Invariants of the graded algebras associated to divisorial valuations dominating a rational surface singularity
Vincent Cossart, Olivier Piltant, Ana J. Reguera

To cite this version:
Vincent Cossart, Olivier Piltant, Ana J. Reguera. Invariants of the graded algebras associated to divisorial valuations dominating a rational surface singularity. 2012. <hal-00731738>

HAL Id: hal-00731738
https://hal.archives-ouvertes.fr/hal-00731738
Submitted on 13 Sep 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
1. Introduction

Let \((R, M)\) be a two-dimensional complete Noetherian local domain and \(K\) its quotient field. Given a divisorial valuation \(\nu\) on \(K^*\) whose valuation ring dominates \(R\), we denote by \(gr_{\nu}R\) its associated algebra. Then, \(gr_{\nu}R\) is finitely generated for all divisorial valuations \(\nu\), if and only if the divisor class group \(\text{Cl}(R, M)\) is a torsion group ([Go], [Cu]) and, if the base field is algebraically closed of characteristic zero, this holds if and only if \((R, M)\) is a rational surface singularity ([Li]).

In this paper we deal with a rational surface singularity \((R, M)\) and divisorial valuations \(\nu\) dominating it. We will assume that the residue field \(R/M\) is algebraically closed. Our purpose is to obtain invariants of the graded algebra \(gr_{\nu}R\), or even more, of the Hilbert-Samuel function of \(gr_{\nu}R\). We develop three main ideas to obtain such invariants. The first one is to embed \(gr_{\nu}R\) in a Veronese algebra. If \(\pi : X \to \text{Spec } R\) is the minimal desingularization among those on which the center of \(\nu\) is a curve \(E\) and \(R'\) is the local ring of the singularity obtained by contracting the center of \(\nu\) in \(X\), then this Veronese algebra is \(gr_{\nu}R' = \mathcal{V}(a)\), where \(-a\) is the self-intersection number of \(E\) (theorem 2.2). The second idea is to understand the period of the Hilbert-Samuel function defined by \(\nu\) on \(R\) (proposition 3.1). The third idea is to determine the self-intersection number of the curve \(E\) from the Hilbert-Samuel function of \(gr_{\nu}R\) (theorem 3.2). In particular, from the Hilbert-Samuel function, we determine whether the divisorial valuation \(\nu\) is essential, i.e. whether is defined by an irreducible component of the exceptional locus of the minimal desingularization of \(\text{Spec } R\).

It follows that, from \(gr_{\nu}R\) we can recover some local information of the dual graph of \(\pi : X \to \text{Spec } R\) around the vertex corresponding to \(E\). For instance, the self-intersection number \(E^2\), the number of exceptional curves \(E_j\) adjacent to \(E\) and the relative position of the intersection points \(E_j \cap E\) in \(E \cong \mathbb{P}^1\) (corollary 3.4). A natural question arises : to know whether the dual graph of the minimal desingularization of a rational surface singularity is determined by the set \(\{gr_{\nu}R\}_\nu\) of all graded algebras associated to divisorial valuations dominating \(R\). In section 4 we show that the answer is no. More precisely, we give two rational surface singularities \(R_1\) and \(R_2\), which are in fact minimal singularities, whose dual graphs for the respective minimal desingularizations are not isomorphic, and such that there exists a one to one correspondence \(\theta : \mathcal{V}_1 \to \mathcal{V}_2\) between the sets \(\mathcal{V}_1\) and \(\mathcal{V}_2\) of divisorial valuations dominating each one, such that, for all \(\nu_1 \in \mathcal{V}_1\), the graded algebras \(gr_{\nu}R_1\) and \(gr_{\theta(\nu_1)}R_2\) are isomorphic (proposition 4.3).
2. Embedding of the graded algebra associated to a divisorial valuation in a Veronese algebra

Let \((R, M)\) be a complete normal ring of dimension two containing an algebraically closed field \(k\) isomorphic to its residue field, and having rational singularity. That is, there exists a resolution of singularities \(X \to S\) of \((S, P) = (\text{Spec } R, M)\) such that \(R^1\pi_*\mathcal{O}_X = 0\).

Let \(\nu\) be a divisorial valuation of the quotient field of \(R\) centered in \(S\). Let \(\text{gr}_\nu R\) be its associated algebra, i.e.

\[
\text{gr}_\nu R = \bigoplus_{n \in \Phi^+} P_n / P_n^+
\]

where \(\Phi^+ = \nu(R \setminus \{0\})\) is the semigroup of \(\nu\) and, for \(n \in \mathbb{N}\), \(P_n := \{h \in R / \nu(h) \geq n\}\) and \(P_n^+ := \{h \in R / \nu(h) > n\}\). In [CPR2] we gave an explicit way to get a finite generating sequence for \(\nu\), i.e. a finite sequence \(\{Q_i\}\), of elements of \(M\) whose initial forms in \(\text{gr}_\nu R\) generate it as \(k\)-algebra. Equivalently, every \(\nu\)-ideal \(I\) is generated by the finite products \(\prod a_i Q_j^\nu\) where \(a_i \in \mathbb{N}\) are such that \(\sum a_i \nu(Q_j) \geq \nu(I)\). We will review that construction. More precisely, we will review the argument in [CPR2] to determine a finite set \(\Sigma \subset \Phi^+\) such that \(\text{gr}_\nu R\) is generated by \(\oplus_{n \in \Sigma} P_n / P_n^+\), and then we will define an embedding of \(\text{gr}_\nu R\) in a Veronese algebra that will allow us to describe generators of \(P_n / P_n^+\), for \(n \in \Sigma\).

Among all resolutions of singularities of \((S, P)\) such that the center of \(\nu\) in \(X\) is a curve, there is a minimal one. Let \(\pi : X \to S\) be this minimal resolution and let \(E\) be the center of \(\nu\) in \(X\). Let \(\{E_\gamma\}_{\gamma \in \Delta}\) be the set of irreducible components of the exceptional locus of \(\pi\). Recall that the dual graph of \(\pi\) is a graph defined from the configuration of the exceptional curves \(\{E_\gamma\}\), as follows : each \(E_\gamma\) is represented by a vertex \(e_\gamma\) and there is a segment joining \(e_\gamma\) and \(e_\gamma'\) if and only if \(E_\gamma \cap E_\gamma' \neq \emptyset\).

The following result, due to Artin, will be applied throughout this work.

([Ar2], p.133) : If \(D\) is a divisor on \(X\) such that \(D \cdot E_\gamma = 0\) for all \(\gamma \in \Delta\), then there exists \(h \in M\) such that \((h)^* = D\), where by \((h)^*\) we mean the total transform in \(X\) of the divisor defined by \(h\) on \(S\).

Let \(E_X\) be the free group generated by the \(E_\gamma\)'s, and \(E_X^+\) the subsemigroup of \(E_X\) of all divisors \(D\) such that \(D \cdot E_\gamma \leq 0\) for all \(\gamma\). From the negativity of the intersection matrix \((E_\gamma, E_\gamma')_{\gamma, \gamma' \in \Delta}\) it follows that all elements of \(E_X^+\) are effective divisors ([Li], p. 238). Let \(I_X\) be the semigroup of \(M\)-primary complete (i.e. integrally closed) ideals \(I\) such that the sheaf \(I_0X\) is invertible, with the usual product of ideals ([Li] th. 7.1) For each ideal \(I \in I_X\), there is a unique divisor \(D_I \in E_X^+\) such that \(I_0X = O_X(-D_I)\). Then, the map

\[
I_X \to E_X^+, \quad I \mapsto D_I
\]

defines an isomorphism of semigroups \((I_X, \cdot) \cong (E_X^+, +)\). Its inverse map is \(E_X^+ \to I_X, \ D \mapsto I_D := \pi_*O_X(-D)_P\) ([Li] prop. 6.2, see also [CPR1] section 1).

For \(n \in \mathbb{N}\), the \(\nu\)-ideal \(P_n\) is equal to \(\pi_*O_X(-nE)_P\). This implies that \(P_n\) belongs to \(I_X\). Let \(D_n\) be the element of \(E_X^+\) corresponding to \(P_n \in I_X\) by the isomorphism in (1), i.e. such that

\[
P_n = \pi_*O_X(-D_n).
\]
Recall that $D_n \in \mathbb{P}^1_\mathbb{C}$ is the Laufé divisor associated to $nE$, obtained by applying the following algorithm: set $D_1 := nE$ and, for $i \geq 1$, let $D_i = \hat{D}_i$ if $\hat{D}_i \in \mathbb{P}^1_\mathbb{C}$, or else $\hat{D}_{i+1} = \hat{D}_i + E_{\gamma_i}$, where $\gamma_i$ is such that $\hat{D}_i \cdot E_{\gamma_i} > 0 \ (\text{[CPR2], prop. 2.1}).$

Let $\hat{D}$ be the extremal divisor in $\mathbb{P}^1_\mathbb{C}$ corresponding to the irreducible exceptional component $E$, i.e. $\hat{D}$ is the minimal element in $\mathbb{P}^1_\mathbb{C}$ such that $\hat{D} \cdot E_\gamma = 0$ for all $E_\gamma \neq E$. Let $\hat{P} = I_{\hat{D}}$ be the associated complete ideal. Then,

2.1. ([CPR] proposition 2.9 and theorem 2.10)

(i) The ideal $\hat{P}$ is a $\nu$-ideal, that is, $\hat{P} = P_p$ where $p = \nu(\hat{P})$.

(ii) For $n \in \Phi^+$ and $r \in \mathbb{N}$, we have

$$P_{r+p+n} = \hat{P}^r \cdot P_n$$

Therefore, if $\Sigma := \{p\} \cup \{n \in \Phi^+ / n - p \notin \Phi^+\}$, then $gr_{\nu}R$ is generated as a $k$-algebra by $\oplus_{n \in \Sigma} P_n / P_n^\nu$.

Now, let $R'$ be the local ring of the rational surface singularity obtained by contracting $E$ in $X$. Then,

**Theorem 2.2.** (Embedding of $gr_{\nu}R$ in a Veronese algebra)

(i) Let $a = -E^2$. Then $gr_{\nu}R'$ is isomorphic to the Veronese algebra $\mathcal{V}(a) := \oplus_{r \in \mathbb{N}} k[T, T']_r$, where $T$ and $T'$ are projective coordinates in $E \cong \mathbb{P}^1$, and by $k[T, T']_\alpha$ we mean the homogeneous polynomials in $T, T'$ of degree $\alpha$.

(ii) The inclusion $R \hookrightarrow R'$ induces an inclusion

$$i : gr_{\nu}R \hookrightarrow gr_{\nu}R' \cong \mathcal{V}(a)$$

such that, if $\{E_j\}_{j \in \Delta}$ are the exceptional curves intersecting $E$ and, for $n \in \Phi^+$, $\{e_j(n)\}_{j \in \Delta}$ are the coefficients in the $E_j$'s of the Laufé divisor $\overline{D}_n$, and $\alpha_n := -\overline{D}_n \cdot E$, then

$$i(P_n / P_n^\nu) = \left( \prod_{j \in \Delta} T_j^{e_j(n)} \right) k[T, T']_{\alpha_n}$$

where, if $E \cap E_j$ has coordinates $(\lambda_j : 1)$ in $E = \text{Spec } k[T, T']$, then $T_j = T + \lambda_j T'$.

**Proof:** Let $n \in \Phi^+$, and let us consider the exact sequence

$$0 \to O_X(-E - \overline{D}_n) \to O_X(-\overline{D}_n) \to O_E \otimes O_X(-\overline{D}_n) \cong O_E(\alpha_n) \to 0.$$ 

Taking global sections, we have

$$0 \to P_n^+ \to P_n \to \Gamma(X, O_E(\alpha_n)) \cong k[T, T']_{\alpha_n}$$

and therefore, an injective morphism

$$\psi_n : P_n / P_n^+ \to k[T, T']_{\alpha_n}.$$ 

Let us prove that $\psi_n$ is surjective. Let $C_1$ and $C_2$ be two nonsingular irreducible curves in $X$ intersecting transversally at two different points not belonging to any $E_\gamma \neq E$. We may take projective coordinates $T, T'$ in $E \cong \mathbb{P}^1$ so that these points are $\{0 : 1\}$ and $\{1 : 0\}$ respectively. Let us consider the divisors

$$D_{n,s} = \overline{D}_n + sC_1 + (\alpha_n - s)C_2 \quad \text{for } 0 \leq s \leq \alpha_n.$$ 

We have $D_{n,s} \cdot E_\gamma \leq 0$ for all $\gamma \in \Delta$, and besides

$$D_{n,s} \cdot E = \overline{D}_n \cdot E + s(C_1 \cdot E) + (\alpha_n - s)(C_2 \cdot E) = -\alpha_n + s + (\alpha_n - s) = 0.$$ 

For each $E_\gamma \neq E$, let $b_{n,s,\gamma} = -D_{n,s} \cdot E_\gamma \in \mathbb{N}$ and let $C_\gamma$ be a nonsingular irreducible curve in $X$ intersecting transversally $E_\gamma$ in a point not belonging to $E_{\gamma'}$ for $\gamma' \neq \gamma$. 


By Artin’s result (see beginning of section 2), there exist $Q_{n,s} \in R$, $0 \leq s \leq \alpha_n$ such that

$$(Q_{n,s})^* = D_{n,s} + \sum_{\gamma} b_{n,s,\gamma} C_{\gamma}.$$  

(7)

This implies that $Q_{n,s} \in P_n \setminus P_n^+$ and that

$$\psi_n(Q_{n,s}) = \lambda \ T^{s} T^{\rho_{n,-s}}$$

for some $\lambda \in k$, $\lambda \neq 0$. Therefore $\psi_n$ is surjective.

Now, for (i), we consider the new rational surface singularity $Spec R'$ and the same valuation $\nu$, whose center is $E$. We have $\nu(R' \setminus \{0\}) = \mathbb{N}$, the extremal divisor corresponding to $E$ is $D_1 = E$ and $D_1 \cdot E = a$. Therefore, the isomorphisms (5) in this case are

$$\psi_n : \mathcal{P}_n \rightarrow \mathcal{P}_n$$

where $\mathcal{P}_n$ and $\mathcal{P}_n^+$ are the corresponding $\nu$-ideals of $R'$. From this (i) follows.

For (ii), let $n \in \mathbb{N}$, and let $Q_{n,s} \in R$, $1 \leq s \leq \alpha_n$, be the elements satisfying (7). From (6) and (7) it follows that

$$(Q_{n,s})^* = nE + \sum_{j \in \Delta_E} c_j(n)E_j + s \ C_1 + (\alpha_n - s) \ C_2 + D'$$

where $D'$ is a divisor whose support does not intersect $E$. Therefore

$$\psi_n'(i(Q_{n,s})) = \lambda_s \left( \prod_{j \in \Delta_E} T^{c_j(n)} \right) T^{s} T^{\rho_{a_n,-s}}$$

for $1 \leq s \leq \alpha_n$

where $\lambda_s \in k \setminus \{0\}$. From this, (ii) follows and we conclude the proof of theorem 2.2.

If $z \Delta_E \geq 1$, i.e. there is at least one exceptional curve $E_j$ adjacent to $E$, we may improve the previous argument with the following one: Let us consider the configuration of exceptional curves $\{E_j\}_{\gamma \in \Delta} \setminus \{E\}$. It has as many irreducible components $\Gamma_j$ as exceptional curves $E_j$ intersecting $E$. For each $\Gamma_j$, let $(S_j, P_j)$ be the rational surface singularity obtained by contracting $\Gamma_j$, and let $Z_j$ be the fundamental cycle for the morphism $X \rightarrow S_j$. Then, we can replace the divisor $D_{n,s}$ in (6) by the following one: if $z \Delta_E \geq 2$, we choose two exceptional curves $E_1, E_2$ adjacent to $E$, and we set

$$D_{n,s} = \overline{D}_n + sZ_1 + (\alpha_n - s)Z_2$$

which is a divisor with exceptional support (while the divisor in (6) depends on the choice of the curves $C_j$). If $z \Delta_E = 1$, then we take the unique exceptional curve $E_1$ adjacent to $E$ and set $D_{n,s} = \overline{D}_n + sZ_1 + (\alpha_n - s)C_2$. The reason why these new divisors $D_{n,s}$ belong to $E_X$ is the following:

**Lemma 2.3.** The coefficient in $E_j$ of $Z_j$ is 1.

**Proof:** To compute the fundamental cycle $Z$ for the morphism $X \rightarrow S$, we may apply Laufer’s algorithm (see [CPR2] prop. 2.1) to any of the divisors $E_\gamma$, i.e. we consider a sequence $E_\gamma = \hat{D}_1 < \ldots < \hat{D}_t = Z$ where $\hat{D}_{i+1} = \hat{D}_i + E_\gamma$, for some $E_\gamma$ with $\hat{D}_r \cdot E_\gamma > 0$. In particular, we may take such a sequence $E_j = \hat{D}_1 < \ldots < \hat{D}_t = Z$ in such a way that the first steps consist of the Laufer’s algorithm to compute $Z_j$, hence, for some $r$, $0 \leq r < t$, we have $\hat{D}_r = Z_j$ and $E_\gamma = E$. By [La], theorem 4.2, we have $\hat{D}_i \cdot E_i = 1$ for all $i$, $0 \leq i < t$, therefore the coefficient of $Z_j$ in $E_j$ is $Z_j \cdot E = 1$. 

Remark 2.4. Lemma 2.3 has appeared in [Ok] lemma 3.5, and it is a key point in Okuma’s work [Ok]. The techniques in theorem 2.2 above have been applied in [Re1], [Re2].

Remark 2.5. The above proof gives an explicit way to determine elements \( \{Q_{n,s}\}_{n=0}^{\alpha_n} \) whose initial forms define a basis of \( \mathcal{P}_n / \mathcal{P}_n^+ \), for each \( n \in \Phi^+ \). More precisely, such that \( iQ_{n,s} = T^{\alpha_n s} (\prod_{j \in \Delta_E} T_j^{c_j(n_1+n_2)}) \). We will proceed as above for all \( n \in \Sigma \). Besides note that, if \( n \in \Sigma \) admits a decomposition \( n = n_1 + n_2 \) where \( n_1, n_2 \in \Sigma \), then

\[
(9) \quad i(Q_{n_1,s_1}Q_{n_2,s_2}) = \left( \prod_{j \in \Delta_E} T_j^{c_j(n_1+n_2)} \right) T^{s_1 + s_2} T^{\alpha_n s_1 + \alpha_n s_2 - s_1 - s_2} \left( \prod_{j \in \Delta_E} T_j^{b_j} \right)
\]

where \( b_j \in \mathbb{N} \) is the coefficient in \( E_j \) of \( \overline{\mathcal{D}}_{n_1} + \overline{\mathcal{D}}_{n_2} - \overline{\mathcal{D}}_{n_1+n_2} \). Therefore, to obtain a generating sequence for \( \nu \), for each \( n \in \Phi^+ \), we determine elements \( \{Q_{n,s}\}_{n} \) whose image by \( i \) is a basis of

\[
( \prod_{j \in \Delta_E} T_j^{c_j(n_1+n_2)}) k[T,T]/\langle \{T^{s_1 + s_2} T^{\alpha_n s_1 + \alpha_n s_2 - s_1 - s_2} ( \prod_{j \in \Delta_E} T_j^{b_j}) \rangle_{n=n_1+n_2, 0 \leq s_1 \leq \alpha_n, i=1,2}\n\]

Then the union of such elements is a finite generating sequence for \( \nu \).

Remark 2.6. Let \( R \) be a regular local ring and \( \nu \) a divisorial valuation dominating \( R \). Let \( X \) be the minimal nonsingular surface dominating Spec \( R \) such that the center of \( \nu \) in \( X \) is a curve \( E \), and let \( Q_{g+1} \in R \) define a curve whose strict transform in \( X \) is transversal to \( E \) in a point not belonging to any other exceptional curve. Let \( \overline{\mathcal{E}}_{g+1} := \nu(Q_{g+1}) \), and let \( \overline{\mathcal{E}}_0, \ldots, \overline{\mathcal{E}}_g \) be a minimal system of generators of \( \Phi^+ = \nu(R \setminus \{0\}) \). Then, with the notation in 2.1 and 2.2, \( p = \overline{\mathcal{E}}_{g+1} \), \( \alpha_n = 0 \) for all \( n \in \Sigma \setminus \{\overline{\mathcal{E}}_{g+1}\} \) and \( a_n^{g+1} = 1 \). Any \( n \in \Sigma \setminus \{\overline{\mathcal{E}}_0, \ldots, \overline{\mathcal{E}}_g\} \) decomposes as \( n = n_1 + n_2 \) for some \( n_1, n_2 \in \Sigma \). Therefore, to obtain a generating sequence for \( \nu \), we have to define a generator \( Q_i \) of \( \mathcal{P}_{\overline{\mathcal{E}}_i} / \mathcal{P}_{\overline{\mathcal{E}}_i}^+ \) for \( 0 \leq i \leq g \) as in (7) and (8), and, if \( \overline{\mathcal{E}}_g / \overline{\mathcal{E}}_{g+1} \) then also \( Q_{g+1} \) for \( \mathcal{P}_{\overline{\mathcal{E}}_{g+1}} / \mathcal{P}_{\overline{\mathcal{E}}_{g+1}}^+ \). If \( \overline{\mathcal{E}}_g / \overline{\mathcal{E}}_{g+1} \), the \( \overline{\mathcal{E}}_{g+1} \) has two different expressions \( \overline{\mathcal{E}}_{g+1} = n_0 \overline{\mathcal{E}}_0 + \ldots + a_g \overline{\mathcal{E}}_g \) in terms of \( \overline{\mathcal{E}}_0, \ldots, \overline{\mathcal{E}}_g \) from which, by (9), it follows that we do not have to add any element in \( \mathcal{P}_{\overline{\mathcal{E}}_{g+1}} / \mathcal{P}_{\overline{\mathcal{E}}_{g+1}}^+ \) (see [Sp], theorem 8.6).

Example 2.7. Let \( S \) be the blowing up of the ideal \( I = (x^6, y) \cdot (x^2, y + x) \cdot (x^2, y - x) \) of the regular local ring \( k[x,y][x,y] \). The surface \( S \) has one singular point \( P \). Let \( \pi : X \rightarrow S \) be the minimal desingularization of \( (S,P) \). The dual graph of \( \pi \) is

```
\begin{figure}[h]
\centering
\begin{tikzpicture}
\node (1) at (-0.5,0) {$e_1$};
\node (2) at (0,0) {$e_2$};
\node (3) at (0.5,0) {$e_3$};
\node (4) at (1,0) {$e_4$};
\node (5) at (1.5,0) {$e_5$};
\draw (1) -- (2);
\draw (2) -- (3);
\draw (3) -- (4);
\draw (4) -- (5);
\end{tikzpicture}
\caption{fig. 1}
\end{figure}
```

where \( e_i \) represents an exceptional curve \( E_i \), and we have \( E_1^2 = -4 \), \( E_2^2 = E_3^2 = E_4^2 = E_5^2 = -2 \). Let us consider the divisorial valuation \( \nu \) defined by \( E_2 \). Then
\[ D_1 = E_1 + E_2 + E_3 + E_4 + E_5 \]
\[ D_2 = E_1 + 2E_2 + 2E_3 + 2E_4 + E_5 \]
\[ D_3 = E_1 + 3E_2 + 3E_3 + 2E_4 + E_5 \]
\[ \tilde{D} = D_4 = E_1 + 4E_2 + 3E_3 + 2E_4 + E_5 \]

where we have represented with weighted arrows the intersection of each divisor \( D_n \) with the \( E_i \) corresponding to the basis of the arrow.

Therefore, to obtain a generating sequence for \( \nu \), we need elements \( Q_1 \in P_1/P_1^+ \), \( Q_2 \in P_2/P_2^+ \), \( Q_3 \in P_3/P_3^+ \) and \( Q_{40}, Q_{44} \in P_4/P_4^+ \) whose images by the embedding \( gr_\nu R \hookrightarrow V(2) \) in theorem 2.2 are \( TT', TT'^3, TT'^5, TT'^7 \) respectively. Since \( x, y, x, y, x, y \) are coordinates in \( O_{S,P} \), we may take

\[ Q_1 = x, \quad Q_2 = y, \quad Q_3 = \frac{y^2}{y-x} = \frac{(y+x) + \frac{x^2}{y-x}}{y-x}, \]
\[ Q_{40} = \frac{x^6}{y}, \quad Q_{44} = \frac{y^3}{(y-x)(y+x)} = \frac{1}{2} \left( \frac{x^2}{y-x} + \frac{1}{2} \frac{x^2}{y-x} \right) \]

Therefore, \( gr_\nu R \) is isomorphic to

\[ k[\overline{Q}_1, \overline{Q}_2, \overline{Q}_3, \overline{Q}_{4,0}, \overline{Q}_{4,4}] \rightarrow \begin{pmatrix} \overline{Q}_2 - \overline{Q}_3(\overline{Q}_2 - \overline{Q}_1) \\ \overline{Q}_1 - \overline{Q}_{4,0}\overline{Q}_2 \\ \overline{Q}_2 - \overline{Q}_{4,4}(\overline{Q}_2^2 - \overline{Q}_1^2) \end{pmatrix} \]

where \( \overline{Q}_i \) has degree \( i \), for \( 1 \leq i \leq 3 \), and \( \overline{Q}_{4,0} \) and \( \overline{Q}_{4,4} \) have degree 4.

**3. Some invariants of the singularity recovered from the graded algebra associated to a divisorial valuation**

In this section we will describe some invariants of \( (S,P) \) which can be recovered from the graded algebra \( gr_\nu R \) of a divisorial valuation \( \nu \) dominating \( R \). Most of them will be determined by the Hilbert-Samuel function of \( gr_\nu R \).

The Hilbert-Samuel function defined by \( \nu \) is the function

\[ \mathbb{N} \rightarrow \mathbb{N}, \quad n \mapsto h(n) := l(R/P_n). \]

Therefore, it is an equivalent data to the Samuel function defined by \( \nu \),

\[ \Phi^+ \rightarrow \mathbb{N}, \quad n \mapsto l(P_n/P_n^+) = \alpha_n + 1 \]

where \( \alpha_n := -D_n \cdot E \) (see equality (4)).

The following result improves theorem 4.2 in [CPR2].

**Proposition 3.1.** For all \( n \in \mathbb{N} \) we have

\[ l(R/P_n) = Q(n) + \varphi(n) \]
where \( Q(n) \) is a polynomial of degree two in \( n \) and \( \varphi(n) \) is a periodic function of period \( p = \nu(\tilde{P}) \) for \( n >> 0 \).

Proof: If, for any \( \mathbb{Q} \)-Cartier divisor \( B \) on \( X \), we set \( \chi(B) := -\frac{1}{2}B \cdot (B + K) \), where \( K \) is a canonical divisor on \( X \), then

\[
Q(n) := \chi \left( \frac{n}{p} \tilde{D} \right)
\]

is a polynomial of degree two in \( n \). Besides, if we denote by \( c(n) \) the coefficient of \( \tilde{D} \) in \( E \), then equality (3) implies that, for \( n >> 0 \), the function

\[
\mathbb{N} \to (\mathbb{E}_X)_\mathbb{Q} \quad n \mapsto \tilde{D}_n := \tilde{D}_n - \frac{c(n)}{p} \tilde{D}
\]

is periodic and its period divides \( p \), where \( (\mathbb{E}_X)_\mathbb{Q} := \sum_{\gamma \in \Delta} \mathbb{Q} \gamma \). We have

\[
l(R/P_n) = Q(n) + \chi(\tilde{D}_n).
\]

(see [CPR2], theorem 4.2). Therefore, \( l(R/P_n) \) is expressed as stated in the proposition, where \( \varphi(n) := \chi(\tilde{D}_n) \) is a periodic function, for \( n >> 0 \), whose period divides \( p \). We have to prove that the period of \( \varphi \) is \( p \).

For \( n \in \Phi^+ \), let

\[
L(n) := Q(n + 1) - Q(n) \quad \psi(n) := \varphi(n + 1) - \varphi(n).
\]

Then, for \( n \in \mathbb{N} \), \( l(P_n/P_n^+) = L(n) + \psi(n) \) and

\[
\alpha_n + 1 = L(n) + \psi(n) \quad \text{for } n \in \Phi^+.
\]

The function \( \psi(n) \) is periodic for \( n >> 0 \), and its period divides the period of \( \varphi \), hence it divides \( p \).

Let \( q \) be the period of \( \psi \), thus \( q \) divides \( p \). We have

\[
L(n) = \chi \left( \frac{n + 1}{p} \tilde{D} \right) - \chi \left( \frac{n}{p} \tilde{D} \right) = \chi \left( \frac{1}{p} \tilde{D} \right) + \left( \frac{n}{p} \tilde{D} \right)
\]

hence,

\[
\psi(n) = -\tilde{D}_n \cdot E + 1 - L(n) = -\tilde{D}_n \cdot E + 1 - \chi \left( \frac{1}{p} \tilde{D} \right) + \frac{n}{p} \tilde{D} \cdot E.
\]

Therefore, \( \psi(n + q) = \psi(n) \) is equivalent to

\[
(\tilde{D}_{n+q} - \tilde{D}_n) \cdot E = \frac{q}{p} \tilde{D} \cdot E.
\]

This equality for \( n >> 0 \) implies that

\[
\tilde{D}_{rp+q} \cdot E = \left( r + \frac{q}{p}k \right) \tilde{D} \cdot E \quad \text{for } r >> 0, \ 1 \leq k \leq \frac{p}{q},
\]

In fact, given \( r >> 0 \), the previous equality for \( k = \frac{q}{p} \) is consequence of (3) and, by inverse recurrence, if it holds for \( k + 1 \), then (10) implies that it also holds for \( k \).

For \( r >> 0 \), let us consider that \( \mathbb{Q} \)-divisor

\[
D' := \tilde{D}_{rp+q} - \left( r + \frac{q}{p} \right) \tilde{D}.
\]

By (11), \( D' \) belongs to the cone \( (\mathbb{E}_X^+)_\mathbb{Q} \) in \( (\mathbb{E}_X)_\mathbb{Q} \) defined by \( \mathbb{E}_X^+ \). Since, for \( r >> 0 \), we have \( rp + q \in \Phi^+ \), the coefficient of \( D' \) in \( E \) is \( r + q + \left( r + \frac{q}{p} \right) p = 0 \). This implies that \( D' = 0 \), i.e. \( \tilde{D}_{rp+q} = \left( r + \frac{q}{p} \right) \tilde{D} \). Then \( \frac{q}{p} \tilde{D} \) is a divisor and, by the definition of \( \tilde{D} \), it follows that \( p \) divides \( q \), hence \( q = p \).
Theorem 3.2. Let \( n \in \mathbb{N} \) be any multiple of \( p = \nu(P) \) such that \( n-1 \in \Phi^+ \). Then
\[
\mathcal{D}_n = \mathcal{D}_{n-1} + E.
\]
(12)

Therefore,
\[
-E^2 = \alpha_n - \alpha_{n-1}.
\]
(13)

Proof: First, let us show that \( \text{Supp}(\mathcal{D}_n - \mathcal{D}_{n-1}) \) is connected. This will hold for any \( n \in \Phi^+ \) such that \( n-1 \in \Phi^+ \). Let \( \{E_{\gamma}\}_{\gamma \in \Delta'} \) be the configuration of exceptional curves defined by the connected component of \( \text{Supp}(\mathcal{D}_n - \mathcal{D}_{n-1}) \) which contains \( E \).

Let \( \mathcal{D} \in \mathcal{E}_X \) be such that \( \mathcal{D} = \mathcal{D}_n \) on \( \cup_{\gamma \in \Delta'} E_{\gamma} \) and \( \mathcal{D} = \mathcal{D}_{n-1} \) on \( \cup_{\gamma \in \Delta \setminus \Delta'} E_{\gamma} \). Let us show that \( \mathcal{D} \in \mathcal{E}_X \). Then, since \( \mathcal{D}_{n-1} + E \leq \mathcal{D} \leq \mathcal{D}_n \), we will have \( \mathcal{D} = \mathcal{D}_n \) and hence, for any \( \gamma \in \Delta \setminus \Delta' \), the coefficient of \( (\mathcal{D}_n - \mathcal{D}_{n-1}) \) in \( E_{\gamma} \) will be 0. Therefore \( \text{Supp}(\mathcal{D}_n - \mathcal{D}_{n-1}) \) will be connected. By the definition of \( \Delta' \), for any \( \gamma \in \Delta' \) we have
\[
\mathcal{D} \cdot E_{\gamma} = \mathcal{D}_n \cdot E_{\gamma} \leq 0.
\]

If \( \gamma \notin \Delta' \) is such that \( E_{\gamma} \) is not adjacent to any \( E_{\gamma'} \) for \( \gamma' \in \Delta' \), then
\[
\mathcal{D} \cdot E_{\gamma} = \mathcal{D}_{n-1} \cdot E_{\gamma} \leq 0.
\]

Finally, if \( \gamma \notin \Delta' \) is such that \( E_{\gamma} \) is adjacent to some \( E_{\gamma'} \), for \( \gamma' \in \Delta' \), then the coefficient of \( \mathcal{D} \) in \( E_{\gamma} \) is equal to the coefficient of \( \mathcal{D}_n \) in \( E_{\gamma} \). Let \( \{b_{\beta}\}_{\beta \in \Delta_{E_{\gamma}}} \) be the coefficients of \( \mathcal{D}_n - \mathcal{D}_{n-1} \) in the \( E_{\beta} \)'s adjacent to \( E_{\gamma} \). Then
\[
\mathcal{D} \cdot E_{\gamma} = \mathcal{D}_n \cdot E_{\gamma} - \sum_{\beta \in \Delta_{E_{\gamma}}} b_{\beta} \leq \mathcal{D}_n \cdot E_{\gamma} \leq 0
\]
since \( b_{\beta} \geq 0 \) for all \( \beta \in \Delta_{E_{\gamma}} \). Thus, \( \text{Supp}(\mathcal{D}_n - \mathcal{D}_{n-1}) \) is connected.

Now, let \( n \) be as in the statement of the theorem, and let \( \Delta' \subseteq \Delta \) be such that \( \text{Supp}(\mathcal{D}_n - \mathcal{D}_{n-1}) = \cup_{\gamma \in \Delta'} E_{\gamma} \). Let \( S' \) be the surface obtained by contracting \( \cup_{\gamma \in \Delta'} E_{\gamma} \). Then, \( S' \) has only a singular point which is a rational surface singularity. Therefore, for any effective divisor \( D \) with support in \( \cup_{\gamma \in \Delta'} E_{\gamma} \) we have \( p_n(D) \leq 0 \) ([Ar1], 1.7).

Let us apply this to the divisor \( D = \mathcal{D}_n - \mathcal{D}_{n-1} \). For any \( \gamma \in \Delta' \) we have
\[
D \cdot E_{\gamma} = (\mathcal{D}_n - \mathcal{D}_{n-1}) \cdot E_{\gamma} = \begin{cases} 
-\mathcal{D}_{n-1} \cdot E_{\gamma} \geq 0 & \text{if } E_{\gamma} \neq E \\
E^2 + \sum_{j \in \Delta_{E_{\gamma} \cap \Delta'}} b_j & \text{if } E_{\gamma} = E
\end{cases}
\]
where, for any \( \gamma \in \Delta' \), \( b_{\gamma} \) is the coefficient of \( \mathcal{D}_n - \mathcal{D}_{n-1} \) in \( E_{\gamma} \), in particular, the \( \{b_{\gamma}\}_{\gamma \in \Delta_{\gamma} \cap \Delta'} \) are the nonzero coefficients of \( \mathcal{D}_n - \mathcal{D}_{n-1} \) in the \( E_{\gamma} \)'s adjacent to \( E \) (in the above equality we use the fact that \( n \) is a multiple of \( p \), since it implies that \( \mathcal{D}_n \cdot E_{\gamma} = 0 \) for \( E_{\gamma} \neq E \)). From the above equality, it follows that
\[
D^2 = \sum_{\gamma \in \Delta', E_{\gamma} \neq E} b_{\gamma}(D \cdot E_{\gamma}) + (D \cdot E) \geq D \cdot E = E^2 + \sum_{j \in \Delta_{E_{\gamma} \cap \Delta'}} b_j.
\]

Let \( K \) be a canonical divisor on \( X \). Since \( p_n(E_{\gamma}) = 0 \) for all \( \gamma \in \Delta \), by the adjunction formula we have \( K \cdot E_{\gamma} = -2 - E_{\gamma}^2 \), and, since \( X \) is the minimal desingularization of \( S \) where the center of \( n \) is a curve \( E \), then \(-E_{\gamma}^2 \geq 2 \) for \( E_{\gamma} \neq E \). Thus,
\[
D \cdot K = \sum_{\gamma \in \Delta'} b_{\gamma}(E_{\gamma} \cdot K) = \sum_{\gamma \in \Delta'} b_{\gamma}(-2 - E_{\gamma}^2) \geq -2 - E^2.
\]

By the adjunction formula, we have
\[
p_n(D) = 1 + \frac{1}{2} D \cdot (D + K) \geq 1 + \frac{1}{2} \left( E^2 + \sum_{j \in \Delta_{E_{\gamma} \cap \Delta'}} b_j - 2 - E^2 \right) = \frac{1}{2} \sum_{j \in \Delta_{E_{\gamma} \cap \Delta'}} b_j.
\]
Besides, from all the graded algebras $\{\text{Hilbert-Samuel function of the graded algebra defined by matrix (12)}\}$, by the adjunction formula, $l(R/\tilde{P}) = -\frac{1}{2} \tilde{D} \cdot (\tilde{D} + K)$. We have $\tilde{D}^2 = -p\alpha_p$, hence we obtain $\tilde{D} \cdot K$. Then $\text{cor}(\tilde{D} \cdot K) = -\alpha_p$. Therefore, from all the Hilbert-Samuel functions defined by the divisorial valuations dominating $(S, P)$, we recover:

(i) The number $n$ of exceptional curves $E_j$ which are adjacent to $E$.

(ii) For each $n \in \Phi^+$, the coefficients $\{c_j(n)\}_{j \in \Delta E}$ of $\tilde{D}_n$ in the $E_j$’s adjacent to $E$.

(iii) The relative position of the intersection points $\{E_j \cap E\}_{j \in \Delta E}$ in $E \cong \mathbb{P}^1$.

Therefore, from all the Hilbert-Samuel functions defined by the divisorial valuations dominating $(S, P)$, we recover:

(iv) The integer $a = -E^2$.

Moreover, from the graded algebra $gr_\nu R$, the following is determined:

(v) The semigroup $\Phi^+ = \nu(R \setminus \{0\})$ and the function $\Phi^+ \to \mathbb{N}$, $n \mapsto \alpha_n = -\tilde{D}_n \cdot E$.

(vi) The equation for $E$ in the fundamental cycle $Z$ for $\pi : X \to S$.

(vii) The integer $p = \nu(\tilde{P})$.

(viii) The number of irreducible exceptional curves in the minimal desingularization of $(S, P)$, and their self-intersections.

(ix) The determinant of the intersection matrix of the exceptional curves for the minimal desingularization of $(S, P)$.

Corollary 3.3. From the Hilbert-Samuel function of the graded algebra $gr_\nu R$ associated to a divisorial valuation $\nu$ dominating $R$, we can determine whether the valuation $\nu$ is essential, i.e. it is defined by an irreducible component of the exceptional locus of the minimal desingularization.

Proof: It follows directly from theorem 3.2, since $E$ is essential if and only if $-E^2 \geq 2$.

Corollary 3.4. Let $(S, P)$ be a rational surface singularity, $\nu$ a divisorial valuation dominating $(S, P)$ and $\pi : X \to S$ the minimal desingularization such that the center of $\nu$ in $X$ is a curve $E$. Let $\{E_{\gamma}\}_{\gamma \in \Delta}$ be the irreducible components of the exceptional locus of $\pi$. The following invariants of $(S, P)$ are determined from the Hilbert-Samuel function of the graded algebra defined by $\nu$:

(i) The semigroup $\Phi^+ = \nu(R \setminus \{0\})$ and the function $\Phi^+ \to \mathbb{N}$, $n \mapsto \alpha_n = -\tilde{D}_n \cdot E$.

(ii) The coefficient of $E$ in the fundamental cycle $Z$ for $\pi : X \to S$.

(iii) The integer $p = \nu(\tilde{P})$.

(iv) The coefficient of $E$ in the unique canonical divisor of $X$ with exceptional support for $\pi$.

(v) The integer $a = -E^2$.

Therefore, from all the Hilbert-Samuel functions defined by the divisorial valuations dominating $(S, P)$, we recover:

(i) The number $n$ of exceptional curves in the minimal desingularization of $(S, P)$, and their self-intersections.

(ii) The multiplicity of $(S, P)$.

(iii) The determinant of the intersection matrix of the exceptional curves for the minimal desingularization of $(S, P)$.

Besides, from all the graded algebras $\{gr_\nu R\}_\nu$ where $\nu$ is any divisorial valuation dominating $(S, P)$ we also obtain:

(vi) For each exceptional curve $E$ for the minimal desingularization, the integer $v_E = \sharp \Delta E$ and the relative position of the intersection points $\{E_j \cap E\}_{j \in \Delta E}$.

Proof: (i) is clear. For (ii) it suffices to note that the coefficient in $E$ of $Z$ is the smallest element in $\Phi^+$. From proposition 3.1 and theorem 3.2, (iii) and (v) follow. For (iv), let $K$ be a canonical divisor on $X$. By the negativity of the intersection matrix $(E_{\gamma} \cdot E_{\gamma'})_{\gamma, \gamma' \in \Delta}$, there exists a unique $\mathbb{Q}$-Cartier divisor $K_0$ with exceptional support for $\pi$ such that $K_0 \cdot E_{\gamma} = K \cdot E_{\gamma}$ for all $\gamma \in \Delta$ being this integer equal to $-2 - E^2_{\gamma}$ by the adjunction formula. On the other hand, the integer $p = \nu(\tilde{P})$ is determined by the Hilbert-Samuel function of $gr_\nu R$ and, by the adjunction formula, $l(R/\tilde{P}) = -\frac{1}{2} \tilde{D} \cdot (\tilde{D} + K)$. We have $\tilde{D}^2 = -p\alpha_p$, hence we obtain $\tilde{D} \cdot K$. Then $\text{cor}(\tilde{D} \cdot K) = -\alpha_p$. Therefore, from all the graded algebras $\{\text{Hilbert-Samuel function of the graded algebra defined by matrix (12)}\}$, by the adjunction formula, $l(R/\tilde{P}) = -\frac{1}{2} \tilde{D} \cdot (\tilde{D} + K)$. We have $\tilde{D}^2 = -p\alpha_p$, hence we obtain $\tilde{D} \cdot K$. Thus $K_0$ is the unique canonical divisor with exceptional support for $\pi$, and (iv) follows.
Suppose that we know, not only the Hilbert-Samuel function, but also the graded algebra $gr_{\nu}R$ associated to $\nu$. Let us consider the fraction field of $gr_{\nu}R$, let us fix $n \in \mathbb{N}^+$, and let
\[
\Omega = \{(s_1, s_2) \in (P_n/P^+_{\kappa})^2 / \frac{s_1}{s_2} \text{ is an } \alpha_n \text{-power}\}.
\]
Then, for any embedding $j : gr_{\nu}R \hookrightarrow V(a)$ where $a = -E^2$, known by (v), the greatest common divisor of $j(s_1), j(s_2)$, where $(s_1, s_2) \in \Omega$, determines $\prod_{j \in \Delta_n} T_{j}^{c_{\gamma}(a)}$ in equality (4) modulo a unit. From this, (vi), (vii) and (viii) follow.

Now, suppose that we know all the Hilbert-Samuel functions of the divisorial valuations $\nu$ dominating $(S, P)$. By (v), the data (ix) is determined. Let us consider the set $\{h_{\gamma}\}_{\gamma \in \Delta_n}$ of all the Hilbert-Samuel functions of the essential valuations and, for each $h_{\gamma}$, let $z_{\gamma}$ be the coefficient of $Z$ in the corresponding exceptional curve $E_{\gamma}$, and $\alpha_{2v}^{\gamma} = -Z \cdot E_{\gamma}$, which are determined from $h_{\gamma}$. Then the multiplicity of $(S, P)$ is
\[
\text{mult } S = -Z^2 = \sum_{\gamma} z_{\gamma} \alpha_{2v}^{\gamma}
\]
(see [Ar2] theorem 4). For (xi) it suffices to note that, if $p_{\gamma}$ is the integer in (iii) obtained from $h_{\gamma}$, then the determinant of the intersection matrix $(E_{\gamma} \cdot E_{\gamma'})_{\gamma, \gamma' \in \Delta_n}$ is equal to the smallest common multiple of $\{\alpha_{2v}^{\gamma}\}_{\gamma \in \Delta_n}$. Finally, if we know all graded algebras $\{gr_{\nu}R\}_\nu$, then, by (v), we may take the ones defined by an essential valuation, and hence (xii) follows from (vi) and (viii).

4. The dual graph is not recovered from the graded algebras

Given a normal surface singularity $(S, P)$ over an algebraically closed field of characteristic zero, from the set $\{gr_{\nu}R\}_\nu$ of all graded algebras associated to divisorial valuations $\nu$ dominating $(S, P)$, we can determine whether $(S, P)$ is a rational surface singularity. In fact, for any normal surface singularity $(S, P)$ over any field $k$, $(S, P)$ satisfies the property that, for all $\nu$, the graded algebra $gr_{\nu}R$ is finitely generated if and only if its group $Cl(S, P)$ of classes of divisors is a torsion group ([Go], [Cu]) and, if the base field $k$ is algebraically closed of characteristic zero, this is equivalent to $(S, P)$ being a rational surface singularity ([Li]).

Besides, we have shown in the last section that, from the data $\{gr_{\nu}R\}_\nu$ of all graded algebras associated to divisorial valuations dominating a rational surface singularity $(S, P)$, we can recover the ones associated to the essential valuations, i.e. the divisorial valuations defined by an irreducible component $E$ of the exceptional locus of the minimal desingularization of $(S, P)$. Moreover, we also recover the self-intersection number of these exceptional curves $E$, the number of other exceptional curves in the minimal desingularization which are adjacent to $E$, and some other invariants (corollary 3.4). From these observations, a natural question arises:

**Question 4.1.** Is the dual graph of the minimal desingularization of a rational surface singularity determined by the set $\{gr_{\nu}R\}_\nu$ of all graded algebras associated to divisorial valuations dominating $(S, P)$?

We will show in this section that, in general, *the answer to this question is no*. More precisely, we will give two minimal singularities (i.e. rational surface singularities whose fundamental cycle is reduced), $(S_1, P_1)$ and $(S_2, P_2)$, whose dual graphs of the respective minimal desingularizations are not isomorphic, and such that there
exists a one to one correspondence $\theta : \mathcal{V}_1 \rightarrow \mathcal{V}_2$ between the sets $\mathcal{V}_1$ and $\mathcal{V}_2$ of divisorial valuations dominating $(S_1, P_1)$ and $(S_2, P_2)$ respectively, such that, for all $\nu_1 \in \mathcal{V}_1$, the graded algebras associated respectively to $\nu_1$ and $\theta(\nu_1)$ are isomorphic (proposition 4.3).

Let $\pi : X \rightarrow S$ be a desingularization and let $\{E_{\gamma}\}_\gamma$ be the exceptional curves of $\pi$. By weighted dual graph of $\pi : X \rightarrow S$ we mean the weighted graph obtained from the dual graph of $\pi$ by adding, for each $\gamma$, the integer $-E_1^{2\gamma} = \gamma \Delta_{E_{\gamma}}$ is the number of exceptional curves adjacent to $E_{\gamma}$, then $a_{\gamma} + 1 \geq v_{\gamma}$ for any desingularization of a rational surface singularity ([La], th. 4.2). A given weighted graph is the weighted dual graph of a minimal desingularization of a minimal surface singularity if and only if $a_{\gamma} \geq v_{\gamma}$.

A cyclic quotient singularity is characterized by the shape of the weighted dual graph for its minimal desingularization, which is

\[
-a_1 \quad -a_2 \quad \ldots \ldots \ldots \quad -a_{n-1} - a_n
\]

where $a_i \geq 2$. Therefore, a cyclic quotient singularity is characterized by the property that $v_E = 2$ for all essential curves $E$ except for two of them, for which $v_E = 1$. Hence, it is characterized by the the set $\{gr_{\nu} R\}_\nu$ of all graded algebras associated to divisorial valuations $\nu$ dominating $(S, P)$ which dominate it (corollary 3.4).

**Proposition 4.2.** Let $(S, P)$ be a cyclic quotient singularity. Then question 4.1 has an affirmative answer. More precisely, if $(S_2, P_2)$ is a rational surface singularity such that there exists a one to one correspondence $\theta : \mathcal{V}_1 \rightarrow \mathcal{V}_2$, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are the sets of divisorial valuations dominating $(S, P)$ and $(S_2, P_2)$ respectively, such that, for all $\nu \in \mathcal{V}_1$, $\text{gr}_{\nu} O_{S, P} \cong \text{gr}_{\theta(\nu)} O_{S_2, P_2}$, then $(S, P)$ and $(S_2, P_2)$ are isomorphic.

**Proof:** Let $\{E_i\}_{i=1}^n$ be the irreducible components of the exceptional locus of the minimal desingularization of $(S, P)$, where $E_i$ is represented by the $i$-th vertex in figure 4, hence $-E_1^2 = a_1 \geq 2$. Let $d_n$ be the determinant of the intersection matrix $(E_i \cdot E_j)_{1 \leq i, j \leq n}$ and let $d_{n-1}$ be the $(n-1) \times (n-1)$-minor $(E_i \cdot E_j)_{1 \leq i, j \leq n-1}$. Then, since $a_i \geq 2$ for $1 \leq i \leq n$, the sequence $(a_1, \ldots, a_n)$ is uniquely determined by the expression as continued fraction of $d_n/d_{n-1}$. Besides, $d_{n-1} = \frac{p}{\alpha_p}$ where $p$ and $\alpha_p$ are the data associated to the valuation corresponding to the exceptional $E_n$, hence determined from its Hilbert-Samuel function. Thus, from (i), (iii), (vi) and (xi) of corollary 3.4 it follows that the weighted dual graphs of the minimal desingularizations of $(S, P)$ and $(S_2, P_2)$ are the same, and $(S, P)$, isomorphic to $(S_2, P_2)$, is the toric singularity defined by the cone $\langle (1, 0), (d_n - d_{n-1}, d_n) \rangle \subset \mathbb{R}^2$ (see [Od], lemma 1.22 and corol. 1.23).

**Corollary 4.3.** Let $(S, P)$ be a normal surface singularity over an algebraically closed field of characteristic zero. From the set $\{\text{gr}_{\nu} R\}_\nu$ of all graded algebras associated to the divisorial valuations $\nu$ dominating $(S, P)$, we can determine whether $(S, P)$ is a cyclic quotient singularity. Moreover, a cyclic quotient singularity is determined up to isomorphism by the collection of graded algebras $\{\text{gr}_{\nu} R\}_\nu$ associated to the divisorial valuations $\nu$ dominating it.
Let us consider the following two weighted dual graphs, that will be called respectively \( G_1 \) and \( G_2 \),

\[
\begin{array}{ccc}
F & -3 & F' \\
| & & |
\end{array} \quad \begin{array}{ccc}
F & -3 & F' \\
| & & |
\end{array}
\]

where \( F \) and \( F' \) are respectively

\[
\begin{array}{ccc}
F & -2 & F' \\
\downarrow & & \downarrow \\
-4 & -3 & -2
\end{array}
\]

and they are joined in figure 6 by the small segments in the corners. Both \( G_1 \) and \( G_2 \) satisfy \( a_\gamma \geq \psi_\gamma \) for all \( \gamma \). Therefore, there exist two minimal singularities \((S_1, P_1)\) and \((S_2, P_2)\) whose weighted dual graphs for their minimal desingularizations are respectively \( G_1 \) and \( G_2 \). Moreover, we may also ask \((S_1, P_1)\) and \((S_2, P_2)\) to satisfy the following property: Let \( \{E_\alpha\}_{\alpha \in \Delta} \) (resp. \( \{E'_\alpha\}_{\alpha \in \Delta} \)) be the exceptional curves of the minimal desingularization of \((S, P)\) (resp. \((S_2, P_2)\)) and \( \{E_\beta\} \) (resp. \( \{E'_\beta\} \)) be the curves corresponding to the \(-3\)-vertices in fig. 6, and, for \( j = 1, 2 \), let \( \alpha_{j,1}, \alpha_{j,2} \in \Delta \setminus \{\beta_1, \beta_2\} \) be such that \( E_{\alpha_{j,1}} \) and \( E_{\alpha_{j,2}} \) (resp. \( E'_{\alpha_{j,1}} \) and \( E'_{\alpha_{j,2}} \)) intersect \( E_{\beta_j} \) (resp. \( E'_{\beta_j} \)). Then, for \( j = 1, 2 \), the relative position of the three points \( E_{\alpha_{j,1}} \cap E_{\beta_j}, E_{\alpha_{j,2}} \cap E_{\beta_j}, E_{\beta_j} \cap E_{\beta_j} \) in \( E_{\beta_j} \cong \mathbb{P}^1 \) is the same as the relative position of the three points \( E'_{\alpha_{j,1}} \cap E'_{\beta_j}, E'_{\alpha_{j,2}} \cap E'_{\beta_j}, E'_{\beta_j} \cap E'_{\beta_j} \) in \( E'_{\beta_j} \cong \mathbb{P}^1 \), and moreover, this relative position for \( j = 1 \) is the same as the one for \( j = 2 \).

Let \( R_1 := \mathcal{O}_{S_1, P_1} \) and \( R_2 = \mathcal{O}_{S_2, P_2} \), and, for \( i = 1, 2 \), let \( \mathcal{V}_i \) be the set of divisorial valuations dominating \( R_i \). Then,

**Proposition 4.4.** There exists a one to one correspondence \( \theta : \mathcal{V}_1 \to \mathcal{V}_2 \) such that, for all \( \nu_1 \in \mathcal{V}_1 \), the graded algebra \( \text{gr}_\nu R_1 \) and \( \text{gr}_{\theta(\nu_1)} R_2 \) are isomorphic. But the dual graphs of the minimal desingularizations of the singularities \((S_1, P_1)\) and \((S_2, P_2)\) are not isomorphic.

In order to prove proposition 4.3, let us first study a way to obtain the Hilbert-Samuel function in terms of some local data, which holds under certain conditions, in particular for minimal surface singularities. Let \( \pi : X \to S \) be a desingularization of a rational surface singularity \((S, P)\), and let \( \{E_\gamma\}_{\gamma \in \Delta} \) be the irreducible components of the exceptional locus of \( \pi \). Let \( E \) be one of them, \( \{E_j\}_{j \in \Delta_E} \) the exceptional curves which are adjacent to \( E \), and \( a_j := -E_j^2 \). Let \( \{\Gamma_j\}_{j \in \Delta_E} \) be the connected components of \( \bigcup_{\gamma \in \Delta} E_\gamma \setminus E \) where, for each \( j \in \Delta_E \), \( E_j \) is contained in \( \Gamma_j \). For each \( j \in \Delta_E \), let \( \Sigma_j \) the configuration of curves obtained from \( \Gamma_j \) by substituting \( E_j \) by another curve \( E'_j \) also isomorphic to \( \mathbb{P}^1 \) and with the same intersection numbers with the other curves in \( \Gamma_j \), but with self-intersection \(-(a_j - 1)\). Suppose that there exists a rational surface singularity \((Y_j, Q_j)\) and a desingularization \( \pi_j : X_j \to Y_j \) such that the configuration of exceptional curves for \( \pi_j \) is \( \Sigma_j \), in particular this implies that \( a_j \geq 1 \). It happens, for instance, if \((S, P)\) is a minimal surface singularity, \( X \) is a desingularization of \((S, P)\), and \( E \) is an irreducible component of its
Let $h_j$ be the Hilbert-Samuel function of the valuation $\nu_j$ on $S_j$ defined by $E_j$. Then

**Lemma 4.5.** The Hilbert-Samuel function $h$ of the valuation on $(S, P)$ defined by $E$ and, moreover, the coefficients $(c_j(n))_{j \in \Delta_E}$ in the $E_j$’s adjacent to $E$ of the divisors $\{D_n\}_{n \in \Phi^+}$ associated to this valuation, can be determined from $\{h_j\}_{j \in \Delta_E}$ and $E^2$.

**Proof:** Let $p = \nu(\overline{P})$ and let $r > 0$ be such that $rp$ is greater or equal to the conductor of the semigroup of $\Phi^+$. For $n \in \mathbb{N} \cup \{0\} \setminus \Phi^+$, let

$$D_n := \overline{D}_{rP+n} - r\overline{D}$$

(see (3)) and, for $n \in \Phi^+$, let $D_n := \overline{D}_n$. For any $n \in \mathbb{N}$, let $\alpha_n := -D_n \cdot E$. Then, the function $\mathbb{N} \to \mathbb{N}$, $n \mapsto \alpha_n - \alpha_{n-1}$ is a periodic function (for all $n \in \mathbb{N}$ of period $p$ (proof of proposition 3.1).

For any $j \in \Delta_E$, let us do the same construction as before, obtaining thus divisors $\{D_n^j\}_{n \in \mathbb{N} \cup \{0\}}$ on $\Sigma_j$ and a function $\mathbb{N} \to \mathbb{N}$, $n \mapsto \alpha_n^j$, such that $n \mapsto \alpha_n^j - \alpha_n^{j-1}$ is periodic, let $p_j$ be its period. We will show by induction on $n$ that for all $n \in \mathbb{N}$, the coefficients $(c_j(n))_{j \in \Delta_E}$, and hence the integer $\alpha_n$, are determined from $\{h_j\}_{j \in \Delta_E}$ and $a := -E^2$.

Let us first show that $(c_j(1))_{j \in \Delta_E}$ are determined from $\{h_j\}_{j \in \Delta_E}$ and $a$. Fix $j \in \Delta_E$. If $D^j = \sum_{E_i \in \Sigma_j} c_i \cdot E_i$ is a divisor on $\Sigma_j$, let us set $D^j|_{\gamma_j} = \sum_{E_i \in \Sigma_j, E_i \neq E_j} c_i \cdot E_i + c_j \cdot E_j$, which is a divisor on $\Gamma_j$ hence also on $X$. Since $D_{rp} = r\overline{D}$, the divisor on $\Gamma_j$ obtained by restriction of $D_{rp}$ has intersection 0 with all $E_j \in \Gamma_j \setminus \{E_j\}$, hence it is equal to $D_{sp_j}|_{\gamma_j}$ for some $s \in \mathbb{N}$, in particular $c_j(rp) = sp_j$. Then,

$$0 = D_{rp} \cdot E_j = \left(D_{sp_j}|_{\gamma_j} + rpE\right) \cdot E_j = -\alpha_{sp_j}^j + rp - sp_j,$$

Therefore, for $k \geq 1$,

$$\left(D_{sp_j+k}|_{\gamma_j} + (rp+1)E\right) \cdot E_j = -\alpha_{sp_j+k}^j + rp - sp_j - (k-1) = -\left(\alpha_{sp_j+k}^j - \alpha_{sp_j}^j\right) - (k-1)$$

and this implies that the smallest $k$ such that $-(\alpha_{sp_j+k}^j - \alpha_{sp_j}^j) - (k-1) \leq 0$ is the coefficient in $E_j$ of $D_{rp+1} - D_{rp}$, i.e. it is $c_j(1)$.

Now, let $n \geq 1$ and let us suppose that we know $(c_j(n))_{j \in \Delta_E}$. For any $j \in \Delta_E$, we have

$$0 \geq D_n \cdot E_j = \left(D_{c_j(n)}^j|_{\gamma_j} + nE\right) \cdot E_j = -\alpha_{c_j(n)}^j + n - c_j(n)$$

and, for $k \geq 0$,

$$\left(D_{c_j(n)+k}^j|_{\gamma_j} + (n+1)E\right) \cdot E_j = -\alpha_{c_j(n)+k}^j + n - c_j(n) - (k-1).$$

Hence, $c_j(n+1) = c_j(n) + \min\{k \mid -\alpha_{c_j(n)+k}^j + n - c_j(n) - (k-1) \leq 0\}$. From this, the lemma follows.

Now, let us consider the weighted dual graphs

$$G_\Sigma: \begin{array}{cccc}
-2 & -2 & -2 & -2 \\
\epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \\
-2 & -2 & -2 & -2 \\
\epsilon_5 & \epsilon_6 & \epsilon_7 & \epsilon_8
\end{array}$$

$$G_\Sigma': \begin{array}{cccc}
-3 & -2 & -2 \\
\epsilon_1' & \epsilon_2' & \epsilon_3' & \epsilon_4' \\
-3 & -2 & -2 \\
\epsilon_5' & \epsilon_6' & \epsilon_7' & \epsilon_8'
\end{array}$$

fig. 8
obtained from the weighted graphs in figures 6 and 7 by the process described before lemma 4.4. There exist two minimal surface singularities \((Y, Q)\) and \((Y', Q')\) whose weighted dual graphs for their minimal desingularizations are \(G_Y\) and \(G_{Y'}\) respectively. Let \(\{E_i\}_{i=1}^5\) and \(\{E'_j\}_{j=1}^5\) be the irreducible components of the exceptional locus of the minimal desingularization of \((Y, Q)\) and \((Y', Q')\) respectively, which are represented by \(\{e_i\}_{i=1}^5\) and \(\{e'_j\}_{j=1}^5\) in figure 8. Let \(\nu\) and \(\nu'\) be the valuations on \((Y, Q)\) and \((Y', Q')\) respectively determined by \(E_4\) and \(E'_4\), and let \(h\) and \(h'\) be the respective Hilbert-Samuel functions.

**Lemma 4.6.** The Hilbert-Samuel functions \(h : \mathbb{N} \to \mathbb{N}\) and \(h' : \mathbb{N} \to \mathbb{N}\) are the same.

**Proof:** The exceptional divisors \(\{D_n\}_n\) and \(\{D'_n\}_n\) for the valuations \(\nu\) and \(\nu'\) respectively are

\[
\begin{align*}
D_1 &= E_1 + E_2 + E_3 + E_4 + E_5 & D'_1 &= E'_1 + E'_2 + E'_3 + E'_4 + E'_5 \\
D_2 &= E_1 + 2E_2 + 2E_3 + 2E_4 + E_5 & D'_2 &= E'_1 + 2E'_2 + E'_3 + 2E'_4 + E'_5 \\
D_3 &= E_1 + 2E_2 + 3E_3 + 3E_4 + E_5 & D'_3 &= E'_1 + 2E'_2 + E'_3 + 3E'_4 + 2E'_5 \\
\tilde{D} &= D_4 = E_1 + 2E_2 + 3E_3 + 4E_4 + E_5 & \tilde{D}' &= D'_4 = E'_1 + 2E'_2 + E'_3 + 4E'_4 + 2E'_5.
\end{align*}
\]

Hence, the period of both \(h\) and \(h'\) is 4 and we have

\[
\begin{align*}
-\tilde{D}_1 \cdot E_4 &= -\tilde{D}'_1 \cdot E'_4 = 0 \\
-\tilde{D}_2 \cdot E_4 &= -\tilde{D}'_2 \cdot E'_4 = 1 \\
-\tilde{D}_3 \cdot E_4 &= -\tilde{D}'_3 \cdot E'_4 = 2 \\
-\tilde{D}_4 \cdot E_4 &= -\tilde{D}'_4 \cdot E'_4 = 4.
\end{align*}
\]

Since \(h(n+1) - h(n) = -\tilde{D}_n \cdot E_4 + 1\) and the same holds for \(h'\), we conclude the result.

**End of proof of proposition 4.3:** Let \(\nu_1\) be a divisorial valuation dominating \((S_1, P_1)\). Let \(\pi_1 : X_1 \to S_1\) be the minimal desingularization of \((S_1, P_1)\) where the center of \(\nu_1\) is a curve \(E\), and let \(\{E_j\}_{j \in \Delta_E}\) be the exceptional curves for \(\pi\) adjacent to \(E\). From lemmas 4.4 and 4.5 it follows that there exists a valuation \(\nu_2\) dominating \((S_2, P_2)\) such that, if \(\pi_2 : X_2 \to S_2\) is the minimal desingularization of \((S_2, P_2)\) where the center of \(\nu_2\) is a curve \(E'\), and \(\{E'_j\}_{j \in \Delta_{E'}}\) are the exceptional curves for \(\pi\) adjacent to \(E'\), then we may identify \(\Delta_E\) and \(\Delta_{E'}\), the coefficients \(\{e_j(n)\}_{j \in \Delta_E}\) and \(\{e'_j(n)\}_{j' \in \Delta_{E'}}\) of the divisors \(D_n\) and \(D'_n\) for \(\nu_1\) and \(\nu_2\) respectively coincide, and hence the Hilbert-Samuel functions \(h_1\) and \(h_2\) of \(\nu_1\) and \(\nu_2\) are the same.

Now, note that \(v_E \leq 3\) by the definition of \((S_1, P_1)\) and of \(X_1\), and that \(v_E = 3\) if and only if \(E\) is either \(E_{\beta_1}\) or \(E_{\beta_2}\), and hence \(E'\) is either \(E'_{\beta_1}\) or \(E'_{\beta_2}\). Hence, we may order \(\{E_j\}_{j \in \Delta_E}\) and \(\{E'_j\}_{j' \in \Delta_{E'}}\) and take coordinates \(T_1, T_2\) in \(E\) (resp \(E'\)) such that \(E_j \cap E\) and \(E'_j \cap E'\) have the same coordinates for all \(j \in \Delta_E = \Delta_{E'}\). Then, from (4) in theorem 2.2 it follows that

\[
gr_{\nu_1} R_1 \cong \oplus_{n \in \mathbb{N}} \left(T^{e_1(n)}_1 T^{e_2(n)}_2 T^{e_3(n)}_3\right) k[T_1, T_2]_{h_1(n+1) - h_1(n) - 1} \cong gr_{\nu_2} R_2
\]

and we conclude the result.

**References.**


Vincent Cossart,
Laboratoire de Mathématiques LMV UMR8100, Univ. de Versailles, Saint-Quentin
45 avenue des États-Unis, 78035 Versailles Cedex (France)
E-mail : cossart@math.uvsq.fr

Olivier Piltant,
CNRS, LMV, UMR8100, CNRS-UVSQ, Université de Versailles, Saint-Quentin
45 avenue des États-Unis, F-78035 Versailles Cedex (France)
E-mail : piltant@math.uvsq.fr

Ana J. Reguera,
Dep. de Álgebra, Geometría y Topología, Fac. Ciencias, Universidad de Valladolid,
Paseo Belén 7, 47011 Valladolid, Spain.
E-mail : areguera@agt.uva.es