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Scattering theory for graphs isomorphic to a homogeneous tree at infinity

Yves Colin de Verdière*
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Abstract

We describe the spectral theory of the adjacency operator of a graph which is isomorphic to a regular tree at infinity. Using some combinatorics, we reduce the problem to a scattering problem for a finite rank perturbation of the adjacency operator on a regular tree. We develop this scattering theory using the classical recipes for Schrödinger operators in Euclidian spaces.

1 Introduction

The aim of this paper is to describe in an explicit way the spectral theory of the adjacency operator on an infinite graph Γ which, outside of a finite sub-graph Γ_0, looks like a regular tree $\mathbb{T}_q$ of degree $q+1$. We mainly adapt the case of the Schrödinger operators as presented in [12, 11]. The proofs are often simpler here and the main results are similar. This paper can be read as an introduction to the scattering theory for differential operators on smooth manifolds. Even if we do not find our results in the literature, there is probably nothing really new for experts in the scattering theory of Schrödinger operators, except the combinatorial part in Section 5.

The main result is an explicit spectral decomposition: the Hilbert space $l^2(\Gamma)$ splits into a sum of two invariant subspaces $l^2(\Gamma) = \mathcal{H}_{\rm ac} \oplus \mathcal{H}_{\rm pp}$. The first one

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is an absolutely continuous part isomorphic to a closed sub-space of that of the regular tree of degree $q+1$, while the second one is finite dimensional and we have an upper bound on its dimension. The absolutely continuous part of the spectral decomposition is given in terms of explicit generalized eigenfunctions whose behavior at infinity is described in terms of a scattering matrix.

We first introduce the setup, then we recall the spectral decomposition of the adjacency operator A_0 of a regular tree $\mathbb{T}_q$ by using the Fourier-Helgason transform. In Section 3, we consider a Schrödinger operator $A = A_0 + W$ on $\mathbb{T}_q$, where W is a compactly supported non local potential. We build the generalized eigenfunctions for A, define a deformed Fourier-Helgason transform and get a spectral decomposition of A (Theorem 4.3). In section 4, we derive a similar spectral decomposition of the adjacency operator of any graph Γ asymptotic to a regular tree $\mathbb{T}_q$ by proving the following combinatorial result (Theorem 4.2): any such graph Γ is isomorphic to a connected component of a graph $\hat{\Gamma}$ which is obtained from $\mathbb{T}_q$ by a finite number of modifications. This implies that the adjacency operator of $\hat{\Gamma}$ is a finite rank perturbation of the adjacency operator of $\mathbb{T}_q$. In section 5, we investigate some consequences of the scattering theory developed in section 3: we write the point-to-point correlations of scattered waves in terms of the Green's function, we define the transmission coefficients, connect them to the scattering matrix, and get an explicit expression of them in terms of a Dirichlet-to-Neumann operator. For the sake of clarity, this part has been postponed, since it is not necessary to prove Theorem 4.2.

2 The setup: graphs asymptotic to a regular tree

Let us consider a connected graph $\Gamma = (V_\Gamma, E_\Gamma)$ with V_Γ the set of *vertices* and E_Γ the set of *edges*. We write $x \sim y$ for $\{x, y\} \in E_\Gamma$.

Definition 2.1 *Let $q \geq 2$ be a fixed integer. We say that the infinite connected graph Γ is* asymptotic to a regular tree of degree $q+1$ *if there exists a finite sub-graph Γ_0 of Γ such that $\Gamma' := \Gamma \setminus \Gamma_0$ is a disjoint union of a finite number of trees T_l, $l = 1, \cdots, L$, rooted at a vertex x_l linked to Γ_0 and so that all vertices of T_l different from x_l are of degree $q+1$. The trees T_l, $l = 1, \cdots, L$, are called the* ends *of Γ.*

Equivalently, Γ is infinite, has a finite number of cycles and a maximal sub-tree of Γ has all vertices of degree $q+1$ except a finite number of them.

Definition 2.2 *We define the* edge boundary *$(\partial_e \Gamma_0)$ of Γ_0 as the set of edges of Γ connecting a vertex of Γ_0 to a vertex of Γ', namely one of the x_l's. We denote by $|x|_{\Gamma_0}$ the combinatorial distance of $x \in V_\Gamma$ to Γ_0.*

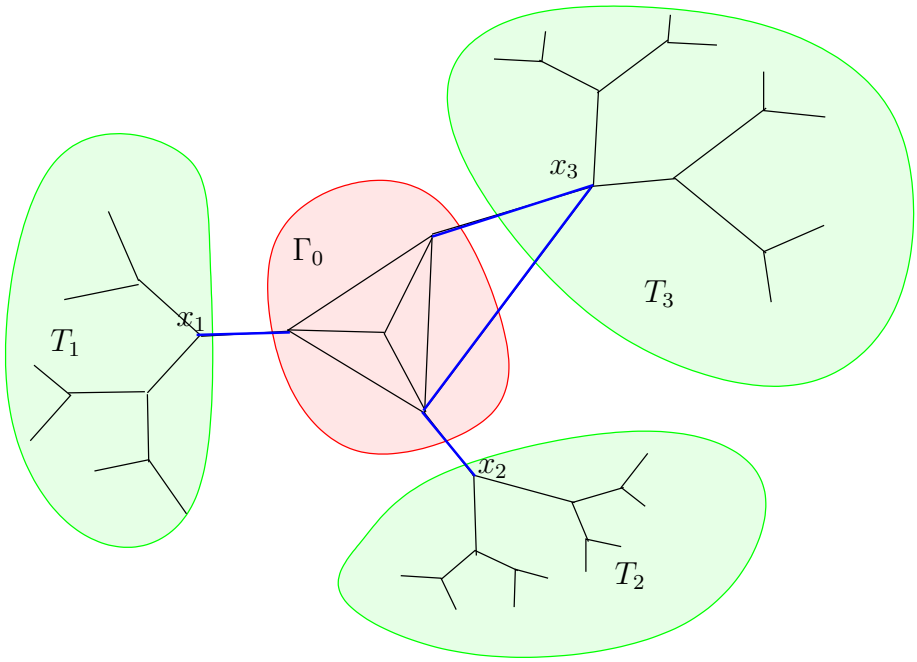


Figure 1: *A graph* Γ *asymptotic to a regular 2-tree with* $L = 3$*; the edge boundary* $\partial_e\Gamma_0$ *has 4 edges.*

In particular, for $l = 1, \cdots, L$, $|x_l|_{\Gamma_0} = 1$.

The space of complex-valued functions on V_Γ is denoted

$$C(\Gamma) = \{f : V_\Gamma \longrightarrow \mathbb{C}\}$$

and $C_0(\Gamma) \subset C(\Gamma)$ is the subspace of functions with finite support. We define also

$$l^2(\Gamma) = \{f \in C(\Gamma);\ \sum_{x \in V_\Gamma} |f|^2(x) < \infty\}.$$

It is a Hilbert space when equipped with the inner product:

$$\langle f, g\rangle = \sum_{x \in V_\Gamma} \overline{f(x)}.g\,(x)\ \ .$$

Let us emphasize that we take the physicist's notation, as in [12] for example: our inner product is conjugate-linear in the first vector and linear in the second. On $C_0(\Gamma)$, we define the adjacency operator A_Γ by the formula:

$$(A_\Gamma f)\,(x) = \sum_{y \sim x} f\,(y) \tag{1}$$

The operator A_Γ is bounded on $l^2(\Gamma)$ if and only if the degree of the vertices of Γ is bounded, which is the case here. In that case, the operator A_Γ is self-

adjoint; otherwise, the operator A_Γ defined on $C_0(\Gamma)$ could have several self-adjoint extensions.

For any λ outside the spectrum of A_Γ, we denote by $R_\Gamma(\lambda) : l^2(\Gamma) \to l^2(\Gamma)$ the resolvent $(\lambda - A_\Gamma)^{-1}$ and by $G_\Gamma(\lambda, x, y)$ with $x, y \in V_\Gamma$ the matrix of $R_\Gamma(\lambda)$, also called the *Green's function*.

3 The spectral decomposition of the adjacency matrix of the tree $\mathbb{T}_q$ and the Fourier-Helgason transform

3.1 Points at infinity

Let $\mathbb{T}_q = (V_q, E_q)$ be the regular tree of degree $q+1$ and let us choose an origin, also called a root, O. We denote by $|x|$ the combinatorial distance of the vertex x to the root. The set of points at infinity denoted Ω_O is the set of infinite simple paths starting from O. We will say that a sequence $y_n \in V_q$ tends to $\omega \in \Omega_O$ if, for n large enough, y_n belongs to the path ω and is going to infinity along that path. If x is another vertex of V_q, the sets Ω_O and Ω_x are canonically identified by considering paths which coïncide far from O and x. There is a canonical probability measure $d\sigma_O$ on Ω_O: $d\sigma_O$ is the unique probability measure on Ω_O which is invariant by the automorphisms of $\mathbb{T}_q$ leaving O fixed. Later on we will always denote by Ω the set of points at infinity, because the root is fixed. For the tree $\mathbb{T}_q$, the *Busemann function* $x \to b_\omega(x)$ associated to the point $\omega \in \Omega_O$ is defined as follows: let us denote by x_ω the last point lying on ω in the geodesic path joining O to x, (take $x_\omega = O$ in the case where O belongs to the geodesic from x to ω), and let us set $b_\omega(x) = |x_\omega| - d(x, x_\omega)$. The level sets of b_ω are the *horocycles* associated to ω. We notice that the function $b_\omega(x)$ increases by one for one of x's neighbors, namely the one of the ray from x to ω, and decreases for the others. Thus the function $b_\omega(x)$ goes to $+\infty$ as x tends to ω. As x tends to $\omega' \neq \omega$, the function $b_\omega(x)$ tends to $-\infty$, whereas the quantity $b_\omega(x) + |x|$ remains bounded, since it tends to $2|x_\omega|$.

3.2 The spectral Riemann surface

Let us define the Riemann surface $S = \mathbb{R}/\tau\mathbb{Z} \times i\mathbb{R}$ with $\tau = 2\pi/\log q$. We denote by $S^0 = \mathbb{R}/\tau\mathbb{Z}$ the circle $\Im s = 0$, and we set $I_q := [-2\sqrt{q}, +2\sqrt{q}]$.

Definition 3.1 *For any $s \in S$, we set : $\lambda_s = q^{\frac{1}{2}+is} + q^{\frac{1}{2}-is}$.*

Proposition 3.1 *The map $\tilde{\Lambda} : s \to \lambda_s$ is holomorphic from S to $\mathbb{C}$. It maps bijectively the physical sheet $S^+ = \{s \in S \mid \Im s > 0\}$ onto $\mathbb{C} \setminus I_q$. By this map the circle S^0 is a double covering of I_q.*

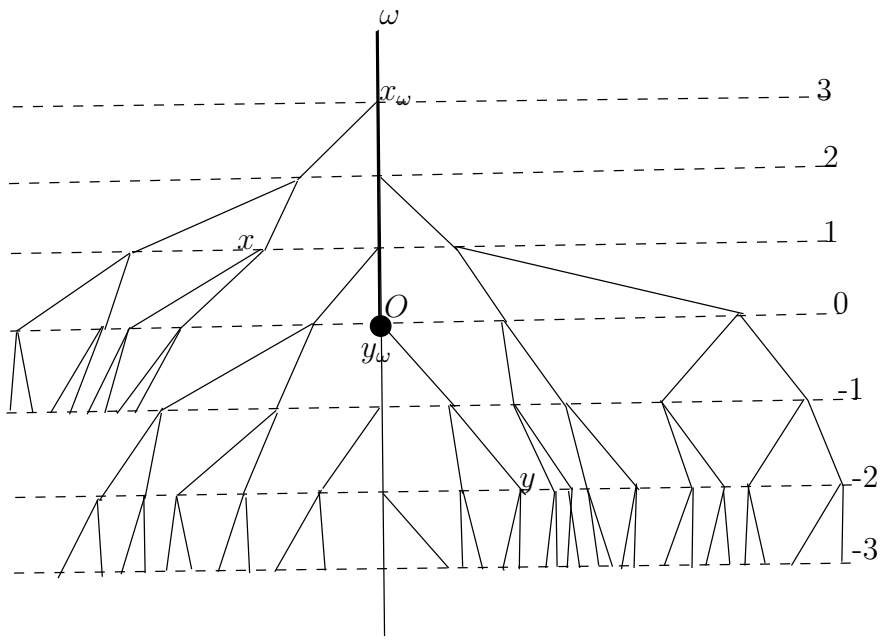


Figure 2: *A regular tree with $q = 2$ and some level-sets of a Buseman function*

Definition 3.2 *If J is a subset of $\mathbb{C}$, we denote by $\hat{J}$ the pre-image of J by the map $\tilde{\Lambda}$, i.e. $\hat{J}$ is the subset of S defined by $\hat{J} := \{s \in S \mid \lambda_s \in J\}$.*

3.3 Calculation of the Green's function

The results of this section are classical, see for example the paper by P. Cartier [2]. We denote by A_0 (resp. G_0) the adjacency operator (resp. the Green's function) on $\mathbb{T}_q$. We will compute explicitly $G_0(\lambda, x, y)$. Let us recall that the regular tree is 2-point regular: for any $x, y, x', y' \in V(\mathbb{T}_q)$ so that $d(x, y) = d(x', y')$, there exists an automorphism J of $\mathbb{T}_q$ so that $J(x) = x'$ and $J(y) = y'$. The Green's function $G(\lambda, x, y)$ satisfies $G(\lambda, Jx, Jy) = G(\lambda, x, y)$ for any automorphism J of $\mathbb{T}_q$. Hence, $G(\lambda, x, y)$ is a function of the distance $d(x, y)$. It is therefore enough to compute $G_0(\lambda, O, x)$ for an $x \in V_q$, that is the value $f(x)$ of the l^2 solution of

$$(\lambda - A_0)f = \delta_O \ , \tag{2}$$

where $f(x)$ depends only on the distance $|x|$ to the origin O. So we set $f(x) = u_k$ if $|x| = k$, $k \in \mathbb{N}$, and rewrite equation (2) as follows:

i) $\lambda u_k - q u_{k+1} - u_{k-1} = 0 \quad$ for $\quad k \geq 1$

ii) $\lambda u_0 - (q+1)u_1 = 1$

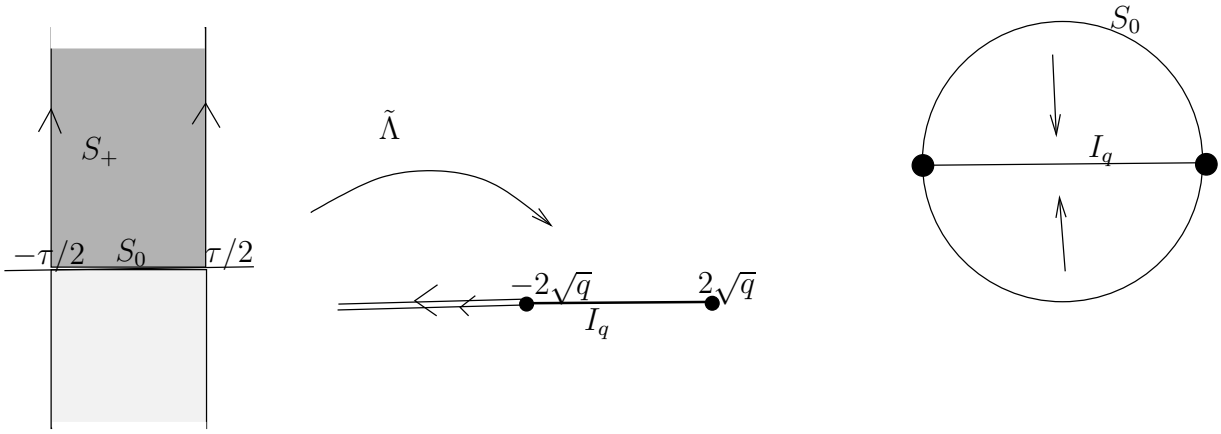


Figure 3: *The surface S, the map $\tilde{\Lambda}$ from S to $\mathbb{C}$, and the double cover of S_0 over I_q.*

iii) $\sum_{n=0}^{\infty}(q+1)q^{n-1}u_n^2 < +\infty$

The last condition stands for f to be in $l^2(\mathbb{T}_q)$.

- If $\lambda \notin I_q$, the equation
$$q\alpha^2 - \lambda\alpha + 1 = 0$$
admits an unique solution α such that $|\alpha| < 1/\sqrt{q}$. From i) and iii), we get that $u_k = C\alpha^k$ and the constant C is determined by ii) :
$$C = C_\lambda = \frac{1}{\lambda - (q+1)\alpha} \ .$$
Therefore we have
$$G_0(\lambda, O, x) = \frac{2q\alpha^{|x|}}{\lambda(q-1) + (q+1)F(\lambda)}$$
where $F(\lambda)$ denotes the determination of $\sqrt{\lambda^2 - 4q}$ in $\mathbb{C}\setminus I_q$ equivalent to λ as λ tends to infinity. Thus using the invariance of the Green's function by the group of automorphisms of the tree, we see that the Green's function $G_0(\lambda, x, y)$ is a function of the distance $d(x, y)$ and we have, for any $x, y \in V(\mathbb{T}_q)$,
$$G_0(\lambda, x, y) = C_\lambda \alpha^{d(x,y)} \ . \tag{3}$$
The operator of matrix $G_0(\lambda, ., .)$ is clearly bounded in $l^2(\mathbb{T}_q)$ and λ is not in the spectrum of A_0.

- If $\lambda \in I_q$, there is no l^2 solution of Equation (2). Therefore we cannot solve $(\lambda - A_0)f = \delta_O$ in l^2, the resolvent does not exist and λ is in the spectrum of A_0.

Using the parameter $s \in S^+$, we have

$$\alpha = q^{-\frac{1}{2}+is}, \qquad C_{\lambda_s} := C(s) = \frac{1}{q^{\frac{1}{2}-is} - q^{-\frac{1}{2}+is}} \quad \text{and} \quad F(\lambda_s) = q^{\frac{1}{2}-is} - q^{\frac{1}{2}+is} \ .$$

Theorem 3.1 *The spectrum of A_0 is the interval $I_q = [-2\sqrt{q}, +2\sqrt{q}]$.*

The Green's function of the tree $\mathbb{T}_q$ is given, for $s \in S^+$ by

$$G_0(\lambda_s, x, y) = C(s) q^{(-\frac{1}{2}+is)d(x,y)} = \frac{q^{(-\frac{1}{2}+is)d(x,y)}}{q^{\frac{1}{2}-is} - q^{-\frac{1}{2}+is}} \ . \tag{4}$$

As a function of s, the Green's function extends meromorphically to S with two poles $-i/2$ and $-i/2 + \tau/2$.

Moreover we have, for any $x \in V_q$ and any y belonging to the ray from x_ω to ω,

$$G_0(\lambda_s, x, y) = G_{rad}(\lambda_s, y) q^{(\frac{1}{2}-is)b_\omega(x)} \tag{5}$$

with

$$G_{rad}(\lambda_s, y) = C(s) q^{(-\frac{1}{2}+is)|y|} \tag{6}$$

Proof.–

The last result comes from the definition $b_\omega(x) = |x_\omega| - d(x, x_\omega)$.

□

3.4 The density of states

Let us recall how to introduce a notion of spectral measure (also called density of states) on the graph Γ. For a given continuous function $\phi : \mathbb{R} \to \mathbb{R}$, we associate by the functional calculus an operator $\phi(A_\Gamma)$ on $l^2(\Gamma)$, which has a matrix $[\phi(A_\Gamma)](x, x')$. We consider then, for any $x \in V_\Gamma$, the linear form on $C(\mathbb{R}, \mathbb{R})$

$$L_x(\phi) = [\phi(A_\Gamma)](x, x) \ .$$

L_x is positive and satisfies $L_x(1) = 1$, so we have $L_x(\phi) = \int_{\mathbb{R}} \phi de_x$ where de_x is a probability measure on $\mathbb{R}$, supported by the spectrum of A_Γ which is called the spectral measure of Γ at the vertex x.

The density of states of $\mathbb{T}_q$ is given by the

Theorem 3.2 *(See for example [4]) The spectral measure de_x of $\mathbb{T}_q$ is independent of the vertex x and is given by*

$$de_x(\lambda) := de(\lambda) = \frac{(q+1)\sqrt{4q - \lambda^2}}{2\pi\left((q+1)^2 - \lambda^2\right)} d\lambda \tag{7}$$

Proof.–

For the sake of clarity, we recall the main ingredients:

1) an explicit computation of the diagonal entries of the Green's function

$$G_0(\lambda, x, x) = \frac{2q}{\lambda(q-1) - (q+1)F(\lambda)}$$

where $F(\lambda)$ denotes as previously the determination of $\sqrt{\lambda^2 - 4q}$ in $\mathbb{C}/I_q$ (with $I_q = [-2\sqrt{q}, 2\sqrt{q}]$) equivalent to λ for great values of λ.

2) The expression of the spectral measure via Stone formula

$$de(\lambda) = \frac{-1}{2i\pi} \left(G(\lambda + i0, x, x) - G(\lambda - i0, x, x) \right) dt \ . \tag{8}$$

□

The previous density of states is the weak limit for the densities of a graph asymptotic to a regular tree. More precisely we have

Theorem 3.3 *Let Γ be as in definition 2.1. Consider the adjacency operator A_Γ defined by (1), denote $A := A_\Gamma$ for simplicity. When x tends to infinity, the densities of states $d\mu_x^A(\lambda)$ of Γ converge weakly to the density of states $de(\lambda)$ of $\mathbb{T}_q$ defined by (7).*

Proof.–

It is enough to compute the limits of $\int \lambda^n d\mu_x^A$ for n fixed and $x \to \infty$. By definition, we have $\int t^n d\mu_x^A = [A^n](x, x)$, and

$$[A^n](x, x) = \sum a_{x,x_1} a_{x_1,x_2} \cdots a_{x_{n-1},x}$$

where the sum is on loops $\gamma = (x, x_1, x_2, \cdots, x_{n-1}, x)$ of length n based at x. If we assume that $|x|_{\Gamma_0} > n/2$, the loops do not meet Γ_0 and therefore $[A^n](x, x) = [A_0^n](x, x)$.

□

3.5 The Fourier-Helgason transform

Let us recall the definition of the Fourier-Helgason transform on the tree $\mathbb{T}_q$ with the root O.

Definition 3.3 *For any $f \in C_0(\mathbb{T}_q)$, the* Fourier-Helgason transform $\mathcal{FH}(f)$ *is the function defined by the finite sum*

$$\mathcal{FH}(f)(\omega, s) := \hat{f}(\omega, s) = \sum_{x \in V_q} f(x) q^{(1/2+is)b_\omega(x)} \ . \tag{9}$$

for any $\omega \in \Omega_O$ and any $s \in S$.

Definition 3.4 *For any $\omega \in \Omega_O$ and any $s \in S$ we define the "incoming plane wave" $e_0(\omega, s)$ as the function $x \to e_0(x; \omega, s)$, where*

$$\forall x \in V_q, \quad e_0(x, \omega, s) = q^{(1/2-is)b_\omega(x)}.$$

For $s \in S_0$, such a plane wave is a generalized eigenfunction for the adjacency operator A_0 on $\mathbb{T}_q$ in the sense that it satisfies

$$(\lambda_s - A_0)e_0(x, \omega, s) = 0 \ \ (\lambda_s = 2\sqrt{q}\cos(s \log q)) \ , \tag{10}$$

but is not in l^2.

If we restrict ourselves to $s \in S^0$, definition 3.3 writes

$$\hat{f}(\omega, s) = \langle e_0(\omega, s), f \rangle = \sum_{x \in V_\Gamma} f\,(x)\,\overline{e_0(x, \omega, s)} \ , \tag{11}$$

and the completeness of the set $\{e_0(\omega, s), \ s \in S^0, \ \omega \in \Omega\}$ is expressed by the following inversion formula (see [CMS]):

Theorem 3.4 *For any $f \in C_0(\mathbb{T}_q)$, the following inverse transform holds*

$$f(x) = \int_{S^0} \int_\Omega e_0(x, \omega, s)\hat{f}(\omega, s) d\sigma_O(\omega) d\mu(s) \tag{12}$$

where

$$d\mu(s) = \frac{(q+1)\log q}{\pi} \frac{\sin^2(s \log q)}{q + q^{-1} - 2\cos(2s \log q)} |ds| \ . \tag{13}$$

Moreover the Fourier-Helgason transform extends to a unitary map from $l^2(\mathbb{T}_q)$ into $L^2(\Omega \times S^0, d\sigma_O \otimes d\mu)$.

The Fourier-Helgason transform is not surjective: its range is the subspace $L^2_{\text{even}}(\Omega \times S^0, d\sigma_O \otimes d\mu)$ of the functions F of $L^2(\Omega \times S^0, d\sigma_O \otimes d\mu)$ which satisfy the symmetry condition (see, for example, [9] or [10])

$$\int_\Omega e_0(x, \omega, s) F(\omega, s) d\sigma_O(\omega) = \int_\Omega e_0(x, \omega, -s) F(\omega, -s) d\sigma_O(\omega) \ .$$

The Fourier-Helgason transform provides a spectral resolution of A_0: if $\phi : \mathbb{R} \to \mathbb{R}$ is continuous,

$$\phi(A_0) = (\mathcal{F}\mathcal{H})^{-1} \phi(\lambda_s) \mathcal{F}\mathcal{H} \ ,$$

where $\phi(\lambda_s)$ denotes the operator of multiplication by that function on $L^2_{\text{even}}(\Omega \times S^0, d\sigma_0 \otimes d\mu)$.

Corollary 3.1 *From the inverse Fourier-Helgason transform formula (12) we find back the expression of the spectral measure of $\mathbb{T}_q$ (see Theorem 3.2).*

Proof.–

By homogeneity of the tree $\mathbb{T}_q$, for any continuous function $\phi : \mathbb{R} \to \mathbb{R}$, $[\phi(A_0)](x,x)$ is independent of x. Using (12), we get

$$[\phi(A_0)](O,O) = \int_\Omega \int_{S^0} \phi(\lambda_s)\overline{e_0(O,\omega,s)}e_0(O,\omega,s)\, d\sigma_O(\omega)d\mu(s) \;= \int_{S^0} \phi(\lambda_s)\, d\mu(s).$$

Let us perform the change of variables

$$s = f_q(\lambda) := \frac{1}{\log q} \arccos \frac{\lambda}{2\sqrt{q}}\ . \tag{14}$$

Using (13) and the fact that, by the map $s \to \lambda_s$, the circle S^0 is a double covering of I_q, we write

$$\begin{aligned}[\phi(A_0)](x,x) \;&= 2\frac{(q+1)\log q}{\pi} \int_{I_q} \phi(\lambda) \frac{1-\lambda^2/4q}{q+q^{-1}+2-\lambda^2/q} f_q'(\lambda)d\lambda \\ &= 2\frac{(q+1)}{4\pi} \int_{I_q} \frac{\sqrt{4q-\lambda^2}}{(q+1)^2-\lambda^2}\phi(\lambda)d\lambda\ ,\end{aligned}$$

which actually implies formula (7).

□

4 A scattering problem for a Schrödinger operator with a compactly supported non local potential

We are concerned here with the scattering on $\mathbb{T}_q$ between the adjacency operator A_0 and the Schrödinger operator $A = A_0 + W$, where W is a compactly supported non local potential. More precisely the Hermitian matrix (also denoted W) associated to this potential is supported by $K \times K$ where K is a finite part of V_q. We assume in what follows that K is chosen minimal, so that:

$$K = \{x \in V_q \mid \exists y \in V_q \text{ with } W_{x,y} \neq 0\}\ .$$

Let us first describe the spectral theory of A: it follows from [12], Sec. XI 3, and from the fact that A is a finite rank perturbation of A_0 (see also Section 4.3) that the Hilbert space $l^2(\mathbb{T}_q)$ admits an orthogonal decomposition into two subspaces invariant by A: $l^2(\mathbb{T}_q) = \mathcal{H}_{\rm ac} \oplus \mathcal{H}_{\rm pp}$ where

- $\mathcal{H}_{\rm ac}$ is the isometric image of $l^2(\mathbb{T}_q)$ by the *wave operator*

$$\Omega^+ = s - \lim_{-\infty} e^{itA} e^{-itA_0} \ .$$

  We have $A_{|\mathcal{H}_{\rm ac}} = \Omega^+ A_0 (\Omega^+)^\star$, so that the corresponding part of the spectral decomposition is isomorphic to that of A_0 which is an absolutely continuous spectrum on the interval I_q.

- The space $\mathcal{H}_{\rm pp}$ is finite dimensional, admits an orthonormal basis of l^2 eigenfunctions associated to a finite set of eigenvalues, some of them may be embedded in the continuous spectrum I_q.

We will denote by $P_{\rm ac}$ and $P_{\rm pp}$ the orthogonal projections on both subspaces.

In order to make the spectral decomposition more explicit, we will introduce suitable generalized eigenfunctions of A. These generalized eigenfunctions are particular solutions of

$$(\lambda_s - A) e(.,\omega,s) = 0 \ , \tag{15}$$

meaning not l^2 solutions, but only point-wise solutions. For the adjacency operator A_0, we have seen that these generalized eigenfunctions, called the " plane waves" are given by the $e_0(\omega,s)$'s with $s \in S^0$ and $\omega \in \Omega_O$ (see definition 3.4) and give the Fourier-Helgason transform which is the spectral decomposition of A_0 (Theorem 3.4).

We are going to prove a similar eigenfunction expansion theorem for A, using generalized eigenfunctions of A. We will mainly adapt the presentation of [12], Sec. XI.6, for Schrödinger operators in $\mathbb{R}^3$ (see also [11]). Our first goal is to build the generalized eigenfunctions $x \to e(x,\omega,s)$ also denoted $e(\omega,s)$. We will derive and solve the so-called *Lippmann-Schwinger equation*. This is an integral equation that $e(\omega,s)$ will satisfy.

4.1 Formal derivation of the Lippmann-Schwinger equation

Let us proceed first in a *formal way* by transferring the functions $e_0(\omega,s)$ by the wave operator: if $e(\omega,s)$ is the image of $e_0(\omega,s)$ by the wave operator Ω^+ in some sense (they are not in l^2!), then we should have $e_0(\omega,s) = \lim_{t\to-\infty} e^{itA_0} e^{-itA} e(\omega,s)$

$$= \lim_{t\to-\infty} [e(\omega,s) - i \int_0^t e^{iuA_0} W e^{-iuA} e(\omega,s) du]$$

$$= e(\omega,s) - i \lim_{\varepsilon\to 0} \int_0^{-\infty} e^{iuA_0} W e^{-iu\lambda_s} e^{\varepsilon u} e(\omega,s) du$$

$$= e(\omega,s) + \lim_{\varepsilon\to 0} [(A_0 - (\lambda_s + i\varepsilon))^{-1} W e(\omega,s)] \ .$$

So $e(\omega,s)$ should obey the following "Lippmann-Schwinger-type" equation

$$e(\omega, s) = e_0(\omega, s) + G_0(\lambda_s) W e(\omega, s) \ . \tag{16}$$

4.2 Existence and uniqueness of the solution for the modified "Lippmann-Schwinger-type" equation

Let $\chi \in C_0(\mathbb{T}_q)$ be a compactly supported real-valued function so that $W\chi = \chi W = W$. For example χ can be the characteristic function of K. We first introduce a modified "Lippmann-Schwinger-type" equation. If $e(\omega, s)$ obeys (16) and $a(\omega, s) = \chi e(\omega, s)$, then a obeys

$$a(\omega, s) = \chi e_0(\omega, s) + \chi G_0(\lambda_s) W a(\omega, s) \ . \tag{17}$$

We have the following result :

Proposition 4.1 *Let $\chi \in C_0(\mathbb{T}_q)$ be a compactly supported real-valued function so that $W\chi = \chi W = W$. Set*

$$\hat{\mathcal{E}} =: \{s \in S^0; \ker(\mathrm{Id} - \chi G_0(\lambda_s) W) \neq 0\} \tag{18}$$

1. *The set $\hat{\mathcal{E}}$ is finite and independent of the choice of χ.*
2. *If $s \notin \hat{\mathcal{E}}$, then (17) has a unique solution $a(\omega, s) \in C_0(\mathbb{T}_q)$ and the function $e(\omega, s) = e_0(\omega, s) + G_0(\lambda_s) W a(\omega, s)$ is the unique solution of the Lippmann-Schwinger equation (16).*
3. *The set $\hat{\mathcal{E}}$ is invariant by $s \to -s$ and consequently it is the pre-image by $\tilde{\Lambda} : s \to \lambda_s$ of a subset of I_q which we denote by $\mathcal{E}$.*

Proof.–

We first prove 2). Let $L_{s,\chi}$ be the finite rank operator on $l^2(\mathbb{T}_q)$ defined by $L_{s,\chi} = \chi G_0(\lambda_s) W$. The map $s \to L_{s,\chi}$ extends holomorphically to $\Im s > -\frac{1}{2}$. Equation (17) takes the form

$$a(., \omega, s) = \eta(., \omega, s) + L_{s,\chi} a(., \omega, s), \tag{19}$$

where $\eta(., \omega, s) \in C_0(\mathbb{T}_q)$. By the analytic Fredholm theorem ([12], p. 101), there exists a finite subset $\hat{\mathcal{E}}$ of S^0, defined by $\hat{\mathcal{E}} =: \{s \in S^0; \ker(\mathrm{Id} - L_{s,\chi}) \neq 0\}$, so that equation (17) has a unique solution $a(\omega, s) \in C_0(\mathbb{T}_q)$ whenever $s \notin \hat{\mathcal{E}}$. The second assertion of 2) comes from the fact that $Wa(\omega, s) = W\chi e(\omega, s) = We(\omega, s)$.

Let us now prove 1): the "minimal" χ is $\chi_W = \mathbf{1}_{\mathrm{K}}$. If a is a non trivial solution of $a - \chi_W G_0(\lambda_s) W a = 0$, and $\chi\chi_W = \chi_W$, a is also solution of $a - \chi G_0(\lambda_s) W a = 0$.

Conversely, if $a-\chi G_0(\lambda_s)Wa=0$, we have $\chi_W a-\chi_W G_0(\lambda_s)W\chi_W a=0$. We have to prove that $\chi_W a\neq 0$. If $\chi_W a=0$, we would have $Wa=0$ and $a=0$.

To prove 3) it is enough to notice that for any $s\notin\hat{\mathcal{E}}$, we have $L_s=L_{-s}$.

□

4.3 The set $\mathcal{E}$ and the pure point spectrum

Proposition 4.2 *If $(A-\lambda)f=0$ with $\lambda\in I_q$ and $f\in l^2(\mathbb{T}_q)$, then $\mathrm{Supp}(f)\subset\hat{K}$ where $\hat{K}$ is the smallest subset of V_q so that $\mathrm{Supp}(W)\subset\hat{K}\times\hat{K}$ and all connected components of $\mathbb{T}_q\setminus\hat{K}$ are infinite.*

Proof.–

We will proceed by contradiction. Let $x\in V_q\setminus\hat{K}$ be so that $f(x)\neq 0$. Let us define an infinite sub-tree $\mathbb{T}_x$ of $\mathbb{T}_q$ as follows: let $y_\alpha,\alpha=1,\cdots,a$ be the vertices of $\mathbb{T}_q$ which satisfy $y_\alpha\sim x$ and y_α is closer to $\hat{K}$ than x. Then $\mathbb{T}_x$ is the connected component of x in the graph obtained from $\mathbb{T}_q$ by removing the edges $\{x,y_\alpha\}$ for $\alpha=1,\cdots,a$. Let us consider the "averaged" function

$$n\in\mathbb{N}\to\bar{f}_x(n):=\frac{1}{q^n}\sum_{z\in\mathbb{T}_x,\ d(x,z)=n}f(z)\ .$$

Then $\bar{f}_x$ satisfies the ordinary difference equation $\lambda g(n)-qg(n+1)-g(n-1)=0$.

We thus get a contradiction, since this equation has no non-zero l^2 solution when λ is in I_q and hence $f(x)=\bar{f}_x(0)=0$.

□

Corollary 4.1 $\#\{\sigma_{\mathrm{pp}}(A)\cap I_q\}\leq\#\hat{K}$.

This holds because any eigenfunction associated to an eigenvalue in $\{\sigma_{\mathrm{pp}}(A)\cap I_q\}$ is supported in $\hat{K}$ and the dimension of the vector space of functions supported in $\hat{K}$ is $\#\hat{K}$.

Theorem 4.1 *If $s\in S^0$, $(A-\lambda_s)f=0$ and $f\in l^2(\mathbb{T}_q)\setminus 0$, then $s\in\hat{\mathcal{E}}$.*

Conversely, if $s\in\hat{\mathcal{E}}\subset S^0$, there exists $f\neq 0$ so that $(A-\lambda_s)f=0$ and $f(x)=O\left(q^{-|x|/2}\right)$.

Proof.–

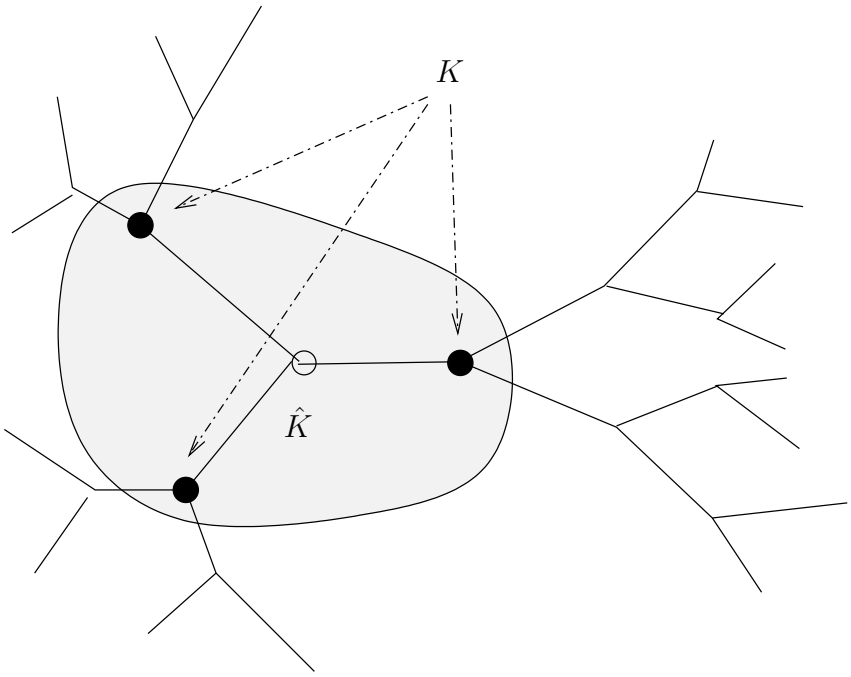


Figure 4: *A simple example with $\hat{K}$ strictly larger than K*

Due to Proposition 4.2, the support of such an f is included in $\hat{K}$ and $(\lambda_s - A_0)f = Wf$. We can apply $G_0(\lambda_s)$ to both sides of the equation, (although λ_s is in the spectrum of A_0) because the functions on both sides are finitely supported. On the lefthandside we have $G_0(\lambda_s)(\lambda_s - A_0)f = f$: this is true for $\lambda_s \notin I_q$ because I_q is the spectrum of A_0 and hence by continuity ($G_0(\lambda_s)$ extends holomorphically near S^0) for every λ_s since f is compactly supported. Hence applying $G_0(\lambda_s)$ to both sides yields $f = G_0(\lambda_s)Wf$. Due to proposition 4.1, we can choose for χ the characteristic function of $\hat{K}$, so we get $f - \chi G_0(\lambda_s)Wf = 0$. We have a non trivial solution of $a - L_s a = 0$, namely $a = f$.

Conversely, let us start from a, a non trivial solution of $a - L_s a = 0$ and define $f = G_0(\lambda_s)Wa$. Then

$$(\lambda_s - A)f = (\lambda_s - A_0)G_0(\lambda_s)Wa - WG_0(\lambda_s)Wa$$

and $(\lambda_s - A_0)G_0(\lambda_s)Wa = Wa$ by analytic extension from $s \in S^+$. Hence, using $W\chi = W$,

$$(\lambda_s - A)f = Wa - W\chi G_0(\lambda_s)Wa = Wa - Wa = 0 \ .$$

From the definition of f, we get that f is a finite linear combination of the functions $G_0(\lambda_s, ., y),\ y \in \mathrm{Supp}(W)$ and we can use Equation (6) to get the bound in x.

□

Theorem 4.2 *The pure point spectrum $\sigma_{\rm pp}(A)$ of A splits into 3 parts*

$$\sigma_{\rm pp}(A) = \sigma^-_{\rm pp}(A) \cup \sigma^+_{\rm pp}(A) \cup \sigma^0_{\rm pp}(A)$$

where $\sigma^-_{\rm pp}(A) = \sigma_{\rm pp}(A) \cap]-\infty, -2\sqrt{q}[$, $\sigma^+_{\rm pp}(A) = \sigma_{\rm pp}(A) \cap]2\sqrt{q}, +\infty[$, and $\sigma^0_{\rm pp}(A) = \sigma_{\rm pp}(A) \cap I_q$. We have $\#\sigma^\pm_{\rm pp}(A) \leq \#{\rm Supp}(W)$ and $\#\sigma^0_{\rm pp}(A) \leq \#\hat{K}$.

The first estimate comes from the mini-max principle and the fact that W is a rank N perturbation of A_0 with $N = \#{\rm Supp}(W)$. The second one is already proved.

The reader could ask if there can really be some compactly supported eigenfunctions. They can exist as shown by the following 2 examples.

Example 4.1 *Γ is a tree with root O and $W_{x,0} = W_{0,x} = -1$ for any $x \sim O$. All other entries of W vanish. Then if $H = A_\Gamma + W$, $f = \delta(0)$, we have $Hf = 0$.*

Example 4.2 *The graph Γ is the union of a cycle with 4 vertices $\{1, 2, 3, 4\}$ and a tree whose root is attached to 2 neighboring vertices of the cycle. If $f(p) = (-1)^p$ on the cycle and 0 on all other vertices, $A_\Gamma f = 0$.*

However the proof of the following result is left to the reader:

Proposition 4.3 *If Γ is an infinite tree, then A_Γ has no compactly supported eigenfunction.*

4.4 The deformed Fourier-Helgason transform

Definition 4.1 *We define the deformed Fourier-Helgason transform $\mathcal{FH}_{\rm sc}$ of $f \in C_0(\mathbb{T}_q)$ as the function $\hat{f}_{\rm sc}$ on $\Omega \times (S^0 \setminus \hat{\mathcal{E}})$ defined by*

$$\hat{f}_{\rm sc}(\omega, s) = \langle e(\omega, s), f \rangle = \sum_{x \in V_\Gamma} f(x)\, \overline{e(x, \omega, s)}\ . \tag{20}$$

We want to prove the following

Theorem 4.3 *For any $f \in C_0(\mathbb{T}_q)$ and any closed interval $J \subset I_q \setminus \mathcal{E}$, if we denote by $\hat{J}$ the inverse image of J by $s \to \lambda_s$, the following inverse transform holds*

$$P_J f(x) = \int_{\hat{J}} \int_\Omega e(x, \omega, s) \hat{f}_{\rm sc}(\omega, s) d\sigma_O(\omega) d\mu(s)\ . \tag{21}$$

Moreover, $f \to \hat{f}_{\rm sc}$ extends to an isometry from $\mathcal{H}_{\rm ac}$ onto $L^2_{\rm even}(\Omega \times S^0, d\sigma_O \otimes d\mu)$.

4.4.1 The relation of the deformed Fourier-Helgason transform with the resolvent

Denoting, with a slight abuse of notation, for $s \in S^+$, by $G(s)$ the operator $(\lambda_s - A)^{-1}$ and similarly by $G_0(s)$ the operator $(\lambda_s - A_0)^{-1}$, we have the resolvent equation

$$G(s) = G_0(s) + G_0(s)WG(s) \tag{22}$$

For $\sigma \in S^0$ and s in S^+, we set

$$h(s;\omega,\sigma) = (\lambda_s - \lambda_\sigma)G(s)e_0(\omega,\sigma) \ ,$$

where the right hand side is a convergent series which identifies to $(\lambda_s - \lambda_\sigma)$-times the inverse Fourier-Helgason transform of $y \to G(s;x,y)$.

Using the definition of G_0 and (10) we have

$$(\lambda_s - \lambda_\sigma)G_0(s)e_0(\omega,\sigma) = e_0(\omega,\sigma) + A_0G_0(s)e_0(\omega,\sigma) - G_0(s)[\lambda_\sigma e_0(\omega,\sigma)] = e_0(\omega,\sigma) \ ,$$

so equation (22) for G gives an integral equation for h

$$h(s;\omega,\sigma) = e_0(\omega,\sigma) + G_0(s)Wh(s;\omega,\sigma)$$

and, if $p(s;\omega,\sigma) = \chi h(s;\omega,\sigma)$,

$$p(s;\omega,\sigma) = \chi e_0(\omega,\sigma) + \chi G_0(s)Wp(s;\omega,\sigma) \ . \tag{23}$$

The key fact is the relation between (23) and the modified "Lippmann-Schwinger-type" equation (17). If $s \in S^+$ is fixed and $\sigma = s$, then the equation for $p(s;\omega,s)$ is identical to equation (17) for $a(\omega,s)$.

This can be used to prove

Lemma 4.1 *Let us consider $f \in C_0(\mathbb{T}_q)$, $\omega \in \Omega$ and $s \in S^+$. Then the function*

$$\Phi(s;\omega,\sigma) = \sum_{x \in V_q} \overline{h(x;s;\omega,\sigma)} f(x), \qquad \forall \sigma \in S^0$$

has a holomorphic extension in σ to S^+ and

$$\Phi(s;\omega,s) = \sum_{x \in V_q} \overline{e(x,\omega,s)} f(x) = \hat{f}_{\rm sc}(\omega,s) \ .$$

We thus have related $\hat{f}_{\rm sc}$ to the resolvent.

4.4.2 End of the proof of Theorem 4.3

Let $\lambda_s = \Lambda + i\varepsilon$ with $\Lambda \in I_q \backslash \mathcal{E}$ and $\varepsilon > 0$, and $s \in S^+$ (this implies $0 < \Re s < \tau/2$). Up to a factor of $(\Lambda + i\varepsilon - \lambda_\sigma)$, $(\omega, \sigma) \to h(x; s; \omega, \sigma)$ is the inverse Fourier-Helgason transform of $y \to G(\lambda_s, x, y)$; so the Plancherel theorem implies (after multiplying by $\overline{f(x)} f(y)$) that

$$(\lambda_s - \overline{\lambda}_s) \sum_{z \in V_q} \overline{G(\overline{\lambda}_s, x, z)} G(\overline{\lambda}_s, z, y) \overline{f(x)} f(y) = ...$$

$$....2i\varepsilon \int_{S^0} \int_\Omega \frac{h(x; s; \omega, \sigma) \overline{h(y; s; \omega, \sigma) f(x)} f(y)}{|\lambda_\sigma - \Lambda|^2 + \varepsilon^2} d\sigma_O(\omega) d\mu(\sigma)$$

If we sum over all x's and y's, we obtain for the left-hand side

$$(\lambda_s - \overline{\lambda_s}) \langle G(\overline{\lambda_s}) f, G(\overline{\lambda_s}) f \rangle = (\lambda_s - \overline{\lambda_s}) \langle f | G(\lambda_s) G(\overline{\lambda_s}) f \rangle = \langle f | [G(\lambda_s) - G(\overline{\lambda}_s)] f \rangle$$

whereas the right-hand side becomes

$$\int_{S^0} \int_\Omega \frac{2i\varepsilon}{|\lambda_\sigma - \Lambda|^2 + \varepsilon^2|} |\Phi(s; \omega, \sigma)|^2 d\sigma_O(\omega) d\mu(\sigma) \ .$$

We thus conclude that, for any closed sub-interval J of I_q disjoint from $\mathcal{E}$,

$$\frac{1}{2\pi i} \int_J \langle f | [G(\Lambda + i\varepsilon) - G(\Lambda - i\varepsilon)] f \rangle d\Lambda = \frac{1}{\pi} \int_J d\Lambda \int_{S^0} \int_\Omega \frac{\varepsilon}{|\lambda_\sigma - \Lambda|^2 + \varepsilon^2} |\Phi(s, \omega, \sigma)|^2 d\sigma_O(\omega) d\mu(\sigma)$$

As $\varepsilon \to 0$, Stone's formula implies that the left-hand side approaches $\|P_J f\|^2$. Moreover the measures

$$dl_\varepsilon = \frac{\varepsilon d\Lambda}{\pi[|\lambda_\sigma - \Lambda|^2 + \varepsilon^2]}$$

converge weakly to $\delta(\Lambda - \lambda_\sigma)$ as $\varepsilon \to 0^+$. So one has to put $\Lambda = \lambda_\sigma$, which implies $\sigma = \pm s$.

Thus one gets that the right-hand side tends to $\int_{\hat{J}} \int_\Omega |\hat{f}_{\rm sc}(\omega, \sigma)|^2 |d\sigma_O(\omega) d\mu(\sigma)$, where $\hat{J}$ is the inverse image of J by $s \to \lambda_s$.

5 The spectral theory for a graph asymptotic to a regular tree

We are concerned here with the spectral theory of the adjacency matrix of a graph Γ asymptotic to a regular tree of degree $q + 1$, in the sense of Definition 2.1, which we recall here:

Definition 5.1 *Let $q \geq 2$ be a fixed integer. We say that the infinite graph Γ is* asymptotic to a regular tree of degree $q+1$ *if Γ is connected and there exists a finite connected sub-graph Γ_0 of Γ such that $\Gamma' := \Gamma \setminus \Gamma_0$ is a disjoint union of a finite number of trees T_l, $l = 1, \cdots, L$, rooted at a vertex x_l linked to Γ_0 and so that all vertices of T_l different from x_l are of degree $q+1$. The trees T_l, $l = 1, \cdots, L$, are called the* ends *of Γ.*

We want to reduce the spectral theory of A_Γ to the situation studied in Section 4. For that, we need a preliminary combinatorial study which could be of independent interest.

5.1 Some combinatorics

We need the following combinatorial result:

Theorem 5.1 *If Γ is asymptotic to a regular tree of degree $q+1$, then Γ is isomorphic to a connected component of a graph $\hat{\Gamma}$ which can be obtained from $\mathbb{T}_q$ by adding and removing a finite number of edges.*

Remark 5.1 *By removing a finite number of edges, one could assume that Γ is a tree. Then the result is quite elementary if the degree of all vertices of Γ is $\leq q+1$: it is then enough to add infinite regular trees to the vertices of degrees $< q+1$ in order to get the final result. This argument, suggested by the referee, is not enough to give a complete proof.*

In order to prove Theorem 5.1, we first introduce an integer $\nu(\Gamma)$ associated to the graph Γ; the integer ν is a combinatorial analogue of the regularized total curvature of a Riemannian surface S which is of constant curvature $\equiv K_0$ near infinity, namely $\int_S (K - K_0) |d\sigma|$.

Definition 5.2 *If Γ is asymptotic to a regular tree of degree $q+1$, we define $\nu(\Gamma)$ by*

$$\nu(\Gamma) = \sum_{x \in V_\Gamma} (q + 1 - d(x)) + 2b_1 \ ,$$

where $d(x)$ is the degree of the vertex x and b_1 is the first Betti number of Γ or equivalently the number of edges to be removed from Γ in order to get a tree.

Note that, if T is a maximal sub-tree of Γ, $\nu(T) = \nu(\Gamma)$.

We will need the

Lemma 5.1 *If, for $r \geq 2$, $B_r = \{x \in V_\Gamma \mid |x|_{\Gamma_0} \leq r\}$, then we have*

$$\nu(\Gamma) = (q-1)m - M + 2 \ ,$$

where m is the number of inner vertices of B_r and M the number of boundary vertices (i.e. connected to a vertex of $\Gamma \setminus B_r$) of B_r.

Proof.– Each of the M boundary vertices has q neighbors in $\Gamma \setminus B_r$ and one in B_r. From Euler formula applied to the sub-graph $\Gamma \cap B_r$ which is connected by the assumption on Γ_0, we get

$$1 - b_1 = (m + M) - \frac{1}{2}\left(\sum_{|x|_{\Gamma_0} \leq r-1} d(x) + M \right) .$$

Thus

$$\nu(\Gamma) = \sum_{|x|_{\Gamma_0} \leq r-1} (q + 1 - d(x)) + 2b_1$$

is equal to

$$\nu(\Gamma) = (q+1)m - (2m + M - 2 + 2b_1) + 2b_1 .$$

□

We will also need the:

Lemma 5.2 *Let F be a finite tree whose all vertices are of degree $q+1$ except the ends which are of degree 1. Let M be the number of ends and m the number of inner vertices of F. We have the relation*

$$M = 2 + (q-1)m . \tag{24}$$

Conversely, for each choice of (m, M) satisfying Equation (24), there exists such a tree F.

Proof.– From Euler formula applied to F, we get $1 = |V_F| - |E_F|$. Moreover $|V_F| = m + M$. Let us choose a root inside F and orient the edges from that root. Then we count the edges by partitioning them with their m possible origins; this gives $|E_F| = (q+1) + (m-1)q$.

Conversely, the statement is true for $m = 1,\ M = q+1$ and we proceed by induction on m by adding q edges to a boundary vertex and the corresponding q boundary vertices, we have $m \to m+1,\ M \to M + (q-1)$. □

Lemma 5.3 *If Γ is asymptotic to a regular tree of degree $q+1$, Γ can be obtained from a tree $\mathbb{T}_q$ by removing and adding a finite number of edges if and only if $\nu(\Gamma) = 0$.*

Proof.– All the changes will take place inside the sub-graph B_r. If we denote by M the number of boundary vertices and m the number of inner vertices of B_r, we have, using $\nu(\Gamma) = 0$ and Lemma 5.1, $M = 2 + (q-1)m$. We replace the graph B_r by a tree F whose existence is stated in Lemma 5.2. The vertices of both graphs are the same and all vertices of the new graph have degree $q+1$. Hence the new graph is a regular tree $\mathbb{T}_q$. □

We will now make some modifications of Γ in order to get a new graph $\hat{\Gamma}$ with $\nu(\hat{\Gamma}) = 0$.

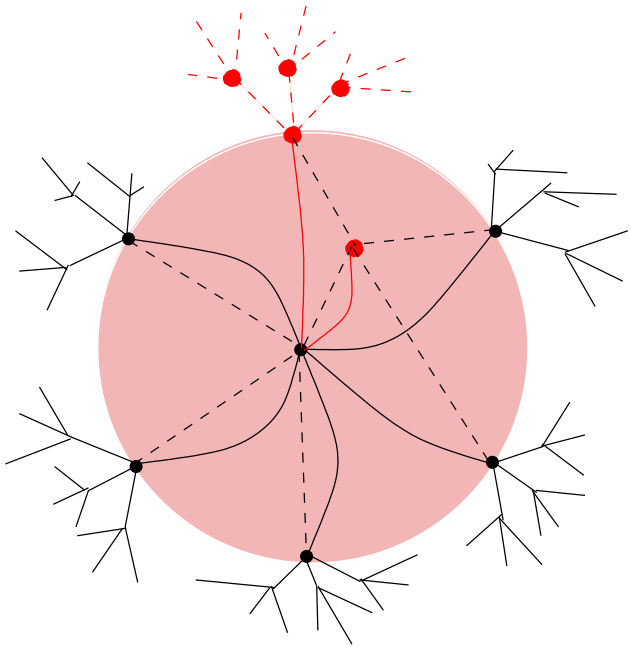


Figure 5: *Changing the graph with $\nu = 0$ into $\mathbb{T}_q$: the dashed edges are the new edges, the continuous one the old edges. The picture is done in the same situation as in Figure 6.*

Lemma 5.4 *If $\Gamma' = M_1(\Gamma)$ is defined by adding to Γ a vertex and an edge connecting that vertex to a vertex of Γ_0, then $\nu(\Gamma') = \nu(\Gamma) + q - 1$.*

If $\Gamma'' = M_2(\Gamma)$ is defined by adding to Γ a tree whose root x is of degree q and all other vertices of degree $q+1$ and connecting x by an edge to a vertex of Γ_0, Γ'' is asymptotic to a regular tree of degree $q+1$ and $\nu(\Gamma') = \nu(\Gamma) - 1$.

This Lemma is quite easy to check.

Proof of Theorem 5.1.– Let us now write $\nu(\Gamma) = N'' - (q-1)N'$ with $N' \geq 0$ and $N'' \geq 0$. By performing N' times the move M_1 and N'' times the move M_2, we arrive to a graph $\tilde{\Gamma}$ with $\nu(\tilde{\Gamma}) = 0$. Let $\hat{\Gamma}$ be the graph obtained by removing from $\tilde{\Gamma}$ the $(N' + N'')$ edges not in E_Γ, one of whose vertices is in Γ_0. The graph $\hat{\Gamma}$ is clearly asymptotic to a regular tree of degree $(q+1)$ and Γ is a connected component of $\hat{\Gamma}$.

It remains to prove that, by removing and adding a finite number of edges to $\hat{\Gamma}$, we get a tree $\mathbb{T}_q$: this is the content of Lemma 5.3. □

5.2 The spectral theory of Γ

From Theorem 5.1, we can identify the set of vertices of Γ to a subset of the set of vertices of $\hat{\Gamma}$ which is the same as the set of vertices of $\mathbb{T}_q$. We deduce the

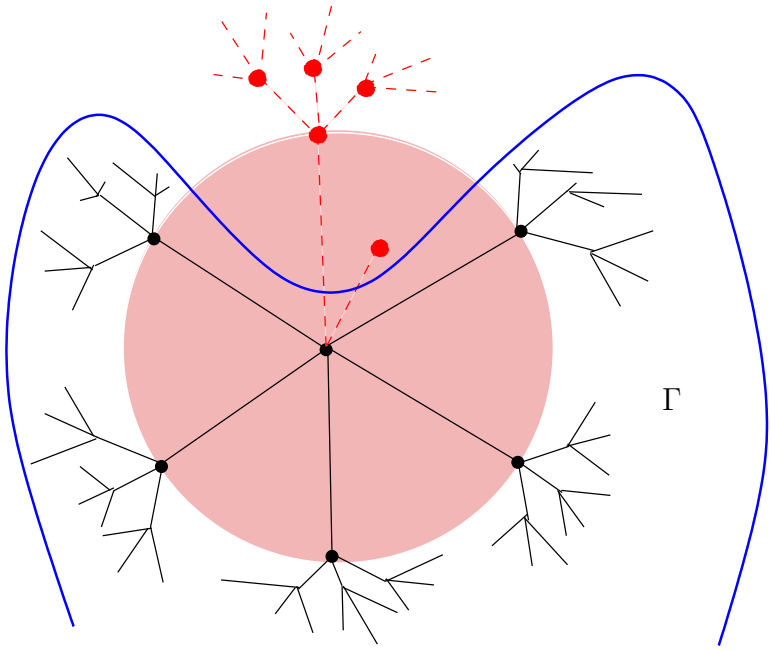


Figure 6: *The construction in the proof of Theorem 5.1; the graph* Γ *has* $q = 3$, $\nu = -1$, $N' = N'' = 1$, $m = 2$, $M = 6$.

existence of a Hilbert space $\mathcal{H}$ so that $l^2(\mathbb{T}_q) = l^2(\Gamma) \oplus \mathcal{H}$ and this decomposition is invariant by $A_{\hat{\Gamma}}$. Moreover $A_{\hat{\Gamma}}$ is a finite rank perturbation of $A_0 = A_{\mathbb{T}_q}$. This will allow us to describe the spectral theory of A_Γ by using the results of Section 4.

In order to get the spectral decomposition of A_Γ in terms of the spectral decomposition of $A_{\hat{\Gamma}}$ given in Section 4.4, we will need the

Lemma 5.5 *Let* $A_{\hat{\Gamma}} = A_{\mathbb{T}_q} + W$ *with* $\mathrm{Support}(W) \subset K \times K$ *and* K *finite. Let* Γ *be an unbounded connected component of* $\hat{\Gamma}$ *and* ω *a point at infinity of* Γ. *Then, for any* $s \notin \hat{\mathcal{E}}$, *we have*

$$\mathrm{support}(e(., s, \omega)) \subset V_\Gamma \ .$$

Conversely, if ω' *is a point at infinity of* $\hat{\Gamma}$ *which is not a point at infinity of* Γ *then*

$$\mathrm{support}(e(., s, \omega')) \cap V_\Gamma = \emptyset \ .$$

Proof.– We will apply the general Theorem 4.3 in our combinatorial context. Let us prove the first assertion, the proof of the second is similar. It is enough to prove it for $s \in S^+$ close to S_0 and hence λ_s not in the spectrum of A_0, because $s \to e(x, s, \omega)$ is meromorphic on S. We have then (Equation (16))

$$e(s, \omega) = e_0(s, \omega) + G_0(\lambda_s) W e(s, \omega) \ .$$

From the explicit expression of e_0 (see Definition 3.4), we get that the first term belongs to $l^2(\hat{\Gamma} \setminus \Gamma)$, and so does the second one, as the image of a compactly supported function by the resolvent for $\lambda_s \notin I_q = \text{spectrum}(A_0)$ (recall that the resolvent is continuous in l^2).

This proves that the restriction of $e(., s, \omega)$ to $V_{\hat{\Gamma}} \setminus V_\Gamma$ is an l^2 eigenfunction, with eigenvalue λ_s, of $A_{\hat{\Gamma}}$. Since $A_{\hat{\Gamma}}$ has no eigenvalue λ_s for $s \in S^+$ close to S_0, it follows that $e(x, s, \omega)$ vanishes for $s \in S^+$ close to S_0 and $x \notin V_\Gamma$. □

Theorem 5.1 allows to consider the set Ω of points at infinity of Γ as a subset of the set $\hat{\Omega}$ of the points at infinity of $\hat{\Gamma}$. The space $l^2(\hat{\Gamma})$ splits as a direct sum $l^2(\Gamma) \oplus l^2(\hat{\Gamma} \setminus \Gamma)$ which is preserved by the adjacency matrix. Lemma 5.5 shows that the support of the generalized eigenfunctions $e(., s, \omega)$ for $\omega \in \Omega$ is included in V_Γ. Using this, we can state the spectral decomposition of A_Γ as an immediate corollary of Theorem 4.3.

Theorem 5.2 *The Hilbert space $l^2(\Gamma)$ splits into a finite dimensional part $\mathcal{H}_{\text{pp}}$ and an absolutely continuous part $\mathcal{H}_{\text{ac}}$. This decomposition is preserved by A_Γ. If $f \in C_0(\Gamma)$ and, for $\omega \in \Omega$, $\hat{f}(s, \omega) = \langle e(., s, \omega), f \rangle$, then the map $f \to \hat{f}$ extends to an isometry from $\mathcal{H}_{\text{ac}}$ onto $L^2_{\text{even}}(S_0 \times \Omega, d\sigma_0 \otimes d\mu)$ which intertwines the action of A_γ with the multiplication by λ_s.*

6 Other features of the scattering theory in the setting of section 3

We are again concerned here with the scattering theory on $\mathbb{T}_q$ between the adjacency operator A_0 and the Schrödinger operator $A = A_0 + W$, where W is a compactly supported non local potential. Let us recall that

$$K = \{x \in V_q \mid \exists y \in V_q \text{ with } W_{x,y} \neq 0\} .$$

6.1 Correlation of scattered plane waves

In the paper [5], the first author computed the point-point correlations of the plane waves for a scattering problem in $\mathbb{R}^d$ in terms of the Green's function: for a fixed spectral parameter, plane waves are viewed as random waves parametrised by the direction of their incoming part. The motivation comes from passive imaging in seismology, a method developped by Michel Campillo's seismology group in Grenoble, as described for example in the papers [6, 7]. Following a similar method, we will compute the correlation of plane waves for our graphs viewed as random waves parametrised by points at infinity.

From Theorem 4.3 we get, for any $\phi \in C_0(\mathbb{T}_q)$ such that supp $\phi \in I_q \setminus \mathcal{E}$, the following formula for the kernel of $\phi(A)$:

$$[\phi(A)]_{x,y} = \int_\Omega \int_{S^0 \setminus \hat{\mathcal{E}}} \Phi(\lambda_s) \overline{e_(x,\omega,s)} e_(y,\omega,s) \; d\sigma_O(\omega) d\mu(s)$$

Taking $\phi = 1_I$, the characteristic function of some interval $I = [a, \lambda] \subset I_q \setminus \mathcal{E}$, we get:

$$[\Pi]_I(x,y) = 2 \int_\Omega \int_{f_q(\lambda)}^{f_q(a)} \overline{e(x,\omega,s)} e(y,\omega,s) \; d\sigma_O(\omega) d\mu(s)$$

where we set $f_q(t) = \frac{1}{\log q} \text{Arccos} \frac{t}{2\sqrt{q}}$ as in (14), and where we use the fact that, by the map $s \to \lambda_s$, the circle S^0 is a double covering of I_q. In particular $f_q(\lambda_s) = s$. In the sequel we note $f_q(\lambda) = s(\lambda)$ for simplicity.

If we consider the plane wave $e(x, \omega, s(\lambda))$ for $\lambda \in I_q \setminus \mathcal{E}$, as a random wave, we can define the point-to-point correlation $C_\lambda^{sc}(x,y)$ of such a random wave in the usual way:

Definition 6.1 *For any $\lambda \in I_q \setminus \mathcal{E}$, the point-to-point correlation $C_\lambda^{sc}(x,y)$ of the random wave $e(x,\omega,s(\lambda))$ is given by*

$$C_\lambda^{sc}(x,y) = \int_\Omega \overline{e(x,\omega,s(\lambda))} e(y,\omega,s(\lambda)) \; d\sigma_O(\omega) \; .$$

Denoting again by G l the Green's function of A: $(\lambda - A)^{-1}[x,y] := G(\lambda, x, y)$ for $Im\lambda > 0$ we prove the

Theorem 6.1 *For any $\lambda \in I_q \setminus \mathcal{E}$ and any vertices x, y the point-to-point correlation can be expressed in terms of the Green's function as*

$$C_\lambda^{sc}(x,y) = -\frac{2(q^2 + 2q + 1 - \lambda^2)}{(q+1)\sqrt{4q - \lambda^2}} \; \Im G(\lambda + i0, x, y) \; .$$

Proof.–

Taking the derivative with respect to λ in equation (6.1) yields:

$$\begin{aligned}
\tfrac{d}{d\lambda} \quad & [\Pi]_I(x,y) = ... \\
= \quad & -2 f_q'(\lambda) \frac{(q+1)\log q}{\pi} \frac{\sin^2(s(\lambda)\log q)}{q + q^{-1} - 2\cos(2s(\lambda)\log q)} \int_\Omega \overline{e(x,\omega,s(\lambda))} e(y,\omega,s(\lambda)) \; d\sigma_O(\omega) \\
= \quad & \frac{q+1}{2\pi} \frac{\sqrt{4q - \lambda^2}}{(q^2 + 2q + 1 - \lambda^2)} \int_\Omega \overline{e(x,\omega,s(\lambda))} e(y,\omega,s(\lambda)) \; d\sigma_O(\omega).
\end{aligned}$$

Thus we have

$$\frac{d}{d\lambda}[\Pi_I](x,y) = \frac{q+1}{2\pi} \frac{\sqrt{4q - \lambda^2}}{(q^2 + 2q + 1 - \lambda^2)} \; C_\lambda^{sc}(x,y) \; .$$

Now we use the resolvent kernel of A : $(\lambda - A)^{-1}[x, y] := G(\lambda, x, y)$ for $Im\lambda > 0$ and Stone formula (8) to write

$$[\Pi]_I(x, y) = -\frac{1}{\pi} \int_a^\lambda \Im G(t + i0, x, y) dt$$

and get the result.

□

6.2 The T- matrix and the S-matrix

The Lippmann-Schwinger eigenfunctions $e(x, \omega, s)$ are especially useful to describe the so-called $S-$matrix ($S = (\Omega^-)^* \Omega^+$). First we introduce the following object:

Definition 6.2 *Let (ω, s) and (ω', s') be in $\Omega \times (S^0 \setminus \hat{\mathcal{E}})$. Define*

$$T(\omega, s; \omega', s') = \langle W e_0(\omega, s), e(\omega', s') \rangle = \sum_{(x,y) \in V_q \times V_q} e(x, \omega', s') \overline{W(x, y) e_0(y, \omega, s)} \ .$$

$T(.;.)$ is called the $T-$matrix.

The goal of this section is to establish a relation between S and T (Theorem 6.2). To get the result we will need the following

Lemma 6.1 *For any $f \in C_0(\mathbb{T}_q)$*

$$\mathcal{FH}_{\rm sc}(\Omega^+ f)(\omega, s) = \hat{f}(\omega, s) \tag{25}$$

Proof.–

Suppose that we can prove $\mathcal{FH}((\Omega^+)^* f) = \mathcal{FH}_{\rm sc}(f) (= \hat{f}_{\rm sc})$, then (25) follows from $\mathcal{FH}_{\rm sc}(\Omega^+ f) = \mathcal{FH}((\Omega^+)^* \Omega^+ f) = \hat{f}$. So, by Plancherel formula it is enough to prove that

$$(f, \Omega^+ g) = \int_{S^0 \times \Omega} \overline{\hat{f}_{\rm sc}(\omega, s)} \hat{g}(\omega, s) d\sigma_O(\omega) d\mu(s) \ . \tag{26}$$

In the sequel, we set $d\Sigma := d\sigma_O(\omega) d\mu(s)$ to simplify notations.

We have $(f, \Omega^+ g) - (f, g) = i \lim_{\varepsilon \to 0} \int_{-\infty}^0 e^{\varepsilon t} (f, e^{itA} W e^{-itA_0} g) dt$.

But

$$(f, e^{itA} h) = \int_{S^0 \times \Omega} \overline{\hat{f}_{\rm sc}(\omega, s)} \ e^{i\lambda_s t} \ \hat{h}_{\rm sc}(\omega, s) \ d\Sigma$$

if either f or h is in $\mathcal{H}_{ac}$. As a result (using definition 4.1)

$$(f, e^{itA}We^{-itA_0}g) = \sum_{x\in V_q}\int_{S^0\times\Omega}\overline{\hat{f}_{\rm sc}(\omega,s)}\, e^{i\lambda_s t}\,(We^{-itA_0}g)(x)\overline{e(x,\omega,s)}d\Sigma.$$

Thus

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{0} e^{\varepsilon t}(f, e^{itA}We^{-itA_0}g)dt$$

$$= \lim_{\varepsilon\to 0}\sum_{x\in V_q}\int_{-\infty}^{0}\int_{S^0\times\Omega}\overline{\hat{f}_{\rm sc}(\omega,s)}\, W(x)\, e^{-it(A_0-\lambda_s+i\varepsilon)}\, g(x)\overline{e(x,\omega,s)}\, d\Sigma$$

$$= -i\lim_{\varepsilon\to 0}\sum_{x\in V_q}\int_{S^0\times\Omega}\overline{\hat{f}_{\rm sc}(\omega,s)}\, W(x)\, [(A_0-\lambda_s+i\varepsilon)^{-1}g](x)\overline{e(x,\omega,s)}\, d\Sigma$$

$$= i\lim_{\varepsilon\to 0}\sum_{x,y\ \in V_q}\int_{S^0\times\Omega}\overline{\hat{f}_{\rm sc}(\omega,s)}\, W(x)\overline{G_0(\lambda_s+i\varepsilon)}(x,y)g(y)\overline{e(x,\omega,s)}\, d\Sigma$$

$$= i\sum_{y\in V_q}\int_{S^0\times\Omega}\overline{\hat{f}_{\rm sc}(\omega,s)}[\sum_{x\in V_q}\overline{G_0(\lambda_s)}(x,y)W(x)\overline{e(x,\omega,s)}]g(y)\ d\Sigma$$

$$= i\sum_{y\in V_q}\int_{S^0\times\Omega}\overline{\hat{f}_{\rm sc}(\omega,s)}[\overline{e(y,\omega,s)} - \overline{e_0(y,\omega,s)}]g(y)\ d\Sigma$$

$$= i(f,g) - i\int_{S^0\times\Omega}\overline{\hat{f}_{\rm sc}(\omega,s)}\hat{g}(\omega,s)\ d\Sigma.$$

In the second line above we used the Lippmann-Schwinger equation and at the last step we used the isometric property of the deformed Fourier Helgason transform (Theorem 4.3).

□

We are ready to prove the following

Theorem 6.2 *For any f and $g \in C_0(\mathbb{T}_q)$*

$$(f,(S-I)g) = -2\pi i\int_{(S^0\times\Omega)^2} T(\omega,s;\omega',s')\overline{\hat{f}(\omega,s)}\delta(\lambda_s-\lambda_{s'})\hat{g}(\omega',s')\ d\Sigma d\Sigma'$$

where we set: $d\Sigma d\Sigma' := d\sigma_O(\omega)d\mu(s)d\sigma_O(\omega')d\mu(s')$, (see Appendix A for the precise definition of the measure $\delta(\lambda_s-\lambda_{s'})d\mu(s)d\mu(s')$).

This can be written symbolically by

$$S(\omega,s;\omega',s') = \delta(s-s') - 2\pi i T(\omega,s;\omega',s')\delta(\lambda_s-\lambda_{s'})\ . \tag{27}$$

Proof.–

From the definition of S we get that

$$(f,(S-I)g) = ((\Omega^- - \Omega^+)f, \Omega^+ g) = \lim_{T\to\infty} \int_T^{-T} (e^{itA}(iW)e^{-itA_0}f, \Omega^+ g)dt$$

$$= (-i)\lim_{\varepsilon\to 0} \int_{-\infty}^{+\infty} e^{-\varepsilon|t|}(e^{itA}We^{-itA_0}f, \Omega^+ g)dt = (-i)\lim_{\varepsilon\to 0} \int_{-\infty}^{+\infty} e^{-\varepsilon|t|}\mathcal{L}(t)dt\,,$$

with

$$\mathcal{L}(t) = \int_{S^0\times\Omega} \overline{\mathcal{FH}_{\rm sc}(e^{itA}We^{-itA_0}f)}(\omega',s')\mathcal{FH}_{\rm sc}(\Omega^+ g)(\omega',s')\; d\Sigma'\;. \tag{28}$$

In the last step we used the isometric property of $\mathcal{FH}_{\rm sc}$ and the fact that $\Omega^+ g \in \mathcal{H}_{ac}$. Moreover we have $\mathcal{FH}_{\rm sc}(\Omega^+ g)(\omega',s') = \hat{g}(\omega',s')$ (Lemma 6.1) and $\mathcal{FH}_{\rm sc}(e^{itA}We^{-itA_0}f)(\omega',s') = e^{i\lambda_{s'}t}\mathcal{FH}_{\rm sc}(We^{-itA_0}f)(\omega',s')$
$= e^{i\lambda_{s'}t}\sum_{x\in V_q}(We^{-itA_0}f)(x)\overline{e(x,\omega',s')}$
$= \sum_{x,y\in V_q}\int_{S^0\times\Omega} e^{i(\lambda_{s'}-\lambda_s)t}W(x,y)e_0(y,\omega,s)\hat{f}(\omega,s)\overline{e(x,\omega',s')}\; d\Sigma$. Thus the expression in (28) is

$$\mathcal{L}(t) = \sum_{x,y\in V_q}\int_{(S^0\times\Omega)^2} e^{i(\lambda_s-\lambda_{s'})t-\varepsilon|t|}\overline{V(x,y)e_0(y,\omega,s)}\,\overline{\hat{f}(\omega,s)}e(x,\omega',s')\hat{g}(\omega',s')\; d\Sigma d\Sigma'$$

and after doing the t-integration we get,

$$(f,(S-I)g) = (-i)\lim_{\varepsilon\to 0}\int_{(S^0\times\Omega)^2} T(\omega,s;\omega',s')\frac{2\varepsilon}{(\lambda_s-\lambda_{s'})^2+\varepsilon^2}\overline{\hat{f}(\omega,s)}\hat{g}(\omega',s')\; d\Sigma d\Sigma'\,.$$

We conclude by noticing as previously that the measures

$$dl_\varepsilon = \frac{2\varepsilon d\mu(s)d\mu(s')}{(\lambda_s-\lambda_{s'})^2+\varepsilon^2}$$

converge weakly to $2\pi\delta(\lambda_s-\lambda_{s'})d\mu(s)d\mu(s')$ as $\varepsilon\to 0^+$.

□

A consequence of the relation between T and S is the unitarity relation for T:

Theorem 6.3 *Suppose $\alpha \notin \mathcal{E}$. Then for any s and $s' \in S^+$ with $\lambda_s = \lambda_{s'} = \alpha$, and for any $(\omega,\omega') \in \Omega\times\Omega$,*

$$\Im T(\omega,s;\omega',s') = \pi\int_{S^0\times\Omega} \overline{T(\omega'',s'';\omega,s)}T(\omega'',s'';\omega',s')\delta(\lambda_{s''}-\alpha)d\Sigma'' \tag{29}$$

Proof.–

By Theorem 6.2 we have

$$\overline{(\hat{Sf})}(\omega,s) = \hat{f}(\omega,s) - 2\pi i \int_{S^0\times\Omega} T(\omega,s;\omega',s')\hat{f}(\omega',s')\delta(\lambda_s-\lambda_{s'})d\Sigma' \ .$$

The adjoint of the map $M : \hat{f} \to (\hat{Sf})$ is clearly given by

$$(M^*(g))(\omega,s) = g(\omega,s)+2\pi i \int_{S^0\times\Omega} \overline{T(\omega',s';\omega,s)}g(\omega',s')\delta(\lambda_s-\lambda_{s'})d\Sigma' \ .$$

The relation $M^*M = I$, which follows from $S^*S = I$, implies that (29) holds.

□

6.3 The S-matrix and the asymptotics of the deformed plane waves

Next result explicits the link between the asymptotic behavior of the generalized eigenfunctions and the coefficients of the scattering matrix.

Theorem 6.4 *There exist "transmission coefficients" $\tau(s,\omega,\omega')$ so that the solution of the Lippmann-Schwinger equation (16) writes*

$$e(x;\omega,s) = e_0(x;\omega,s) + \tau(s,\omega,\omega')q^{(-\frac{1}{2}+is)|x|}$$

for any x close enough to ω' , and these coefficients are related to the scattering matrix by the following formula

$$S(\omega',-s;\omega,s) = -\frac{2i\pi}{C(s)}\tau(s,\omega,\omega') \tag{30}$$

with $C(s) = \dfrac{1}{q^{\frac{1}{2}-is} - q^{-\frac{1}{2}+is}}$.

Proof.–

From the study of the Lippmann-Schwinger equation (16), we write the decomposition

$$e(x;\omega,s) = e_0(x;\omega,s) + e_{\text{scatt}}(x;\omega,s) \ ,$$

where

$$e_{\text{scatt}}(x;\omega,s) = \sum_{y\in K} G_0(\lambda_s,x,y)g(y;\omega,s)$$

where $g(y;\omega,s)=\sum_{z\in K}W(y,z)e(z,\omega,s)$.
Let us look at the asymptotic behaviour of $e_{\rm scatt}(x;\omega,s)$ as $x\to\omega'$. We have seen (Theorem 3.1) that the Green's function $G_0(\lambda_s;x,y)$ satisfies equation (4)

$$G_0(\lambda_s;x,y)=C(s)q^{(-\frac{1}{2}+is)d(x,y)};$$

then (5) and (6) imply that, if $x\to\omega'$,

$$e_{\rm scatt}(x;\omega,s)=\tau(s,\omega,\omega')q^{(-\frac{1}{2}+is)|x|}\ ,$$

with

$$\tau(s,\omega,\omega')=C(s)\sum_{y\in K}g(y;\omega,s)q^{(\frac{1}{2}-is)b_{\omega'}(y)}=C(s)\sum_{(y,z)\in K\times K}e(z,\omega,s)\overline{W(z,y)}e_0(y,\omega',s)\ .$$

Noticing that $e_0(y,\omega',s)=\overline{e_0(y,\omega',-s)}$ we get that

$$\tau(s,\omega,\omega')=C(s)T(\omega',-s;\omega,s)\ .$$

and from (27) we derive formula (30).

□

Remark 6.1 *For any $y\in K$ we have $b_\omega(y)=b_{\omega'}(y)$ if ω and ω' belong to the same end of $\mathbb{T}_q\setminus K$. This implies that the function $\omega'\to\tau(s,\omega,\omega')$ is in fact constant in each end of $\mathbb{T}_q\setminus K$, so that the transmission coefficient $\tau(s,\omega,\omega')$ can be written as a function $\tau(s,\omega,l)$. Moreover the reduced Lippmann-Schwinger equation depends only on the restriction of e_0 to K, this implies that the function $\omega\to\tau(s,\omega,l)$ is also constant in each end of $\mathbb{T}_q\setminus K$. Finally, we get an $L\times L$ matrix depending on s, denoted by*

$$\tilde{S}(s)=(S(l',-s,l,s))_{l,l'}=-\frac{2i\pi}{C(s)}\left(\tau(s,l,l')\right)_{l,l'}$$

.

6.4 Computation of the transmission coefficients in terms of the Dirichlet-to Neumann operator

In this section, we compute the transmission coefficients following the method of [13]. Let us recall that

$$K=\{x\in V_q\mid\exists y\in V_q\text{ with }W_{x,y}\neq 0\}\ .$$

We recall that $|x|$ denotes the combinatorial distance of the vertex x to the root O of $\mathbb{T}_q$. Let us set $B_{n-1}=\{x\in V_\Gamma\mid|x|\leq n-1\}$, where n is chosen so that

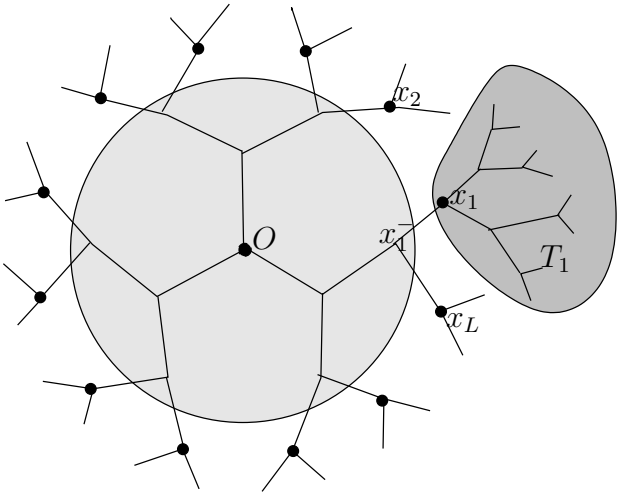


Figure 7: *The tree $\mathbb{T}_2$, the ball B_{n-1} and the end T_1 for $n = 3$.*

$n-2$ is the supremum of $|x|$ for x in K. We denote by T_l the ends of $\mathbb{T}_q \setminus B_{n-1}$ $(1 \leq l \leq L)$, by x_l the root of T_l and by Ω_l the boundary of T_l, which consists in the set of all geodesic rays starting from x_l and staying into T_l. The set of the roots $\{x_l \mid l = 1, \cdots, L\}$ is the circle of radius n and $L = (q+1)q^{n-1}$. From now on, we consider a fixed l $(1 \leq l \leq L)$, a fixed geodesic ray ω in Ω_l, and the associated "incoming plane wave"

$$\forall x \in V_q, \quad e_0(x, \omega, s) = q^{(1/2 - is)b_\omega(x)},$$

where $s \in S^0$. We recall that such a plane wave is a generalized eigenfunction for the adjacency operator A_0 on $\mathbb{T}_q$ in the sense that it satisfies

$$(\lambda_s - A_0)e_0(x, \omega, s) = 0 \quad (\lambda_s = 2\sqrt{q}\cos(s \log q)) \ ,$$

but is not in l^2. We are looking for solutions

$$e(.; \omega, s) = e_0(.; \omega, s) + e_{scat}(.; \omega, s), \quad x \in V_q$$

of the equation

$$(\lambda_s - A)e(.; \omega, s) = 0, \tag{31}$$

where the scattered wave $e_{scat}(.; \omega, s)$ satisfies:

$$e_{scat}(x, \omega, s) = \tau(s, \omega, l')\Phi_s(x) \quad \text{if} \quad x \in V(T_{l'}) \ , \tag{32}$$

where

$$\Phi_s(x) = q^{(-1/2 + is)|x|}$$

(the so-called radiation condition) and the coefficients $\tau(s, \omega, l')$ are the transmission coefficients. These radial waves are generalized eigenfunctions of A_0 in the

sense defined previously. We want to get an explicit expression of the transmission vector

$$\overrightarrow{\tau(s,\omega)} := (\tau(s,\omega,1),\cdots,\tau(s,\omega,l'),\cdots,\tau(s,\omega,L)). \tag{33}$$

As we shall see, the transmission vector does not depend on the choice of the geodesic ray ω, it is uniquely determined by the choice of l; we define

$$\overrightarrow{\tau(s,l)} := \overrightarrow{\tau(\mathbf{s},\omega)}, \quad \forall \omega \in \Omega_l \ . \tag{34}$$

We thus recover the result of the previous section, with the following relation

$$\forall l,l' \quad \tau(s,l,l') = -\frac{C(s)}{2i\pi} S(l',-s,l,s) \ .$$

We begin with noticing that $b_\omega(x_{l'})$ does not depend of $\omega \in \Omega_l$ for any $l' \in \{1,\cdots,L\}$. We set

$$\overrightarrow{A_l} := (\alpha^{-b_\omega(x_1)},\cdots,\alpha^{-b_\omega(x_L)}) = (\alpha^{-b_l(x_1)},\cdots,\alpha^{-b_l(x_L)}) \quad (\alpha = q^{-1/2+is}),$$

and denote by $\overrightarrow{E_l}$ the vector in $\mathbb{R}^L$ having all null coordinates excepted the l-th coordinate, which is equal to 1.
We will prove the following

Theorem 6.5 *Consider the integer n so that B_{n-2} is the smallest ball containing the finite graph K.*

Set $\Gamma = B_n$, $\partial\Gamma = \{x_{l'}, 1 \le l' \le L\}$, denote by $\widehat{A_n}$ the restriction of A to B_n in the sense that $\widehat{A_n} = (A_{x,y})_{(x,y)\in B_n}$, define I_n in the same way, set $B = \widehat{A_n} - \lambda_s I_n$ and denote by $\mathcal{DN}_s$ the corresponding Dirichlet-to Neumann operator (see Definition 6.4, Appendix B).

Then $\mathcal{DN}_s$ and the transmission vector $\overrightarrow{\tau(s,l)}$ defined by (32),(33) and (34) exist for any

$$s \notin \mathcal{E}_0 = \{s \in S^0 \ ; \ \lambda_s \in \sigma(\widehat{A}_{n-1})\}$$

and

$$(\tau(s,l,l')) = -\alpha^{-2n}\left[\frac{1}{C(s)}\left(\mathcal{DN}_s + q^{1/2+is} I\right)^{-1} + \mathcal{A}\right] \ ,$$

with $\widehat{A}_{n-1} = (A_{x,y})_{(x,y)\in B_{n-1}}$, $\mathcal{A} = (\mathcal{A}_{l,l'}) = (\alpha^{d(x_l,x_{l'})})$, $\alpha = q^{-1/2+is}$.

Proof.–

Let us recall that we have fixed l $(1 \le l \le L)$ and a geodesic ray ω in Ω_l. From now on we write $e(x)$ instead of $e(x,\omega,s)$ for any $x \in V_q$ for simplicity.
Equation (31) splits into 3 expressions, depending on where x is taken.

- If $x \notin B_n$ the equation is already verified, since A coincide with A_0 on each end $T_{l'}$.

- if $x \in B_{n-1}$, the equation becomes the Dirichlet problem $Ae(x) = \lambda_s e(x), \quad x \in B_{n-1}$, $e_{|\partial\Gamma} = f$, and is uniquely solvable for any s outside $\mathcal{E}_0$, the prescribed values of the function f at the boundary being given by the L-vector

$$\vec{e_l} =: (e(x_1)...e(x_L))$$

to be determined in the sequel.

- if $x \in \partial\Gamma = \{x_{l'}, 1 \leq l' \leq L\} = B_n \setminus B_{n-1}$, then x is one of the roots $x_{l'}$, and equation (31) writes:

$$\forall l' \in \{1, \cdots, L\}, \quad e(x_{l'}^-) + \sum_{x \sim x_{l'}, x \in T_{l'}} e(x) = \lambda_s e(x_{l'}) \qquad (35)$$

where $x_{l'}^-$ is the unique interior neighbor of $x_{l'}$, (see figure 7), and where we used that the potential W vanishes outside B_{n-2}.

According to definition 6.4 (Appendix B), the Dirichlet-to Neumann operator $\mathcal{DN}_s$ corresponding to $B = \widehat{A_n} - \lambda_s I_n$ writes

$$\mathcal{DN}_s(\vec{e_l})(x_{l'}) = e(x_{l'}^-) - \lambda_s e(x_{l'}) \quad \forall l' \in \{1, \cdots, L\},$$

Therefore, if $s \notin \mathcal{E}_0$, (35) can be rewritten as follows

$$\forall l' \in \{1, \cdots, L\}, \quad \mathcal{DN}_s(\vec{e_l})(x_{l'}) + \sum_{x \sim x_{l'}, x \in T_{l'}} e(x) = 0 \qquad (36)$$

Now it remains to compute $\sum_{x \sim x_{l'}, x \in T_{l'}} e(x)$. We have, for $x \in T_{l'}$, $e(x) = q^{(1/2-is)b_\omega(x)} + \tau(s, \omega, l')\Phi_s(x)$. According to the expression of the radial function Φ_s, we get, for any $s \notin \mathcal{E}_0$ and $l' \in \{1, \cdots, L\}$, that

$$\begin{aligned} \Phi_s(x_{l'}) &= q^{(-1/2+is)|x_{l'}|} = q^{n(-1/2+is)} \\ \Phi_s(x) &= q^{(-1/2+is)(n+1)} \quad \forall x \in T_{l'} \quad x \sim x_{l'}. \end{aligned}$$

Let us write the set $N_l = \{x \in T_l, \quad x \sim x_l\}$ as $N_l = \{y_l\} \cup \tilde{N}_l$, where y_l belongs to the infinite path ω whereas the $q-1$ vertices of $\tilde{N}_l$ do not. Then, using the properties of the Busemann function we have

$$\begin{aligned} b_\omega(x_l) &= b_l(x_l) = |x_l| = n \\ b_\omega(y_l) &= n+1 \\ b_\omega(x) &= b_l(x_l) - 1 = n-1, \quad \forall x \in \tilde{N}_l \\ b_\omega(x) &= b_l(x_{l'}) - 1 \quad \forall x \in N_{l'}, \qquad l' \neq l \ . \end{aligned}$$

So we get for any $l' \in \{1, \cdots, L\}$ and after setting $\alpha = q^{-1/2+is}$,

$$\begin{aligned} e(x_{l'}) &= \alpha^{-b_l(x_{l'})} + \tau(s,\omega,l')\alpha^n \\ e(x) &= \alpha^{-b_l(x_{l'})+\varepsilon} + \tau(s,\omega,l')\alpha^{n+1} \end{aligned}$$

with

$$\begin{aligned} \varepsilon &= 1 \quad \forall x \in \tilde{N}_l \\ \varepsilon &= 1, \quad \forall\ x \in N_{l'} \quad l' \neq l \\ \varepsilon &= -1 \quad \text{if} \quad x = y_l\ . \end{aligned}$$

Hence we have

$$\sum_{x \sim x_{l'}, x \in T_{l'}} e(x) = q\alpha^{-b_l(x_{l'})+1} + q\tau(s,\omega,l')\alpha^{n+1} \quad \text{if} \quad l' \neq l$$

$$\sum_{x \sim x_l, x \in T_l} e(x) = (q-1)\alpha^{-b_l(x_l)+1} + \alpha^{-b_l(x_l)-1} + q\tau(s,\omega,l')\alpha^{n+1}\ .$$

These equations can be summarised, for any $l' \in \{1, \cdots, L\}$, as

$$\sum_{x \sim x_{l'}, x \in T_{l'}} e(x) = \alpha^{-b_l(x_{l'})+1}[q + \delta_{ll'}(\alpha^{-2}-1)] + q\tau(s,\omega,l')\alpha^{n+1}$$

so that equation (36) gives, for any $l' \in \{1, \cdots, L\}$

$$\mathcal{DN}_s(\vec{e_l})(x_{l'}) + \alpha^{-b_l(x_{l'})+1}[q + \delta_{ll'}(\alpha^{-2}-1)] + q\tau(s,\omega,l')\alpha^{n+1} = 0\ . \quad (37)$$

Let us set $\mathcal{DN}_s(\vec{e_l}) = (\mathcal{DN}_s(\vec{e_l})(x_1), ...\mathcal{DN}_s(\vec{e_l})(x_L))$,
and write $\vec{e_l} = (e(x_1), ...e(x_L)) = \vec{A_l} + \alpha^n \overrightarrow{\tau(s,\omega)}$ (recall that
$\vec{A_l}$ and $\overrightarrow{\tau(s,\omega)}$ are $L-$vectors having respectively $\alpha^{-b_l(x_{l'})}$ and $\tau(s,\omega,l')$
as their $l'-$coordinate).

Substituting in (37) and denoting by $\vec{E_l}$ the vector in $\mathbb{R}^L$ having all null coordinates except $x_l = 1$, we get

$$\mathcal{DN}_s[\vec{A_l} + \alpha^n \overrightarrow{\tau(s,\omega)}] + q\alpha\vec{A_l} + \alpha^{-n}(\alpha^{-1} - \alpha)\vec{E_l} + q\overrightarrow{\tau(s,\omega)}\alpha^{n+1} = 0$$

which yields

$$\alpha^n \left(\mathcal{DN}_s + q\alpha I\right) \overrightarrow{\tau(s,\omega)} = \alpha^{-n}(\alpha - \alpha^{-1})\vec{E_l} - \left(q\alpha I + \mathcal{DN}_s\right) \vec{A_l}\ .$$

Using the expression of α and $C(s)$, we have then

$$\alpha^n \left(\mathcal{DN}_s + q^{1/2+is} I\right) \overrightarrow{\tau(s,\omega)} = -\frac{\alpha^{-n}}{C(s)} \vec{E_l} - \left(q^{1/2+is} I + \mathcal{DN}_s\right) \vec{A_l}\ .$$

Since the matrix $\mathcal{DN}_s$ is real symmetric, $\mathcal{DN}_s + q^{1/2+is} I$ is an invertible matrix for any $s \in S^0$ so that $\lambda_s \notin \sigma(\widehat{A}_{n-1})$, and

$$\overrightarrow{\tau(s,\omega)} = -\frac{\alpha^{-2n}}{C(s)} \left(\mathcal{DN}_s + q^{1/2+is} I\right)^{-1} \vec{E}_l - \alpha^{-n} \vec{A}_l \ .$$

We conclude the proof by noticing that, for any $l' \in \{1, \cdots, L\}$, $b_l(x_{l'}) = n - d(x_l, x_{l'})$.

□

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Appendix A: delta measures

The goal of this Appendix is to define in a precise way the meaning of the measures $d\mu = \delta(S = 0)d\nu$ where $d\nu = a(x)dx$ is absolutely continuous w.r. to the Lebesgue measure in $\mathbb{R}^d$ and S is a C^1 real valued function so that dS does not vanish on the hyper-surface $S = 0$. The measure $d\mu = \delta(S = 0)d\nu$ is supported by the hyper-surface $\Sigma := \{S = 0\}$.

We can assume that $\mathbb{R}^d$ and the hyper-surface $S = 0$ are oriented, so that we can play with differential forms instead of measures.

The proof of the following Lemma is left to the reader:

Lemma 6.2 *There exists a differential form β defined in some neighborhood of Σ so that $a dx_1 \wedge \cdots \wedge dx_d = dS \wedge \beta$. Moreover the restriction of β to Σ is uniquely defined.*

Definition 6.3 *If ν is the measure on Σ associated to the restriction of β to Σ. we can view ν as a measure on $\mathbb{R}^d$ denoted $d\nu = \delta(S = 0)d\mu$.*

We can view $d\nu$ as weak limits: if $f : \mathbb{R} \to \mathbb{R}^+$ is a positive L^1 function of integral 1 and $f_\varepsilon(t) = \varepsilon^{-1} f(\varepsilon^{-1} t)$ the measure $\delta(S = 0)d\mu$ is the weak limit as $\varepsilon \to 0$ of the measures $d\nu_\varepsilon = f_\varepsilon(t)d\mu$ (For the proof, take local coordinates so that $S = x_1$).

Usual choices are f_1 the characteristic function of the interval $[-\frac{1}{2}, \frac{1}{2}]$ and $f_2(t) = \frac{1}{\pi} \frac{1}{1+t^2}$.

Appendix B: the Dirichlet-to-Neumann operator $\mathcal{DN}$ on a finite graph

Let $\Gamma = (V, E)$ be a connected finite graph and let $\partial\Gamma$ be a subset of V called the "boundary of Γ". Let $B = (b_{i,j}) : \mathbb{R}^V \to \mathbb{R}^V$ be a symmetric matrix associated

to Γ, namely

$$b_{i,j} = 0 \quad \text{if } i \neq j \quad \text{and} \quad \{i,j\} \notin E.$$

After setting $V_0 = V \setminus \partial\Gamma$, we define $B_0 : \mathbb{R}^{V_0} \to \mathbb{R}^{V_0}$ as the restriction of B to the functions which vanish on $\partial\Gamma$.

We have the following

Lemma 6.3 *Assume that B_0 is invertible. Then, for any given $f \in C(\partial\Gamma)$, there exists a unique solution $F \in C(\Gamma)$ of the Dirichlet problem*

$$(D_f) \ : \ F_{|\partial\Gamma} = f \text{ and } BF(l) = 0 \text{ if } l \in V_0 \ .$$

The *Dirichlet-to-Neumann* operator $\mathcal{DN}$ associated to B is the linear operator from $C(\partial\Gamma)$ to $C(\partial\Gamma)$ defined as follows:

Definition 6.4 *Assume that B_0 is invertible. Let $f \in C(\partial\Gamma)$, and F be the unique solution of the Dirichlet problem (D_f). Then, the Dirichlet-to-Neumann operator form $\mathcal{DN} : \mathbb{R}^{\partial\Gamma} \to \mathbb{R}^{\partial\Gamma}$ is defined as follows: if $l \in \partial\Gamma$,*

$$\mathcal{DN}(f)(l) = \sum_{i=1}^{m} b_{l,i} F(i) (= BF(l)) \ .$$

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