INVARIANTS OF THE GRADED ALGEBRAS ASSOCIATED TO DIVISORIAL VALUATIONS DOMINATING A RATIONAL SURFACE SINGULARITY

V. COSSART, O. PILTANT, A. J. REGUERA

1. INTRODUCTION

Let (R, M) be a two dimensional complete Noetherian local domain and K its quotient field. Given a divisorial valuation ν on K^* whose valuation ring dominates R , we denote by $gr_\nu R$ its associated algebra. Then, $gr_\nu R$ is finitely generated for all divisorial valuations ν , if and only if the divisor class group $Cl(R, M)$ is a torsion group ([Go], [Cu]) and, if the base field is algebraically closed of characteristic zero, this holds if and only if (R, M) is a rational surface singularity ([Li]).

In this paper we deal with a rational surface singularity (R, M) and divisorial valuations ν dominating it. We will assume that the residue field R/M is algebraically closed. Our purpose is to obtain invariants of the graded algebra $gr_\nu R$, or even more, of the Hilbert-Samuel function of $gr_\nu R$. We develop three main ideas to obtain such invariants. The first one is to embed $gr_\nu R$ in a Veronese algebra. If $\pi : X \rightarrow Spec R$ is the minimal desingularization among those on which the center of ν is a curve E and R' is the local ring of the singularity obtained by contracting the center of ν in X , then this Veronese algebra is $gr_\nu R' = \mathcal{V}(a)$, where $-a$ is the self-intersection number of E (theorem 2.2). The second idea is to understand the period of the Hilbert-Samuel function defined by ν on R (proposition 3.1). The third idea is to determine the self-intersection number of the curve E from the Hilbert-Samuel function of $gr_\nu R$ (theorem 3.2). In particular, from the Hilbert-Samuel function, we determine whether the divisorial valuation ν is essential, i.e. whether is defined by an irreducible component of the exceptional locus of the minimal desingularization of $Spec R$.

It follows that, from $gr_\nu R$ we can recover some local information of the dual graph of $\pi : X \rightarrow Spec R$ around the vertex corresponding to E . For instance, the self-intersection number E^2 , the number of exceptional curves E_j adjacent to E and the relative position of the intersection points $E_j \cap E$ in $E \cong \mathbb{P}^1$ (corollary 3.4). A natural question arises : to know whether the dual graph of the minimal desingularization of a rational surface singularity is determined by the set $\{gr_\nu R\}_\nu$ of all graded algebras associated to divisorial valuations dominating R . In section 4 we show that the answer is no. More precisely, we give two rational surface singularities R_1 and R_2 , which are in fact minimal singularities, whose dual graphs for the respective minimal desingularizations are not isomorphic, and such that there exists a one to one correspondence $\theta : \mathcal{V}_1 \rightarrow \mathcal{V}_2$ between the sets $\mathcal{V}_1$ and $\mathcal{V}_2$ of divisorial valuations dominating each one, such that, for all $\nu_1 \in \mathcal{V}_1$, the graded algebras $gr_{\nu_1} R_1$ and $gr_{\theta(\nu_1)} R_2$ are isomorphic (proposition 4.3).

2. EMBEDDING OF THE GRADED ALGEBRA ASSOCIATED TO A DIVISORIAL VALUATION IN A VERONESE ALGEBRA

Let (R, M) be a complete normal ring of dimension two containing an algebraically closed field k isomorphic to its residue field, and having rational singularity. That is, there exists a resolution of singularities $X \rightarrow S$ of $(S, P) = (\text{Spec } R, M)$ such that $R^1 \pi_* \mathcal{O}_X = 0$.

Let ν be a divisorial valuation of the quotient field of R centered in R . Let $gr_\nu R$ be its associated algebra, i.e.

$$gr_\nu R = \bigoplus_{n \in \Phi^+} P_n / P_n^+$$

where $\Phi^+ = \nu(R \setminus \{0\})$ is the semigroup of ν and, for $n \in \mathbb{N}$, $P_n := \{h \in R / \nu(h) \geq n\}$ and $P_n^+ := \{h \in R / \nu(h) > n\}$. In [CPR2] we gave an explicit way to get a *finite generating sequence* for ν , i.e. a finite sequence $\{Q_i\}_i$ of elements of M whose initial forms in $gr_\nu R$ generate it as k -algebra. Equivalently, every ν -ideal I is generated by the finite products $\prod_j Q_j^{a_j}$ where $a_j \in \mathbb{N}$ are such that $\sum_j a_j \nu(Q_j) \geq \nu(I)$. We will review that construction. More precisely, we will review the argument in [CPR2] to determine a finite set $\Sigma \subset \Phi^+$ such that $gr_\nu R$ is generated by $\bigoplus_{n \in \Sigma} P_n / P_n^+$, and then we will define an embedding of $gr_\nu R$ in a Veronese algebra that will allow us to describe generators of P_n / P_n^+ , for $n \in \Sigma$.

Among all resolutions of singularities of (S, P) such that the center of ν in X is a curve, there is a minimal one. Let $\pi : X \rightarrow S$ be this minimal resolution and let E be the center of ν in X . Let $\{E_\gamma\}_{\gamma \in \Delta}$ be the set of irreducible components of the exceptional locus of π . Recall that the *dual graph* of π is a graph defined from the configuration of the exceptional curves $\{E_\gamma\}_\gamma$ as follows : each E_γ is represented by a vertex e_γ and there is a segment joining e_γ and $e_{\gamma'}$ if and only if $E_\gamma \cap E_{\gamma'} \neq \emptyset$.

The following result, due to Artin, will be applied throughout this work.

([Ar2], p.133) : If D is a divisor on X such that $D \cdot E_\gamma = 0$ for all $\gamma \in \Delta$, then there exists $h \in M$ such that $(h)^* = D$, where by $(h)^*$ we mean the total transform in X of the divisor defined by h on S .

Let $\mathbb{E}_X$ be the free group generated by the E_γ 's, and $\mathbb{E}_X^+$ the subsemigroup of $\mathbb{E}_X$ of all divisors D such that $D \cdot E_\gamma \leq 0$ for all γ . From the negativity of the intersection matrix $(E_\gamma \cdot E_{\gamma'})_{\gamma, \gamma' \in \Delta}$ it follows that all elements of $\mathbb{E}_X^+$ are effective divisors ([Li], p. 238). Let $\mathbb{I}_X$ be the semigroup of M -primary complete (i.e. integrally closed) ideals I such that the sheaf IO_X is invertible, with the usual product of ideals ([Li] th. 7.1) For each ideal $I \in \mathbb{I}_X$, there is a unique divisor $D_I \in \mathbb{E}_X^+$ such that $IO_X = \mathcal{O}_X(-D_I)$. Then, the map

$$(1) \quad \mathbb{I}_X \rightarrow \mathbb{E}_X^+, \quad I \mapsto D_I$$

defines an isomorphism of semigroups $(\mathbb{I}_X, \cdot) \cong (\mathbb{E}_X^+, +)$. Its inverse map is $\mathbb{E}_X^+ \rightarrow \mathbb{I}_X$, $D \mapsto I_D := \pi_* \mathcal{O}_X(-D)_P$ ([Li] prop. 6.2, see also [CPR1] section 1).

For $n \in \mathbb{N}$, the ν -ideal P_n is equal to $\pi_* \mathcal{O}_X(-nE)_P$. This implies that P_n belongs to $\mathbb{I}_X$. Let $\bar{D}_n$ be the element of $\mathbb{E}_X^+$ corresponding to $P_n \in \mathbb{I}_X$ by the isomorphism in (1), i.e. such that

$$(2) \quad P_n = \pi_* \mathcal{O}_X(-\bar{D}_n).$$

Recall that $\overline{D}_n \in \mathbb{E}_X^+$ is the *Laufer divisor* associated to nE , obtained by applying the following algorithm : set $\widehat{D}_1 := nE$ and, for $i \geq 1$, let $\overline{D}_n = \widehat{D}_i$ if $\widehat{D}_i \in \mathbb{E}_X^+$, or else $\widehat{D}_{i+1} = \widehat{D}_i + E_{\gamma_i}$ where γ_i is such that $\widehat{D}_i \cdot E_{\gamma_i} > 0$ ([CPR2], prop. 2.1).

Let $\widetilde{D}$ be the extremal divisor in $\mathbb{E}_X^+$ corresponding to the irreducible exceptional component E , i.e. $\widetilde{D}$ is the minimal element in $\mathbb{E}_X^+$ such that $\widetilde{D} \cdot E_\gamma = 0$ for all $E_\gamma \neq E$. Let $\tilde{P} = I_{\widetilde{D}}$ be the associated complete ideal. Then,

2.1. ([CPR] proposition 2.9 and theorem 2.10)

- (i) The ideal $\tilde{P}$ is a ν -ideal, that is, $\tilde{P} = P_p$ where $p = \nu(\tilde{P})$.
- (ii) For $n \in \Phi^+$ and $r \in \mathbb{N}$, we have

$$(3) \quad P_{rp+n} = \tilde{P}^r \cdot P_n$$

Therefore, if $\Sigma := \{p\} \cup \{n \in \Phi^+ \mid n - p \notin \Phi^+\}$, then $gr_\nu R$ is generated as a k -algebra by $\oplus_{n \in \Sigma} P_n / P_n^+$.

Now, let R' be the local ring of the rational surface singularity obtained by contracting E in X . Then,

Theorem 2.2. (*Embedding of $gr_\nu R$ in a Veronese algebra*)

- (i) Let $a = -E^2$. Then $gr_\nu R'$ is isomorphic to the Veronese algebra $\mathcal{V}(a) := \bigoplus_{r \in \mathbb{N}} k[T, T']_{ra}$, where T and T' are projective coordinates in $E \cong \mathbb{P}^1$, and by $k[T, T']_\alpha$ we mean the homogeneous polynomials in T, T' of degree α .
- (ii) The inclusion $R \hookrightarrow R'$ induces an inclusion

$$i : gr_\nu R \hookrightarrow gr_\nu R' \cong \mathcal{V}(a)$$

such that, if $\{E_j\}_{j \in \Delta_E}$ are the exceptional curves intersecting E and, for $n \in \Phi^+$, $\{c_j(n)\}_{j \in \Delta_E}$ are the coefficients in the E_j 's of the Laufer divisor $\overline{D}_n$, and $\alpha_n := -\overline{D}_n \cdot E$, then

$$(4) \quad i(P_n / P_n^+) = \left(\prod_{j \in \Delta_E} T_j^{c_j(n)} \right) k[T, T']_{\alpha_n}$$

where, if $E \cap E_j$ has coordinates $(\lambda_j : 1)$ in $E = \text{Spec } k[T, T']$, then $T_j = T + \lambda_j T'$.

Proof : Let $n \in \Phi^+$, and let us consider the exact sequence

$$0 \rightarrow \mathcal{O}_X(-E - \overline{D}_n) \rightarrow \mathcal{O}_X(-\overline{D}_n) \rightarrow \mathcal{O}_E \otimes \mathcal{O}_X(-\overline{D}_n) \cong \mathcal{O}_E(\alpha_n) \rightarrow 0.$$

Taking global sections, we have

$$0 \rightarrow P_n^+ \rightarrow P_n \rightarrow \Gamma(X, \mathcal{O}_E(\alpha_n)) \cong k[T, T']_{\alpha_n}$$

and therefore, an injective morphism

$$(5) \quad \psi_n : P_n / P_n^+ \rightarrow k[T, T']_{\alpha_n}.$$

Let us prove that ψ_n is surjective. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two nonsingular irreducible curves in X intersecting transversally E in two different points not belonging to any $E_\gamma \neq E$. We may take projective coordinates T, T' in $E \cong \mathbb{P}^1$ so that these points are $(0 : 1)$ and $(1 : 0)$ respectively. Let us consider the divisors

$$(6) \quad D_{n,s} = \overline{D}_n + s \mathcal{C}_1 + (\alpha_n - s) \mathcal{C}_2 \quad \text{for } 0 \leq s \leq \alpha_n.$$

We have $D_{n,s} \cdot E_\gamma \leq 0$ for all $\gamma \in \Delta$, and besides

$$D_{n,s} \cdot E = \overline{D}_n \cdot E + s(\mathcal{C}_1 \cdot E) + (\alpha_n - s)(\mathcal{C}_2 \cdot E) = -\alpha_n + s + (\alpha_n - s) = 0.$$

For each $E_\gamma \neq E$, let $b_{n,s,\gamma} = -D_{n,s} \cdot E_\gamma \in \mathbb{N}$ and let $\mathcal{C}_\gamma$ be a nonsingular irreducible curve in X intersecting transversally E_γ in a point not belonging to $E_{\gamma'}$ for $\gamma' \neq \gamma$.

By Artin's result (see beginning of section 2), there exist $Q_{n,s} \in R$, $0 \leq s \leq \alpha_n$ such that

$$(7) \quad (Q_{n,s})^* = D_{n,s} + \sum_{\gamma} b_{n,s,\gamma} \mathcal{C}_{\gamma}.$$

This implies that $Q_{n,s} \in P_n \setminus P_n^+$ and that

$$\psi_n(Q_{n,s}) = \lambda T^s T'^{\alpha_n - s}$$

for some $\lambda \in k$, $\lambda \neq 0$. Therefore ψ_n is surjective.

Now, for (i), we consider the new rational surface singularity $\text{Spec } R'$ and the same valuation ν , whose center is E . We have $\nu(R' \setminus \{0\}) = \mathbb{N}$, the extremal divisor corresponding to E is $\overline{D}'_1 = E$ and $\overline{D}'_1 \cdot E = a$. Therefore, the isomorphisms (5) in this case are

$$\psi'_n : P'_n / (P'_n)^+ \rightarrow k[T, T']_{na} \quad \text{for every } n \in \mathbb{N}$$

where P'_n and $(P'_n)^+$ are the corresponding ν -ideals of R' . From this (i) follows.

For (ii), let $n \in \Phi^+$, and let $Q_{n,s} \in R$, $1 \leq s \leq \alpha_n$, be the elements satisfying (7). From (6) and (7) it follows that

$$(Q_{n,s})^* = nE + \sum_{j \in \Delta_E} c_j(n) E_j + s \mathcal{C}_1 + (\alpha_n - s) \mathcal{C}_2 + D'$$

where D' is a divisor whose support does not intersect E . Therefore

$$\psi'_n(i(Q_{n,s})) = \lambda_s \left(\prod_{j \in \Delta_E} T_j^{c_j(n)} \right) T^s T'^{\alpha_n - s} \quad \text{for } 1 \leq s \leq \alpha_n$$

where $\lambda_s \in k \setminus \{0\}$. From this, (ii) follows and we conclude the proof of theorem 2.2.

If $\sharp \Delta_E \geq 1$, i.e. there is at least one exceptional curve E_j adjacent to E , we may improve the previous argument with the following one : Let us consider the configuration of exceptional curves $\{E_{\gamma}\}_{\gamma \in \Delta} \setminus \{E\}$. It has as many irreducible components Γ_j as exceptional curves E_j intersecting E . For each Γ_j , let (S_j, P_j) be the rational surface singularity obtained by contracting Γ_j , and let Z_j be the fundamental cycle for the morphism $X \rightarrow S_j$. Then, we can replace the divisor $D_{n,s}$ in (6) by the following one : if $\sharp \Delta_E \geq 2$, we choose two exceptional curves E_1, E_2 adjacent to E , and we set

$$(8) \quad D_{n,s} = \overline{D}_n + sZ_1 + (\alpha_n - s)Z_2$$

which is a divisor with exceptional support (while the divisor in (6) depends on the choice of the curves $\mathcal{C}_i$). If $\sharp \Delta_E = 1$, then we take the unique exceptional curve E_1 adjacent to E and set $D_{n,s} = \overline{D}_n + sZ_1 + (\alpha_n - s)\mathcal{C}_2$. The reason why these new divisors $D_{n,s}$ belong to $\mathbb{E}_X^+$ is the following :

Lemma 2.3. *The coefficient in E_j of Z_j is 1.*

Proof : To compute the fundamental cycle Z for the morphism $X \rightarrow S$, we may apply Laufer's algorithm (see [CPR2] prop. 2.1) to any of the divisors E_{γ} , i.e. we consider a sequence $E_{\gamma} = \widehat{D}_1 < \dots < \widehat{D}_t = Z$ where $\widehat{D}_{i+1} = \widehat{D}_i + E_{\gamma_i}$ for some E_{γ_i} with $\widehat{D}_i \cdot E_{\gamma_i} > 0$. In particular, we may take such a sequence $E_j = \widehat{D}_1 < \dots < \widehat{D}_t = Z$ in such a way that the first steps consist of the Laufer's algorithm to compute Z_j , hence, for some r , $0 \leq r < t$, we have $\widehat{D}_r = Z_j$ and $E_{\gamma_r} = E$. By [La], theorem 4.2, we have $\widehat{D}_i \cdot E_{\gamma_i} = 1$ for all i , $0 \leq i < t$, therefore the coefficient of Z_j in E_j is $Z_j \cdot E = 1$.

Remark 2.4. Lemma 2.3 has appeared in [Ok] lemma 3.5, and it is a key point in Okuma's work [Ok]. The techniques in theorem 2.2 above have been applied in [Re1],[Re2].

Remark 2.5. The above proof gives an explicit way to determine elements $\{Q_{n,s}\}_{s=0}^{\alpha_n}$ whose initial forms define a basis of P_n / P_n^+ , for each $n \in \Phi^+$. More precisely, such that $i(Q_{n,s}) = T^s T'^{\alpha_n - s} (\prod_{j \in \Delta_E} T_j^{c_j(n)})$. We will proceed as above for all $n \in \Sigma$. Besides note that, if $n \in \Sigma$ admits a decomposition $n = n_1 + n_2$ where $n_1, n_2 \in \Sigma$, then

$$(9) \quad i(Q_{n_1, s_1} Q_{n_2, s_2}) = \left(\prod_{j \in \Delta_E} T_j^{c_j(n_1 + n_2)} \right) T^{s_1 + s_2} T'^{\alpha_{n_1} + \alpha_{n_2} - s_1 - s_2} \left(\prod_{j \in \Delta_E} T_j^{b_j} \right)$$

where $b_j \in \mathbb{N}$ is the coefficient in E_j of $\overline{D}_{n_1} + \overline{D}_{n_2} - \overline{D}_{n_1 + n_2}$. Therefore, to obtain a generating sequence for ν , for each $n \in \Phi^+$, we determine elements $\{Q_{n,s}\}_s$ whose image by i is a basis of

$$\left(\prod_{j \in \Delta_E} T_j^{c_j(n)} \right) k[T, T']_{\alpha_n} / \langle \{T^{s_1 + s_2} T'^{\alpha_{n_1} + \alpha_{n_2} - s_1 - s_2} \left(\prod_{j \in \Delta_E} T_j^{b_j} \right)\}_{n=n_1 + n_2, 0 \leq s_i \leq \alpha_{n_i}, i=1,2} \rangle$$

Then the union of such elements is a finite generating sequence for ν .

Remark 2.6. Let R be a regular local ring and ν a divisorial valuation dominating R . Let X be the minimal nonsingular surface dominating $\text{Spec } R$ such that the center of ν in X is a curve E , and let $Q_{g+1} \in R$ define a curve whose strict transform in X is transversal to E in a point not belonging to any other exceptional curve. Let $\overline{\beta}_{g+1} := \nu(Q_{g+1})$, and let $\overline{\beta}_0, \dots, \overline{\beta}_g$ be a minimal system of generators of $\Phi^+ = \nu(R \setminus \{0\})$. Then, with the notation in 2.1 and 2.2, $p = \overline{\beta}_{g+1}$, $\alpha_n = 0$ for all $n \in \Sigma \setminus \{\overline{\beta}_{g+1}\}$ and $\alpha_{\overline{\beta}_{g+1}} = 1$. Any $n \in \Sigma \setminus \{\overline{\beta}_0, \dots, \overline{\beta}_g\}$ decomposes as $n = n_1 + n_2$ for some $n_1, n_2 \in \Sigma$. Therefore, to obtain a generating sequence for ν , we have to define a generator Q_i of $P_{\overline{\beta}_i} / P_{\overline{\beta}_i}^+$ for $0 \leq i \leq g$ as in (7) and (8), and, if $\overline{\beta}_g \nmid \overline{\beta}_{g+1}$ then also Q_{g+1} for $P_{\overline{\beta}_{g+1}} / P_{\overline{\beta}_{g+1}}^+$. If $\overline{\beta}_g \mid \overline{\beta}_{g+1}$, the $\overline{\beta}_{g+1}$ has two different expressions $\overline{\beta}_{g+1} = n_g \overline{\beta}_g = a_0 \overline{\beta}_0 + \dots + a_g \overline{\beta}_g$ in terms of $\overline{\beta}_0 \dots \overline{\beta}_g$ from which, by (9), it follows that we do not have to add any element in $P_{\overline{\beta}_{g+1}} / P_{\overline{\beta}_{g+1}}^+$ (see [Sp], theorem 8.6).

Example 2.7. Let S be the blowing up of the ideal $I = (x^6, y) \cdot (x^2, y + x) \cdot (x^2, y - x)$ of the regular local ring $k[x, y]_{(x, y)}$. The surface S has one singular point P . Let $\pi : X \rightarrow S$ be the minimal desingularization of (S, P) . The dual graph of π is

fig. 1

where e_i represents an exceptional curve E_i , and we have $E_1^2 = -4$, $E_2^2 = E_3^2 = E_4^2 = E_5^2 = -2$. Let us consider the divisorial valuation ν defined by E_2 . Then

$$\begin{array}{lcl}
\overline{D}_1 = E_1 + E_2 + E_3 + E_4 + E_5 & \begin{array}{c} \uparrow -3 \qquad \qquad \qquad \uparrow -1 \\ 1 \quad 1 \quad 1 \quad 1 \quad 1 \end{array} \\
\overline{D}_2 = E_1 + 2E_2 + 2E_3 + 2E_4 + E_5 & \begin{array}{c} \uparrow -2 \quad \uparrow -1 \qquad \qquad \uparrow -1 \\ 1 \quad 2 \quad 2 \quad 2 \quad 1 \end{array} \\
\overline{D}_3 = E_1 + 3E_2 + 3E_3 + 2E_4 + E_5 & \begin{array}{c} \uparrow -1 \quad \uparrow -2 \quad \uparrow -1 \\ 1 \quad 3 \quad 3 \quad 2 \quad 1 \end{array} \\
\widetilde{D} = \overline{D}_4 = E_1 + 4E_2 + 3E_3 + 2E_4 + E_5 & \begin{array}{c} \uparrow -4 \\ 1 \quad 4 \quad 3 \quad 2 \quad 1 \end{array}
\end{array}$$

fig. 2

where we have represented with weighted arrows the intersection of each divisor $\overline{D}_n$ with the E_i corresponding to the basis of the arrow.

Therefore, to obtain a generating sequence for ν , we need elements $Q_1 \in P_1/P_1^+$, $Q_2 \in P_2/P_2^+$, $Q_3 \in P_3/P_3^+$ and $Q_{40}, Q_{44} \in P_4/P_4^+$ whose images by the embedding $gr_\nu R \hookrightarrow \mathcal{V}(2)$ in theorem 2.2 are TT' , TT'^3 , TT'^5 , $T^5T'^3$, TT'^7 respectively. Since

$$x, y, \frac{x^6}{y}, \frac{x^2}{y+x}, \frac{x^2}{y-x}$$

are coordinates in $\mathcal{O}_{S,P}$, we may take

$$\begin{aligned}
Q_1 &= x, & Q_2 &= y, & Q_3 &= \frac{y^2}{y-x} = (y+x) + \frac{x^2}{y-x}, \\
Q_{40} &= \frac{x^6}{y}, & Q_{44} &= \frac{y^3}{(y-x)(y+x)} = y + \frac{1}{2} \frac{x^2}{y+x} + \frac{1}{2} \frac{x^2}{y-x}.
\end{aligned}$$

Therefore, $gr_\nu R$ is isomorphic to

$$k[\overline{Q}_1, \overline{Q}_2, \overline{Q}_3, \overline{Q}_{4,0}, \overline{Q}_{4,4}] \left/ \left(\begin{array}{l} \overline{Q}_2^2 - \overline{Q}_3(\overline{Q}_2 - \overline{Q}_1), \\ \overline{Q}_1^6 - \overline{Q}_{4,0}\overline{Q}_2, \\ \overline{Q}_2^3 - \overline{Q}_{4,4}(\overline{Q}_2^2 - \overline{Q}_1^2) \end{array} \right) \right.$$

where $\overline{Q}_i$ has degree i , for $1 \leq i \leq 3$, and $\overline{Q}_{4,0}$ and $\overline{Q}_{4,4}$ have degree 4.

3. SOME INVARIANTS OF THE SINGULARITY RECOVERED FROM THE GRADED ALGEBRA ASSOCIATED TO A DIVISORIAL VALUATION

In this section we will describe some invariants of (S, P) which can be recovered from the graded algebra $gr_\nu R$ of a divisorial valuation ν dominating R . Most of them will be determined by the Hilbert-Samuel function of $gr_\nu R$.

The Hilbert-Samuel function defined by ν is the function

$$\mathbb{N} \rightarrow \mathbb{N}, \quad n \mapsto h(n) := l(R/P_n).$$

Therefore, it is an equivalent data to the Samuel function defined by ν ,

$$\Phi^+ \rightarrow \mathbb{N}, \quad n \mapsto l(P_n/P_n^+) = \alpha_n + 1$$

where $\alpha_n := -\overline{D}_n \cdot E$ (see equality (4)).

The following result improves theorem 4.2 in [CPR2].

Proposition 3.1. *For all $n \in \mathbb{N}$ we have*

$$l(R/P_n) = Q(n) + \varphi(n)$$

where $Q(n)$ is a polynomial of degree two in n and $\varphi(n)$ is a periodic function of period $p = \nu(\tilde{P})$ for $n \gg 0$.

Proof : If, for any $\mathbb{Q}$ -Cartier divisor B on X , we set $\chi(B) := -\frac{1}{2}B \cdot (B + K)$, where K is a canonical divisor on X , then

$$Q(n) := \chi\left(\frac{n}{p}\tilde{D}\right)$$

is a polynomial of degree two in n . Besides, if we denote by $c(n)$ the coefficient of $\tilde{D}_n$ in E , then equality (3) implies that, for $n \gg 0$, the function

$$\mathbb{N} \rightarrow (\mathbb{E}_X)_{\mathbb{Q}} \quad n \mapsto \tilde{D}_n := \overline{D}_n - \frac{c(n)}{p} \tilde{D}$$

is periodic and its period divides p , where $(\mathbb{E}_X)_{\mathbb{Q}} := \sum_{\gamma \in \Delta} \mathbb{Q}E_{\gamma}$. We have

$$l(R/P_n) = Q(n) + \chi(\tilde{D}_n).$$

(see [CPR2], theorem 4.2). Therefore, $l(R/P_n)$ is expressed as stated in the proposition, where $\varphi(n) := \chi(\tilde{D}_n)$ is a periodic function, for $n \gg 0$, whose period divides p . We have to prove that the period of φ is p .

For $n \in \Phi^+$, let

$$L(n) := Q(n+1) - Q(n) \quad \psi(n) := \varphi(n+1) - \varphi(n).$$

Then, for $n \in \mathbb{N}$, $l(P_n/P_n^+) = L(n) + \psi(n)$ and

$$\alpha_n + 1 = L(n) + \psi(n) \quad \text{for } n \in \Phi^+.$$

The function $\psi(n)$ is periodic for $n \gg 0$, and its period divides the period of φ , hence it divides p .

Let q be the period of ψ , thus q divides p . We have

$$L(n) = \chi\left(\frac{n+1}{p}\tilde{D}\right) - \chi\left(\frac{n}{p}\tilde{D}\right) = \chi\left(\frac{1}{p}\tilde{D}\right) - \left(\frac{1}{p}\tilde{D}\right) \cdot \left(\frac{n}{p}\tilde{D}\right)$$

hence,

$$\psi(n) = -\overline{D}_n \cdot E + 1 - L(n) = -\overline{D}_n \cdot E + 1 - \chi\left(\frac{1}{p}\tilde{D}\right) + \frac{n}{p}\tilde{D} \cdot E.$$

Therefore, $\psi(n+q) = \psi(n)$ is equivalent to

$$(10) \quad (\overline{D}_{n+q} - \overline{D}_n) \cdot E = \frac{q}{p} \tilde{D} \cdot E.$$

This equality for $n \gg 0$ implies that

$$(11) \quad \overline{D}_{rp+kq} \cdot E = \left(r + \frac{q}{p}k\right) \tilde{D} \cdot E \quad \text{for } r \gg 0, 1 \leq k \leq \frac{p}{q}.$$

In fact, given $r \gg 0$, the previous equality for $k = \frac{p}{q}$ is consequence of (3) and, by inverse recurrence, if it holds for $k+1$, then (10) implies that it also holds for k .

For $r \gg 0$, let us consider that $\mathbb{Q}$ -divisor

$$D' := \overline{D}_{rp+q} - \left(r + \frac{q}{p}\right) \tilde{D}.$$

By (11), D' belongs to the cone $(\mathbb{E}_X^+)_{\mathbb{Q}}$ in $(\mathbb{E}_X)_{\mathbb{Q}}$ defined by $\mathbb{E}_X^+$. Since, for $r \gg 0$, we have $rp+q \in \Phi^+$, the coefficient of D' in E is $rp+q - \left(r + \frac{q}{p}\right)p = 0$. This implies that $D' = 0$, i.e. $\overline{D}_{rp+q} = \left(r + \frac{q}{p}\right) \tilde{D}$. Then $\frac{q}{p} \tilde{D}$ is a divisor and, by the definition of $\tilde{D}$, it follows that p divides q , hence $q = p$.

Theorem 3.2. *Let $n \in \mathbb{N}$ be any multiple of $p = \nu(\tilde{P})$ such that $n-1 \in \Phi^+$. Then*

$$(12) \quad \overline{D}_n = \overline{D}_{n-1} + E.$$

Therefore,

$$(13) \quad -E^2 = \alpha_n - \alpha_{n-1}.$$

Proof : First, let us show that $\text{Supp}(\overline{D}_n - \overline{D}_{n-1})$ is connected. This will hold for any $n \in \Phi^+$ such that $n-1 \in \Phi^+$. Let $\{E_\gamma\}_{\gamma \in \Delta'}$ be the configuration of exceptional curves defined by the connected component of $\text{Supp}(\overline{D}_n - \overline{D}_{n-1})$ which contains E . Let $\overline{D} \in \mathbb{E}_X$ be such that $\overline{D} = \overline{D}_n$ on $\cup_{\gamma \in \Delta'} E_\gamma$ and $\overline{D} = \overline{D}_{n-1}$ on $\cup_{\gamma \in \Delta \setminus \Delta'} E_\gamma$. Let us show that $\overline{D} \in \mathbb{E}_X^+$. Then, since $\overline{D}_{n-1} + E \leq \overline{D} \leq \overline{D}_n$, we will have $\overline{D} = \overline{D}_n$ and hence, for any $\gamma \in \Delta \setminus \Delta'$, the coefficient of $(\overline{D}_n - \overline{D}_{n-1})$ in E_γ will be 0. Therefore $\text{Supp}(\overline{D}_n - \overline{D}_{n-1})$ will be connected. By the definition of Δ' , for any $\gamma \in \Delta'$ we have

$$\overline{D} \cdot E_\gamma = \overline{D}_n \cdot E_\gamma \leq 0.$$

If $\gamma \notin \Delta'$ is such that E_γ is not adjacent to any $E_{\gamma'}$ for $\gamma' \in \Delta'$, then

$$\overline{D} \cdot E_\gamma = \overline{D}_{n-1} \cdot E_\gamma \leq 0.$$

Finally, if $\gamma \notin \Delta'$ is such that E_γ is adjacent to some $E_{\gamma'}$, for $\gamma' \in \Delta'$, then the coefficient of $\overline{D}$ in E_γ is equal to the coefficient of $\overline{D}_n$ in E_γ . Let $\{b_\beta\}_{\beta \in \Delta_{E_\gamma}}$ be the coefficients of $\overline{D}_n - \overline{D}_{n-1}$ in the E_β 's adjacent to E_γ . Then

$$\overline{D} \cdot E_\gamma = \overline{D}_n \cdot E_\gamma - \sum_{\beta \in \Delta_{E_\gamma}} b_\beta \leq \overline{D}_n \cdot E_\gamma \leq 0$$

since $b_\beta \geq 0$ for all $\beta \in \Delta_{E_\gamma}$. Thus, $\text{Supp}(\overline{D}_n - \overline{D}_{n-1})$ is connected.

Now, let n be as in the statement of the theorem, and let $\Delta' \subseteq \Delta$ be such that $\text{Supp}(\overline{D}_n - \overline{D}_{n-1}) = \cup_{\gamma \in \Delta'} E_\gamma$. Let S' be the surface obtained by contracting $\cup_{\gamma \in \Delta'} E_\gamma$. Then, S' has only a singular point which is a rational surface singularity. Therefore, for any effective divisor D with support in $\cup_{\gamma \in \Delta'} E_\gamma$ we have $p_a(D) \leq 0$ ([Ar1], 1.7).

Let us apply this to the divisor $D = \overline{D}_n - \overline{D}_{n-1}$. For any $\gamma \in \Delta'$ we have

$$D \cdot E_\gamma = (\overline{D}_n - \overline{D}_{n-1}) \cdot E_\gamma = \begin{cases} -\overline{D}_{n-1} \cdot E_\gamma \geq 0 & \text{if } E_\gamma \neq E \\ E^2 + \sum_{j \in \Delta_E \cap \Delta'} b_j & \text{if } E_\gamma = E \end{cases}$$

where, for any $\gamma \in \Delta'$, b_γ is the coefficient of $\overline{D}_n - \overline{D}_{n-1}$ in E_γ , in particular, the $\{b_j\}_{j \in \Delta' \cap \Delta_E}$ are the nonzero coefficients of $\overline{D}_n - \overline{D}_{n-1}$ in the E_j 's adjacent to E (in the above equality we use the fact that n is a multiple of p , since it implies that $\overline{D}_n \cdot E_\gamma = 0$ for $E_\gamma \neq E$). From the above equality, it follows that

$$D^2 = \sum_{\gamma \in \Delta', E_\gamma \neq E} b_\gamma (D \cdot E_\gamma) + (D \cdot E) \geq D \cdot E = E^2 + \sum_{j \in \Delta_E \cap \Delta'} b_j.$$

Let K be a canonical divisor on X . Since $p_a(E_\gamma) = 0$ for all $\gamma \in \Delta$, by the adjunction formula we have $K \cdot E_\gamma = -2 - E_\gamma^2$, and, since X is the minimal desingularization of S where the center of ν is a curve E , then $-E_\gamma^2 \geq 2$ for $E_\gamma \neq E$. Thus,

$$D \cdot K = \sum_{\gamma \in \Delta'} b_\gamma (E_\gamma \cdot K) = \sum_{\gamma \in \Delta'} b_\gamma (-2 - E_\gamma^2) \geq -2 - E^2.$$

By the adjunction formula, we have

$$p_a(D) = 1 + \frac{1}{2} D \cdot (D + K) \geq 1 + \frac{1}{2} \left(E^2 + \sum_{j \in \Delta_E \cap \Delta'} b_j - 2 - E^2 \right) = \frac{1}{2} \sum_{j \in \Delta_E \cap \Delta'} b_j.$$

Then $p_a(D) \leq 0$ implies that $b_j = 0$ for all E_j adjacent to E . From this, (12) follows. Equality (13) is obtained by computing the intersection numbers with E of the members of (12).

Corollary 3.3. *From the Hilbert-Samuel function of the graded algebra $gr_\nu R$ associated to a divisorial valuation ν dominating R , we can determine whether the valuation ν is essential, i.e. it is defined by an irreducible component of the exceptional locus of the minimal desingularization.*

Proof : It follows directly from theorem 3.2, since E is essential if and only if $-E^2 \geq 2$.

Corollary 3.4. *Let (S, P) be a rational surface singularity, ν a divisorial valuation dominating (S, P) and $\pi : X \rightarrow S$ the minimal desingularization such that the center of ν in X is a curve E . Let $\{E_\gamma\}_{\gamma \in \Delta}$ be the irreducible components of the exceptional locus of π . The following invariants of (S, P) are determined from the Hilbert-Samuel function of the graded algebra defined by ν :*

- (i) *The semigroup $\Phi^+ = \nu(R \setminus \{0\})$ and the function $\Phi^+ \rightarrow \mathbb{N}$, $n \mapsto \alpha_n = -\bar{D}_n \cdot E$.*
- (ii) *The coefficient of E in the fundamental cycle Z for $\pi : X \rightarrow S$.*
- (iii) *The integer $p = \nu(\tilde{P})$.*
- (iv) *The coefficient of E in the unique canonical divisor of X with exceptional support for π .*
- (v) *The integer $a = -E^2$.*

Moreover, from the graded algebra $gr_\nu R$, the following is determined :

- (vi) *The number v_E of exceptional curves E_j which are adjacent to E .*
- (vii) *For each $n \in \Phi^+$, the coefficients $\{c_j(n)\}_{j \in \Delta_E}$ of $\bar{D}_n$ in the E_j 's adjacent to E .*
- (viii) *The relative position of the intersection points $\{E_j \cap E\}_{j \in \Delta_E}$ in $E \cong \mathbb{P}^1$.*

Therefore, from all the Hilbert-Samuel functions defined by the divisorial valuations dominating (S, P) , we recover :

- (ix) *The number of irreducible exceptional curves in the minimal desingularization of (S, P) , and their self-intersections.*
- (x) *The multiplicity of (S, P) .*
- (xi) *The determinant of the intersection matrix of the exceptional curves for the minimal desingularization of (S, P) .*

Besides, from all the graded algebras $\{gr_\nu R\}_\nu$ where ν is any divisorial valuation dominating (S, P) we also obtain :

- (xii) *For each exceptional curve E for the minimal desingularization, the integer $v_E = \sharp \Delta_E$ and the relative position of the intersection points $\{E \cap E_j\}_{j \in \Delta_E}$*

Proof : (i) is clear. For (ii) it suffices to note that the coefficient in E of Z is the smallest element in Φ^+ . From proposition 3.1 and theorem 3.2, (iii) and (v) follow. For (iv), let K be a canonical divisor on X . By the negativity of the intersection matrix $(E_\gamma \cdot E_{\gamma'})_{\gamma, \gamma' \in \Delta}$, there exists a unique $\mathbb{Q}$ -Cartier divisor K_0 with exceptional support for π such that

$$K_0 \cdot E_\gamma = K \cdot E_\gamma \text{ for all } \gamma \in \Delta$$

being this integer equal to $-2 - E_\gamma^2$ by the adjunction formula. On the other hand, the integer $p = \nu(\tilde{P})$ is determined by the Hilbert-Samuel function of $gr_\nu R$ and, by the adjunction formula, $l(R/\tilde{P}) = -\frac{1}{2} \tilde{D} \cdot (\tilde{D} + K)$. We have $\tilde{D}^2 = -p\alpha_p$, hence we obtain $\tilde{D} \cdot K$. But $\text{coef}_E K_0 = -\alpha_p \tilde{D} \cdot K_0 = -\alpha_p \tilde{D} \cdot K \in \mathbb{Z}$. Thus K_0 is the unique canonical divisor with exceptional support for π , and (iv) follows.

Suppose that we know, not only the Hilbert-Samuel function, but also the graded algebra $gr_\nu R$ associated to ν . Let us consider the fraction field of $gr_\nu R$, let us fix $n \in \Phi^+$, and let

$$\Omega = \{(s_1, s_2) \in (P_n/P_n^+)^2 \mid \frac{s_1}{s_2} \text{ is a } \alpha_n - \text{power}\}.$$

Then, for any embedding $j : gr_\nu R \hookrightarrow \mathcal{V}(a)$ where $a = -E^2$, known by (v), the greatest common divisor of $j(s_1)$, $j(s_2)$, where $(s_1, s_2) \in \Omega$, determines $\prod_{j \in \Delta_E} T_j^{c_j(n)}$ in equality (4) modulo a unit. From this, (vi), (vii) and (viii) follow.

Now, suppose that we know all the Hilbert-Samuel functions of the divisorial valuations ν dominating (S, P) . By (v), the data (ix) is determined. Let us consider the set $\{h_\gamma\}_{\gamma \in \Delta_0}$ of all the Hilbert-Samuel functions of the essential valuations and, for each h_γ , let z_γ be the coefficient of Z in the corresponding exceptional curve E_γ , and $\alpha_{z_\gamma}^\gamma = -Z \cdot E_\gamma$, which are determined from h_γ . Then the multiplicity of (S, P) is

$$\text{mult } S = -Z^2 = \sum_{\gamma} z_\gamma \alpha_{z_\gamma}^\gamma$$

(see [Ar2] theorem 4). For (xi) it suffices to note that, if p_γ is the integer in (iii) obtained from h_γ , then the determinant of the intersection matrix $(E_\gamma \cdot E_{\gamma'})_{\gamma, \gamma' \in \Delta_0}$ is equal to the smallest common multiple of $\{\alpha_{p_\gamma}^\gamma\}_{\gamma \in \Delta_0}$. Finally, if we know all graded algebras $\{gr_\nu R\}_\nu$, then, by (v), we may take the ones defined by an essential valuation, and hence (xii) follows from (vi) and (viii).

4. THE DUAL GRAPH IS NOT RECOVERED FROM THE GRADED ALGEBRAS

Given a normal surface singularity (S, P) over an algebraically closed field of characteristic zero, from the set $\{gr_\nu R\}_\nu$ of all graded algebras associated to divisorial valuations ν dominating (S, P) , we can determine whether (S, P) is a rational surface singularity. In fact, for any normal surface singularity (S, P) over any field k , (S, P) satisfies the property that, for all ν , the graded algebra $gr_\nu R$ is finitely generated if and only if its group $Cl(S, P)$ of classes of divisors is a torsion group ([Go], [Cu]) and, if the base field k is algebraically closed of characteristic zero, this is equivalent to (S, P) being a rational surface singularity ([Li]).

Besides, we have shown in the last section that, from the data $\{gr_\nu R\}_\nu$ of all graded algebras associated to divisorial valuations dominating a rational surface singularity (S, P) , we can recover the ones associated to the essential valuations, i.e. the divisorial valuations defined by an irreducible component E of the exceptional locus of the minimal desingularization of (S, P) . Moreover, we also recover the self-intersection number of these exceptional curves E , the number of other exceptional curves in the minimal desingularization which are adjacent to E , and some other invariants (corollary 3.4). From these observations, a natural question arises :

Question 4.1. Is the dual graph of the minimal desingularization of a rational surface singularity determined by the set $\{gr_\nu R\}_\nu$ of all graded algebras associated to divisorial valuations dominating (S, P) ?

We will show in this section that, in general, *the answer to this question is no*. More precisely, we will give two minimal singularities (i.e. rational surface singularities whose fundamental cycle is reduced), (S_1, P_1) and (S_2, P_2) , whose dual graphs of the respective minimal desingularizations are not isomorphic, and such that there

exists a one to one correspondence $\theta : \mathcal{V}_1 \rightarrow \mathcal{V}_2$ between the sets $\mathcal{V}_1$ and $\mathcal{V}_2$ of divisorial valuations dominating (S_1, P_1) and (S_2, P_2) respectively, such that, for all $\nu_1 \in \mathcal{V}_1$, the graded algebras associated respectively to ν_1 and $\theta(\nu_1)$ are isomorphic (proposition 4.3).

Let $\pi : X \rightarrow S$ be a desingularization and let $\{E_\gamma\}_\gamma$ be the exceptional curves of π . By *weighted dual graph* of $\pi : X \rightarrow S$ we mean the weighted graph obtained from the dual graph of π by adding, for each γ , the integer $-E_\gamma^2$ to the vertex e_γ representing E_γ . Note that, if $\pi : X \rightarrow S$ is the minimal desingularization and, for each E_γ , $a_\gamma := -E_\gamma^2$, and $v_\gamma := \#\Delta_{E_\gamma}$ is the number of exceptional curves adjacent to E_γ , then $a_\gamma + 1 \geq v_\gamma$ for any desingularization of a rational surface singularity ([La], th. 4.2). A given weighted graph is the weighted dual graph of a minimal desingularization of a minimal surface singularity if and only if $a_\gamma \geq v_\gamma$.

A cyclic quotient singularity is characterized by the shape of the weighted dual graph for its minimal desingularization, which is

fig. 4

where $a_i \geq 2$. Therefore, a cyclic quotient singularity is characterized by the property that $v_E = 2$ for all essential curves E except for two of them, for which $v_E = 1$. Hence, it is characterized by the set $\{gr_\nu R\}_\nu$ of all graded algebras associated to divisorial valuations ν dominating (S, P) which dominate it (corollary 3.4).

Proposition 4.2. *Let (S, P) be a cyclic quotient singularity. Then question 4.1 has an affirmative answer. More precisely, if (S_2, P_2) is a rational surface singularity such that there exists a one to one correspondence $\theta : \mathcal{V}_1 \rightarrow \mathcal{V}_2$, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are the sets of divisorial valuations dominating (S, P) and (S_2, P_2) respectively, such that, for all $\nu \in \mathcal{V}_1$, $gr_\nu \mathcal{O}_{S,P} \cong gr_{\theta(\nu)} \mathcal{O}_{S_2,P_2}$, then (S, P) and (S_2, P_2) are isomorphic.*

Proof : Let $\{E_i\}_{i=1}^n$ be the irreducible components of the exceptional locus of the minimal desingularization of (S, P) , where E_i is represented by the i -th vertex in figure 4, hence $-E_i^2 = a_i \geq 2$. Let d_n be the determinant of the intersection matrix $(E_i \cdot E_j)_{1 \leq i, j \leq n}$ and let d_{n-1} be the $(n-1) \times (n-1)$ -minor $(E_i \cdot E_j)_{1 \leq i, j \leq n-1}$. Then, since $a_i \geq 2$ for $1 \leq i \leq n$, the sequence $(a_1, \dots, a_n)$ is uniquely determined by the expression as continued fraction of d_n/d_{n-1} . Besides, $d_{n-1} = \frac{p}{\alpha_p}$ where p and α_p are the data associated to the valuation corresponding to the extreme E_n , hence determined from its Hilbert-Samuel function. Thus, from (i), (iii), (vi) and (xi) of corollary 3.4 it follows that the weighted dual graphs of the minimal desingularizations of (S, P) and (S_2, P_2) are the same, and (S, P) , isomorphic to (S_2, P_2) , is the toric singularity defined by the cone $\langle (1, 0), (d_n - d_{n-1}, d_n) \rangle \subset \mathbb{R}^2$ (see [Od], lemma 1.22 and corol. 1.23).

Corollary 4.3. *Let (S, P) be a normal surface singularity over an algebraically closed field of characteristic zero. From the set $\{gr_\nu R\}_\nu$ of all graded algebras associated to the divisorial valuations ν dominating (S, P) , we can determine whether (S, P) is a cyclic quotient singularity. Moreover, a cyclic quotient singularity is determined up to isomorphism by the collection of graded algebras $\{gr_\nu R\}_\nu$ associated to the divisorial valuations ν dominating it.*

Let us consider the following two weighted dual graphs, that will be called respectively G_1 and G_2 ,

fig. 6

where F and F' are respectively

fig. 7

and they are joined in figure 6 by the small segments in the corners. Both G_1 and G_2 satisfy $a_\gamma \geq v_\gamma$ for all γ . Therefore, there exist two minimal singularities (S_1, P_1) and (S_2, P_2) whose weighted dual graphs for their minimal desingularizations are respectively G_1 and G_2 . Moreover, we may also ask (S_1, P_1) and (S_2, P_2) to satisfy the following property : Let $\{E_\alpha\}_{\alpha \in \Lambda}$ (resp. $\{E'_\alpha\}_{\alpha \in \Lambda}$) be the exceptional curves of the minimal desingularization of (S, P) (resp. (S_2, P_2)), let E_{β_1}, E_{β_2} (resp. $E'_{\beta_1}, E'_{\beta_2}$) be the curves corresponding to the -3 -vertices in fig. 6, and, for $j = 1, 2$, let $\alpha_{j,1}, \alpha_{j,2} \in \Lambda \setminus \{\beta_1, \beta_2\}$ be such that $E_{\alpha_{j,1}}$ and $E_{\alpha_{j,2}}$ (resp. $E'_{\alpha_{j,1}}$ and $E'_{\alpha_{j,2}}$) intersect E_{β_j} (resp. E'_{β_j}). Then, for $j = 1, 2$, the relative position of the three points $E_{\alpha_{j,1}} \cap E_{\beta_j}, E_{\alpha_{j,2}} \cap E_{\beta_j}, E_{\beta_1} \cap E_{\beta_2}$ in $E_{\beta_j} \cong \mathbb{P}_k^1$ is the same as the relative position of the three points $E'_{\alpha_{j,1}} \cap E'_{\beta_j}, E'_{\alpha_{j,2}} \cap E'_{\beta_j}, E'_{\beta_1} \cap E'_{\beta_2}$ in $E'_{\beta_j} \cong \mathbb{P}_k^1$, and moreover, this relative position for $j = 1$ is the same as the one for $j = 2$.

Let $R_1 := \mathcal{O}_{S_1, P_1}$ and $R_2 = \mathcal{O}_{S_2, P_2}$, and, for $i = 1, 2$, let $\mathcal{V}_i$ be the set of divisorial valuations dominating R_i . Then,

Proposition 4.4. *There exists a one to one correspondence $\theta : \mathcal{V}_1 \rightarrow \mathcal{V}_2$ such that, for all $\nu_1 \in \mathcal{V}_1$, the graded algebras $gr_{\nu_1} R_1$ and $gr_{\theta(\nu_1)} R_2$ are isomorphic. But the dual graphs of the minimal desingularizations of the singularities (S_1, P_1) and (S_2, P_2) are not isomorphic.*

In order to prove proposition 4.3, let us first study a way to obtain the Hilbert-Samuel function in terms of some local data, which holds under certain conditions, in particular for minimal surface singularities. Let $\pi : X \rightarrow S$ be a desingularization of a rational surface singularity (S, P) , and let $\{E_\gamma\}_{\gamma \in \Delta}$ be the irreducible components of the exceptional locus of π . Let E be one of them, $\{E_j\}_{j \in \Delta_E}$ the exceptional curves which are adjacent to E , and $a_j := -E_j^2$. Let $\{\Gamma_j\}_{j \in \Delta_E}$ be the connected components of $\cup_{\gamma \in \Delta} E_\gamma \setminus E$ where, for each $j \in \Delta_E$, E_j is contained in Γ_j . For each $j \in \Delta_E$, let Σ_j the configuration of curves obtained from Γ_j by substituting E_j by another curve E'_j , also isomorphic to $\mathbb{P}^1$ and with the same intersection numbers with the other curves in Γ_j , but with self-intersection $-(a_j - 1)$. Suppose that there exists a rational surface singularity (Y_j, Q_j) and a desingularization $\pi_j : X_j \rightarrow Y_j$ such that the configuration of exceptional curves for π_j is Σ_j , in particular this implies that $a_j \geq 1$. It happens, for instance, if (S, P) is a minimal surface singularity, X is a desingularization of (S, P) , and E is an irreducible component of its

exceptional divisor. Let h_j be the Hilbert-Samuel function of the valuation ν_j on S_j defined by E'_j . Then

Lemma 4.5. *The Hilbert-Samuel function h of the valuation on (S, P) defined by E and, moreover, the coefficients $\{c_j(n)\}_{j \in \Delta_E}$ in the E_j 's adjacent to E of the divisors $\{\bar{D}_n\}_{n \in \Phi^+}$ associated to this valuation, can be determined from $\{h_j\}_{j \in \Delta_E}$ and E^2 .*

Proof : Let $p = \nu(\tilde{P})$ and let $r \gg 0$ be such that rp is greater or equal to the conductor of the semigroup of Φ^+ . For $n \in \mathbb{N} \cup \{0\} \setminus \Phi^+$, let

$$D_n := \bar{D}_{rp+n} - r\tilde{D}$$

(see (3)) and, for $n \in \Phi^+$, let $D_n := \bar{D}_n$. For any $n \in \mathbb{N}$, let $\alpha_n := -D_n \cdot E$. Then, the function $\mathbb{N} \rightarrow \mathbb{N}$, $n \mapsto \alpha_n - \alpha_{n-1}$ is a periodic function (for all $n \in \mathbb{N}$) of period p (proof of proposition 3.1).

For any $j \in \Delta_E$, let us do the same construction as before, obtaining thus divisors $\{D_n^j\}_{n \in \mathbb{N} \cup \{0\}}$ on Σ_j and a function $\mathbb{N} \rightarrow \mathbb{N}$, $n \mapsto \alpha_n^j$, such that $n \mapsto \alpha_n^j - \alpha_{n-1}^j$ is periodic, let p_j be its period. We will show by induction on n that for all $n \in \mathbb{N}$, the coefficients $\{c_j(n)\}_{j \in \Delta_E}$, and hence the integer α_n , are determined from $\{h_j\}_{j \in \Delta_E}$ and $a := -E^2$.

Let us first show that $\{c_j(1)\}_{j \in \Delta_E}$ are determined from $\{h_j\}_{j \in \Delta_E}$ and a . Fix $j \in \Delta_E$. If $D^j = \sum_{E_\gamma \in \Sigma_j} c_\gamma E_\gamma$ is a divisor on Σ_j , let us set $D^j|_{\Gamma_j} = \sum_{E_\gamma \in \Sigma_j, E_\gamma \neq E_j} c_\gamma E_\gamma + c_j E_j$, which is a divisor on Γ_j hence also on X . Since $D_{rp} = r\tilde{D}$, the divisor on Γ_j obtained by restriction of D_{rp} has intersection 0 with all $E_\gamma \in \Gamma_j \setminus \{E_j\}$, hence it is equal to $D_{sp_j}^j|_{\Gamma_j}$ for some $s \in \mathbb{N}$, in particular $c_j(rp) = sp_j$. Then,

$$0 = D_{rp} \cdot E_j = (D_{sp_j}^j|_{\Gamma_j} + rpE) \cdot E_j = -\alpha_{sp_j}^j + rp - sp_j.$$

Therefore, for $k \geq 1$,

$$(D_{sp_j+k}^j|_{\Gamma_j} + (rp+1)E) \cdot E_j = -\alpha_{sp_j+k}^j + rp - sp_j - (k-1) = -(\alpha_{sp_j+k}^j - \alpha_{sp_j}^j) - (k-1)$$

and this implies that the smallest k such that $-(\alpha_{sp_j+k}^j - \alpha_{sp_j}^j) - (k-1) \leq 0$ is the coefficient in E_j of $D_{rp+1} - D_{rp}$, i.e. it is $c_j(1)$.

Now, let $n \geq 1$ and let us suppose that we know $\{c_j(n)\}_{j \in \Delta_E}$. For any $j \in \Delta_E$, we have

$$0 \geq D_n \cdot E_j = (D_{c_j(n)}^j|_{\Gamma_j} + nE) \cdot E_j = -\alpha_{c_j(n)}^j + n - c_j(n)$$

and, for $k \geq 0$,

$$(D_{c_j(n)+k}^j|_{\Gamma_j} + (n+1)E) \cdot E_j = -\alpha_{c_j(n)+k}^j + n - c_j(n) - (k-1).$$

Hence, $c_j(n+1) = c_j(n) + \min\{k / -\alpha_{c_j(n)+k}^j + n - c_j(n) - (k-1) \leq 0\}$. From this, the lemma follows.

Now, let us consider the weighted dual graphs

fig. 8

obtained from the weighted graphs in figures 6 and 7 by the process described before lemma 4.4. There exist two minimal surface singularities (Y, Q) and (Y', Q') whose weighted dual graphs for their minimal desingularizations are G_Σ and $G_{\Sigma'}$ respectively. Let $\{E_i\}_{i=1}^5$ and $\{E'_i\}_{i=1}^5$ be the irreducible components of the exceptional locus of the minimal desingularization of (Y, Q) and (Y', Q') respectively, which are represented by $\{e_i\}_{i=1}^5$ and $\{e'_i\}_{i=1}^5$ in figure 8. Let ν and ν' be the valuations on (Y, Q) and (Y', Q') respectively determined by E_4 and E'_4 , and let h and h' be the respective Hilbert-Samuel functions.

Lemma 4.6. *The Hilbert-Samuel functions $h : \mathbb{N} \rightarrow \mathbb{N}$ and $h' : \mathbb{N} \rightarrow \mathbb{N}$ are the same.*

Proof : The exceptional divisors $\{\overline{D}_n\}_n$ and $\{\overline{D}'_n\}_n$ for the valuations ν and ν' respectively are

$$\begin{aligned} \overline{D}_1 &= E_1 + E_2 + E_3 + E_4 + E_5 & \overline{D}'_1 &= E'_1 + E'_2 + E'_3 + E'_4 + E'_5 \\ \overline{D}_2 &= E_1 + 2E_2 + 2E_3 + 2E_4 + E_5 & \overline{D}'_2 &= E'_1 + 2E'_2 + E'_3 + 2E'_4 + E'_5 \\ \overline{D}_3 &= E_1 + 2E_2 + 3E_3 + 3E_4 + E_5 & \overline{D}'_3 &= E'_1 + 2E'_2 + E'_3 + 3E'_4 + 2E'_5 \\ \widetilde{D} &= \overline{D}_4 = E_1 + 2E_2 + 3E_3 + 4E_4 + E_5 & \widetilde{D}' &= \overline{D}'_4 = E'_1 + 2E'_2 + E'_3 + 4E'_4 + 2E'_5. \end{aligned}$$

Hence, the period of both h and h' is 4 and we have

$$\begin{aligned} -\overline{D}_1 \cdot E_4 &= -\overline{D}'_1 \cdot E'_4 = 0 \\ -\overline{D}_2 \cdot E_4 &= -\overline{D}'_2 \cdot E'_4 = 1 \\ -\overline{D}_3 \cdot E_4 &= -\overline{D}'_3 \cdot E'_4 = 2 \\ -\overline{D}_4 \cdot E_4 &= -\overline{D}'_4 \cdot E'_4 = 4. \end{aligned}$$

Since $h(n+1) - h(n) = -\overline{D}_n \cdot E_4 + 1$ and the same holds for h' , we conclude the result.

End of proof of proposition 4.3 : Let ν_1 be a divisorial valuation dominating (S_1, P_1) . Let $\pi_1 : X_1 \rightarrow S_1$ be the minimal desingularization of (S_1, P_1) where the center of ν_1 is a curve E , and let $\{E_j\}_{j \in \Delta_E}$ be the exceptional curves for π adjacent to E . From lemmas 4.4 and 4.5 it follows that there exists a valuation ν_2 dominating (S_2, P_2) such that, if $\pi_2 : X_2 \rightarrow S_2$ is the minimal desingularization of (S_2, P_2) where the center of ν_2 is a curve E' , and $\{E'_{j'}\}_{j' \in \Delta_{E'}}$ are the exceptional curves for π adjacent to E' , then we may identify Δ_E and $\Delta_{E'}$, the coefficients $\{c_j(n)\}_{j \in \Delta_E}$ and $\{c_{j'}(n)\}_{j' \in \Delta_{E'}}$ of the divisors $\overline{D}_n$ and $\overline{D}'_n$ for ν_1 and ν_2 respectively coincide, and hence the Hilbert-Samuel functions h_1 and h_2 of ν_1 and ν_2 are the same.

Now, note that $v_E \leq 3$ by the definition of (S_1, P_1) and of X_1 , and that $v_E = 3$ if and only if E is either E_{β_1} or E_{β_2} , and hence E' is either E'_{β_1} or E'_{β_2} . Hence, we may order $\{E_j\}_{j \in \Delta_E}$ and $\{E'_{j'}\}_{j' \in \Delta_{E'}}$ and take coordinates T_1, T_2 in E (resp E') such that $E_j \cap E$ and $E'_{j'} \cap E'$ have the same coordinates for all $j \in \Delta_E = \Delta_{E'}$. Then, from (4) in theorem 2.2 it follows that

$$gr_{\nu_1} R_1 \cong \oplus_{n \in \mathbb{N}} \left(T_1^{c_1(n)} T_2^{c_2(n)} T_3^{c_3(n)} \right) k[T_1, T_2]_{h_1(n+1) - h_1(n) - 1} \cong gr_{\nu_2} R_2$$

and we conclude the result.

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Vincent Cossart,
 Laboratoire de Mathématiques LMV UMR8100, Univ. de Versailles, Saint-Quentin
 45 avenue des États-Unis, 78035 Versailles Cedex (France)
 E-mail : cossart@math.uvsq.fr

Olivier Piltant,
 CNRS, LMV, UMR8100, CNRS-UVSQ, Université de Versailles, Saint-Quentin
 45 avenue des États-Unis, F-78035 Versailles Cedex (France)
 E-mail : piltant@math.uvsq.fr

Ana J. Reguera,
 Dep. de Álgebra, Geometría y Topología, Fac. Ciencias, Universidad de Valladolid,
 Paseo Belén 7, 47011 Valladolid, Spain.
 E-mail : areguera@agt.uva.es