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***Abstract*— In this paper, a hybrid controller design for a continuous-time linear time-invariant (LTI) plant is presented. The idea is to simultaneously design the flow and jump maps with the respective sets of the controller, guaranteeing $\mathcal{H}_\infty$ specifications and decay rate of the plant state of the hybrid closed-loop system. A convex LMI-based design procedure is proposed, generalizing the results in [22].**

***Index Terms*—Hybrid controller synthesis, exponential stability, convergence rate, $\mathcal{H}_\infty$ specification.**

I. INTRODUCTION

Hybrid control theory is being developed to provide more flexible stabilizing and performing controllers, overcoming some intrinsic limitations of the classical theory. Within the general context of hybrid systems, much attention has been devoted in recent years to the study of hybrid (or reset) controllers for improved performance with continuous-time plants. In [1], [2], [15] promising performance analysis for some specific hybrid systems have been presented with respect to rise-time, overshoot, settling-time. In [17] it is shown that the desirable closed-loop behavior may be induced by resetting the controller according to an optimal reset law. Also in [19] optimal techniques for overshoot reduction and maximization of the decay rate have been presented. Both in [17] and [19] (see also [2], [5], [6], [11], [18]) the hybrid part with a suitable reset law is added to a preexisting controller (not necessarily stabilizing in some cases) to enhance the entire closed-loop system.

In this paper, we present an optimal technique to design a hybrid controller, that is, both continuous and discrete parts (including also the flow and jump sets) are simultaneously designed according to an optimal criterion. We point out that, in general, the optimal design of a hybrid controller cannot be achieved by separately designing the continuous and discrete parts according to the desired cost functions and combining the two parts a posteriori, because the obtained hybrid dynamics can negatively affect the optimal indexes and may also lead to instability (see [10], [13]).

The hybrid controller synthesis we propose is multiobjective oriented because we want to overcome some limitations due to the continuous and/or linear theory in the multiobjective domain (see [11] for a discussion on the trade-offs of a single linear controller used for multiple performance purposes). On the other hand, in [12], [20] it has been proved that for LTI plants, there exists no nonlinear (possibly time-varying) controller that yields an $\mathcal{L}_2$ gain lower than the one associated to the optimal linear controller. The main idea of this paper is to carry out with a multiobjective synthesis to achieve a desired convergence rate and $\mathcal{H}_\infty$ specifications through a hybrid controller. It will be shown that, given a desired (and achievable) $\mathcal{L}_2$ gain, the hybrid controller guarantees a convergence rate higher than or equal to the optimal one induced by a convex linear controller design, thus reaching a better trade-off between the two required specifications.

In the current literature, [21] needs to be mentioned where a line search parameter is used to design a linear dynamical feedback controller and a resetting rule satisfying some $\mathcal{H}_\infty$ specification.

Finally, we point out that the results in this paper can be extended to the output feedback case using the construction presented in [6]. For a comprehensive overview of the hybrid systems framework that we use here, the reader can consult [8], [9].

The paper is organized as follows. In Section II the considered plant and the problem we want to solve are defined. In Section III the main results are presented followed by some remarks. In Section IV, we compare our performance to the one obtained with the linear solution. Section V contains some simulations to show the advantages of our technique. Section VI concludes the paper. The proofs of the results are omitted.

Notation. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers. The Euclidean norm is denoted by $|\cdot|$. For a matrix M, $\mathrm{He}(M) = M + M^T$. For any $s \in \mathbb{R}$, the function $\mathrm{dz} : \mathbb{R} \to \mathbb{R}$ is defined by $\mathrm{dz}(s) = 0$ if $|s| \leq 1$ and $\mathrm{dz}(s) = \mathrm{sgn}(s)(|s| - 1)$ if $|s| \geq 1$. Given a matrix Q, $\lambda_{min}(Q)$ (resp. $\lambda_{max}(Q)$) denotes the minimum (resp. the maximum) eigenvalue of Q.

II. PROBLEM STATEMENT

We consider a LTI plant $\mathcal{P}$, represented by

$$\mathcal{P} \left\{ \begin{array}{rcl} \dot{x}_p & = & \bar{A}_p x_p + \bar{B}_p u + \bar{B}_w w \\ z_p & = & \bar{C}_z x_p + \bar{D}_z u + \bar{D}_{zw} w \\ y_p & = & \bar{C}_p x_p + \bar{D}_p u + \bar{D}_w w \end{array} \right. \tag{1}$$

where $x_p \in \mathbb{R}^{n_p}$ is the state of the system, $u \in \mathbb{R}^p$ is the control input, $y_p \in \mathbb{R}^q$ is the measured output (used for the feedback), $w \in \mathbb{R}^r$ is an exogenous input (like disturbances, references) and $z_p \in \mathbb{R}^\nu$ is the performance output.

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Now let us introduce the following assumption to simplify the technique we want to present.

Assumption 1: The plant (1) has $\bar{D}_p = 0$. ◦

Note that Assumption 1 is not very restrictive. In case the system $\mathcal{P}$ has $\bar{D}_p \neq 0$, we can always define $\bar{y}_p := y_p - \bar{D}_p u$ and use $\bar{y}_p$ as new input for the hybrid controller (2).

Let us introduce our hybrid controller architecture

$$\mathcal{C}\begin{cases} \dot{x}_c = \bar{A}_c x_c + \bar{B}_c y_p & \text{if } (x_p,\, x_c) \in \mathcal{F} \text{ or} \\ \dot{\tau} = 1 - \mathrm{dz}\left(\frac{\tau}{\rho}\right) & \qquad \tau \in [0,\, \rho] \\ x_c^+ = K_p x_p & \text{if } (x_p,\, x_c) \in \mathcal{J} \text{ and} \\ \tau^+ = 0 & \qquad \tau \in [\rho,\, 2\rho] \\ u = \bar{C}_c x_c + \bar{D}_c y_p & \end{cases} \tag{2}$$

with $x_c \in \mathbb{R}^{n_c}$, $\tau \in \mathbb{R}$ is the dwell-time logic (depending on the parameter ρ to be selected and on the standard unit deadzone function $\mathrm{dz}(\cdot)$) and $\mathcal{F}$ and $\mathcal{J}$ are the flow and jump sets, respectively, defined as

$$\mathcal{F} = \left\{ \begin{bmatrix} x_p \\ x_c \end{bmatrix} : \begin{bmatrix} x_p \\ x_c \end{bmatrix}^T N_p \begin{bmatrix} x_p \\ x_c \end{bmatrix} \leq -\tilde{\alpha} x_p^T \bar{P}_p x_p - \epsilon |x_c|^2 \right\}, \tag{3a}$$

$$\mathcal{J} = \left\{ \begin{bmatrix} x_p \\ x_c \end{bmatrix} : \begin{bmatrix} x_p \\ x_c \end{bmatrix}^T N_p \begin{bmatrix} x_p \\ x_c \end{bmatrix} \geq -\tilde{\alpha} x_p^T \bar{P}_p x_p - \epsilon |x_c|^2 \right\}, \tag{3b}$$

with

$$N_p := \mathrm{He}\left(\begin{bmatrix} \bar{P}_p A_p & \bar{P}_p B_p \\ 0 & 0 \end{bmatrix} \right). \tag{3c}$$

The parameters of the hybrid controller (2), (3) correspond to $\bar{A}_c$, $\bar{B}_c$, $\bar{C}_c$, $\bar{D}_c$, K_p, $\bar{P}_p = \bar{P}_p^T > 0$, $\tilde{\alpha}$, ϵ and ρ and will be designed in this paper.

Connecting in feedback $\mathcal{C}$ and $\mathcal{P}$ is always possible since Assumption 1 implies well-posedness in the linear sense. Thus, we obtain the hybrid closed-loop system

$$\begin{cases} \begin{bmatrix} \dot{x}_p \\ \dot{x}_c \end{bmatrix} = \mathcal{A}x + \mathcal{B}w & \text{if } x \in \mathcal{F} \text{ or } \tau \in [0,\, \rho] \\ \quad \dot{\tau} = 1 - \mathrm{dz}\left(\frac{\tau}{\rho}\right) & \\ \begin{bmatrix} x_p^+ \\ x_c^+ \end{bmatrix} = \mathcal{A}_r x & \text{if } x \in \mathcal{J} \text{ and } \tau \in [\rho,\, 2\rho] \\ \quad \tau^+ = 0 & \\ z_p = \mathcal{C}x + \mathcal{D}w & \\ y_p = \bar{C}_p x_p + \bar{D}_w w & \end{cases} \tag{4}$$

with $x = [x_p^T\; x_c^T]^T \in \mathbb{R}^n$, $\mathcal{A}_r := \left[\begin{smallmatrix} I & 0 \\ K_p & 0 \end{smallmatrix}\right]$ and

$$\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right) := \left(\begin{array}{cc|c} A_p & B_p & B_{pw} \\ B_c & A_c & B_{cw} \\ \hline \multicolumn{2}{c|}{\mathcal{C}} & \mathcal{D} \end{array}\right)$$

$$:= \left(\begin{array}{cc|c} \bar{A}_p + \bar{B}_p \bar{D}_c \bar{C}_p & \bar{B}_p \bar{C}_c & \bar{B}_w + \bar{B}_p \bar{D}_c \bar{D}_w \\ \bar{B}_c \bar{C}_p & \bar{A}_c & \bar{B}_c \bar{D}_w \\ \hline \bar{C}_z + \bar{D}_z \bar{D}_c \bar{C}_p & \bar{D}_z \bar{C}_c & \bar{D}_{zw} + \bar{D}_z \bar{D}_c \bar{D}_w \end{array}\right).$$

Note that the architecture of the hybrid controller (2) with the flow and jump sets (3) corresponds to the one presented in [5].

System (3), (4) is a hybrid system with inputs which, following the works in [3], [23], is suitably described by ensuring that the hybrid time domain of the input w, state (x_p, x_c, τ) and outputs, all coincide. Therefore when characterizing the $\mathcal{L}_2$ norm of a solution pair (w, x) or of an output, one should use sums and integrals (see also the recent work [14]). Here we take a different route because we are focusing on the ordinary time response of the plant $\mathcal{P}$, which is not hybrid, and we focus on the ordinary-time $\mathcal{L}_2$ norm[1] (or t-$\mathcal{L}_2$ norm for short):

$$\|\xi\|_{2t} = \left(\int_0^\infty |\xi(t,j)|^2 dt \right)^{\frac{1}{2}}, \tag{5}$$

which is well defined due to the presence of the dwell-time logic τ ensuring that all the solutions have unbounded time domain unbounded in the ordinary time direction. A similar approach has been used in [7], [16]. Note that using the norm (5) will enable us to carry out useful comparison to linear controllers inducing an $\mathcal{H}_\infty$ specification on the continuous-time $\mathcal{L}_2$ norm of the plant output. Based on (5), we will denote by t-$\mathcal{L}_2$ gain of (3), (4) from w to z_p the worst case ratio between $\|z_p\|_{2t}$ and $\|w\|_{2t}$ over all w such that $\|w\|_{2t} \neq 0$ whenever (3), (4) starts from zero initial conditions.

While the above commented tools will be used to characterize the external performance of our hybrid closed loop (its response to "external" perturbations), the internal property will be assessed establishing an exponential bound on the trajectories of (3), (4) which only involves the ordinary time. In particular, we will say that (3), (4) with $w = 0$ has t-decay rate α if there exists $M_x > 0$ such that for all initial conditions $(x(0,0), \tau(0,0))$, one has

$$|x(t,j)| \leq M_x \exp(-\alpha t)|x(0,0)|, \quad \forall (t,j) \in \mathrm{dom}(x). \tag{6}$$

This notion of ordinary-time exponential decay will allow us to perform comparisons with the (continuous-time) exponential decay induced by the standard $\mathcal{H}_\infty$ controller. Note that due to the presence of the dwell-time logic τ, ensuring $\rho + t - s \geq \rho(j - k)$ for any pair of hybrid times (t, j), $(s, k) \in \mathrm{dom}(x)$, $(t, j) \geq (s, k)$, the t-decay rate property (6) ensures uniform exponential stability of the x component of (3), (4) in the hybrid sense (see [24]). Nevertheless, we use (6) in our statement because we are actually interested in establishing a (tight) exponential bound for our solution which only involves the ordinary time t. Based on the above observations, the problem that we address in this paper is the following one:

Problem: *Consider the plant $\mathcal{P}$ in (1) under Assumption 1. Design the matrices $\bar{A}_c$, $\bar{B}_c$, $\bar{C}_c$, $\bar{D}_c$, K_p, $\bar{P}_p$, and the positive scalars $\tilde{\alpha}$, ρ and ϵ such that*

i. t-**Decay rate:** *with $w = 0$ and for any initial condition, global exponential stability of the hybrid closed-loop*

[1] To be precise the function in (5) is not a norm because, for example, a solution ξ starting at a nonzero position and jumping in zero at $(t, j) = (0, 0)$ would satisfy $\|\xi\|_{2t} = 0$ (this is not the case for the norms in [3], [14]). Nevertheless we call it norm through the paper due to the intuition that it generalizes the continuous-time norm.

$$\Sigma := \left[\begin{array}{c|c} \Sigma_1 & \Sigma_2 \\ \hline \Sigma_2^T & \Sigma_3 \end{array}\right] := \mathrm{He}\left(\left[\begin{array}{cc|cc} \bar{A}_p Y + \bar{B}_p \hat{C} & \bar{A}_p + \bar{B}_p \hat{D} \bar{C}_p & \bar{B}_w + \bar{B}_p \hat{D} \bar{D}_w & Y \bar{C}_z^T + \hat{C}^T \bar{D}_z^T \\ \hat{A} & W \bar{A}_p + \hat{B} \bar{C}_p & W \bar{B}_w + \hat{B} \bar{D}_w & \bar{C}_z^T + \bar{C}_p^T \hat{D}^T \bar{D}_z^T \\ \hline 0 & 0 & -\frac{\gamma}{2} I & \bar{D}_{zw}^T + \bar{D}_w^T \hat{D}^T \bar{D}_z^T \\ 0 & 0 & 0 & -\frac{\gamma}{2} I \end{array}\right]\right) \tag{7}$$

system (4) (in the sense of [24]) with a t-decay rate $\tilde{\alpha}$ for the x_p component of the solution is ensured;

ii. $\mathcal{H}_\infty$ **specification:** *given any $w \in t\text{-}\mathcal{L}_2$, the t-$\mathcal{L}_2$ gain from w to z_p is less than γ for all initial conditions satisfying* $x(0,0) = 0$.

III. MAIN AND PRELIMINARY RESULTS

A. A preliminary result

Let us first state the lemma given below, which has some interest of its own. The result is an extension of the main result in [19, Theorem 2], which shows that an arbitrarily small twist of the flow and jump sets of [19, Theorem 2] (the $\epsilon|x_c|^2$ term in (3)) is sufficient to obtain a strict Lyapunov function (instead, non-strict ones and a LaSalle type reasoning were required in [19, Theorem 2]). The lemma below also illustrates how the solution proposed in this paper does not require the dwell-time logic in the absence of disturbances and exhibits trivial (that is, at the origin) Zeno solutions. When looking at t-$\mathcal{L}_2$ norms and nonzero w, dwell time is needed to ensure that all hybrid time domains are unbounded in the ordinary time direction t.

In the next statement we use *hybrid controller (2), (3) without dwell time* to denote dynamics (2), (3) without the state τ and where the conditions involving τ are removed from the flow/jump rules (this can be interpreted by selecting $\rho = \infty$ and disregarding the τ subcomponent of the solution). Similarly for (3), (4).

Lemma 1: Given the plant $\mathcal{P}$ in (1) with $w = 0$. If there exist matrices $\bar{P}_p = \bar{P}_p^T > 0$, and $K_p \in \mathbb{R}^{n_p \times n_c}$, and positive scalars α and ϵ such that

$$\mathrm{He}(\bar{P}_p(A_p + B_p K_p)) < -\alpha \bar{P}_p - \epsilon K_p^T K_p, \tag{8}$$

then the *hybrid controller (2), (3) without dwell-time* τ and with $0 < \tilde{\alpha} \le \alpha$, guarantees global exponential stability of the origin in the sense of [24] and t-decay rate $\tilde{\alpha}/2$ for the x_p component of each solution to the *hybrid closed-loop system (3), (4) without dwell-time* τ. ◊

Lemma 1 provides sufficient conditions for global exponential stability of the origin for the hybrid controller (2) with flow and jump sets (3) and without dwell-time τ. Note that in [5] the same architecture is shown to guarantee global exponential stability of the origin relying on a dwell-time, which is unnecessary here for this purpose. Moreover, this lemma provides an estimate of the t-decay rate.

Note that for the simple exponential stability objective we might just use Lemma 1. Since (8) involves a quadratic term in the unknown K_p, one way to design the controller is first to select $\epsilon = 0$, to solve (8) like in [5], [6] and once $\bar{P}_p$ and K_p are obtained, since inequality (8) is strict, it is always possible to find a small enough $\epsilon > 0$ that satisfies (8).

B. Main result

Theorem 1: Given the plant (1) under Assumption 1, assume that there exist $Y = Y^T \in \mathbb{R}^{n_p \times n_p}$, $W = W^T \in \mathbb{R}^{n_p \times n_p}$, $\hat{A} \in \mathbb{R}^{n_p \times n_p}$, $\hat{B} \in \mathbb{R}^{n_p \times q}$, $\hat{C} \in \mathbb{R}^{p \times n_p}$, $\hat{D} \in \mathbb{R}^{p \times q}$ and positive scalars γ and α such that

$$\begin{bmatrix} Y & I \\ I & W \end{bmatrix} > 0, \tag{9a}$$

$$\Sigma < 0, \tag{9b}$$

$$\mathrm{He}\left(\bar{A}_p Y + \bar{B}_p \hat{C}\right) < -\alpha Y. \tag{9c}$$

Based on any solution to (9), define

$$\begin{aligned} \bar{D}_c &= \hat{D}, \\ \bar{C}_c &= (\hat{C} - \bar{D}_c \bar{C}_p Y)(Y - W^{-1})^{-1}, \\ \bar{B}_c &= -W^{-1}\hat{B} + \bar{B}_p \bar{D}_c, \\ \bar{A}_c &= -W^{-1}(\hat{A} + W\bar{B}_c \bar{C}_p Y - W\bar{B}_p \bar{C}_c (Y - W^{-1}) \\ &\qquad - W(\bar{A}_p + \bar{B}_p \bar{D}_c \bar{C}_p) Y)(Y - W^{-1})^{-1}, \\ \bar{P}_p &= Y^{-1}, \\ K_p &= (Y - W^{-1})Y^{-1}. \end{aligned} \tag{10}$$

Then, for each $\tilde{\alpha}$ satisfying $0 < \tilde{\alpha} \le \alpha$, there exists $\epsilon > 0$ such that

$$\mathrm{He}(\bar{P}_p(A_p + B_p K_p)) < -\tilde{\alpha}\bar{P}_p - \epsilon K_p^T K_p. \tag{11}$$

Moreover, for each $\epsilon > 0$ satisfying (11), there exists a $\bar{\rho} > 0$ such that for any $\rho \in (0, \bar{\rho})$ the hybrid controller (2) with the flow and jump sets in (3) guarantees global exponential stability of the origin for the x_p component of the hybrid closed-loop system (3), (4) with t-decay rate $\tilde{\alpha}/2$ and t-$\mathcal{L}_2$ gain smaller than γ. □

Theorem 1 gives an LMI-based convex procedure to design a hybrid controller, of the same order as the plant $\mathcal{P}$, solving the problem at the end of Section II. In this paper we only consider the synthesis of a plant-order optimal controller, which implies $x \in \mathbb{R}^{2n_p}$ in (4). The (α, γ) trade-off in our design can be addressed either by fixing α and solving an eigenvalue problem minimizing γ because constraints (9) are linear in the unknown variables after α has been fixed, or fixing γ and solving the generalized eigenvalue problem arising from (9) (in particular, (9c)).

Inequalities (9a) and (9b) imply the existence of a matrix $P = \left[\begin{smallmatrix} Y & Z \\ Z & Z \end{smallmatrix}\right]^{-1} = P^T > 0$. More specifically, defining $\bar{V}(x) = x^T P x$, it turns out that $\bar{V}$ can be used as a disturbance attenuation Lyapunov function which does not increase at jumps, thus providing the t-$\mathcal{L}_2$ gain of the statement. On the other hand, the hybrid controller (2) has the flow and jump sets (3) based on a Lyapunov-like function $V_p(x_p) := x_p^T \bar{P}_p x_p$ (see also Lemma 1) that under condition (11) guarantees global exponential stability of the hybrid closed-loop system with t-decay rate $\tilde{\alpha}/2$. The main idea behind our construction is to define a reset map able to overlap these two functions without affecting each performance property. The next remark gives further details on this topic.

Remark 1: Under the hypotheses of Theorem 1 and with relation to the design problem at the end of Section II, the two functions $\bar{V}(x)$ and $V_p(x_p)$ mentioned above are such that

- $\bar{V}$ guarantees the $\mathcal{H}_\infty$ specification, arising from the continuous (flow) dynamics of the closed loop;
- V_p guarantees the t-decay rate, by enforcing a jump (from the definition of flow and jump sets) whenever the decay rate condition would be violated;
- $\bar{V}$ and V_p do not increase across jumps;
- $\bar{V}$ and V_p match after each jump (namely $\bar{V}(x^+) = V_p(x_p)$);
- after each jump both functions share the same dynamics (namely $\dot{\bar{V}}(x^+) = \dot{V}_p(x_p)$).

Using the properties above, through the resets we can keep all the trajectories in the region where V_p, therefore $|x_p|$, decreases at the desired rate. At the same time, we can integrate $\bar{V}$ along flows and, since $\bar{V}$ does not increase at jumps, we can add all these integrals to obtain the $\mathcal{H}_\infty$ specification. Note that we are not claiming to use a different Lyapunov function for each objective, and the conservativeness discussed in [22, §IV.A] still holds. However, since the controller state can be reset (this is an extra degree of freedom), the flow and jump sets (affecting the decay rate) can be designed based on the Lyapunov-like function V_p that privileges the decrease in the x_p-direction. Moreover, such a function is built from the function $\bar{V}$ and shares with it some properties as stated above. As a final remark, we should mention that an important degree of freedom is obtained by the fact that we only require the t-decay rate property for the x_p substate (whereas with linear techniques, one would need to focus on the whole state (x_p, x_c)). ⋆

Remark 2: The flow and jump sets (3) depend on the selection of $\tilde{\alpha}$ and ϵ. Note that, as $\tilde{\alpha}$ tends to α, the flow set shrinks and the controller is expected to jump more often. The smallest flow set is obtained for $\tilde{\alpha} = \alpha$. It should be also emphasized that increased values of $\tilde{\alpha}$ are expected to produce smaller values of $\bar{\rho}$, because requiring a faster convergence rate, in general, would reduce the left margin by our inequalities for tolerating the perturbations arising from the dwell-time mechanism. ⋆

IV. COMPARISON TO LINEAR PERFORMANCE

In this section we present the equivalent multiobjective technique for the linear case (see [22], [4]) and then a lemma stating the expected performance difference, concerning the decay rate, between the linear and hybrid approaches.

First let us consider the continuous-time part of controller (2), that is $\bar{A}_c$, $\bar{B}_c$, $\bar{C}_c$ and $\bar{D}_c$ without dwell time. The reader is referred to [22, Theorem 2] and [4] for what is next. Since the quadratic Lyapunov function in the transformed coordinates of Σ is a quadratic form with the matrix $P = \left[\begin{smallmatrix} Y & I \\ I & W \end{smallmatrix}\right]$, then the multiobjective synthesis (optimal with respect to the $\mathcal{L}_2$ gain and decay rate α_L) for the linear (and continuous-time) case is given by solving (9a), (9b) and

$$\Sigma_1 < -\alpha_L \begin{bmatrix} Y & I \\ I & W \end{bmatrix}, \tag{12}$$

where Σ_1 is defined in (7), and computing the linear controller $(\bar{A}_c, \bar{B}_c, \bar{C}_c, \bar{D}_c)$ by using (10).

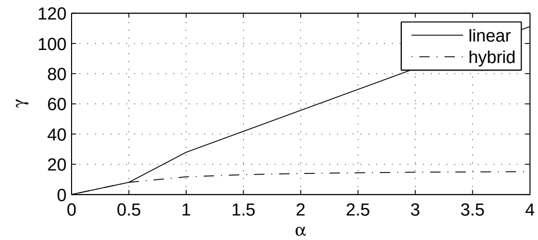

Fig. 1. Trend of γ and α for the linear and hybrid case.

Note that inequality (12) is more restrictive than (9c) and allows us to conclude that the guaranteed decay rate by the continuous-time linear design is $\alpha_L/2$. In the sequel we will use α_L to denote the decay rate for the linear case and to distinguish it from the hybrid decay rate α used in the previous section. With this notation, we can state the following lemma.

Lemma 2: Given the plant $\mathcal{P}$ in (1) under Assumption 1, partition matrix Σ_1 in (7) as $\Sigma_1 := \left[\begin{smallmatrix} \bar{\Sigma}_{11} & \bar{\Sigma}_{12} \\ \bar{\Sigma}_{12}^T & \bar{\Sigma}_{13} \end{smallmatrix}\right]$. If there exists a $\gamma > 0$ such that conditions (9a), (9b) and (12) are satisfied by suitable matrices $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$, Y, W and a scalar α_L, then there exist an $\alpha > \alpha_L$ satisfying (9c). More specifically, given the strictly positive scalar

$$\hat{\tilde{\alpha}} := \lambda_{min}(-\Gamma(\bar{\Sigma}_{13} + \alpha_L W)^{-1}\Gamma^T), \tag{13}$$

with $\Gamma := Y^{-\frac{1}{2}}(\bar{\Sigma}_{12} + \alpha_L I)$, conditions (9) hold with the same solution $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$, Y, W and $\alpha = \alpha_L + \hat{\tilde{\alpha}}$. ◊

Lemma 2 establishes that any solution to the linear design problem (9a), (9b), (12) is also a solution of our construction with a strictly larger decay rate, the gap being at least the quantity $\hat{\tilde{\alpha}} > 0$ defined in (13).

V. SIMULATIONS

A comparison between the linear and hybrid case is presented. The controller in the linear case is designed combining the $\mathcal{H}_\infty$ synthesis and the regional pole placement presented in Section IV (see also [22], [4]). The hybrid controller is obtained with the technique in Theorem 1. Both design syntheses are obtained by fixing α and α_L such a way to cope only with LMI eigenvalue problems rather than generalized eigenvalue problems. Therefore, we compare controllers (linear and hybrid) guaranteeing the same convergence rate (namely $\alpha = \alpha_L$).

Let us consider a DC motor and a load used in [6], approximated by a second order model by neglecting the electrical time constant. The plant can be represented as

$$\left[\begin{array}{c|c|c} \bar{A}_p & \bar{B}_p & \bar{B}_w \\ \hline \bar{C}_z & \bar{D}_z & \bar{D}_{zw} \\ \hline \bar{C}_p & \bar{D}_p & \bar{D}_w \end{array}\right] = \left[\begin{array}{cc|c|c} -2.4 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 10 & 0 \\ \hline 0 & 1 & 0 & 5 \end{array}\right].$$

Note that for the purpose of the simulation we decided to use a performance output z_p which penalizes the control input u coming from the controller and the plant output. The exogenous signal w can affect the state dynamics and the output y_p.

Figure 1 shows the optimal values γ obtained with the linear and the hybrid syntheses as a function of the decay rate. It is easy to see that the hybrid case can induce a certain convergence rate without giving up too much on the achievable t-$\mathcal{L}_2$ gain (in the sequel $\mathcal{L}_2$, since there is no ambiguity (see Section II)). For decay rates larger than $\alpha = 0.5$, the linear synthesis returns larger $\mathcal{L}_2$ gains than the hybrid synthesis, whose $\mathcal{L}_2$ gains show a mild increase.

To show the effectiveness of our method, we propose two design syntheses with $\alpha = \alpha_L = 0.5$ and $\alpha = \alpha_L = 2$, respectively. To show that both items of the problem in Section II are solved, for each synthesis there will be two simulations:

- (**no disturbance**) a simulation with $x(0) := [x_p(0)^T, x_c(0)^T]^T = [-0.7, -4, 0, 0]^T$ and no disturbance; this case illustrates the effectiveness of the estimate of the rate of convergence;
- (**zero initial condition**) a simulation with $x(0) = 0$ and $w(t - t_0) := \exp(-(t - t_0))$, with $t_0 = 1$; this case illustrates the effectiveness of the estimate of the $\mathcal{L}_2$ gain.

Once again, we point out that both linear and hybrid syntheses are obtained for a given speed of convergence. Therefore, we do not expect, a priori, important differences in the speed of convergence between the linear and hybrid case.

A. *Synthesis with* $\alpha = 0.5$

In this case, the syntheses return the linear controller

$$\left[\begin{array}{c|c} \bar{A}_c & \bar{B}_c \\ \hline \bar{C}_c & \bar{D}_c \end{array}\right] := \left[\begin{array}{cc|c} -2.5635 & -1.1005 & 0.5222 \\ 1 & -0.2 & 0.2 \\ \hline -0.0818 & -0.4503 & 0.1611 \end{array}\right],$$

and the hybrid controller

$$\left[\begin{array}{c|c} \bar{A}_c & \bar{B}_c \\ \hline \bar{C}_c & \bar{D}_c \end{array}\right] := \left[\begin{array}{cc|c} -2.5635 & -1.1005 & 0.5222 \\ 1 & -0.2 & 0.2 \\ \hline -0.0818 & -0.4503 & 0.1611 \end{array}\right],$$

$$\left[\, \bar{P}_p \middle| K_p \,\right] := \left[\begin{array}{cc|cc} 0.1661 & 0.3842 & 0.9999 & 0 \\ 0.3842 & 0.8888 & 0 & 0.9999 \end{array}\right]$$

$$\left[\begin{array}{c|c|c} \tilde{\alpha} & \epsilon & \rho \end{array}\right] = \left[\begin{array}{c|c|c} 0.4995 & 0.001 & 0.001 \end{array}\right].$$

Note that in this case the hybrid synthesis returned a hybrid controller whose continuous-time part (that is $(\bar{A}_c, \bar{B}_c, \bar{C}_c, \bar{D}_c)$) matches the controller obtained through the linear synthesis.

TABLE I

	$\Vert z_p\Vert_{2t}/\Vert w\Vert_{2t}$	γ	α
hybrid / linear	8.0533	8.0539	0.5

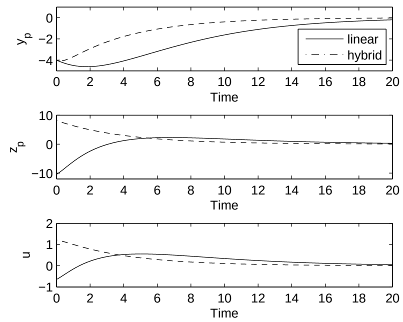

Fig. 2. Hybrid and linear controllers for $\alpha = 0.5$ (**no disturbance**).

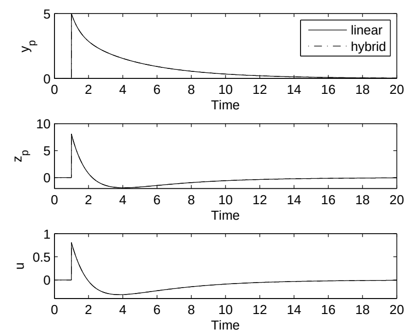

Fig. 3. Hybrid and linear controllers for $\alpha = 0.5$ (**zero initial condition**).

As Figure 1 shows (see also Table I), for $\alpha = 0.5$ the linear and hybrid controllers guarantee the same $\mathcal{L}_2$ gain. Moreover, looking at the transfer functions of both closed-loop systems (for the hybrid case we used the continuous-time part to compute it), it turns out that there is an unstable zero. It is surprising that the effects of the reset action on the reset controller somehow mitigates the negative effects of this bad zero on the transient response (see Figure 2 where the hybrid response shows a reduced undershoot). On the other hand, Figure 3 depicts the fact that in the presence of a disturbance $w \in \mathcal{L}_2$, the hybrid and linear controllers behave essentially in the same way, which is expected, since the γ is the same for both controllers. Table I reports the estimated $\mathcal{L}_2$ gain γ coming from our LMIs, together with the lower bound obtained from computing $\Vert z_p\Vert_{2t}/\Vert w\Vert_{2t}$ for the simulation curves. Clearly this lower bound is smaller than γ.

B. *Synthesis with* $\alpha = 2$

Let us now consider the syntheses obtained with $\alpha = 2$. In this case, the syntheses return the linear controller

$$\left[\begin{array}{c|c} \bar{A}_c & \bar{B}_c \\ \hline \bar{C}_c & \bar{D}_c \end{array}\right] := \left[\begin{array}{cc|c} -759441.3562 & -1756876.5916 & -0.2001 \\ -273818.5690 & -633452.3572 & 1.5878 \\ \hline -393020.4671 & -909208.6041 & -0.1334 \end{array}\right],$$

and the hybrid controller

$$\left[\begin{array}{c|c}\bar{A}_c & \bar{B}_c \\ \hline \bar{C}_c & \bar{D}_c\end{array}\right] := \left[\begin{array}{cc|c}-3.3135 & -3.0708 & 0.7572 \\ 0.9999 & -0.2 & 0.2 \\ \hline -0.4567 & -1.4354 & 0.2786\end{array}\right],$$

$$\left[\,\bar{P}_p \middle| K_p\,\right] := \left[\begin{array}{cc|cc}0.7488 & 1.7324 & 0.9999 & 0 \\ 1.7324 & 4.0081 & 0 & 0.9999\end{array}\right].$$

$$\left[\begin{array}{c|c|c}\tilde{\alpha} & \epsilon & \rho\end{array}\right] = \left[\begin{array}{c|c|c}1.998 & 0.001 & 0.001\end{array}\right].$$

Figure 4 shows that the hybrid controller induces a comparable decay rate to the linear one. Indeed both controllers induce decay rate $\alpha = 2$. Instead, Figure 5 illustrates the $\mathcal{L}_2$ gain improvement arisen from the use of the hybrid solution. In particular, looking at the performance output z_p (middle plot), it is possible to see the improvement with the hybrid controller (see also Table II).

VI. Conclusions

A multiobjective synthesis for a hybrid controller has been presented. It has been shown that the hybrid synthesis can guarantee a better trade-off between $\mathcal{L}_2$ gain and guaranteed decay rate as compared to a linear controller. It has been also proved that for a given γ the hybrid controller can guarantee a strictly larger decay rate as compared to the linear case, even if the improvement can be small.

TABLE II

	$\|z_p\|_{2t}/\|w\|_{2t}$	γ	α
hybrid	13.9302	13.9312	2
linear	41.8168	55.6699	

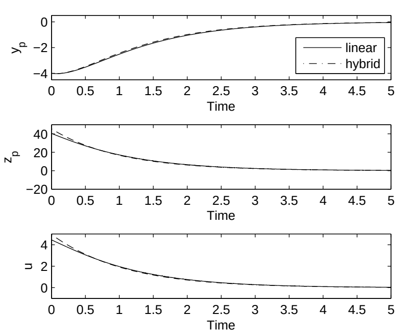

Fig. 4. Hybrid and linear controllers for $\alpha = 2$ (**no disturbance**).

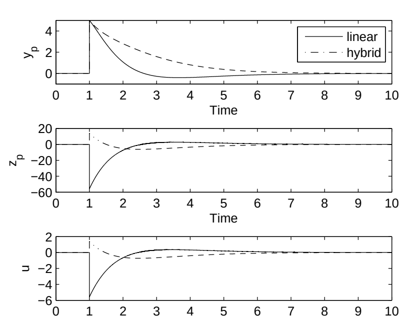

Fig. 5. Hybrid and linear controllers for $\alpha = 2$ (**zero initial condition**).

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