\lim_{t\to T} \frac{X_t^2}{T-t} = 0$, a.s.

We denote by $\mathcal{A}^{\mathcal{H}}(t,x)$ the set of all admissible closing strategies of x at t.

We suppose that any transaction entails a price impact that is linear with respect to the selling rate. Moreover, the impact is assumed to be absolute and only instantaneous. Selling at a rate of ξ_t is thus possible only at the *realized price* of

$$\tilde{S}_t = S_t - \eta \xi_t,$$

where $\eta > 0$ is the price impact parameter.

The final revenues (possibly negative) of the liquidation operation when selling at a rate $(\xi_t)_{t\in[0,T]}$ are given by

$$R_T = \int_0^T \xi_u \tilde{S}_u du.$$

Notice that by the product formula we have

$$\begin{aligned} R_T &= \int_0^T \xi_u S_u du - \eta \int_0^T \xi_u^2 du \\ &= X_0 S_0 + \int_0^T X_u a(u, S_u) du + \int_0^T X_u \sigma dW_u - \eta \int_0^T \xi_u^2 du. \end{aligned} \tag{1}$$

Assumption (A2) guarantees that a position process X, associated to an admissible strategy ξ, is square integrable, and thus taking expectations in (1) we get

$$E(R_T) = X_0 S_0 + E \int_0^T (X_u a(u, S_u) - \eta \xi_u^2) du.$$

We assume that the agent aims at maximizing the expected value of the final revenues. More precisely the target function is given by

$$J(t,x,s,\xi) = E\left[\int_t^T \left(X_u a(u,S_u) - \eta\xi_u^2\right) du \middle| X_t = x, S_t = s\right],$$

for $(t,x,s) \in [0,T] \times \mathbb{R} \times \mathbb{R}$. The value function is defined by

$$V(t,x,s) = \sup_{\xi \in \mathcal{A}^{\mathcal{H}}(t,x)} J(t,x,s,\xi). \tag{2}$$

Notice that the value function can be interpreted as expected execution costs.

Remark 1 If the impact of the liquidation operation on the price dynamics is not only instantaneous, but lasts in the considered period, one can add to the model a so-called perpetual impact factor, depending on the total amount of the position closed up to time t. The form of the realized price dynamics is then

$$\tilde{S}_t = S_t - \eta_t - c(X_0 - X_t),$$

where c is the permanent impact factor.
The final revenues in this case are given by

$$R_T = X_0 S_0 + \frac{1}{2} c X_0^2 + \int_0^T X_u a(u,S_u) du + \int_0^T X_u \sigma dW_u - \eta \int_0^T \xi_u^2 du. \tag{3}$$

The only difference with Equation (1) is the constant term $\frac{1}{2}cX_0^2$, which does not influence the optimization. From a mathematical point of view the problem is identical.

1.2 Optimal closure for linear price drifts

If the price drift coefficient is linear, then the value function turns out to be a quadratic form of both the position size and the price. One can determine the value function and the optimal position process in closed form by using classical stochastic control techniques.

The Hamilton-Jacobi-Bellman Equation associated to the control problem (2) is given by

$$-V_t - a(t,s)V_s - \frac{1}{2}\sigma^2 V_{ss} - a(t,s)x - \sup_{\xi \in \mathbb{R}}[-\xi V_x - \eta\xi^2] = 0, \tag{4}$$

with the singular terminal condition

$$\lim_{t \uparrow T} V(t,x,s) = \begin{cases} 0, & \text{if } x = 0, \\ -\infty, & \text{if } x \neq 0. \end{cases} \tag{5}$$

The first order condition implies that the supremum on the left hand side of (4) is attained by $\xi^* = -\frac{V_x}{2\eta}$. We obtain a simplified HJB equation

$$-V_t - a(t,s)V_s - \frac{1}{2}\sigma^2 V_{ss} - a(t,s)x - \frac{V_x^2}{4\eta} = 0. \tag{6}$$

From now on, we suppose that the price drift coefficient is an affine linear function of the form

$$a(t,s) = \alpha(t) + \beta(t)s,$$

where α and β are functions on $[0,T]$. Notice that this implies

$$S_t = h(0,t)S_0 + \int_0^t h(r,t)\alpha(r)dr + \int_0^t h(r,t)\sigma dW_r,$$

where $h(r,t) = e^{\int_r^t \beta(s)ds}$. Moreover, the value function is a quadratic function of the price and the remaining position, as stated in the following theorem:

Theorem 1 *Assume that $a(t,s) = \alpha(t) + \beta(t)s$, where α and β are bounded. Then the value function satisfies*

$$V(t,x,s) = b(t)x^2 + c(t)xs + d(t)s^2 + e(t)x + f(t)s + g(t), \tag{7}$$

where b, c, d, e, f and g are deterministic functions with explicit expressions provided in the appendix. The optimal position trajectory is given by

$$X_t^* = \frac{T-t}{T}\left(X_0 + \frac{1}{2\eta}\int_0^t [c(u)S_u + e(u)]\frac{T}{T-u}du\right). \tag{8}$$

Theorem 1.2 can be derived from results in [15]. Nevertheless, we present in the appendix a simple, direct and self-contained proof, based on classical verifications arguments (in contrast to variational arguments used in [15]).

2 Informed and non-informed agents

Suppose that there is an expert, e.g. a market analyst or an insider, who has obtained a signal about the asset price at time T. In this section we aim at quantifying the value of the signal from the perspective of the agent having to close the asset position.

For simplicity we suppose in the following that the price process is a Brownian motion without drift; more precisely $S_t = \sigma W_t$. This means that S is a martingale with respect to $(\mathcal{F}_t^W)$, the filtration generated by W.

We model the signal as a random variable $G = S_T + N$, where N is independent of the price process and normally distributed with mean zero. Since G is Gaussian, it is equivalent to the signal sent from a price $S_{T'}$, where $T' \geq T$. One can interpret the difference $\sigma^2(T'-T)$ as the variance of the signal's noise.

If the expert discloses the signal to the agent, then we say that the agent is *informed*. In this case the agent's information flow can be modeled as the following initial enlargement of the Brownian filtration:

$$\mathcal{G}_t = \mathcal{F}_t^W \vee \sigma(S_{T'}), \quad 0 \le t \le T.$$

In case the expert does not pass on the signal, we say that the agent is non-informed. The information flow of the agent is then represented by the natural filtration $(\mathcal{F}_t^W)$. The value function of the informed agent, conditional to $S_{T'}$, is given by

$$V^I(t,x,s) = \sup_{\xi \in \mathcal{A}^{\mathcal{G}}(t,x)} E\left[\int_t^T \left(X_u a(u,S_u) - \eta \xi_u^2\right) du \,\middle|\, X_t = x, S_t = s, S_{T'} \right] \tag{9}$$

and the value function of the non-informed agent by

$$V^N(t,x,s) = \sup_{\xi \in \mathcal{A}^{\mathcal{F}}(t,x)} J(t,x,s,\xi). \tag{10}$$

2.1 A priori and posteriori signal value

One can show that the price dynamics under $(\mathcal{G}_t)$ satisfy

$$dS_t = \sigma dW_t^{\mathcal{G}} + \sigma \frac{S_{T'} - S_t}{T' - t} dt, \tag{11}$$

where $W^{\mathcal{G}}$ is a Brownian motion with respect to $(\mathcal{G}_t)$ (see e.g. [14]). The drift in the $(\mathcal{G}_t)$-dynamics (11) is linear with $\alpha(t) = \frac{S_T'}{T'-t}$ and $\beta(t) = -\frac{1}{T'-t}$. If $T' > T$, then α and β are bounded. To put it differently: once the signal is revealed, the informed agent perceives the price with a linear drift satisfying the assumptions of Subsection 1.2. Therefore, we can use Theorem 1 to calculate, conditional to $S_{T'}$, the *a posteriori* expected execution costs. The a priori costs are given as the expectation of the a posteriori costs.

The next theorem provides the value functions in closed form for both the informed and non-informed agent.

Theorem 2 *Let $T' > T$. The value function of the informed agent is a quadratic function as in* (7) *with coefficients*

$$
\begin{aligned}
b^I(t) &= -\eta \frac{1}{T-t} \\
c^I(t) &= -\frac{1}{2}\frac{T-t}{T'-t} \\
d^I(t) &= \frac{1}{48\eta}\frac{(T-t)^3}{(T'-t)^2} \\
e^I(t) &= \frac{1}{2}\frac{T-t}{T'-t}S_{T'} \\
f^I(t) &= \frac{1}{\eta}\frac{S_{T'}}{T'-t}\left(\frac{1}{8}(T'-T)(T-t) - \frac{1}{24}(T'-t)^2 + \frac{1}{24}\frac{(T'-T)^3}{T'-t}\right) \\
g^I(t) &= \frac{S_{T'}^2}{\eta}\left(\frac{1}{12}(T'-T) - \frac{1}{16}\frac{(T'-T)^2}{T'-t}\right) \\
&\quad -\frac{S_{T'}^2}{\eta}\left(\frac{1}{8}(T'-T)\frac{T-t}{T'-t} - \frac{1}{24}(T-t) - \frac{1}{48}\frac{(T'-T)^3}{(T'-t)^2}\right) \\
&\quad +\frac{\sigma^2}{48\eta}\left(\frac{(T'-T)^3}{T'-t} - \frac{3}{2}(T'-T)^2 + 3(T'-T)^2\ln\left(\frac{T'-t}{T'-T}\right)\right) \\
&\quad +\frac{\sigma^2}{48\eta}\left(-3(T'-T)(T-t) + \frac{1}{2}(T'-t)^2\right).
\end{aligned}
$$

The value function for the non-informed agent is given by $V^N(0,x,s) = -\eta\frac{x^2}{T}$.

Proof The result follows from Theorem 1 and straightforward calculations.

2.2 The expected additional gain

We define the difference $E\left[V^I(0,s,x)\right] - V^N(0,s,x)$ as the *expected additional gain* of the agent having access to the signal. The next result provides an explicit formula for additional gain.

Theorem 3 *Let $T' > T$. The expected additional gain is given by*

$$
E\left[V^I(0,s,x) - V^N(0,s,x)\right] = \frac{\sigma^2}{16\eta}\left((T'-T)^2\ln\left(\frac{T'}{T'-T}\right) - TT' + \frac{3}{2}T^2\right) \tag{12}
$$

The additional gain does *not* depend on the size of the initial position x. This is because a priori the direction of the signal is not clear. The signal can entail a positive or a negative drift. If the agent has a long position in the asset, then a negative drift entails lower expected revenues and a positive drift yields higher expected revenues. In average, this effect cancels out so that the expected additional gain from the signal is independent of the position size.

Observe further that the additional value does also not depend on the price level s. The reason is that in the underlying Bachelier model the size of the price increments do not depend on the starting price.

In the following we denote the expected additional gain by $\Delta(T, T') = E\left[V^I(0,s,x) - V^N(0,s,x)\right]$. Sometimes we write $\Delta(T, T', \sigma, \eta)$ in order to stress its dependence on σ and η.

The following lemma will be necessary to prove Theorem 3.

Lemma 1

$$E\left[\left(\int_0^t \frac{W_T - W_u}{T-u} du\right)^2\right] = 2t(1+\ln(T)) - 2T\ln(T) + 2(T-t)\ln(T-t) \tag{13}$$

Proof The product formula applied to the $(\mathcal{G}_t)$-semimartingales $X_t = \ln(T-t)$ and $Y_t = W_T - W_t$, $t \in [0,T)$, implies

$$-\int_0^t \frac{W_T - W_u}{T-u} du = \ln(T-t)(W_T - W_t) - \ln(T)W_T - \int_0^t \ln(T-u) dW_u, \tag{14}$$

and hence

$$E\left[\left(\int_0^t \frac{W_T - W_u}{T-u} du\right)^2\right] = \ln^2(T-t)(T-t) + \ln^2(T)T^2 + \int_0^t \ln^2(T-u) du$$
$$-2\ln(T-t)\ln(T)(T-t) - 2\ln(T)\int_0^t \ln(T-u) du.$$

A straightforward simplification of the integrals leads to Equation (13).

Proof (Proof of Theorem 3) By Theorem 1 the optimal strategy of the informed agent satisfies

$$\xi_t^* = \frac{X_t^*}{T-t} - \frac{1}{4\eta}\frac{T-t}{T'-t}(S_{T'} - S_t),$$

and the optimal position trajectory is given by

$$X_t^* = \frac{T-t}{T}\left(x + \frac{1}{4\eta} T \int_0^t \frac{S_{T'} - S_u}{T'-u} du\right).$$

The martingale property of the price process implies that the value function satisfies

$$\begin{aligned} &V^I(0,x,s) \\ &= -\eta\frac{x^2}{T} + E\int_0^T \left[\frac{3}{2}X_t^*\frac{S_{T'} - S_t}{T'-t} - \frac{1}{16\eta}\left(\int_0^t \frac{S_{T'} - S_u}{T'-u} du\right)^2 - \frac{1}{16\eta}\frac{(T-t)^2}{(T'-t)^2}(S_{T'} - S_t)^2\right] dt \end{aligned} \tag{15}$$

Observe that

$$\begin{aligned} E(X_t^*(S_{T'} - S_t)) &= (T-t)\frac{1}{4\eta}E\left[(S_{T'} - S_t)\int_0^t \frac{S_{T'} - S_u}{T'-u} du\right] \\ &= (T-t)\frac{1}{4\eta}\int_0^t \frac{E\left[(S_{T'} - S_t)\left((S_{T'} - S_t) + (S_t - S_u)\right)\right]}{T'-u} du \\ &= \frac{\sigma^2}{4\eta}(T-t)(T'-t)\ln\left(\frac{T'}{T'-t}\right). \end{aligned} \tag{16}$$

Moreover, by Lemma 1,

$$E\left[\left(\int_0^t \frac{S_{T'}-S_u}{T'-u}du\right)^2\right] = \sigma^2[2t(1+\ln(T'))-2T'\ln(T')+2(T'-t)\ln(T'-t)] \quad (17)$$

Combining Equation (16) and (17) with (15) yields

$$\begin{aligned} E[V^I(0,x,s)] = &-\eta\frac{x^2}{T}+\frac{3\sigma^2}{8\eta}\int_0^T (T-t)\ln(\frac{T'}{T'-t})dt \\ &-\frac{\sigma^2}{8\eta}\int_0^T [t(1+\ln(T'))-T'\ln(T')+(T'-t)\ln(T'-t)]dt \\ &-\frac{\sigma^2}{16\eta}\int_0^T \frac{(T-t)^2}{(T'-t)}dt. \end{aligned} \quad (18)$$

Notice that

$$\begin{aligned} \int_0^T (T-t)\ln(T'-t)dt = &\frac{1}{2}(T'-T)^2\ln(T'-T)+TT'\ln(T')-\frac{1}{2}(T')^2\ln(T') \\ &+\frac{1}{2}TT'-\frac{3}{4}T^2 \end{aligned}$$

and

$$\begin{aligned} \int_0^T (T'-t)\ln(T'-t)dt = &-\frac{1}{2}(T'-T)^2\ln(T'-T)+\frac{1}{4}(T'-T)^2 \\ &+\frac{1}{2}(T')^2\ln(T')-\frac{1}{4}(T')^2 \end{aligned}$$

and

$$\int_0^T \frac{(T-t)^2}{(T'-t)}dt = (T'-T)^2\ln\left(\frac{T'}{T'-T}\right)-2(T'-T)T+TT'-\frac{1}{2}T^2.$$

A straightforward calculation shows that (18) simplifies to

$$E[V^I(0,x,s)] = -\eta\frac{x^2}{T}+\frac{\sigma^2}{16}\left((T'-T)^2\ln\left(\frac{T'}{T'-T}\right)-TT'+\frac{3}{2}T^2\right).$$

Remark 2 One can alternatively calculate the additional utility by computing the expectation of the coefficients $e^I(0)$, $f^I(0)$ and $g^I(0)$. By simplifying terms one obtains again formula (12).

The expected additional gain $\Delta(T,T')$ converges to a finite value as $T'\downarrow T$. If $T=T'$, then the market would admit arbitrage if there was no price impact. It has been shown that an informed investor can achieve infinite expected utility in a frictionless market (see e.g. [13] and [10]). In our model, in contrast, the price impact excludes arbitrage and implies that the expected additional gain doesn't become infinite when T' is equal to T.

Notice that if we choose $T' = T$, then the drift in the $(\mathcal{G}_t)$-price dynamics (11) is not bounded, and hence the assumptions of Theorem 1 are technically not satisfied. Nevertheless, one can show that the result applies also to this particular case. To this end one needs to make sure that the candidate for the optimal control is admissible.

Proposition 1 *Suppose that $T' = T$. Then the expected additional gain of the informed agent is given by*

$$E[V^I(0,s,x) - V^N(0,s,x)] = \frac{\sigma^2}{32\eta} T^2. \tag{19}$$

The optimal strategy is admissible and the associated position process satisfies $X_t^* = \frac{T-t}{T}\left(x + \frac{1}{4\eta} T \int_0^t \frac{S_T - S_u}{T-u} du\right)$.

Proof The first expression is obtained by taking the limit in $\Delta(T,T')$ as $T' \downarrow T$. To prove the admissibility, notice first that using Equation (14) for any $p > 2$, there exists C_p such that

$$\begin{aligned} &E\left(| \int_0^t \frac{S_T - S_u}{T-u} |^p \right) \\ &\leq C_p \left\{ ln^p(T-t)(T-t)^{p/2} + ln^p(T) T^{p/2} + \left(\int_0^T \ln^2(T-u) du \right)^{p/2} \right\} \end{aligned}$$

This shows that $\left(\int_0^t \frac{S_T - S_u}{T-u} du\right)^2$ is uniformly integrable. We further obtain that $\left(\frac{X_t^*}{T-t}\right)^2$, $0 \leq t \leq T$, is uniformly integrable and $\lim_{t \to T} \frac{X_t^{*\,2}}{T-t} = 0$, a.s. Moreover the process ξ^* is squared integrable.

3 Comparative Statics

We next analyze the impact of the model parameters on the additional gain. We start with the dependence on the signal quality.

3.1 Sensitivity with respect to the signal noise

If the noise of the signal increases, then the additional revenues of the informed agent decrease. This is indeed confirmed by the next result.

Lemma 2 *The expected revenues from additional information decrease as T' increases, i.e. the mapping $f(x) = (x-T)^2 \ln\left(\frac{x}{x-T}\right) - Tx + \frac{3}{2}T^2$ is decreasing on $[T, \infty)$. Moreover, $f(T) = \frac{1}{2}T^2$ and $\lim_{x\to\infty} f(x) = 0$.*

Proof Notice that $f'(x) = 2(x-T)\ln\left(\frac{x}{x-T}\right) + \frac{T^2}{x} - 2T$ and

$$f''(x) = -2\ln\left(1 - \frac{T}{x}\right) - \frac{2T}{x} - \frac{T^2}{x^2}.$$

Since the logarithm is analytic on the open interval $(0,2)$, we further have for $x > T$

$$f''(x) = 2\left(\frac{T}{x} + \frac{1}{2}\frac{T^2}{x^2} + \frac{1}{3}\frac{T^3}{x^3} + \cdots\right) - \frac{2T}{x} - \frac{T^2}{x^2} = \left(\frac{1}{3}\frac{T^3}{x^3} + \cdots\right) \geq 0.$$

Consequently f' is increasing on $[T,\infty)$. Besides observe that $f'(T) = -T$, and

$$\lim_{x\to\infty} f'(x) = 2(x-T)\left(\frac{T}{x} + \frac{1}{2}\frac{T^2}{x^2} + \frac{1}{3}\frac{T^3}{x^3} + \cdots\right) - \frac{2T}{x} - \frac{T^2}{x^2} = 0,$$

which, together with the monotonicity of f', implies $f' \leq 0$ on $[T,\infty)$. The function f, therefore, is decreasing in x.

3.2 Sensitivity with respect to the time horizon

The additional gain increases when the time horizon T increases, while T' stays constant. Indeed, a straightforward computation shows that

$$\frac{\partial^2 \Delta}{\partial T^2} = 2\ln\left(\frac{T'}{T'-T}\right) \geq 0 \text{ for all } T \in [0,T'].$$

Hence the first derivative is increasing. Since $\frac{\partial \Delta}{\partial T}(0,T') = 0$, the first derivative is non-negative and hence Δ is increasing in T.

The increase in expected revenues has three reasons: first the signal becomes more valuable as the difference between T and T' decreases (**information effect**); second there is more time for spreading orders over time and hence one can reduce trading costs (**liquidity effect**); finally the variance of the price over the trading period increases (**variance effect**).

We next aim at analyzing the three effects separately. The additional revenues depend linearly on the volatility squared. We can thus eliminate the variance effect by making σ^2 inversely proportional to T. We define the variance corrected gain by

$$l(T,x) = \Delta(T,x,\sigma/\sqrt{T},\eta) = \frac{\sigma^2}{16\eta}\left(\frac{(x-T)^2}{T}\ln\left(\frac{x}{x-T}\right) - x + \frac{3}{2}T\right),$$

for $0 \leq T \leq x$.

We next aim at analyzing the part of revenue increase that goes back to the liquidity effect. To this end we simultaneously change T and T' such that the information content of the signal remains the same. We appeal to the notion of *mutual information* for measuring the information content of the signal.

Recall that the mutual information between two normally distributed random variables X and Y is given by $I(X,Y) = -\frac{1}{2}\ln(1-\text{corr}^2(X,Y))$ (see e.g. [12]). In particular, for any $\delta > 0$ we have $I(S_T, S_{T+\delta}) = \frac{1}{2}\ln\left(\frac{T+\delta}{T}\right)$.

For $\gamma > 0$ the mutual information $I(S_T, S_{(\gamma+1)T}) = \frac{1}{2}\ln(1+\gamma)$ does not depend on the time horizon T. We can thus interpret

$$h(T) = l(T, (1+\gamma)T)$$

as a variance *and* information (v&i) corrected gain function. The next proposition shows that the v&i corrected gain increases as the time horizon increases. The reason is that the additional time for trading allows to reduce liquidity costs and to make more use of the information advantage.

Proposition 2 (The liquidity effect) *Let $\gamma > 0$. The v&i corrected gain function h is linear, increasing and satisfies $h(0) = 0$.*

Proof Note that

$$\begin{aligned}\frac{16\eta}{\sigma^2}h(T) &= \gamma^2 T\ln\left(\frac{1+\gamma}{\gamma}\right) - \gamma T + \frac{1}{2}T \\ &= \left[\gamma^2\left(\frac{1}{\gamma} - \frac{1}{2}\frac{1}{\gamma^2} + \frac{1}{3}\frac{1}{\gamma^3} - \frac{1}{4}\frac{1}{\gamma^4} + \cdots\right) - \gamma + \frac{1}{2}\right]T \\ &= \gamma^2\left(\frac{1}{3}\frac{1}{\gamma^3} - \frac{1}{4}\frac{1}{\gamma^4} + \cdots\right)T,\end{aligned}$$

which shows that h is non-negative and linearly increasing in T.

Finally we turn to the information effect. By scaling the volatility with $1/\sqrt{T}$ and the price impact parameter with T, we obtain a variance and liquidity (v&l) corrected gain function

$$k(y) = \Delta(y, T', \sigma/\sqrt{y}, \eta y),$$

defined for all $y \in [0, T']$. The function k describes the gain that exclusively goes back to the additional information, as T approaches T'. It is, as expected, increasing:

Proposition 3 (The information effect) *The v&l corrected gain function k increases superlinearly on $[0, T']$ and satisfies $\lim_{x\uparrow T'} k(x) = \frac{1}{2}\frac{\sigma^2}{16\eta}$.*

Proof $k''(x) = 2T'\frac{3T'-2x}{x^4}\ln\left(\frac{T'}{T'-x}\right) - \frac{6T'}{x^3} + \frac{1}{x^2}$ is positive and $k'(0) \le 0$, which implies the first statement. The second is straightforward to show.

Conclusion

The paper studies the optimal liquidation problem under additional information. The kind of additional information chosen here is modeled via an initial enlargement of filtration, sometimes referred to as *strong initial information* (see for example [3], [10], or [9] for an introduction into the subject. See also [11] and [6] for a presentation of other types of additional information). Since the price dynamics under the enlargement have a drift that is linear with respect to the price, we obtain the optimal liquidation strategy in closed form. For any kind of additional information resp. filtration enlargement under which the drift is linear, one can derive explicitly the additional gain by using Theorem 1. For example the additional information studied in the paper of Corcuera et al. [8] (strong noisy information, represented by a signal plus a decreasing noise) leads to a linear drift, and hence fits to our model.
Acknowledgement The authors thank anonymous referees for very useful comments.

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Appendix

The coefficients of the value function in Theorem 1 are given by

$$\begin{aligned}
b(t) &= -\eta\frac{1}{T-t},\\
c(t) &= \frac{T}{T-t}h^{-1}(0,t)\int_t^T \beta(u)\frac{T-u}{T}h(0,u)du,\\
d^{\mathrm{hom}}(t) &= \exp\left(-\int_0^t 2\beta(u)du\right)\\
d(t) &= d^{\mathrm{hom}}(t)\int_t^T \frac{c^2(u)}{4\eta d^{\mathrm{hom}}(u)}du\\
e(t) &= \frac{T}{T-t}\int_t^T (c(u)+1)\alpha(u)\frac{T-u}{T}du,\\
f^{\mathrm{hom}}(t) &= \exp\left(-\int_0^t \beta(u)du\right)\\
f(t) &= f^{\mathrm{hom}}(t)\int_t^T \left(\frac{1}{2\eta}c(u)e(u)+2\alpha(u)d(u)\right)\frac{1}{f^{\mathrm{hom}}(u)}du\\
g(t) &= \int_t^T \left(\frac{e^2(u)}{4\eta}+\alpha(u)f+\sigma^2 d(u)\right)du,
\end{aligned}$$

with $t \in [0,T]$. We remark that Theorem 1 can be derived from Theorem 2 in [15]. The proof given below is completely different though, using classical verification arguments.

Proof (Proof of Theorem 1) Let $w(t,x,s) = b(t)x^2 + c(t)xs + d(t)s^2 + e(t)x + f(t)s + g(t)$. We first show that the value function satisfies $V \le w$. Notice that w is a solution of the HJB Equation (4) and satisfies the terminal condition (5). This follows from the fact that the coefficients satisfy the following ODEs

$$\begin{aligned}
-b_t - \frac{1}{\eta}b^2 &= 0\\
-c_t - \frac{1}{\eta}bc - \beta c - \beta &= 0\\
-d_t - \frac{1}{4\eta}c^2 - 2\beta d &= 0\\
-e_t - \frac{1}{\eta}be - \alpha c - \alpha &= 0\\
-f_t - \frac{1}{2\eta}ce - 2\alpha d - \beta f &= 0\\
-g_t - \frac{1}{4\eta}e^2 - \alpha f - \sigma^2 d &= 0.
\end{aligned}$$

Since the functions α and β are bounded, there exists a constant $C \in \mathbb{R}_+$ such that

$$|c(t)| + |d(t)| + |e(t)| + |f(t)| + |g(t)| \le C(T-t) \tag{20}$$

for all $t \in [0,T]$. Moreover, we have $|b(t)| \le C\frac{1}{T-t}$.

Let $\xi \in \mathcal{A}(t,x)$ be an arbitrary admissible control and let X be its associated position process. Let $\tau < T$. Itô's formula implies

$$\begin{aligned}
w(\tau, X_\tau, S_\tau) = w(t,x,s) &+ \int_t^\tau \frac{1}{2}\sigma^2 w_{ss}(u,X_u,S_u)du + M_\tau\\
&+ \int_t^\tau [w_t(u,X_u,S_u) - w_x(u,X_u,S_u)\xi_u + a(u,S_u)w_s(u,X_u,S_u)]du,
\end{aligned}$$

where $M_s = \int_t^s w_s(u,X_u,S_u)\sigma dW_u$. As $(X_t)_{t\in[0,\tau]}$ is L^2-bounded and all functions b,c,d,e,f,g and their derivatives are bounded on $[t,\tau]$, M is a strict martingale on $[t,\tau]$. Taking expectations, therefore, leads to

$$\begin{aligned}
E(w(\tau,X_\tau,S_\tau)) &= w(t,x,s) + E\left(\int_t^\tau \left(w_t - w_x\xi + aw_s + \frac{1}{2}\sigma^2 w_{ss}\right)(u,X_u,S_u)du\right)\\
&\le w(t,x,s) + E\left(\int_t^\tau (-a(u,S_u)X_u + \eta\xi_u^2)du\right).
\end{aligned} \tag{21}$$

As ξ is square integrable (Condition (A1)). This further implies that we have

$$\lim_{\tau\to T} E\left(\int_t^\tau (-a(u,S_u)X_u + \eta\xi_u^2)du\right) = J(t,x,s,\xi).$$

Moreover, since also $\left(\frac{X_t^2}{T-t}\right)_{t\in[0,T)}$ is uniformly integrable and $\lim_{t\to T}\frac{X_t^2}{T-t} = 0$, we have $\lim_{\tau\to T} E[w(\tau,X_\tau,S_\tau)] = 0$. Inequality (21), therefore, implies

$w(t,x,s) \geq J(t,x,s,\xi)$. Taking the supremum over all admissible controls, one has $V(t,x,s) \leq w(t,x,s)$.

Secondly, we show that the control $(\xi_t^*)_{t\in[0,T]}$ is admissible. Using the majoration (20) on the coefficients c, b and e, one can show that there exists a constant C such that

$$|[c(u)S_u + e(u)]| \frac{T}{(T-u)} \leq C(|S_u| + 1)$$

for all $u \in [0,T]$. With (8) we obtain that $|X_t^*| \leq C(T-t)(1 + \int_0^t |S_u|\, du)$ and hence Condition (A2) is satisfied.
Condition (A1) is a consequence of $\xi_t^2 \leq C(b(t)^2 X_t^2 + b(t)X_t) \leq C$.
Equality holds in Inequality (21) by choosing $\xi = \xi^*$. This proves that $J(t,x,s,\xi^*) = w(t,s,x)$. Thus the proof is complete.