Invariants of the graded algebras associated to divisorial valuations dominating a rational surface singularity
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1. Introduction

Let \((R, M)\) be a two dimensional complete Noetherian local domain and \(K\) its quotient field. Given a divisorial valuation \(\nu\) on \(K^*\) whose valuation ring dominates \(R\), we denote by \(\text{gr}_\nu R\) its associated algebra. Then, \(\text{gr}_\nu R\) is finitely generated for all divisorial valuations \(\nu\), if and only if the divisor class group \(\text{Cl}(R, M)\) is a torsion group ([Go], [Cu]) and, if the base field is algebraically closed of characteristic zero, this holds if and only if \((R, M)\) is a rational surface singularity ([Li]).

In this paper we deal with a rational surface singularity \((R, M)\) and divisorial valuations \(\nu\) dominating it. We will assume that the residue field \(R/M\) is algebraically closed. Our purpose is to obtain invariants of the graded algebra \(\text{gr}_\nu R\), or even more, of the Hilbert-Samuel function of \(\text{gr}_\nu R\). We develop three main ideas to obtain such invariants. The first one is to embed \(\text{gr}_\nu R\) in a Veronese algebra. If \(\pi : X \to \text{Spec } R\) is the minimal desingularization among those on which the center of \(\nu\) is a curve \(E\) and \(R'\) is the local ring of the singularity obtained by contracting the center of \(\nu\) in \(X\), then this Veronese algebra is \(\text{gr}_\nu R' = \mathcal{V}(a)\), where \(-a\) is the self-intersection number of \(E\) (theorem 2.2). The second idea is to understand the period of the Hilbert-Samuel function defined by \(\nu\) on \(R\) (proposition 3.1). The third idea is to determine the self-intersection number of the curve \(E\) from the Hilbert-Samuel function of \(\text{gr}_\nu R\) (theorem 3.2). In particular, from the Hilbert-Samuel function, we determine whether the divisorial valuation \(\nu\) is essential, i.e. whether is defined by an irreducible component of the exceptional locus of the minimal desingularization of \(\text{Spec } R\).

It follows that, from \(\text{gr}_\nu R\) we can recover some local information of the dual graph of \(\pi : X \to \text{Spec } R\) around the vertex corresponding to \(E\). For instance, the self-intersection number \(E^2\), the number of exceptional curves \(E_j\) adjacent to \(E\) and the relative position of the intersection points \(E_j \cap E\) in \(E \cong \mathbb{P}^1\) (corollary 3.4). A natural question arises : to know whether the dual graph of the minimal desingularization of a rational surface singularity is determined by the set \(\{\text{gr}_\nu R\}_\nu\) of all graded algebras associated to divisorial valuations dominating \(R\). In section 4 we show that the answer is no. More precisely, we give two rational surface singularities \(R_1\) and \(R_2\), which are in fact minimal singularities, whose dual graphs for the respective minimal desingularizations are not isomorphic, and such that there exists a one to one correspondence \(\theta : \mathcal{V}_1 \to \mathcal{V}_2\) between the sets \(\mathcal{V}_1\) and \(\mathcal{V}_2\) of divisorial valuations dominating each one, such that, for all \(\nu_1 \in \mathcal{V}_1\), the graded algebras \(\text{gr}_{\nu_1} R_1\) and \(\text{gr}_{\theta(\nu_1)} R_2\) are isomorphic (proposition 4.3).
2. Embedding of the graded algebra associated to a divisorial valuation in a Veronese algebra

Let \((R, M)\) be a complete normal ring of dimension two containing an algebraically closed field \(k\) isomorphic to its residue field, and having rational singularity. That is, there exists a resolution of singularities \(X \to S\) of \((S, P) = (\text{Spec } R, M)\) such that \(R^1\pi_*\mathcal{O}_X = 0\).

Let \(\nu\) be a divisorial valuation of the quotient field of \(R\) centered in \(R\). Let \(gr_\nu R\) be its associated algebra, i.e.

\[
gr_\nu R = \bigoplus_{n \in \Phi^+} P_n / P_n^+
\]

where \(\Phi^+ = \nu(R \setminus \{0\})\) is the semigroup of \(\nu\) and, for \(n \in \mathbb{N}\), \(P_n := \{h \in R / \nu(h) \geq n\}\) and \(P_n^+ := \{h \in R / \nu(h) > n\}\). In [CPR2] we gave an explicit way to get a finite generating sequence for \(\nu\), i.e. a finite sequence \(\{Q_j\}\), of elements of \(M\) whose initial forms in \(gr_\nu R\) generate it as \(k\)-algebra. Equivalently, every \(\nu\)-ideal \(I\) is generated by the finite products \(\prod_i Q_i^{a_i}\) where \(a_i \in \mathbb{N}\) are such that \(\sum a_i \nu(Q_j) \geq \nu(I)\). We will review that construction. More precisely, we will review the argument in [CPR2] to determine a finite set \(\Sigma \subset \Phi^+\) such that \(gr_\nu R\) is generated by \(\oplus_{n \in \Sigma} P_n / P_n^+\), and then we will define an embedding of \(gr_\nu R\) in a Veronese algebra that will allow us to describe generators of \(P_n / P_n^+\), for \(n \in \Sigma\).

Among all resolutions of singularities of \((S, P)\) such that the center of \(\nu\) in \(X\) is a curve, there is a minimal one. Let \(\pi : X \to S\) be this minimal resolution and let \(E\) be the center of \(\nu\) in \(X\). Let \(\{E_\gamma\}_{\gamma \in \Delta}\) be the set of irreducible components of the exceptional locus of \(\pi\). Recall that the dual graph of \(\pi\) is a graph defined from the configuration of the exceptional curves \(\{E_\gamma\}\) as follows : each \(E_\gamma\) is represented by a vertex \(e_\gamma\) and there is a segment joining \(e_\gamma\) and \(e_{\gamma'}\) if and only if \(E_\gamma \cap E_{\gamma'} \neq \emptyset\).

The following result, due to Artin, will be applied throughout this work.

([Ar2], p.133) : If \(D\) is a divisor on \(X\) such that \(D \cdot E_\gamma = 0\) for all \(\gamma \in \Delta\), then there exists \(h \in M\) such that \((h)^* = D\), where by \((h)^*\) we mean the total transform in \(X\) of the divisor defined by \(h\) on \(S\).

Let \(E_X\) be the free group generated by the \(E_\gamma\)’s, and \(E_X^+\) the subsemigroup of \(E_X\) of all divisors \(D\) such that \(D \cdot E_\gamma \leq 0\) for all \(\gamma\). From the negativity of the intersection matrix \((E_\gamma, E_{\gamma'})_{\gamma, \gamma' \in \Delta}\) it follows that all elements of \(E_X^+\) are effective divisors ([Li], p. 238). Let \(I_X\) be the semigroup of \(M\)-primary complete (i.e. integrally closed) ideals \(I\) such that the sheaf \(IO_X\) is invertible, with the usual product of ideals ([Li] th. 7.1) For each ideal \(I \in I_X\), there is a unique divisor \(D_I \in E_X^+\) such that \(IO_X = \mathcal{O}_X(-D_I)\). Then, the map

\[
I_X \to E_X^+, \quad I \mapsto D_I
\]

defines an isomorphism of semigroups \((I_X, \cdot) \cong (E_X^+, +)\). Its inverse map is \(E_X^+ \to I_X\), \(D \mapsto I_D := \pi_*\mathcal{O}_X(-D)_P\) ([Li] prop. 6.2, see also [CPR1] section 1).

For \(n \in \mathbb{N}\), the \(\nu\)-ideal \(P_n\) is equal to \(\pi_*\mathcal{O}_X(-nE)_P\). This implies that \(P_n\) belongs to \(I_X\). Let \(\overline{D}_n\) be the element of \(E_X^+\) corresponding to \(P_n \in I_X\) by the isomorphism in (1), i.e. such that

\[
P_n = \pi_*\mathcal{O}_X(-\overline{D}_n).
\]
Recall that $\mathcal{D}_n \in \mathbb{P}_X^n$ is the Laufer divisor associated to $nE$, obtained by applying the following algorithm: set $\mathcal{D}_1 := nE$ and, for $i \geq 1$, let $\mathcal{D}_n = \mathcal{D}_i$ if $\mathcal{D}_i \in \mathbb{P}_X^n$, or else $\mathcal{D}_{i+1} = \mathcal{D}_i + E_{\gamma_i}$ where $\gamma_i$ is such that $\mathcal{D}_i \cdot E_{\gamma_i} > 0$ ([CPR2], prop. 2.1).

Let $\hat{D}$ be the extremal divisor in $\mathbb{P}_X^n$ corresponding to the irreducible exceptional component $E$, i.e. $\hat{D}$ is the minimal element in $\mathbb{P}_X^n$ such that $\hat{D} \cdot E_{\gamma} = 0$ for all $E_{\gamma} \neq E$. Let $\hat{P} = I_{\hat{D}}$ be the associated complete ideal. Then,

2.1. ([CPR] proposition 2.9 and theorem 2.10)

(i) The ideal $\hat{P}$ is a $\nu$-ideal, that is, $\hat{P} = P_p$ where $p = \nu(\hat{P})$.

(ii) For $n \in \Phi^+$ and $r \in \mathbb{N}$, we have

$$P_{p+n} = \hat{P}^r \cdot P_n$$

Therefore, if $\Sigma := \{p\} \cup \{n \in \Phi^+ / n - p \notin \Phi^+\}$, then $gr_{\nu} R$ is generated as a $k$-algebra by $\oplus_{n \in \Sigma} P_n / P_n^r$.

Now, let $R'$ be the local ring of the rational surface singularity obtained by contracting $E$ in $X$. Then,

**Theorem 2.2. (Embedding of $gr_{\nu} R$ in a Veronese algebra)**

(i) Let $a = -E^2$. Then $gr_{\nu} R'$ is isomorphic to the Veronese algebra $\mathcal{V}(a) := \oplus_{r \in \mathbb{N}} k[T, T']^r_{\alpha}$, where $T$ and $T'$ are projective coordinates in $E \cong \mathbb{P}^1$, and by $k[T, T']_{\alpha}$ we mean the homogeneous polynomials in $T, T'$ of degree $\alpha$.

(ii) The inclusion $R \hookrightarrow R'$ induces an inclusion

$$i : gr_{\nu} R \hookrightarrow gr_{\nu} R' \cong \mathcal{V}(a)$$

such that, if $\{E_j\}_{j \in \Delta_{\mathcal{D}}}$ are the exceptional curves intersecting $E$ and, for $n \in \Phi^+$, $\{c_j(n)\}_{j \in \Delta_{\mathcal{D}}}$ are the coefficients in the $E_j$'s of the Laufer divisor $\mathcal{D}_n$, and $\alpha_n := -\mathcal{D}_n \cdot E$, then

$$i(P_n / P_n^+ = \left( \prod_{j \in \Delta_{\mathcal{D}}} \left[ T^c_j(n) \right] \right) k[T, T']_{\alpha_n}$$

where, if $E \cap E_j$ has coordinates $(\lambda_j : 1)$ in $E = Spec k[T, T']$, then $T_j = T + \lambda_j T'$.

**Proof:** Let $n \in \Phi^+$, and let us consider the exact sequence

$$0 \rightarrow \mathcal{O}_X(-E - \mathcal{D}_n) \rightarrow \mathcal{O}_X(-\mathcal{D}_n) \rightarrow \mathcal{O}_E \otimes \mathcal{O}_X(-\mathcal{D}_n) \cong \mathcal{O}_E(\alpha_n) \rightarrow 0$$

Taking global sections, we have

$$0 \rightarrow P_n^+ \rightarrow P_n \rightarrow \Gamma(X, \mathcal{O}_E(\alpha_n)) \cong k[T, T']_{\alpha_n}$$

and therefore, an injective morphism

$$\psi_n : P_n / P_n^+ \rightarrow k[T, T']_{\alpha_n}$$

Let us prove that $\psi_n$ is surjective. Let $C_1$ and $C_2$ be two nonsingular irreducible curves in $X$ intersecting transversally $E$ in two different points not belonging to any $E_{\gamma} \neq E$. We may take projective coordinates $T, T'$ in $E \cong \mathbb{P}^1$ so that these points are $(0 : 1)$ and $(1 : 0)$ respectively. Let us consider the divisors

$$D_{n,s} = \mathcal{D}_n + sC_1 + (\alpha_n - s)C_2$$

for $0 \leq s \leq \alpha_n$.

We have $D_{n,s} \cdot E_{\gamma} \leq 0$ for all $\gamma \in \Delta$, and besides

$$D_{n,s} \cdot E = \mathcal{D}_n \cdot E + s(C_1 \cdot E) + (\alpha_n - s)(C_2 \cdot E) = -\alpha_n + s + (\alpha_n - s) = 0$$

For each $E_{\gamma} \neq E$, let $b_{n,s,\gamma} = -D_{n,s} \cdot E_{\gamma} \in \mathbb{N}$ and let $C_{\gamma}$ be a nonsingular irreducible curve in $X$ intersecting transversally $E_{\gamma}$ in a point not belonging to $E_{\gamma'}$ for $\gamma' \neq \gamma$. 


By Artin’s result (see beginning of section 2), there exist $Q_{n,s} \in R$, $0 \leq s \leq \alpha_n$ such that
\begin{equation}
(Q_{n,s})^* = D_{n,s} + \sum_{\gamma} b_{n,s,\gamma} C_{\gamma}.
\end{equation}

This implies that $Q_{n,s} \in P_n \setminus P_n^+$ and that
\[ \psi_n(Q_{n,s}) = \lambda T^s T^{\alpha_n-s} \]
for some $\lambda \in k$, $\lambda \neq 0$. Therefore $\psi_n$ is surjective.

Now, for (i), we consider the new rational surface singularity $\text{Spec } R'$ and the same valuation $\nu$, whose center is $E$. We have $\nu(R' \setminus \{0\}) = \mathbb{N}$, the extremal divisor corresponding to $E$ is $\mathcal{D}_1 = E$ and $\mathcal{D}_1 \cdot E = a$. Therefore, the isomorphisms (5) in this case are
\[ \psi'_n : P'_n \rightarrow k[T, T']_{n, a} \]
for every $n \in \mathbb{N}$

where $P'_n$ and $(P'_n)^+$ are the corresponding $\nu$-ideals of $R'$. From this (i) follows.

For (ii), let $n \in \mathbb{N}$, and let $Q_{n,s} \in R$, $1 \leq s \leq \alpha_n$, be the elements satisfying (7). From (6) and (7) it follows that
\[ (Q_{n,s})^* = nE + \sum_{j \in \Delta_E} c_j(n)E_j + sC_1 + (\alpha_n - s)C_2 + D' \]
where $D'$ is a divisor whose support does not intersect $E$. Therefore
\[ \psi'_n(i(Q_{n,s})) = \lambda_s \left( \prod_{j \in \Delta_E} T^{c_j(n)} \right) T^{sT^\alpha_n-s} \text{ for } 1 \leq s \leq \alpha_n \]
where $\lambda_s \in k \setminus \{0\}$. From this, (ii) follows and we conclude the proof of theorem 2.2.

If $\sharp \Delta_E \geq 1$, i.e. there is at least one exceptional curve $E_j$ adjacent to $E$, we may improve the previous argument with the following one : Let us consider the configuration of exceptional curves $\{E_j\}_{j \in \Delta} \setminus \{E\}$. It has as many irreducible components $\Gamma_j$ as exceptional curves $E_j$ intersecting $E$. For each $\Gamma_j$, let $(S_j, P_j)$ be the rational surface singularity obtained by contracting $\Gamma_j$, and let $Z_j$ be the fundamental cycle for the morphism $X \to S_j$. Then, we can replace the divisor $D_{n,s}$ in (6) by the following one : if $\sharp \Delta_E \geq 2$, we choose two exceptional curves $E_1, E_2$ adjacent to $E$, and we set
\begin{equation}
D_{n,s} = \mathcal{D}_n + sZ_1 + (\alpha_n - s)Z_2
\end{equation}
which is a divisor with exceptional support (while the divisor in (6) depends on the choice of the curves $C_j$). If $\sharp \Delta_E = 1$, then we take the unique exceptional curve $E_1$ adjacent to $E$ and set $D_{n,s} = \mathcal{D}_n + sZ_1 + (\alpha_n - s)C_2$. The reason why these new divisors $D_{n,s}$ belong to $E_X$ is the following :

Lemma 2.3. The coefficient in $E_j$ of $Z_j$ is 1.

Proof : To compute the fundamental cycle $Z$ for the morphism $X \to S$, we may apply Laufer’s algorithm (see [CPR2] prop. 2.1) to any of the divisors $E_j$, i.e. we consider a sequence $E_0 = \hat{D}_1 < \ldots < \hat{D}_t = Z$ where $\hat{D}_{i+1} = \hat{D}_i + E_{\gamma_i}$ for some $E_{\gamma_i}$ with $\hat{D}_i \cdot E_{\gamma_i} > 0$. In particular, we may take such a sequence $E_j = \hat{D}_1 < \ldots < \hat{D}_t = Z$ in such a way that the first steps consist of the Laufer’s algorithm to compute $Z_j$, hence, for some $r$, $0 \leq r < t$, we have $\hat{D}_r = Z_j$ and $E_{\gamma_i} = E$. By [La], theorem 4.2, we have $\hat{D}_i \cdot E_{\gamma_i} = 1$ for all $i$, $0 \leq i < t$, therefore the coefficient of $Z_j$ in $E_j$ is $Z_j \cdot E = 1$. 
Remark 2.4. Lemma 2.3 has appeared in [Ok] lemma 3.5, and it is a key point in Okuma’s work [Ok]. The techniques in theorem 2.2 above have been applied in [Re1],[Re2].

Remark 2.5. The above proof gives an explicit way to determine elements \( \{Q_{n,s}\}_{s=0}^{\alpha_n} \) whose initial forms define a basis of \( P_n / P_n^+ \), for each \( n \in \Phi^+ \). More precisely, such that \( i(Q_{n,s}) = T^\alpha \cdot \nu^{-\alpha} \cdot \prod_j \{ T_j^{(n+1)} \} \). We will proceed as above for all \( n \in \Sigma \). Besides note that, if \( n \in \Sigma \) admits a decomposition \( n = n_1 + n_2 \) where \( n_1, n_2 \in \Sigma \), then

\[
(9) \quad i(Q_{n_1,s_1} Q_{n_2,s_2}) = \left( \prod_{j \in \Delta E} T_j^{(n_1+n_2)} \right) T^{n_1+s_2} \prod_{j \in \Delta E} T_j^{b_j}
\]

where \( b_j \in \mathbb{N} \) is the coefficient in \( E_j \) of \( \partial_{n_1} \cdot \partial_{n_2} \cdot \partial_{n_1+n_2} \). Therefore, to obtain a generating sequence for \( \nu \), for each \( n \in \Phi^+ \), we determine elements \( \{Q_{n,s}\}_s \) whose image by \( i \) is a basis of

\[
\left( \prod_{j \in \Delta E} T_j^{(n)} \right) k[T,T] / \langle \{ T^{n_1+s_2} \prod_{j \in \Delta E} T_j^{b_j} \}_{n = n_1+n_2, 0 \leq s_1, s_2, i = 1, 2} \rangle
\]

Then the union of such elements is a finite generating sequence for \( \nu \).

Remark 2.6. Let \( R \) be a regular local ring and \( \nu \) a divisorial valuation dominating \( R \). Let \( X \) be the minimal nonsingular surface dominating \( Spec \ R \) such that the center of \( \nu \) in \( X \) is a curve \( E \), and let \( Q_{g+1} \in R \) define a curve whose strict transform in \( X \) is transversal to \( E \) in a point not belonging to any other exceptional curve. Let \( \mathcal{B}_{g+1} := \nu(Q_{g+1}) \), and let \( \mathcal{B}_0, \ldots, \mathcal{B}_g \) be a minimal system of generators of \( \Phi^+ = \nu(R \setminus \{0\}) \). Then, with the notation in 2.1 and 2.2. \( p = \mathcal{B}_{g+1}, \alpha_n = 0 \) for all \( n \in \Sigma \setminus \{\mathcal{B}_{g+1}\} \) and \( \alpha_n^{\mathcal{B}_{g+1}} = 1 \). Any \( n \in \Sigma \setminus \{\mathcal{B}_0, \ldots, \mathcal{B}_g\} \) decomposes as \( n = n_1 + n_2 \) for some \( n_1, n_2 \in \Sigma \). Therefore, to obtain a generating sequence for \( \nu \), we have to define a generator \( Q_i \) of \( P_{\mathcal{B}_i} / P_{\mathcal{B}_i}^+ \) for \( 0 \leq i \leq g \) as in (7) and (8), and, if \( \mathcal{B}_g \mid \mathcal{B}_{g+1} \) then also \( Q_{g+1} \) for \( P_{\mathcal{B}_{g+1}} / P_{\mathcal{B}_{g+1}}^+ \). If \( \mathcal{B}_g \mid \mathcal{B}_{g+1} \), the \( \mathcal{B}_{g+1} \) has two different expressions \( \mathcal{B}_{g+1} = n_0 \mathcal{B}_0 + \ldots + n_g \mathcal{B}_g \) in terms of \( \mathcal{B}_0, \ldots, \mathcal{B}_g \) from which, by (9), it follows that we do not have to add any element in \( P_{\mathcal{B}_{g+1}} / P_{\mathcal{B}_{g+1}}^+ \) (see [Sp], theorem 8.6).  

Example 2.7. Let \( S \) be the blowing up of the ideal \( I = (x^2, y) \cdot (x^2, y + x) \cdot (x^2, y - x) \) of the regular local ring \( k[x,y] / (x,y) \). The surface \( S \) has one singular point \( P \). Let \( \pi : X \to S \) be the minimal desingularization of \( (S, P) \). The dual graph of \( \pi \) is

\[
\begin{array}{cccccc}
  & e_1 & e_2 & e_3 & e_4 & e_5 \\
\end{array}
\]

fig. 1

where \( e_i \) represents an exceptional curve \( E_i \), and we have \( E_1^2 = -4, E_2^2 = E_3^2 = E_4^2 = E_5^2 = 2 \). Let us consider the divisorial valuation \( \nu \) defined by \( E_2 \). Then
\[ D_1 = E_1 + E_2 + E_3 + E_4 + E_5 \]
\[ D_2 = E_1 + 2E_2 + 2E_3 + 2E_4 + E_5 \]
\[ D_3 = E_1 + 3E_2 + 3E_3 + 2E_4 + E_5 \]
\[ \tilde{D} = D_4 = E_1 + 4E_2 + 3E_3 + 2E_4 + E_5 \]

where we have represented with weighted arrows the intersection of each divisor \( D_n \) with the \( E_i \) corresponding to the basis of the arrow.

Therefore, to obtain a generating sequence for \( \nu \), we need elements \( Q_1 \in P_1/P_1^{+} \), \( Q_2 \in P_2/P_2^{+} \), \( Q_3 \in P_3/P_3^{+} \) and \( Q_{40}, Q_{44} \in P_4/P_4^{+} \) whose images by the embedding \( gr_{\nu} R \hookrightarrow V(2) \) in theorem 2.2 are \( T T', T T^3, T T^5, T^5 T^3, T T^7 \) respectively. Since \( x, y, x \), \( y \), \( x \), \( y \) are coordinates in \( \mathcal{O}_{S,P} \), we may take

\[ Q_1 = x, \quad Q_2 = y, \quad Q_3 = \frac{y^2}{y - x} = \frac{(y + x) + \frac{x^2}{y - x}}{y - x}, \]
\[ Q_{40} = \frac{x^6}{y}, \quad Q_{44} = \frac{y^3}{(y - x)(y + x)} = y + \frac{1}{2} \frac{x^2}{y + x} + \frac{1}{2} \frac{x^2}{y - x}. \]

Therefore, \( gr_{\nu} R \) is isomorphic to

\[ k[\overline{Q}_1, \overline{Q}_2, \overline{Q}_3, \overline{Q}_{4,0}, \overline{Q}_{4,4}] \]

\[ \left\{ \begin{array}{l}
\overline{Q}_2 - \overline{Q}_3(\overline{Q}_2 - \overline{Q}_1), \\
\overline{Q}_1 - \overline{Q}_{4,0}\overline{Q}_2, \\
\overline{Q}_1 - \overline{Q}_{4,4}(\overline{Q}_2^2 - \overline{Q}_1^2) 
\end{array} \right. \]

where \( \overline{Q}_i \) has degree \( i \), for \( 1 \leq i \leq 3 \), and \( \overline{Q}_{4,0} \) and \( \overline{Q}_{4,4} \) have degree 4.

3. SOME INVARIANTS OF THE SINGULARITY RECOVERED FROM THE GRADED ALGEBRA ASSOCIATED TO A DIVISORIAL VALUATION

In this section we will describe some invariants of \((S, P)\) which can be recovered from the graded algebra \( gr_{\nu} R \) of a divisorial valuation \( \nu \) dominating \( R \). Most of them will be determined by the Hilbert-Samuel function of \( gr_{\nu} R \).

The Hilbert-Samuel function defined by \( \nu \) is the function

\[ \mathbb{N} \to \mathbb{N}, \quad n \mapsto h(n) := l(R/P_n). \]

Therefore, it is an equivalent data to the Samuel function defined by \( \nu \),

\[ \Phi^+ \to \mathbb{N}, \quad n \mapsto l(P_n/P_n^+) = \alpha_n + 1 \]

where \( \alpha_n := -D_n \cdot E \) (see equality (4)).

The following result improves theorem 4.2 in [CPR2].

**Proposition 3.1.** For all \( n \in \mathbb{N} \) we have

\[ l(R/P_n) = Q(n) + \varphi(n) \]
where $Q(n)$ is a polynomial of degree two in $n$ and $\varphi(n)$ is a periodic function of period $p = \nu(\widetilde{P})$ for $n >> 0$.

Proof: If, for any $\mathbb{Q}$-Cartier divisor $B$ on $X$, we set $\chi(B) := -\frac{1}{2}B \cdot (B + K)$, where $K$ is a canonical divisor on $X$, then

$$Q(n) := \chi\left(\frac{n}{p} \widetilde{D}\right)$$

is a polynomial of degree two in $n$. Besides, if we denote by $c(n)$ the coefficient of $\overline{D}_n$ in $E$, then equality (3) implies that, for $n >> 0$, the function

$$N \rightarrow (E_X)_\mathbb{Q} \quad n \mapsto \overline{D}_n := \overline{D}_n - \frac{c(n)}{p} \widetilde{D}$$

is periodic and its period divides $p$, where $(E_X)_\mathbb{Q} := \sum_{\gamma \in \Delta} \mathbb{Q}E_{\gamma}$. We have

$$l(R/P_n) = Q(n) + \chi(\tilde{D}_n).$$

(see [CPR2], theorem 4.2). Therefore, $l(R/P_n)$ is expressed as stated in the proposition, where $\varphi(n) := \chi(\tilde{D}_n)$ is a periodic function, for $n >> 0$, whose period divides $p$. We have to prove that the period of $\varphi$ is $p$.

For $n \in \Phi^+$, let

$$L(n) := Q(n + 1) - Q(n) \quad \psi(n) := \varphi(n + 1) - \varphi(n).$$

Then, for $n \in \mathbb{N}$, $l(P_n/P_n^+) = L(n) + \psi(n)$ and

$$\alpha_n + 1 = L(n) + \psi(n) \quad \text{for } n \in \Phi^+.$$

The function $\psi(n)$ is periodic for $n >> 0$, and its period divides the period of $\varphi$, hence it divides $p$.

Let $q$ be the period of $\psi$, thus $q$ divides $p$. We have

$$L(n) = \chi\left(\frac{n + 1}{p} \widetilde{D}\right) - \chi\left(\frac{n}{p} \widetilde{D}\right) = \chi\left(\frac{1}{p} \widetilde{D}\right) - \left(\frac{1}{p} \widetilde{D}\right) \cdot \left(\frac{n}{p} \widetilde{D}\right)$$

hence,

$$\psi(n) = -\overline{D}_n \cdot E + 1 - L(n) = -\overline{D}_n \cdot E + 1 - \chi\left(\frac{1}{p} \widetilde{D}\right) + \frac{n}{p} \widetilde{D} \cdot E.$$

Therefore, $\psi(n + q) = \psi(n)$ is equivalent to

$$\overline{D}_{n+q} - \overline{D}_n \cdot E = \frac{q}{p} \widetilde{D} \cdot E. \quad (10)$$

This equality for $n >> 0$ implies that

$$\overline{D}_{rp+q} \cdot E = \left(r + \frac{q}{p}k\right) \widetilde{D} \cdot E \quad \text{for } r >> 0, \quad 1 \leq k \leq \frac{p}{q}. \quad (11)$$

In fact, given $r >> 0$, the previous equality for $k = \frac{q}{p}$ is consequence of (3) and, by inverse recurrence, if it holds for $k + 1$, then (10) implies that it also holds for $k$.

For $r >> 0$, let us consider that $\mathbb{Q}$-divisor

$$D' := \overline{D}_{qp+q} - \left(r + \frac{q}{p}\right) \widetilde{D}.$$

By (11), $D'$ belongs to the cone $(E_X)_\mathbb{Q}$ in $(E_X)_\mathbb{Q}$ defined by $E_X^+$. Since, for $r >> 0$, we have $rp + q \in \Phi^+$, the coefficient of $D'$ in $E$ is $rp + q - \left(r + \frac{q}{p}\right) p = 0$. This implies that $D' = 0$, i.e. $\overline{D}_{rp+q} = \left(r + \frac{q}{p}\right) \widetilde{D}$. Then $\frac{q}{p} \widetilde{D}$ is a divisor and, by the definition of $\widetilde{D}$, it follows that $p$ divides $q$, hence $q = p$. 

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Theorem 3.2. Let \( n \in \mathbb{N} \) be any multiple of \( p = \nu(\tilde{P}) \) such that \( n - 1 \in \Phi^+ \). Then
\[
\mathcal{D}_n = \mathcal{D}_{n-1} + E.
\]
Therefore,
\[
-E^2 = \alpha_n - \alpha_{n-1}.
\]

Proof: First, let us show that \( \text{Supp}(\mathcal{D}_n - \mathcal{D}_{n-1}) \) is connected. This will hold for any \( n \in \Phi^+ \) such that \( n - 1 \in \Phi^+ \). Let \( \{E_\gamma\}_{\gamma \in \Delta'} \) be the configuration of exceptional curves defined by the connected component of \( \text{Supp}(\mathcal{D}_n - \mathcal{D}_{n-1}) \) which contains \( E \). Let \( \mathcal{D} \in \mathbb{P}_X \) be such that \( \mathcal{D} = \mathcal{D}_n \) on \( \cup_{\gamma \in \Delta'} E_\gamma \) and \( \mathcal{D} = \mathcal{D}_{n-1} \) on \( \cup_{\gamma \in \Delta \setminus \Delta'} E_\gamma \). Let us show that \( \mathcal{D} \in \mathbb{P}_X \). Then, since \( \mathcal{D}_{n-1} + E \leq \mathcal{D} \leq \mathcal{D}_n \), we will have \( \mathcal{D} = \mathcal{D}_n \) and hence, for any \( \gamma \in \Delta \setminus \Delta' \), the coefficient of \((\mathcal{D}_n - \mathcal{D}_{n-1})\) in \( E_\gamma \) will be 0. Therefore \( \text{Supp}(\mathcal{D}_n - \mathcal{D}_{n-1}) \) will be connected. By the definition of \( \Delta' \), for any \( \gamma \in \Delta \) we have
\[
\mathcal{D} \cdot E_\gamma = \mathcal{D}_n \cdot E_\gamma = 0.
\]
If \( \gamma \notin \Delta' \) is such that \( E_\gamma \) is not adjacent to any \( E_{\gamma'} \) for \( \gamma' \in \Delta' \), then
\[
\mathcal{D} \cdot E_\gamma = \mathcal{D}_{n-1} \cdot E_\gamma \leq 0.
\]
Finally, if \( \gamma \notin \Delta' \) is such that \( E_\gamma \) is adjacent to some \( E_{\gamma'} \), for \( \gamma' \in \Delta' \), then the coefficient of \( \mathcal{D} \) in \( E_{\gamma'} \) is equal to the coefficient of \( \mathcal{D}_n \) in \( E_{\gamma'} \). Let \( \{b_{\beta}\}_{\beta \in \Delta_E} \) be the coefficients of \( \mathcal{D}_n - \mathcal{D}_{n-1} \) in the \( E_{\beta} \)'s adjacent to \( E_{\gamma} \). Then
\[
\mathcal{D} \cdot E_{\gamma} = \mathcal{D}_n \cdot E_{\gamma} - \sum_{\beta \in \Delta_E} b_{\beta} \leq \mathcal{D}_n \cdot E_\gamma \leq 0
\]
\( b_{\beta} \geq 0 \) for any \( \beta \in \Delta_E \). Thus, \( \text{Supp}(\mathcal{D}_n - \mathcal{D}_{n-1}) \) is connected.

Now, let \( n \) be as in the statement of the theorem, and let \( \Delta' \subseteq \Delta \) such that \( \text{Supp}(\mathcal{D}_n - \mathcal{D}_{n-1}) = \cup_{\gamma \in \Delta'} E_{\gamma} \). Let \( S' \) be the surface obtained by contracting \( \cup_{\gamma \in \Delta'} E_{\gamma} \). Then, \( S' \) has only a singular point which is a rational surface singularity. Therefore, for any effective divisor \( D \) with support in \( \cup_{\gamma \in \Delta'} E_{\gamma} \), we have \( p_n(D) \leq 0 \) ([Ar1], 1.7).

Let us apply this to the divisor \( D = \mathcal{D}_n - \mathcal{D}_{n-1} \). For any \( \gamma \in \Delta' \) we have
\[
D \cdot E_{\gamma} = (\mathcal{D}_n - \mathcal{D}_{n-1}) \cdot E_{\gamma} = \left\{ \begin{array}{ll}
-D_{n-1} \cdot E_{\gamma} \geq 0 & \text{if } E_{\gamma} \neq E \\
E^2 + \sum_{j \in \Delta_E \cap \Delta'} b_j & \text{if } E_{\gamma} = E
\end{array} \right.
\]
where, for any \( \gamma \in \Delta' \), \( b_{\gamma} \) is the coefficient of \( \mathcal{D}_n - \mathcal{D}_{n-1} \) in \( E_{\gamma} \), in particular, the \( \{b_{\gamma}\}_{\gamma \in \Delta_E \cap \Delta'} \) are the nonzero coefficients of \( \mathcal{D}_n - \mathcal{D}_{n-1} \) in the \( E_{\gamma} \)’s adjacent to \( E \) (in the above equality we use the fact that \( n \) is a multiple of \( p \), since it implies that \( \mathcal{D}_n \cdot E_\gamma = 0 \) for \( E_\gamma \neq E \)). From the above equality, it follows that
\[
D^2 = \sum_{\gamma \in \Delta', E_{\gamma} \neq E} b_{\gamma}(D \cdot E_{\gamma}) + (D \cdot E) \geq D \cdot E = E^2 + \sum_{j \in \Delta_E \cap \Delta'} b_j.
\]
Let \( K \) be a canonical divisor on \( X \). Since \( p_n(E_{\gamma}) = 0 \) for all \( \gamma \in \Delta \), by the adjunction formula we have \( K \cdot E_{\gamma} = -2 - E^2_{\gamma} \), and, since \( X \) is the minimal desingularization of \( S \) where the center of \( \nu \) is a curve \( E \), then \( -E^2_{\gamma} \geq 2 \) for \( E_{\gamma} \neq E \). Thus,
\[
D \cdot K = \sum_{\gamma \in \Delta'} b_{\gamma}(E_{\gamma} \cdot K) = \sum_{\gamma \in \Delta'} b_{\gamma}(-2 - E^2_{\gamma}) \geq -2 - E^2.
\]
By the adjunction formula, we have
\[
p_n(D) = 1 + \frac{1}{2} D \cdot (D + K) \geq 1 + \frac{1}{2} \left( E^2 + \sum_{j \in \Delta_E \cap \Delta'} b_j - 2 - E^2 \right) = \frac{1}{2} \sum_{j \in \Delta_E \cap \Delta'} b_j.
\]
Then $p_0(D) \leq 0$ implies that $b_j = 0$ for all $E_j$ adjacent to $E$. From this, (12) follows. Equality (13) is obtained by computing the intersection numbers with $E$ of the members of (12).

**Corollary 3.3.** From the Hilbert-Samuel function of the graded algebra $gr \nu R$ associated to a divisorial valuation $\nu$ dominating $R$, we can determine whether the valuation $\nu$ is essential, i.e. it is defined by an irreducible component of the exceptional locus of the minimal desingularization.

**Proof:** It follows directly from theorem 3.2, since $E$ is essential if and only if $-E^2 \geq 2$.

**Corollary 3.4.** Let $(S, P)$ be a rational surface singularity, $\nu$ a divisorial valuation dominating $(S, P)$ and $\pi : X \to S$ the minimal desingularization such that the center of $\nu$ in $X$ is a curve $E$. Let $\{E_\gamma\}_{\gamma \in \Delta}$ be the irreducible components of the exceptional locus of $\pi$. The following invariants of $(S, P)$ are determined from the Hilbert-Samuel function of the graded algebra defined by $\nu$:

1. The semigroup $\Phi^+ = \nu(R \setminus \{0\})$ and the function $\Phi^+ : \mathbb{N} \to \mathbb{N}, n \mapsto \alpha_n = -\bar{T}_n \cdot E$.
2. The coefficient of $E$ in the fundamental cycle $Z$ for $\pi : X \to S$.
3. The integer $p = \nu(\bar{P})$.
4. The coefficient of $E$ in the unique canonical divisor of $X$ with exceptional support for $\pi$.
5. The integer $a = -E^2$.

Moreover, from the graded algebra $gr \nu R$, the following is determined:

(a) The number $v_E$ of exceptional curves $E_j$ which are adjacent to $E$.
(b) For each $n \in \Phi^+$, the coefficients $\{c_j(n)\}_{j \in \Delta}$ of $\bar{T}_n$ in the $E_j$’s adjacent to $E$.
(c) The relative position of the intersection points $\{E_j \cap E\}_{j \in \Delta}$ in $E \cong \mathbb{P}^1$.

Therefore, from all the Hilbert-Samuel functions defined by the divisorial valuations dominating $(S, P)$, we recover:

(a) The number of irreducible exceptional curves in the minimal desingularization of $(S, P)$, and their self-intersections.
(b) The multiplicity of $(S, P)$.
(c) The determinant of the intersection matrix of the exceptional curves for the minimal desingularization of $(S, P)$.

Besides, from all the graded algebras $(gr \nu R)_\nu$ where $\nu$ is any divisorial valuation dominating $(S, P)$ we also obtain:

(a) For each exceptional curve $E$ for the minimal desingularization, the integer $v_E = 2\Delta_E$ and the relative position of the intersection points $\{E \cap E_j\}_{j \in \Delta}$.

**Proof:** (i) is clear. For (ii) it suffices to note that the coefficient in $E$ of $Z$ is the smallest element in $\Phi^+$. From proposition 3.1 and theorem 3.2, (iii) and (v) follow. For (iv), let $K$ be a canonical divisor on $X$. By the negativity of the intersection matrix $(E_\gamma \cdot E_{\gamma'})_{\gamma, \gamma' \in \Delta}$, there exists a unique $\mathbb{Q}$-Cartier divisor $K_0$ with exceptional support for $\pi$ such that

$$K_0 \cdot E_\gamma = K \cdot E_\gamma \quad \text{for all } \gamma \in \Delta$$

being this integer equal to $-2 - E_{\gamma}^2$ by the adjunction formula. On the other hand, the integer $p = \nu(\bar{P})$ is determined by the Hilbert-Samuel function of $gr \nu R$ and, by the adjunction formula, $l(R/\bar{P}) = -\frac{1}{2} \bar{D} \cdot (\bar{D} + K)$. We have $\bar{D}^2 = -pep$, hence we obtain $\bar{D} \cdot K$. But $\text{coeff}_{E_\gamma}K_0 = -\alpha_p \bar{D} \cdot K_0 = -\alpha_p \bar{D} \cdot K \in \mathbb{Z}$. Thus $K_0$ is the unique canonical divisor with exceptional support for $\pi$, and (iv) follows.
Suppose that we know, not only the Hilbert-Samuel function, but also the graded algebra $gr_\nu R$ associated to $\nu$. Let us consider the fraction field of $gr_\nu R$, let us fix $n \in \Phi^+$, and let

$$\Omega = \{(s_1, s_2) \in (P_n/P_n^+)^2 / \frac{s_1}{s_2} \text{ is an } \alpha_n \text{-power}\}.$$  

Then, for any embedding $j : gr_\nu R \hookrightarrow \mathcal{V}(a)$ where $a = -E^2$, known by (v), the greatest common divisor of $j(s_1), j(s_2)$, where $(s_1, s_2) \in \Omega$, determines $\prod_{j \in \Delta_\nu} T_j^{c_j(a)}$ in equality (4) modulo a unit. From this, (vi), (vii) and (viii) follow.

Now, suppose that we know all the Hilbert-Samuel functions of the divisorial valuations $\nu$ dominating $(S, P)$. By (v), the data (ix) is determined. Let us consider the set $\{h_\gamma\}_{\gamma \in \Delta_\nu}$ of all the Hilbert-Samuel functions of the essential valuations and, for each $h_\gamma$, let $z_\gamma$ be the coefficient of $Z$ in the corresponding exceptional curve $E_\gamma$, and $\alpha^2_\gamma = -Z \cdot E_\gamma$, which are determined from $h_\gamma$. Then the multiplicity of $(S, P)$ is

$$\text{mult } S = -Z^2 = \sum_{\gamma} z_\gamma \alpha^2_\gamma$$  

(see [Ar2] theorem 4). For (xi) it suffices to note that, if $p_\gamma$ is the integer in (iii) obtained from $h_\gamma$, then the determinant of the intersection matrix $(E_\gamma \cdot E_\gamma')_{\gamma, \gamma' \in \Delta_\nu}$ is equal to the smallest common multiple of $\{\alpha^2_\gamma\}_{\gamma \in \Delta_\nu}$. Finally, if we know all graded algebras $\{gr_\nu R\}_\nu$, then, by (v), we may take the ones defined by an essential valuation, and hence (xii) follows from (vi) and (viii).

4. THE DUAL GRAPH IS NOT RECOVERED FROM THE GRDED ALGEBRAS

Given a normal surface singularity $(S, P)$ over an algebraically closed field of characteristic zero, from the set $\{gr_\nu R\}_\nu$ of all graded algebras associated to divisorial valuations $\nu$ dominating $(S, P)$, we can determine whether $(S, P)$ is a rational surface singularity. In fact, for any normal surface singularity $(S, P)$ over any field $k$, $(S, P)$ satisfies the property that, for all $\nu$, the graded algebra $gr_\nu R$ is finitely generated if and only if its group $Cl(S, P)$ of classes of divisors is a torsion group ([Go], [Cu]) and, if the base field $k$ is algebraically closed of characteristic zero, this is equivalent to $(S, P)$ being a rational surface singularity ([Li]).

Besides, we have shown in the last section that, from the data $\{gr_\nu R\}_\nu$ of all graded algebras associated to divisorial valuations dominating a rational surface singularity $(S, P)$, we can recover the ones associated to the essential valuations, i.e. the divisorial valuations defined by an irreducible component $E$ of the exceptional locus of the minimal desingularization of $(S, P)$. Moreover, we also recover the self-intersection number of these exceptional curves $E$, the number of other exceptional curves in the minimal desingularization which are adjacent to $E$, and some other invariants (corollary 3.4). From these observations, a natural question arises:

**Question 4.1.** Is the dual graph of the minimal desingularization of a rational surface singularity determined by the set $\{gr_\nu R\}_\nu$ of all graded algebras associated to divisorial valuations dominating $(S, P)$?

We will show in this section that, in general, *the answer to this question is no.* More precisely, we will give two minimal singularities (i.e. rational surface singularities whose fundamental cycle is reduced), $(S_1, P_1)$ and $(S_2, P_2)$, whose dual graphs of the respective minimal desingularizations are not isomorphic, and such that there
exists a one to one correspondence \( \theta : \mathcal{V}_1 \rightarrow \mathcal{V}_2 \) between the sets \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) of divisorial valuations dominating \((S_1, P_1)\) and \((S_2, P_2)\) respectively, such that, for all \( \nu_1 \in \mathcal{V}_1 \), the graded algebras associated respectively to \( \nu_1 \) and \( \theta(\nu_1) \) are isomorphic (proposition 4.3). Let \( \pi : X \rightarrow S \) be a desingularization and let \( \{E_\gamma\}_\gamma \) be the exceptional curves of \( \pi \). By weighted dual graph of \( \pi : X \rightarrow S \) we mean the weighted graph obtained from the dual graph of \( \pi \) by adding, for each \( \gamma \), the integer \(-E_\gamma^2\) to the vertex \( e_\gamma \) representing \( E_\gamma \). Note that, if \( \pi : X \rightarrow S \) is the minimal desingularization and, for each \( E_\gamma \), \( a_\gamma := -E_\gamma^2 \), and \( v_\gamma := \sharp E_\gamma \) is the number of exceptional curves adjacent to \( E_\gamma \), then \( a_\gamma + 1 \geq v_\gamma \) for any desingularization of a rational surface singularity ([La], th. 4.2). A given weighted graph is the weighted dual graph of a minimal desingularization of a minimal surface singularity if and only if \( a_\gamma \geq v_\gamma \).

A cyclic quotient singularity is characterized by the shape of the weighted dual graph for its minimal desingularization, which is

\[
-a_1 - a_2 \ldots \ldots \ldots -a_{n-1} - a_n
\]

where \( a_i \geq 2 \). Therefore, a cyclic quotient singularity is characterized by the property that \( v_E = 2 \) for all essential curves \( E \) except for two of them, for which \( v_E = 1 \). Hence, it is characterized by the the set \( \{gr_\nu R_\nu\} \) of all graded algebras associated to divisorial valuations \( \nu \) dominating \((S, P)\) which dominate it (corollary 3.4).

**Proposition 4.2.** Let \((S, P)\) be a cyclic quotient singularity. Then question 4.1 has an affirmative answer. More precisely, if \((S_2, P_2)\) is a rational surface singularity such that there exists a one to one correspondence \( \theta : \mathcal{V}_1 \rightarrow \mathcal{V}_2 \), where \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) are the sets of divisorial valuations dominating \( (S, P) \) and \((S_2, P_2)\) respectively, such that, for all \( \nu \in \mathcal{V}_1 \), \( gr_\nu O_{S,P} \cong gr_\theta(\nu) O_{S_2,P_2} \), then \((S, P)\) and \((S_2, P_2)\) are isomorphic.

**Proof:** Let \( \{E_i\}_{i=1}^p \) be the irreducible components of the exceptional locus of the minimal desingularization of \((S, P)\), where \( E_i \) is represented by the \( i \)-th vertex in figure 4, hence \(-E_i^2 = a_i \geq 2 \). Let \( d_n \) be the determinant of the intersection matrix \((E_i \cdot E_j)_{1 \leq i,j \leq n}\) and let \( d_{n-1} \) be the \((n-1) \times (n-1)\)-minor \((E_i \cdot E_j)_{1 \leq i,j \leq n-1}\). Then, since \( a_i \geq 2 \) for \( 1 \leq i \leq n \), the sequence \((a_1, \ldots, a_n)\) is uniquely determined by the expression as continued fraction of \( d_n/d_{n-1} \). Besides, \( d_{n-1} = \frac{L_p}{p} \) where \( p \) and \( \alpha_p \) are the data associated to the valuation corresponding to the extreme \( E_n \), hence determined from its Hilbert-Samuel function. Thus, from (i), (iii), (vi) and (xi) of corollary 3.4 it follows that the weighted dual graphs of the minimal desingularizations of \((S, P)\) and \((S_2, P_2)\) are the same, and \((S, P)\), isomorphic to \((S_2, P_2)\), is the toric singularity defined by the cone \(<(1,0), (d_n - d_{n-1}, d_n) > \subset \mathbb{R}^2 \) (see [Od], lemma 1.22 and corol. 1.23).

**Corollary 4.3.** Let \((S, P)\) be a normal surface singularity over an algebraically closed field of characteristic zero. From the set \( \{gr_\nu R_\nu\} \) of all graded algebras associated to the divisorial valuations \( \nu \) dominating \((S, P)\), we can determine whether \((S, P)\) is a cyclic quotient singularity. Moreover, a cyclic quotient singularity is determined up to isomorphism by the collection of graded algebras \( \{gr_\nu R_\nu\} \) associated to the divisorial valuations \( \nu \) dominating it.
Let us consider the following two weighted dual graphs, that will be called respectively $G_1$ and $G_2$,

\[
\begin{array}{c}
\begin{array}{c}
F' \\
F
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
F' \\
F
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
F' \\
F
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
F' \\
F
\end{array}
\end{array}
\]

fig. 6

where $F$ and $F'$ are respectively

\[
\begin{array}{c}
\begin{array}{c}
-2 \quad -2 \quad -2 \\
-3 \quad -3 \quad -3
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
-2 \quad -2 \quad -2 \\
-3 \quad -3 \quad -3
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
-2 \quad -2 \quad -2 \\
-3 \quad -3 \quad -3
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
-2 \quad -2 \quad -2 \\
-3 \quad -3 \quad -3
\end{array}
\end{array}
\]

fig. 7

and they are joined in figure 6 by the small segments in the corners. Both $G_1$ and $G_2$ satisfy $a_j \geq v_\gamma$ for all $\gamma$. Therefore, there exist two minimal singularities $(S_1, P_1)$ and $(S_2, P_2)$ whose weighted dual graphs for their minimal desingularizations are respectively $G_1$ and $G_2$. Moreover, we may also ask $(S_1, P_1)$ and $(S_2, P_2)$ to satisfy the following property: Let $\{E_{\gamma}\}_{\alpha \in \Delta}$ (resp. $\{E'_{\gamma}\}_{\alpha \in \Delta}$) be the exceptional curves of the minimal desingularization of $(S, P)$ (resp. $(S_2, P_2)$). Let $E_{\beta_1}, E_{\beta_2}$ (resp. $E'_{\beta_1}, E'_{\beta_2}$) be the curves corresponding to the $-3$-vertices in fig. 6, and, for $j = 1, 2$, let $\alpha_{j,1}, \alpha_{j,2} \in \Lambda \setminus \{\beta_1, \beta_2\}$ be such that $E_{\alpha_{j,1}}$ and $E_{\alpha_{j,2}}$ (resp. $E'_{\alpha_{j,1}}$ and $E'_{\alpha_{j,2}}$) intersect $E_{\beta_j}$ (resp. $E'_{\beta_j}$). Then, for $j = 1, 2$, the relative position of the three points $E_{\alpha_{j,1}} \cap E_{\beta_j}, E_{\alpha_{j,2}} \cap E_{\beta_j}$, $E_{\beta_1} \cap E_{\beta_2}$ in $E_{\beta_j} \cong \mathbb{P}^1_k$ is the same as the relative position of the three points $E'_{\alpha_{j,1}} \cap E'_{\beta_j}, E'_{\alpha_{j,2}} \cap E'_{\beta_j}$, $E'_{\beta_1} \cap E'_{\beta_2}$ in $E'_{\beta_j} \cong \mathbb{P}^1_k$, and moreover, this relative position for $j = 1$ is the same as the one for $j = 2$.

Let $R_1 := O_{S_1, P_1}$ and $R_2 = O_{S_2, P_2}$, and, for $i = 1, 2$, let $V_i$ be the set of divisorial valuations dominating $R_i$. Then,

**Proposition 4.4.** There exists a one to one correspondence $\theta : V_1 \rightarrow V_2$ such that, for all $v_1 \in V_1$, the graded algebra $gr_{v_1} R_1$ and $gr_{\theta(v_1)} R_2$ are isomorphic. But the dual graphs of the minimal desingularizations of the singularities $(S_1, P_1)$ and $(S_2, P_2)$ are not isomorphic.

In order to prove proposition 4.3, let us first study a way to obtain the Hilbert-Samuel function in terms of some local data, which holds under certain conditions, in particular for minimal surface singularities. Let $\pi : X \rightarrow S$ be a desingularization of a rational surface singularity $(S, P)$, and let $\{E_\gamma\}_{\gamma \in \Delta}$ be the irreducible components of the exceptional locus of $\pi$. Let $E$ be one of them, $\{E_j\}_{j \in \Delta_E}$ the exceptional curves which are adjacent to $E$, and $a_j := -E_j^2$. Let $\{\Gamma_j\}_{j \in \Delta_E}$ be the connected components of $\cup_{\gamma \in \Delta} E_\gamma \setminus E$ where, for each $j \in \Delta_E$, $E_j$ is contained in $\Gamma_j$. For each $j \in \Delta_E$, let $\Sigma_j$ be the configuration of curves obtained from $\Gamma_j$ by substituting $E_j$ by another curve $E'_j$, also isomorphic to $\mathbb{P}^1_k$ and with the same intersection numbers with the other curves in $\Gamma_j$, but with self-intersection $-(a_j - 1)$. Suppose that there exists a rational surface singularity $(Y_j, Q_j)$ and a desingularization $\pi_j : X_j \rightarrow Y_j$ such that the configuration of exceptional curves for $\pi_j$ is $\Sigma_j$, in particular this implies that $a_j \geq 1$. It happens, for instance, if $(S, P)$ is a minimal surface singularity, $X$ is a desingularization of $(S, P)$, and $E$ is an irreducible component of its
Lemma 4.5. The Hilbert-Samuel function $h$ of the valuation on $(S, P)$ defined by $E$ and, moreover, the coefficients $(c_j(n))_{j\in\Delta_E}$ in the $E_j$’s adjacent to $E$ of the divisors $\{D_n\}_{n\in\Phi^+}$ associated to this valuation, can be determined from $(h_j)_{j\in\Delta_E}$ and $E^2$.

Proof: Let $p = \nu(\tilde{P})$ and let $r > 0$ be such that $rp$ is greater or equal to the conductor of the semigroup of $\Phi^+$. For $n \in \mathbb{N} \cup \{0\} \setminus \Phi^+$, let

$$D_n := D_{rP+n} - r\tilde{D}$$

(see (3)) and, for $n \in \Phi^+$, let $D_n := D_n$. For any $n \in \mathbb{N}$, let $\alpha_n := -D_n \cdot E$. Then, the function $\mathbb{N} \to \mathbb{N}$, $n \mapsto \alpha_n - \alpha_{n-1}$ is a periodic function (for all $n \in \mathbb{N}$) of period $p$ (proof of proposition 3.1).

For any $j \in \Delta_E$, let us do the same construction as before, obtaining thus divisors $\{D^j_n\}_{n\in\mathbb{N}\cup\{0\}}$ on $\Sigma_j$ and a function $\mathbb{N} \to \mathbb{N}$, $n \mapsto \alpha^j_n$, such that $n \mapsto \alpha^j_n - \alpha^j_{n-1}$ is periodic, let $p_j$ be its period. We will show by induction on $n$ that for all $n \in \mathbb{N}$, the coefficients $(c_j(n))_{j\in\Delta_E}$, and hence the integer $\alpha_n$, are determined from $(h_j)_{j\in\Delta_E}$ and $a := -E^2$.

Let us first show that $(c_j(1))_{j\in\Delta_E}$ are determined from $(h_j)_{j\in\Delta_E}$ and $a$. Fix $j \in \Delta_E$. If $D^j = \sum_{E_i \in \Sigma_j} c_i E_i$ is a divisor on $\Sigma_j$, let us set $D^j|\Gamma_j = \sum_{E_i \in \Sigma_j, E_i \neq E_j} c_i E_i + c_j E_j$, which is a divisor on $\Gamma_j$ hence also on $X$. Since $D_{rp} = r\tilde{D}$, the divisor on $\Gamma_j$ obtained by restriction of $D_{rp}$ has intersection 0 with all $E_j \in \Gamma_j \setminus \{E_j\}$, hence it is equal to $D_{sp_j}|\Gamma_j$ for some $s \in \mathbb{N}$, in particular $c_j(rp) = sp_j$. Then,

$$0 = D_{rp} \cdot E_j = (D^j_{sp_j}|\Gamma_j + rpE) \cdot E_j = -\alpha^j_{sp_j} + rp - sp_j.$$ Therefore, for $k \geq 1$,

$$0 = D_{rp} \cdot E_j = (D^j_{sp_j+k}|\Gamma_j + (rp+1)E) \cdot E_j = -\alpha^j_{sp_j+k} + rp - sp_j - (k-1) = -(\alpha^j_{sp_j+k} - \alpha^j_{sp_j}) - (k-1)$$

and this implies that the smallest $k$ such that $-(\alpha^j_{sp_j+k} - \alpha^j_{sp_j}) - (k-1) \leq 0$ is the coefficient in $E_j$ of $D_{rp+k} - D_{rp}$, i.e. it is $c_j(1)$.

Now, let $n \geq 1$ and let us suppose that we know $(c_j(n))_{j\in\Delta_E}$. For any $j \in \Delta_E$, we have

$$0 \geq D_n \cdot E_j = (D^j_{c_j(n)}|\Gamma_j + nE) \cdot E_j = -\alpha^j_{c_j(n)} + n - c_j(n)$$

and, for $k \geq 0$,

$$0 \geq D_{c_j(n)+k} \cdot E_j = (D^j_{c_j(n)+k}|\Gamma_j + (n+1)E) \cdot E_j = -\alpha^j_{c_j(n)+k} + n - c_j(n) - (k-1).$$

Hence, $c_j(n+1) = c_j(n) + \min\{k \mid -\alpha^j_{c_j(n)+k} + n - c_j(n) - (k-1) \leq 0\}$. From this, the lemma follows.

Now, let us consider the weighted dual graphs

\[
\begin{array}{cccccc}
G_\Sigma & -2 & -2 & -2 & -2 & -2 \\
\quad e_1 \quad & e_2 \quad & e_3 \quad & e_4 \quad & e_5 \quad & e^*_1 \\
\quad e_6 \quad & e_7 \quad & e_8 \quad & e_9 \quad & e^{*}_2 \\
\end{array}
\]

fig. 8

exceptional divisor. Let $h_j$ be the Hilbert-Samuel function of the valuation $\nu_j$ on $S_j$ defined by $E_j$. Then
obtained from the weighted graphs in figures 6 and 7 by the process described before lemma 4.4. There exist two minimal surface singularities $(Y, Q)$ and $(Y', Q')$ whose weighted dual graphs for their minimal desingularizations are $G_2$ and $G_{2'}$, respectively. Let $\{E_j\}_{j=1}^5$ and $\{\tilde{E}_j\}_{j=1}^5$ be the irreducible components of the exceptional locus of the minimal desingularization of $(Y, Q)$ and $(Y', Q')$ respectively, which are represented by $\{e_1\}_{j=1}^5$ and $\{\tilde{e}_1\}_{j=1}^5$ in figure 8. Let $\nu$ and $\nu'$ be the valuations on $(Y, Q)$ and $(Y', Q')$ respectively determined by $E_4$ and $\tilde{E}_4$, and let $h$ and $h'$ be the respective Hilbert-Samuel functions.

**Lemma 4.6.** The Hilbert-Samuel functions $h : \mathbb{N} \to \mathbb{N}$ and $h' : \mathbb{N} \to \mathbb{N}$ are the same.

**Proof:** The exceptional divisors $\{\mathcal{D}_n\}_n$ and $\{\mathcal{D'}_n\}_n$ for the valuations $\nu$ and $\nu'$ respectively are

\[
\begin{align*}
\mathcal{D}_1 &= E_1 + E_2 + E_3 + E_4 + E_5 \\
\mathcal{D}_2 &= E_1 + 2E_2 + 2E_3 + 2E_4 + E_5 \\
\mathcal{D}_3 &= E_1 + 2E_2 + 3E_3 + 3E_4 + E_5 \\
\mathcal{D} &= \mathcal{D}_4 = E_1 + 2E_2 + 3E_3 + 4E_4 + E_5
\end{align*}
\]

\[
\begin{align*}
\mathcal{D'}_1 &= E'_1 + E'_2 + E'_3 + E'_4 + E'_5 \\
\mathcal{D'}_2 &= E'_1 + 2E'_2 + E'_3 + 2E'_4 + E'_5 \\
\mathcal{D'}_3 &= E'_1 + 2E'_2 + E'_3 + 3E'_4 + 2E'_5 \\
\mathcal{D'} &= \mathcal{D'}_4 = E'_1 + 2E'_2 + E'_3 + 4E'_4 + 2E'_5.
\end{align*}
\]

Hence, the period of both $h$ and $h'$ is 4 and we have

\[
\begin{align*}
-\mathcal{D}_1 \cdot E_4 &= -\mathcal{D'}_1 \cdot E'_4 = 0 \\
-\mathcal{D}_2 \cdot E_4 &= -\mathcal{D'}_2 \cdot E'_4 = 1 \\
-\mathcal{D}_3 \cdot E_4 &= -\mathcal{D'}_3 \cdot E'_4 = 2 \\
-\mathcal{D}_4 \cdot E_4 &= -\mathcal{D'}_4 \cdot E'_4 = 4.
\end{align*}
\]

Since $h(n+1) - h(n) = -\mathcal{D}_n \cdot E_4 + 1$ and the same holds for $h'$, we conclude the result.

**End of proof of proposition 4.3:** Let $\nu_1$ be a divisorial valuation dominating $(S_1, P_1)$. Let $\pi_1 : X_1 \to S_1$ be the minimal desingularization of $(S_1, P_1)$ where the center of $\nu_1$ is a curve $E$, and let $\{E_j\}_{j \in \Delta_E}$ be the exceptional curves for $\pi$ adjacent to $E$. From lemmas 4.4 and 4.5 it follows that there exists a valuation $\nu_2$ dominating $(S_2, P_2)$ such that, if $\pi_2 : X_2 \to S_2$ is the minimal desingularization of $(S_2, P_2)$ where the center of $\nu_2$ is a curve $E'$, and $\{E'_j\}_{j \in \Delta_{E'}}$ are the exceptional curves for $\pi$ adjacent to $E'$, then we may identify $\Delta_E$ and $\Delta_{E'}$, the coefficients $\{c_j(n)\}_{j \in \Delta_E}$ and $\{c'_j(n)\}_{j' \in \Delta_{E'}}$ of the divisors $\mathcal{D}_n$ and $\mathcal{D'}_n$ for $\nu_1$ and $\nu_2$ respectively coincide, and hence the Hilbert-Samuel functions $h_1$ and $h_2$ of $\nu_1$ and $\nu_2$ are the same.

Now, note that $v_E \leq 3$ by the definition of $(S_1, P_1)$ and of $X_1$, and that $v_{E'} = 3$ if and only if $E$ is either $E_{31}$ or $E_{32}$, and hence $E'$ is either $E'_{31}$ or $E'_{32}$. Hence, we may order $\{E_j\}_{j \in \Delta_E}$ and $\{E'_{j'}\}_{j' \in \Delta_{E'}}$, and take coordinates $T_1, T_2$ in $E$ (resp $E'$) such that $E_j \cap E$ and $E'_{j'} \cap E'$ have the same coordinates for all $j \in \Delta_E = \Delta_{E'}$. Then, from (4) in theorem 2.2 it follows that

\[
\text{gr}_{\nu_1} R_1 \cong \oplus_{n \in \mathbb{N}} \left(T_1^{c_1(n)} T_2^{c_2(n)} T_3^{c_3(n)}\right) k[T_1, T_2]_{h_1(n+1) - h_1(n)-1} \cong \text{gr}_{\nu_2} R_2
\]

and we conclude the result.

**References.**


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