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Jean Dolbeault, Marta Garcia-Huidobro, Raul Manásevich

QUALITATIVE PROPERTIES AND EXISTENCE OF SIGN CHANGING SOLUTIONS WITH COMPACT SUPPORT FOR AN EQUATION WITH A p-LAPLACE OPERATOR

Jean Dolbeault [1]
Ceremade (UMR CNRS no. 7534), Université Paris-Dauphine,
Place de Lattre de Tassigny, 75775 Paris Cédex 16, France

Marta García-Huidobro [2]
Facultad de Matemáticas
Pontificia Universidad Católica de Chile, Casilla 306 Correo 22, Santiago, Chile

Rául Manásevich [3]
DIM & CMM (UMR CNRS no. 2071), FCFM,
Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile

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Abstract. We consider radial solutions of an elliptic equation involving the p-Laplace operator and prove by a shooting method the existence of compactly supported solutions with any prescribed number of nodes. The method is based on a change of variables in the phase plane corresponding to an asymptotic Hamiltonian system and provides qualitative properties of the solutions.

1. Introduction

In this paper we shall consider classical radial sign-changing solutions of

$$\Delta_p u + f(u) = 0 \tag{1}$$

on $\mathbb{R}^N$ with $p > 1$. Radial solutions to (1) satisfy the problem

$$\left(r^{N-1}\,\phi_p(u')\right)' + r^{N-1}\,f(u) = 0\;,\quad u'(0) = 0\;. \tag{2}$$

Here, for any $s \in \mathbb{R}\setminus\{0\}$, $\phi_p(s) := |s|^{p-2}\,s$ and $\phi_p(0) = 0$. Also $'$ denotes the derivative with respect to $r = |x| \geq 0$, $x \in \mathbb{R}^N$ and for radial functions as it is usual we shall write $u(x) = u(r)$. We will assume henceforth that $N > p$. By a (classical) solution

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of (2), we mean a function u in $C^1([0,\infty))$ such that $u'(0)=0$ and $|u'|^{p-2}\,u'$ is in $C^1(0,\infty)$.

It is well known that equations involving quasilinear operators (p-Laplace, mean curvature) may have positive solutions with compact support, see for example [CG02], [GST00], and [GHMS+00]. We are interested here in qualitative properties of the solutions to problem (2) that have a prescribed number of zeros. They satisfy the problem

$$\begin{gathered}\big(r^{N-1}\,\phi_p(u')\big)'+r^{N-1}\,f(u)=0\;,\quad r>0\;,\\ u'(0)=0\;,\quad \lim_{r\to\infty}u(r)=0\;.\end{gathered}\tag{3}$$

As we shall see in Section 2, under condition (H3) below, such solutions have compact support.

We assume the following conditions on f.

(H1) f is continuous on $\mathbb{R}$, locally Lipschitz on $\mathbb{R}\setminus\{0\}$, with $f(0)=0$.

(H2) There exist two constants $a>0$ and $b<0$ such that f is strictly decreasing on (b,a), and a (resp. b) is a local minimum (resp. local maximum) of f.

(H3) The function $F(u):=\int_0^u f(s)\,ds$ is such that $u\mapsto|F(u)|^{-1/p}$ is locally integrable near 0. More generally, we will assume that the function $u\mapsto|F(x_0)-F(u)|^{-1/p}$ is locally integrable near $x_0\neq0$ whenever x_0 is a local maximum of F.

(H4) For any u_0 such that $f(u_0)=0$, $F(u_0)<0$.

(H5) The function $u\mapsto f(u)$ is nondecreasing for large values of u and satisfies

$$\liminf_{|u|\to\infty}\frac{f(u)}{|u|^{p-2}\,u}=\infty\;.$$

(H6) For some $\theta\in(0,1)$, we have

$$\liminf_{|x|\to\infty}\frac{F(\theta\,x)}{x\,f(x)}>\frac{N-p}{Np}>0\;.$$

By the last two conditions, our problem is (p)-superlinear and subcritical. As a consequence of the previous assumptions, there exist two constants $B<0<A$ such that

(i) $F(s)<0$ for all $s\in(B,A)\setminus\{0\}$, $F(B)=F(A)=0$ and $f(s)>0$ for all $s>A$ and $f(s)<0$ for all $s<B$,

(ii) F is strictly increasing in (A,∞) and strictly decreasing in $(-\infty,B)$,

(iii) $F(s)$ is bounded below by $-\bar F=\min_{s\in[B,A]}F(s)$ for some $\bar F>0$,

(iv) $\lim_{|s|\to\infty}F(s)=\infty$.

This paper is organized as follows. Our approach is based on a shooting method and a change of variables which is convenient to count the number of nodes. In

Section 2 we state and prove a version of the compact support principle for sign changing solutions. In Section 3, we consider the initial value problem (4), and establish some qualitative properties of the solutions. Most of these properties are interesting by themselves: see for instance Theorem 3.6. Section 4 is devoted to *generalized polar coordinates* that allows us to write the initial value problem (4) as a suitable system of equations, see (19), that describes the evolution on the *phase space* for the asymptotic Hamiltonian system corresponding to the limiting regime as $r \to \infty$. From this system we can estimate the number of rotations of solutions around the origin, in the phase space at high levels of the energy and relate it with the number of sign changes of the solution of (3). In Section 5 we state and prove our existence results that essentially says that for any $k \in \mathbb{N}$ there is a solution to (3) with k nodes that has compact support. This result differs from [BDO03] in the sense that it holds for the p-Laplace operator for any $p > 1$ and the nonlinearity f is an arbitrary superlinear and subcritical function satisfying assumptions (H1) $-$ (H6). It also differs from the recent results of [CGY12] in the sense that the change of coordinates of Section 4 gives a detailed qualitative description of the dependence of the solutions in the shooting parameter $\lambda = u(0)$. When λ varies, the number of nodes changes of at most one and we can estimate the size of the support of compactly supported solutions: see Section 6 for more details and precise statements. Finally we state two already known results in the Appendix, for completeness. The first one deals with existence of solutions to the initial value problem (4) on $[0, \infty)$. The second one shows where uniqueness of the flow defined by (19) holds on the phase space; for a proof we refer to [CGY12].

The case $p = 2$ has been studied in [BDO03] for a special nonlinearity. Assumption (H3) is the sharp condition for the existence of solutions with compact support; see [PSZ99]. If $u \mapsto |F(u)|^{-1/p}$ is not locally integrable, then Hopf's lemma holds according to [Váz84], and there is no solution with compact support. How to adapt the known results on the *compact support principle* to solutions that change sign is relatively easy by extending the results of [SZ99]. See [BBC75, CEF96, BDO03] in case $p = 2$ and [Váz84, SZ99, PSZ99, PS00, FQ02, GHMS$^+$00] in the general case.

We shall refer to [GHMZ97] and to [CGY12] respectively for multiplicity and existence results; earlier references can be found in these two papers. Consequences of a possible asymmetry of F are not detailed here: see, *e.g.,* [FM01] for such questions. There is a huge literature on sign changing solutions and we can quote [HRS11, KLS09, KLS11, KK09, KW10, LS08, MT05, Ma07, NT04, NT08, Tan07] for results in this direction, which are based either on shooting methods or on bifurcation theory but do not take advantage of the representation of the equation in the generalized polar coordinates.

Our main tool in this paper is indeed the change of variables of Section 4, which can be seen as the canonical change of coordinates corresponding either to $N = 1$ and $f(u) = |u|^{p-2}\, u$, or to the asymptotic Hamiltonian system in the limit $r \to +\infty$: see [FM01, DGM01] for earlier contributions.

We end this introduction with a piece of notation and some definitions. We shall denote by F and Φ_p the primitives of f and ϕ_p respectively, such that $F(0) = \Phi_p(0) = 0$. Thus for any $u \in \mathbb{R}$,

$$F(u) = \int_0^u f(s)\, ds \quad \text{and} \quad \Phi_p(u) = \frac{1}{p}\,|u|^p \ .$$

We shall say that the function u has a *double zero* at a point r_0 if $u(r_0) = 0$ and $u'(r_0) = 0$ simultaneously. We call *nodes* of a solution the zeros which are contained in the interior of the support of the solution and where the solution changes sign: for instance, a solution with zero node is a nonnegative solution, eventually with compact support.

2. Compact support principle

The following result is an extension to sign changing solutions of the *compact support principle,* which is usually stated only for nonnegative solutions. See for instance [CEF96, PSZ99]. Our result shows a compact support property of all solutions converging to 0 at infinity, without sign condition and generalizes a result for the case $p = 2$ that can be found in [BDO03].

Lemma 2.1. *Assume that f satisfies assumptions* (H1)*,* (H2) *and* (H3)*. Then any bounded solution u of (3) has compact support.*

Proof. Let us set

$$\mathcal{A} = \int_0^a \frac{ds}{[p'\,(-F(s))]^{\frac{1}{p}}}\ ,$$

where $p' = p/(p-1)$. Defining $\bar u$ on $(0, \mathcal{A})$ implicitly by

$$r = \int_{\bar u(r)}^a \frac{ds}{[p'\,(-F(s))]^{\frac{1}{p}}}\ ,$$

we first have that

$$\frac{1}{p'}|\bar u'|^p + F(\bar u) = 0\ ,$$

and, by differentiation, that $\bar u$ satisfies

$$(\phi_p(\bar u'))' + f(\bar u) = 0\ .$$

It is straightforward to check that $\bar u(0) = a$ and $\bar u(\mathcal{A}) = 0$, so that $\bar u'(\mathcal{A}) = 0$ as well. We may then extend $\bar u$ to $(\mathcal{A}, +\infty)$ by 0.

Let u be a bounded solution of (3) such that $\lim_{r\to\infty} u(r) = 0$. Then there exists $R > 0$ such that

$$b < u(r) < a \quad \forall\, r > R\ .$$

Let

$$w(r) := \bar u(r - R) \quad \forall\, r \geq R\ .$$

Then either $u(r) \leq w(r)$ for any $r \geq R$, and, as a consequence, $u(r) \leq w(r) \leq 0$ for any $r \geq R + \mathcal{A}$, or there exists $r_0 > R$ such that $u(r_0) > w(r_0)$. Assume that this last case holds. Since $(u-w)(R) \le 0$ and $\lim_{r\to\infty}(u-w)(r) = 0$, with no restriction we may assume that

$$(u-w)(r_0) = \max_{r\in[R,\infty)} (u-w) > 0\ .$$

Hence, there exists a positive ε such that

$$(u-w)(r) > 0 \quad \forall\, r \in [r_0, r_0+\varepsilon)\ .$$

From the equations satisfied by u and w,

$$\begin{aligned}
&(r^{N-1}\,\phi_p(u'))' + r^{N-1}\, f(u) = 0\ ,\\
&(r^{N-1}\,\phi_p(w'))' + r^{N-1}\, f(w) = (N-1)\, r^{N-2}\, \phi_p(w')\ ,
\end{aligned}$$

by integrating from r_0 to $r \in (r_0, r_0+\varepsilon)$, and by taking into account the fact that $(u-w)'(r_0) = 0$, we get

$$\begin{aligned}
&r^{N-1}\,\phi_p(u'(r)) - r^{N-1}\,\phi_p(w'(r))\\
&= -\int_{r_0}^{r} s^{N-1} \underbrace{\Big(f(u(s)) - f(w(s))\Big)}_{<0 \text{ because } u(s)>w(s)}\, ds - (N-1)\int_{r_0}^{r} s^{N-2} \underbrace{\phi_p(w'(s))}_{\le 0 \text{ because } w'\le 0}\, ds\ ,
\end{aligned}$$

which proves that $u' > w'$ on $(r_0, r_0+\varepsilon)$. This obviously contradicts the assumption that $u-w$ achieves its maximum at $r = r_0$.

Summarizing, we have proved that $u(r) \le w(r)$ for any $r \ge R$, and, as a consequence,

$$u(r) \le 0 \quad \forall\, r \ge R + \mathcal{A}\ .$$

Similarly, we observe that $\tilde u(r) := -u(r)$ is a solution of

$$(r^{N-1}\,\phi_p(\tilde u'))' + r^{N-1} \tilde f(\tilde u) = 0\ , \quad \tilde u'(0) = 0\ , \quad \lim_{r\to\infty} \tilde u(r) = 0\ ,$$

where

$$\tilde f(s) := -f(-s)$$

has the same properties as f, except that the interval (b,a) has to be replaced by the interval $(-a,-b)$. With obvious notations, we obtain that

$$\tilde u(r) \le \tilde w(r) \quad \forall\, r \ge R + \mathcal{B}\ ,$$

for a certain positive $\mathcal{B}$ and where $\tilde w$ is a nonnegative solution of

$$(\phi_p(w'))' + \tilde f(w) = 0 \quad \text{on} \quad (R, R+\mathcal{B})\ ,$$

such that $\tilde w(R) = -b$, $\tilde w(R+\mathcal{B}) = \tilde w'(R+\mathcal{B}) = 0$, and $\tilde w(r) = 0$ for any $r \ge R + \mathcal{B}$. This proves that

$$u(r) \ge 0 \quad \forall\, r \ge R + \mathcal{B}\ ,$$

which completes the proof:

$$u \equiv 0 \quad \text{on} \quad (R + \max\{\mathcal{A}, \mathcal{B}\}, \infty)\ .$$

$\square$

3. Properties of the solutions

To deal with problem (3), we will use a shooting method and consider the initial value problem

$$\begin{aligned} \left(r^{N-1}\,\phi_p(u')\right)' + r^{N-1}\,f(u) = 0\,, &\quad r > 0\,,\\ u(0) = \lambda > 0\,, &\quad u'(0) = 0\,. \end{aligned} \tag{4}$$

To emphasize the dependence of the solution to (4) in the shooting parameter λ, we will denote it u_λ. Solutions to (4) exist and are globally defined on $[0,\infty)$; see a proof of this fact in Appendix A. By Proposition A.2, these solutions are uniquely defined until they reach a double zero or a point r_0 with $u'(r_0) = 0$ and such that $u(r_0)$ is a relative maxima of F.

To be used in our next results, to a solution $u_\lambda(r)$ of (2), we associate the energy function

$$E_\lambda(r) := \frac{|u_\lambda'(r)|^p}{p'} + F(u_\lambda(r))\,, \tag{5}$$

where $p' = p/(p-1)$. The following proposition shows several properties of the solution u_λ to (4) that are needed to prove Theorem 5.1.

Proposition 3.1. *Let f satisfy* (H1) *through* (H5) *and let u_λ be a solution of* (4).

(*i*) *The energy E_λ is nonincreasing and bounded, hence the limit*

$$\lim_{r\to\infty} E_\lambda(r) = \mathcal{E}_\lambda$$

is finite.

(*ii*) *There exists $C_\lambda > 0$ such that $|u_\lambda(r)| + |u_\lambda'(r)| \le C_\lambda$ for all $r \ge 0$.*

(*iii*) *If u_λ reaches a double zero at some point $r_0 > 0$, then u_λ does not change sign on $[r_0,\infty)$. Moreover, if $u_\lambda \not\equiv 0$ for $r \ge r_0$, then there exists $r_1 \ge r_0$ such that $u_\lambda(r) \ne 0$, and $E_\lambda(r) < 0$ for all $r > r_1$ and $u_\lambda \equiv 0$ on $[r_0, r_1]$.*

(*iv*) *If $\lim_{r\to\infty} u_\lambda(r)$ exists, then there exists a zero ℓ of f such that*

$$\lim_{r\to\infty} u_\lambda(r) = \ell \quad \text{and} \quad \lim_{r\to\infty} u_\lambda'(r) = 0\,.$$

Proof. Let $u_\lambda(r)$ be any solution of (4). As

$$E_\lambda'(r) = -\,\frac{(N-1)}{r}\,|u_\lambda'(r)|^p\,,$$

and $N \ge p > 1$, we have that E_λ is decreasing in r. Moreover, we have that

$$F(\lambda) \ge F(u_\lambda(r)) \ge -\,\bar{F}$$

and thus (*i*) and (*ii*) follow by recalling that from (H5) we get $\lim_{|s|\to\infty} F(s) = \infty$.

Assume next that u_λ reaches a double zero at some point $r_0 > 0$. Then $E_\lambda(r_0) = 0$ implying that $E_\lambda(r) \le 0$ for all $r \ge r_0$. If u_λ is not constantly equal to 0 for $r \ge r_0$, then $E_\lambda(r_1) < 0$ for some $r_1 > r_0$ and thus, by the monotonicity of E_λ, $E_\lambda(r) < 0$ for all $r \ge r_1$. Moreover u_λ cannot have the value 0 again (because at the zeros of u_λ we have $E_\lambda \ge 0$). This proves (*iii*) by taking the infimum on all r_1 with the above properties.

Finally, if $\lim_{r\to\infty} u_\lambda(r) = \ell$, then from the equation in (4) and applying L'Hôpital's rule twice, we obtain that

$$\begin{aligned} 0 = \lim_{r\to\infty} \frac{u_\lambda(r)-\ell}{r^{p'}} &= -\lim_{r\to\infty} \frac{r^{\frac{N-1}{p-1}}\,|u_\lambda'(r)|}{p' r^{\frac{N-1}{p-1}} r^{p'-1}} \\ &= -\frac{1}{p'}\left(\lim_{r\to\infty} \frac{r^{N-1}\,|u_\lambda'(r)|^{p-1}}{r^N}\right)^{p'-1} \\ &= -\frac{1}{p'}\left(\lim_{r\to\infty} \frac{r^{N-1}\,f(u_\lambda)}{N\,r^{N-1}}\right)^{p'-1} = -\frac{1}{p'}\left(\frac{f(\ell)}{N}\right)^{p'-1}. \end{aligned}$$

Next, from the definition in (5), it follows that $\lim_{r\to\infty}|u'(r)| = \left(p'\left(\mathcal{E}_\lambda - F(\ell)\right)\right)^{1/p}$. Assume that $\lim_{r\to\infty}|u'(r)| := m > 0$. Then given $0<\varepsilon<m$ there is $r_0>0$ such that $u'(r) > m-\varepsilon > 0$ or $u'(r) < -m+\varepsilon < 0$, for all $r \geq r_0$. Hence either $u(r) > u(r_0) + (m-\varepsilon)(r-r_0)$ or $u(r) < u(r_0) + (-m+\varepsilon)(r-r_0)$, for all $r > r_0$, which is impossible because $\lim_{r\to\infty} E_\lambda(r) = \mathcal{E}_\lambda$ is finite, and (iv) follows. □

Proposition 3.2. *Let f satisfy* (H1)-(H5) *and let u_λ be a solution of* (4)*. Then u_λ has at most a finite number of sign changes.*

Proof. The result is true if u reaches a double zero. Let us prove it by contradiction. If $\{z_n\}$ is a sequence of zeros accumulating at some double zero r_0, then for each $n\in\mathbb{N}$, there exists a unique point $r_n \in (z_n, z_{n+1})$ at which u_λ reaches its maximum or minimum value. At these points, using that $E_\lambda(r_n) \geq E_\lambda(z_n) \geq 0$, we must have that

$$|u_\lambda(r_n)| \geq \min\{|B|, A\}.$$

As we also have that $u_\lambda(r_n) \to u_\lambda(r_0) = 0$, we obtain a contradiction.

This proves that u_λ has only a finite number of zeros on $(0, r_0)$, and by Proposition 3.1(iii), we know that u_λ cannot change sign on (r_0,∞). Hence, without loss of generality we may assume that u does not have any double zero. By the above argument, we also know that zeros cannot accumulate.

Next, we argue by contradiction and suppose that there is an infinite sequence (tending to infinity) of simple zeros of u. Then $E_\lambda(r) \geq 0$ for all $r>0$. We denote by $\{z_n^+\}$ the zeros for which $u'(z_n^+)>0$ and by $\{z_n^-\}$ the zeros for which $u'(z_n^-)<0$. We have

$$0 < z_1^- < z_1^+ < z_2^- < \cdots < z_n^+ < z_{n+1}^- < z_{n+1}^+ < \cdots$$

Between z_n^- and z_n^+ there is a minimum r_n^m where $u(r_n^m) < 0$ and between z_n^+ and z_{n+1}^- there is a maximum r_n^M where $u(r_n^M) > 0$. As $E_\lambda(r_n^M)$, $E_\lambda(r_n^m) \geq 0$, it must be that $u(r_n^m) < B$ and $u(r_n^M) > A$.

We claim that *there exists $T>0$ and $n_0\in\mathbb{N}$ such that the distance between two consecutive zeros is less than T for all $n \geq n_0$.*

Indeed, let a^+ be the largest positive zero of f (b^- the smallest negative zero of f). Set

$$d = A - a^+\,, \quad b_1 = a^+ + \frac{d}{4}\,, \quad b_2 = A - \frac{d}{4}\,.$$

Let $r_{1,n} \in (z_n^+, r_n^M)$ be the unique point where $u(r_{1,n}) = b_1$, and let $r_{2,n} \in (z_n^+, r_n^M)$ be the unique point where $u(r_{2,n}) = b_2$. Then $z_n^+ < r_{1,n} < r_{2,n}$. For $r \in (z_n^+, r_{2,n})$, $u(r) \in (0, b_2) \subset (0, B^+)$, hence $F(u(r)) < 0$ and thus

$$\frac{|u'|^p}{p'} \geq |F(u(r))|$$

implying that

$$\frac{u'(r)}{|F(u(r))|^{1/p}} \geq (p')^{1/p} \quad \text{for all } r \in (z_n^+, r_{2,n}) \ ,$$

and thus (from (H3))

$$\int_0^{b_2} \frac{du}{|F(u)|^{1/p}} \geq (p')^{1/p} \, (r_{2,n} - z_n^+) \tag{6}$$

Next, from the equation we have that for $r \in [r_{2,n}, r_n^M]$,

$$\begin{aligned} |(\phi_p(u'))'(r)| &= \left| \frac{(N-1)}{r} \, \phi_p(u'(r)) + f(u(r)) \right| \\ &\geq f(u(r)) - \frac{(N-1)}{r} \, \phi_p(C_\lambda) \\ &\geq f(b_2) - \frac{(N-1)}{r} \, \phi_p(C_\lambda) \\ &\geq \frac{1}{2} \, f(b_2) \quad \text{for all } r \geq \tfrac{2\,(N-1)\,\phi_p(C_\lambda)}{f(b_2)} \ . \end{aligned}$$

Hence, choosing n_0 such that $z_n^+ \geq \frac{2\,(N-1)\,\phi_p(C_\lambda)}{f(b_2)}$ for all $n \geq n_0$, we have that

$$|(\phi_p(u'))'(r)| \geq \frac{1}{2} \, f(b_2) \quad \text{for all } r \in [r_{2,n}, r_n^M]$$

and therefore

$$\phi_p(C_\lambda) \geq \phi_p(u'(r_{2,n})) - \phi_p(u'(r_n^M)) = (\phi_p(u'))'(\xi)(r_n^M - r_{2,n}) \geq \frac{1}{2} \, f(b_2)(r_n^M - r_{2,n})$$

implying that

$$(r_n^M - r_{2,n}) \leq \frac{2\,\phi_p(C_\lambda)}{f(b_2)} \ . \tag{7}$$

From (6) and (7) we conclude that

$$r_n^M - z_n^+ \leq \frac{1}{(p')^{1/p}} \int_0^{b_2} \frac{du}{|F(u)|^{1/p}} + \frac{2\,\phi_p(C_\lambda)}{f(b_2)} := T_1 \ .$$

A similar argument over the interval $[r_n^M, z_{n+1}^-]$ yields

$$z_{n+1}^- - r_n^M \leq T_1 \ ,$$

implying

$$z_{n+1}^- - z_n^+ \leq 2\,T_1$$

and finally, the same complete argument over the interval $[z_n^-, z_n^+]$ yields

$$z_n^+ - z_n^- \leq 2\,\overline{T}_1$$

for some $\overline{T}_1$ which will depend on b^- and B only and the claim follows with $T = \max\{2\,\overline{T}_1, 2\,T_1\}$.

We can now prove the proposition. Observe that $u(r) \in [b_1, b_2]$ for $r \in [r_{1,n}, r_{2,n}]$ and thus

$$|u'(r)|^p \geq p'\,|F(u(r))| \geq p'\,|F(b_2)| \tag{8}$$

and from the mean value theorem

$$b_2 - b_1 \leq C_\lambda\,(r_{2,n} - r_{1,n})\ ,$$

hence

$$r_{2,n} - r_{1,n} \geq \frac{b_2 - b_1}{C_\lambda}\ . \tag{9}$$

Then,

$$\begin{aligned}
\infty > E_\lambda(z_{n_0}^+) - E_\lambda(\infty) &= (N-1)\int_{z_{n_0}^+}^{\infty} \frac{|u'(t)|^p}{t}\,dt \qquad (10)\\
&\geq (N-1)\sum_{k=n_0}^{\infty}\int_{r_{1,k}}^{r_{2,k}} \frac{|u'(t)|^p}{t}\,dt\\
\text{from (8)} \quad &\geq (N-1)\sum_{k=n_0}^{\infty} p'\,|F(b_2)|\,(r_{2,k} - r_{1,k})\frac{1}{r_{2,k}}\\
\text{from (9)} \quad &\geq p'\,|F(b_2)|\,\frac{b_2 - b_1}{C_\lambda}\sum_{k=n_0}^{\infty}\frac{1}{r_{2,k}}\ .
\end{aligned}$$

But, setting $s_{2k-1} = r_{1,n_0+k-1}$, $s_{2k} = r_{2,n_0+k-1}$, we have that $s_1 < s_2 < s_3 < \cdots$ and for any i, $s_{i+1} - s_i \leq 3\,T$. Hence $s_n - s_1 \leq 3\,(n-1)\,T$, implying that

$$s_n \leq s_1 + 3\,(n-1)\,T$$

and thus

$$\frac{1}{s_n} \geq \frac{1}{s_1 + 3\,(n-1)\,T}\ .$$

Therefore,

$$\sum_{k=n_0}^{\infty}\frac{1}{r_{2,k}} = \sum_{k=1}^{\infty}\frac{1}{r_{2,n_0+k-1}} = \sum_{k=1}^{\infty}\frac{1}{r_{2,n_0+k-1}} = \sum_{k=1}^{\infty}\frac{1}{s_{2k}} \geq \sum_{k=1}^{\infty}\frac{1}{s_1 + 3\,(2\,k-1)\,T} = \infty$$

contradicting the finiteness of the left hand side in (10) and the proposition follows. □

Corollary 3.3. *Under the assumptions of Proposition 3.2, the only solutions u_λ of (4) satisfying $E_\lambda(r) \geq 0$ for all $r \geq 0$ are those that reach a double zero at some point $r_0 > 0$ and $u_\lambda(r) \equiv 0$ for all $r \geq r_0$.*

Proof. Let u_λ be a solution of (4) such that $E_\lambda(r) \geq 0$ for all $r \geq 0$, and assume that it does not reach any double zero. By Proposition 3.2, u_λ has at most a finite number of (simple) zeros. Without loss of generality we may assume that $u_\lambda(r) > 0$ for $r > r_0$, for some $r_0 > 0$.

If u_λ is eventually monotone, then $\lim_{r\to\infty} u_\lambda(r) = \ell$ exists, and thus by Proposition 3.1(iv), ℓ is a zero of f and $u'_\lambda \to 0$. By assumption (H3) *i.e.* the compact support assumption, we know that $\ell \neq 0$. Hence $\lim_{r\to\infty} E_\lambda(r) = F(\ell) < 0$ because of (H4), implying that $E_\lambda(r) < 0$ for r sufficiently large.

If u_λ has an infinite sequence of critical points, then in particular it has a first positive minimum at some point $r_1 > 0$. From the equation, $f(u_\lambda(r_1)) \leq 0$ and thus $0 < u_\lambda(r_1) < A$, and thus $E_\lambda(r_1) = F(u_\lambda(r_1)) < 0$ implying that $E_\lambda(r) < 0$ for all $r \geq r_1$.

Therefore, in both cases u_λ must reach a first double zero at some $r_0 > 0$. As E_λ decreases, it follows that $E_\lambda(r) = 0$ for all $r \geq r_0$, and in particular, by differentiation,

$$\Big((\phi_p(u'_\lambda))' + f(u_\lambda)\Big)\, u'_\lambda(r) = 0 \quad \text{for all } r \geq r_0\ ,$$

hence

$$-\frac{N-1}{r}\, |u'_\lambda(r)|^p = 0 \quad \text{for all } r \geq r_0\ ,$$

implying that $u'_\lambda(r) = 0$ for all $r \geq r_0$, thus $u_\lambda(r) = 0$ for all $r \geq r_0$. □

Proposition 3.4. *Let f satisfy* (H1)-(H5) *and let u_λ be a solution of* (4). *Let $\{s_n\}$ be any sequence in $[0,\infty)$ that tends to ∞ as $n \to \infty$ and define the sequence of real functions $\{v_n\}$ by*

$$v_n(r) = u_\lambda(r + s_n)\ .$$

Then $\{v_n\}$ contains a subsequence that converges pointwise to a continuous function u_λ^∞, with uniform convergence on compact sets of $[0,\infty)$. Furthermore the function u_λ^∞ is a solution to the asymptotic equation

$$(\phi_p(u'))' + f(u) = 0\ . \tag{11}$$

Thus it satisfies

$$(\phi_p(u_\lambda^{\infty\prime}(r)))' + f(u_\lambda^\infty(r)) = 0\ ,$$

for all $r \in [0,\infty)$.

Proof. Let u_λ be any solution to (4). We know that there exist two constants c_λ^1 and c_λ^2 such that

$$u_\lambda(r) \leq c_\lambda^1\ , \quad u'_\lambda(r) \leq c_\lambda^2\ , \quad \text{for all } r \geq 0\ .$$

Let now $\{s_n\}$ be any sequence in $[0,\infty)$ that tends to ∞ as $n \to \infty$ and define the sequence of real functions $\{v_n\}$ by

$$v_n(r) = u_\lambda(r + s_n)\ .$$

Then

$$v_n(r) \leq c_\lambda^1\ , \quad v'_n(r) \leq c_\lambda^2\ , \quad \text{for all } r \geq 0\ .$$

Hence, for any $s, t > 0$, and all $n \in \mathbb{N}$,

$$|v_n(s) - v_n(t)| \le c_\lambda^2 \, |s - t| \; .$$

Then, from Ascoli's theorem (see [Roy88, Theorem 30]), $\{v_n\}$ contains a subsequence, denoted the same, that converges pointwise to a continuous function u_λ^∞, with uniform convergence on compact sets of $[0, \infty)$.

It is clear that each function v_n satisfies

$$((r + s_n)^{N-1} \, \phi_p(v_n'(r)))' + (r + s_n)^{N-1} f(v_n(r)) = 0 \; ,$$

and hence

$$\phi_p(v_n'(r)) = \phi_p(v_n'(0)) - \int_0^r \Big(\frac{t + s_n}{r + s_n}\Big)^{N-1} f(v_n(r)) = 0 \; .$$

By passing to a subsequence if necessary we can assume that $\phi_p(v_n'(0)) \to a$ as $n \to \infty$. Let now $T > 0$, then since $\{f(v_n\}$ converges uniformly in $[0, T]$ to $f(u_\lambda^\infty)$, we find that v_n' converges uniformly to a continuous function z given by

$$z(r) = \phi_{p'}\Big(a - \int_0^r f(u_\lambda^\infty(t)) \; dt\Big) \; .$$

Hence z' exists and is continuous. Furthermore from

$$v_n(r) = v_n(0) + \int_0^r v_n'(t) \; dt \; ,$$

letting $n \to \infty$, we obtain that

$$u_\lambda^\infty(r) = u_\lambda^\infty(0) + \int_0^r z(t) \; dt \; .$$

Hence u_λ^∞ is continuously differentiable and $u_\lambda^{\infty\prime}(r) = z'(r)$, for all $r \in [0, T]$. Combining, we obtain

$$\phi_p(u_\lambda^{\infty\prime}(r)) = a - \int_0^r f(u_\lambda^\infty(t)) \; dt \; ,$$

that implies first that $a = \phi_p(u_\lambda^{\infty\prime}(0))$, and then that

$$(\phi_p(u_\lambda^{\infty\prime}(r)))' + f(u_\lambda^\infty(r)) = 0 \; .$$

This argument show indeed that u_λ^∞ is a solution to (11) for all $r \in [0, \infty)$. $\square$

Proposition 3.5. $\lim\limits_{r \to \infty} E_\lambda(r) = \mathcal{E}_\lambda = F(\ell)$, *where* ℓ *is a zero of* f.

Proof. Let $T>0$ be arbitrary but fixed. Then

$$\begin{aligned} E_\lambda(k_0T)-\mathcal{E}_\lambda &= (N-1)\int_{k_0T}^{\infty}\frac{|u'|^p}{t}\,dt \\ &= (N-1)\sum_{k=k_0}^{\infty}\int_{k\,T}^{(k+1)\,T}\frac{|u'(t)|^p}{t}\,dt \\ &= (N-1)\sum_{k=k_0}^{\infty}\int_0^T\frac{|u'(s+k\,T)|^p}{s+k\,T}\,ds \\ &\geq (N-1)\sum_{k=k_0}^{\infty}\frac{1}{(k+1)\,T}\int_0^T|u'(s+k\,T)|^p\,ds\;. \end{aligned}$$

As the left hand side of this inequality is finite, it must be that

$$\liminf_{k\to\infty}\int_0^T|u'(s+k\,T)|^p\,ds=0\;,$$

hence there is a subsequence $\{n_k\}$ of natural numbers such that

$$\lim_{k\to\infty}\int_0^T|u'(s+n_k\,T)|^p\,ds=0\;.$$

From Proposition 3.4,

$$v_k(r):=u(r+n_k\,T)$$

has a subsequence, still denoted the same, such that

$$\lim_{k\to\infty}v_k(r)=v(r)\quad\text{and}\quad\lim_{k\to\infty}v_k'(r)=v'(r)$$

uniformly in compact intervals, where v is a solution of

$$(\phi_p(v'))'+f(v)=0\;.$$

Hence,

$$\int_0^T|v'(s)|^p ds=0\;,$$

implying that v is a constant, say $v(r)\equiv v_0$. From the equation satisfied by v, $f(v_0)=0$. On the other hand,

$$\frac{|v_k'(r)|^p}{p'}+F(v_k(r))=\frac{|u'(r+n_k\,T)|^p}{p'}+F(u(r+n_k\,T))=E_\lambda(r+n_k\,T)\to\mathcal{E}_\lambda$$

as $k\to\infty$ and thus

$$F(v_0)=\mathcal{E}_\lambda\;.$$

□

Although not necessary for the proof of our existence results in Theorem 5.1 in our next result we give sufficient conditions for the limit of $u_\lambda(r)$ to exists as $r\to\infty$.

Theorem 3.6. *Let f satisfy* (H1) *through* (H5)*, and assume furthermore that f has only one positive zero at a^+ and only one negative zero at b^-. Then either*

$$\lim_{r\to\infty} u_\lambda(r) \quad \textit{exists and equals either } a^+ \textit{ or } b^- \ ,$$

or $u_\lambda(r) \equiv 0$ for all $r \geq r_0$ for some $r_0 > 0$.

If f has more than one positive or negative zero and if we assume that

$$\int_{x_0} \frac{ds}{|F(s) - F(x_0)|^{1/p}} < \infty \quad \textit{whenever } x_0 \textit{ is a local maximum of } F \ , \tag{12}$$

then $\lim_{r\to\infty} u_\lambda(r)$ exists and it is either a nonzero zero of f or $u_\lambda(r) \equiv 0$ for all $r \geq r_0$, for some $r_0 > 0$.

Proof. We first give the proof for the case f has only one positive zero at a^+ and only one negative zero at b^-.

By Proposition 3.2 we can assume without loss of generality that u_λ remains positive for $r > r_0$, for some $r_0 > 0$. If u_λ has only a finite number of critical points, then it is eventually monotone and thus it converges as $r \to \infty$. Then the result follows from Proposition 3.1(iv).

Hence we are left with the case in which u_λ has an infinite sequence of maxima at $\{r_n^M\}$ and an infinite sequence of minima at $\{r_n^m\}$, with both $u_\lambda(r_n^M)$, $u_\lambda(r_n^m) > 0$. From the equation, the maxima occur with $f(u_\lambda(r_n^M)) \geq 0$, hence $u_\lambda(r_n^M) > a^+$ (strict inequality due to Proposition A.2 in Appendix A) and for the same reason, the minima occur with $f(u_\lambda(r_n^m)) \leq 0$ with $u_\lambda(r_n^m) < a^+$.

As E_λ is decreasing, we must have that $u_\lambda(r_n^m)$ increases (thus $u_\lambda(r_n^m)$ is bounded away from 0) to a positive limit $\ell_1 \in (0, a^+]$, and $u_\lambda(r_n^M)$ decreases to a limit $\ell_2 \in [a^+, A]$. Moreover,

$$\liminf_{r\to\infty} u_\lambda(r) = \lim_{n\to\infty} u_\lambda(r_n^m) = \ell_1 \ , \quad \text{and} \quad \limsup_{r\to\infty} u_\lambda(r) = \lim_{n\to\infty} u_\lambda(r_n^M) = \ell_2 \ .$$

Thus $E_\lambda(r_n^m) = F(u_\lambda(r_n^m)) \to F(\ell_1)$ and $E_\lambda(r_n^M) = F(u_\lambda(r_n^M)) \to F(\ell_2)$, implying $0 \neq F(\ell_1) = F(\ell_2)$.

From Proposition 3.5, $\lim_{r\to\infty} E_\lambda(r)$ is either $F(0) = 0$ or $F(a^+)$. Since $0 \neq F(\ell_1)$, the limit must be $F(a^+)$, and thus $F(\ell_1) = F(\ell_2) = F(a^+)$, and the only possibility is that $\ell_1 = \ell_2 = a^+$ proving the first part of the theorem.

In order to prove the second part of the theorem, for simplicity we consider f with three positive zeros u_1, u_2 and u_3, but the arguments clearly hold for the general case. In this case F has two minimum points at u_1 and u_3, and one maximum point at u_2, and the limit of the energy can be any of the three values $F(u_1)$, $F(u_3)$ or $F(u_2)$.

Claim 1: If $\mathcal{E}_\lambda$ is a relative minima of F, then the solution u_λ converges as $r \to \infty$.

For the relative minima there are two cases: $F(u_1) = F(u_3)$ and $F(u_1) > F(u_3)$.

In the first case (shown in Figure 1), we can prove that if E_λ converges to $L = F(u_1) = F(u_3)$, then the solution u_λ either converges to u_1 or it converges to u_3. Indeed, we can assume that $u_\lambda(r) > 0$ for $r \geq r_0$. If u_λ has an infinite sequence of

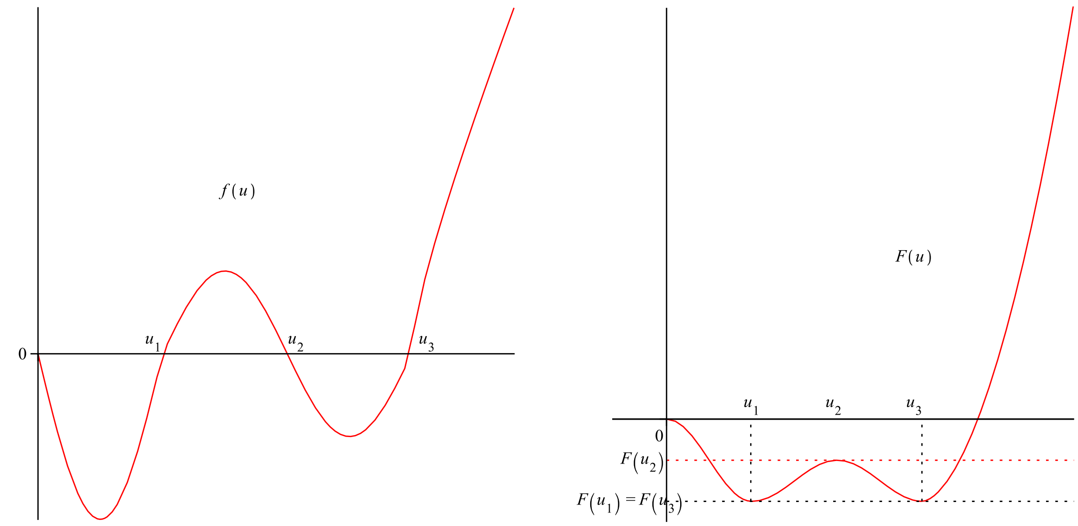

FIGURE 1. Case of f with three positive zeros and $F(u_1) = F(u_3)$.

minima at $\{r_n^m\}$ and an infinite sequence of maxima at $\{r_n^M\}$ (tending to infinity), then by setting

$$\ell_1 = \liminf_{r\to\infty} u_\lambda(r) = \lim_{n\to\infty} u_\lambda(r_n^m) \ , \quad \ell_2 = \limsup_{r\to\infty} u_\lambda(r) = \lim_{n\to\infty} u_\lambda(r_n^M) \ ,$$

we must have that

$$F(\ell_1) = F(\ell_2) = L = F(u_1) \ ,$$

so if $\ell_1 \neq \ell_2$, then $\ell_1 = u_1$ and $\ell_2 = u_3$. But then the solution u_λ crosses the value u_2 at an infinite sequence $\{r_{2,n}\}$ tending to infinity and

$$F(u_1) = \lim_{n\to\infty} E_\lambda(r_{2,n}) = \lim_{n\to\infty} \frac{|u'_\lambda(r_{2,n})|^p}{p'} + F(u_2) \ ,$$

implying that

$$\lim_{n\to\infty} \frac{|u'_\lambda(r_{2,n})|^p}{p'} = F(u_1) - F(u_2) < 0 \ ,$$

which is a contradiction. Hence $\ell_1 = \ell_2$ and the claim follows.

The second case is a little more involved. The following two cases may occur:

$$\text{(a)} \quad \lim_{r\to\infty} E_\lambda(r) = F(u_3) \quad \text{or} \quad \text{(b)} \quad \lim_{r\to\infty} E_\lambda(r) = F(u_1) \ .$$

The case (a) is simple because in this case $F(\ell_1) = F(\ell_2) = L = F(u_3)$ and the only possibility is that $\ell_1 = \ell_2 = u_3$.

In the second case we claim that $\ell_1 = \ell_2 = u_1$. If this is not true, then there are two possibilities: (i) $\ell_1 = u_1$ and ℓ_2 as in Figure 2, or (ii) ℓ_1 and ℓ_2 are as in the same figure. The first case is simple because again the solution u_λ must cross the value u_2 at an infinite sequence $\{r_{2,n}\}$ tending to infinity and we arrive to the same contradiction as above.

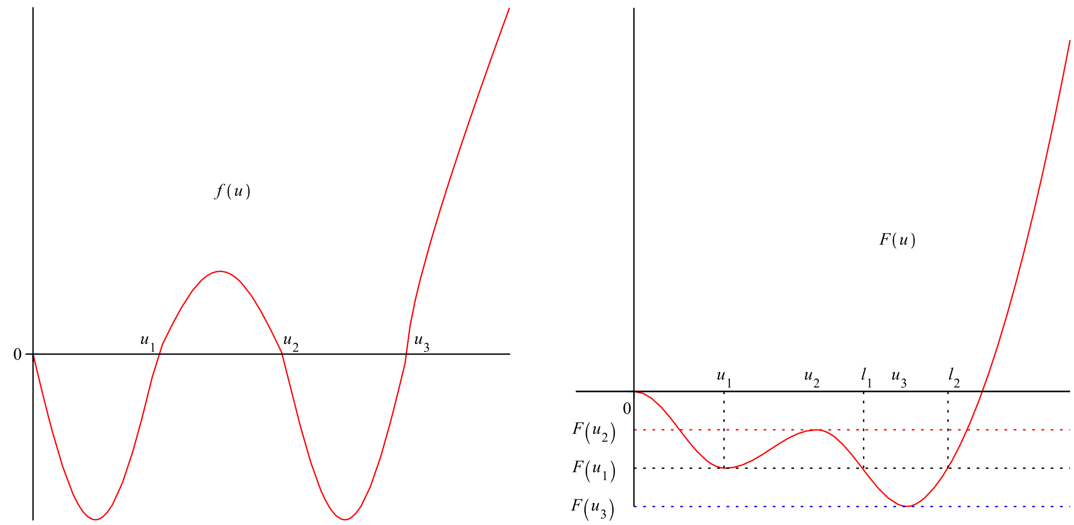

FIGURE 2. Case of f with three positive zeros and $F(u_1) > F(u_3)$.

For the second case, we proceed as in the proof of Proposition 3.2 and prove that the distance between any two consecutive critical points is bounded above. We set

$$b_1 = \frac{\ell_1 + u_3}{2}\ , \quad b_2 = \frac{\ell_2 + u_3}{2}\ ,$$

and let $r_{1,n} \in (r_n^m, r_n^M)$ be the unique point where $u_\lambda(r_{1,n}) = b_1$, and $r_{2,n} \in (r_n^m, r_n^M)$ be the unique point where $u_\lambda(r_{2,n}) = b_2$. See Figure 3.

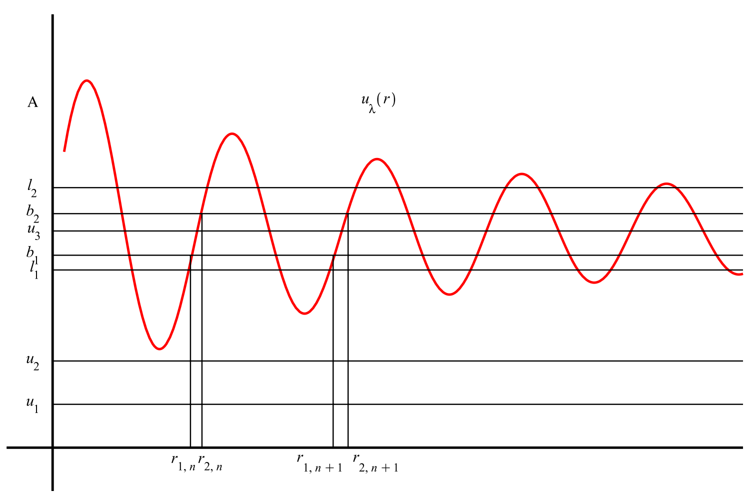

FIGURE 3. Definition of the points $r_{1,n}$, $r_{2,n}$.

As the sequence $\{u_\lambda(r_n^m)\}$ increases to ℓ_1, we may assume that $u_\lambda(r_n^m) \geq (u_2+\ell_1)/2$, and thus, $|f(u_\lambda(r))|$ is bounded below by some positive constant c_1 for all $r \in [r_n^m, r_{1,n}]$.

From the equation we have that for $r \in [r_n^m, r_{1,n}]$,

$$\begin{aligned}|(\phi_p(u_\lambda'))'(r)| &= \Big| \frac{(N-1)}{r}\, \phi_p(u_\lambda'(r)) + f(u_\lambda(r)) \Big| \\ &\geq |f(u_\lambda(r))| - \frac{(N-1)}{r}\, \phi_p(C_\lambda) \\ &\geq c_1 - \frac{(N-1)}{r}\, \phi_p(C_\lambda) \\ &\geq \frac{c_1}{2} \quad \text{for all } r \geq \frac{2\,(N-1)\,\phi_p(C_\lambda)}{c_1}\ .\end{aligned}$$

Hence, choosing n_0 such that $r_n^m \geq \frac{2\,(N-1)\,\phi_p(C_\lambda)}{c_1}$ for all $n \geq n_0$, we have that

$$|(\phi_p(u_\lambda'))'(r)| \geq \frac{c_1}{2} \quad \text{for all } r \in [r_n^m, r_{1,n}]$$

and therefore

$$\phi_p(C_\lambda) \geq \phi_p(u_\lambda'(r_{1,n})) - \phi_p(u_\lambda'(r_n^m)) = (\phi_p(u_\lambda'))'(\xi)(r_{1,n} - r_n^m) \geq \frac{c_1}{2}\,(r_{1,n} - r_n^m)$$

implying that

$$r_{1,n} - r_n^m \leq \frac{2\,\phi_p(C_\lambda)}{c_1}\ . \tag{13}$$

Similarly, for $r \in [r_{2,n}, r_n^M]$, using now that in this interval $f(u_\lambda(r))$ is bounded from below by a positive constant c_2, we conclude that there is $n_1 \geq n_0$ such that

$$r_n^M - r_{2,n} \leq \frac{2\,\phi_p(C_\lambda)}{c_2} \tag{14}$$

for all $n \geq n_1$.

Finally we estimate $r_{2,n} - r_{1,n}$. In the interval $[r_{1,n}, r_{2,n}]$, $u_\lambda(r) \in [b_1, b_2]$ and $F(u_\lambda(r)) \leq max\{F(b_1), F(b_2)\} < F(u_1)$, hence there exists a positive constant c_3 such that

$$F(u_1) - F(u_\lambda(r)) \geq c_3\ ,$$

hence, using that E_λ decreases to $F(u_1)$, we have that

$$|u_\lambda'(r)| \geq (p'\,c_3)^{1/p}\ .$$

Integrating this last inequality over $[r_{1,n}, r_{2,n}]$, we obtain that

$$r_{2,n} - r_{1,n} \leq \frac{b_2 - b_1}{(p'\,c_3)^{1/p}}\ . \tag{15}$$

Hence, from (13), (14) and (15), we conclude that for all $n \geq n_1$

$$r_n^M - r_n^m \leq T$$

where

$$T = \frac{2\,\phi_p(C_\lambda)}{c_2} + \frac{b_2 - b_1}{(p'\,c_3)^{1/p}} + \frac{2\,\phi_p(C_\lambda)}{c_1}\ .$$

Again, from the mean value theorem

$$b_2 - b_1 \le C_\lambda \left(r_{2,n} - r_{1,n}\right) \,,$$

hence, as before, we obtain the contradiction

$$\begin{aligned} F(\lambda) - F(u_1) > E_\lambda(r^m_{n_1}) - E_\lambda(\infty) &= (N-1)\int_{r^m_{n_1}}^{\infty} \frac{|u'_\lambda(t)|^p}{t}\, dt \\ &\ge (N-1)\sum_{k=n_1}^{\infty} \int_{r_{1,k}}^{r_{2,k}} \frac{|u'_\lambda(t)|^p}{t}\, dt \\ &\ge (N-1)\sum_{k=n_1}^{\infty} p'\, c_3\, (r_{2,k} - r_{1,k}) \frac{1}{r_{2,k}} \\ &\ge p'\, c_3\, \frac{b_2 - b_1}{C_\lambda} \sum_{k=n_1}^{\infty} \frac{1}{r_{2,k}} = \infty \,. \end{aligned}$$

Therefore, case (ii) cannot happen and Claim 1 follows.

Claim 2: If $\mathcal{E}_\lambda$ is a relative maxima of F, then the solution u_λ converges as $r \to \infty$. See Figure 4.

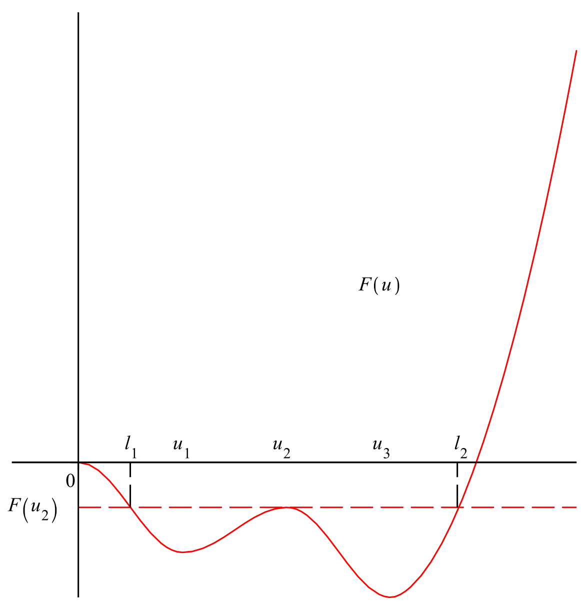

FIGURE 4. If $\ell_1 \neq \ell_2$, then $0 < \ell_1 < u_1$ and $u_3 < \ell_2 < A$.

From (12),

$$\int_{u_2} \frac{du}{|F(u_2) - F(u)|^{1/p}} \quad \text{is convergent .} \tag{16}$$

Then the same arguments used in the proof above can be used to establish the convergence of u_λ. Indeed, we let

$$r_{1,n}: \quad u_\lambda(r_{1,n}) = \frac{\ell_2 + u_3}{2}, \quad r_{2,n}: \quad u_\lambda(r_{2,n}) = \frac{u_2 + u_3}{2}\,,$$

and for $r \in [r_{1,n}, r_{2,n}]$, we have $u_\lambda(r) \in [\frac{u_2+u_3}{2}, \frac{\ell_2+u_3}{2}]$ and thus

$$|u_\lambda'(r)| \geq (p')^{1/p}\,(F(u_2) - F(u_\lambda(r)) \geq c_0 > 0$$

for some positive constant c_0 implying that

$$\frac{\ell_2 - u_2}{2} \geq c_0\,(r_{2,n} - r_{1,n})$$

and similarly, by setting

$$\bar r_{1,n}: \quad u_\lambda(\bar r_{1,n}) = \frac{u_1 + u_2}{2}\,, \quad \bar r_{2,n}: \quad u_\lambda(\bar r_{2,n}) = \frac{u_2 + \ell_1}{2}\,,$$

we have that

$$\frac{u_2 - \ell_1}{2} \geq c_0\,(\bar r_{2,n} - \bar r_{1,n})\,.$$

For $r \in [r_{1,n}, \bar r_{1,n}]$ we use (16) to obtain that

$$(p')^{1/p}\,(\bar r_{1,n} - r_{1,n}) \leq \int_{\frac{u_1+u_2}{2}}^{\frac{u_3+u_2}{2}} \frac{du}{|F(u_2) - F(u)|^{1/p}}\,.$$

The bounds for $r_{2,n} - r_n^M$ and $r_{n+1}^m - \bar r_{2,n}$ is obtained as above using that $|f(u)|$ is bounded below in those intervals and using the equation to obtain

$$\phi_p(C_\lambda) \geq c_0\,(r_{2,n} - r_n^M) \quad \text{and} \quad \phi_p(C_\lambda) \geq c_0\,(r_{n+1}^m - \bar r_{2,n})$$

for some positive constant c_0.

We conclude that the distance between two consecutive critical points is bounded and we end the argument as we did at the end of Proposition 3.2. $\square$

4. A change of coordinates and a lower bound on the angular velocity in the phase space

In this section, we reformulate the problem in the phase space associated to the Hamiltonian system obtained in the (p)-linear case (that is, for $f(u) = |u|^{p-2}\,u)$ in the asymptotic regime corresponding to $r \to \infty$. By computing a lower bound on the angular velocity around the origin, this will allow us to estimate the number of sign changes of the solutions, see Section 5. First, let us explain how to change coordinates.

Setting $v = \phi_p(u')$, or equivalently $u' = \phi_{p'}(v)$, problem (4) is equivalent to the following first order system.

$$\begin{cases} u' = \phi_q(v) \ , \\ \\ v' = -\frac{N-1}{r}\, v - f(u) \ , \\ \\ u(0) = \lambda \ , \quad v(0) = 0 \ . \end{cases} \tag{17}$$

Here $q = p'$ stands for the Hölder conjugate of p. We consider also the auxiliary problem

$$\begin{cases} \dfrac{dx}{dt} = -\phi_q(y) \ , \\ \\ \dfrac{dy}{dt} = \phi_p(x) \ , \\ \\ x(0) = 1 \ , \quad y(0) = 0 \ . \end{cases}$$

The auxiliary problem describes the asymptotics of (17) as $r \to \infty$, that is, when the $\frac{N-1}{r}\, v$ term is neglected in case of a (p)-linear function $f(u) = |u|^{p-2}\, u$. It is well known, see [dPEM89], that solutions to this last systems are $2\,\pi_p = 2\,\pi_q$ periodic. Furthermore, with the notation of [dPEM89], we can define

$$\sin_q(t) := y(t) \quad \text{and} \quad \cos_q(t) := x(t) = \phi_q\left(\frac{d}{dt}\,\sin_q(t)\right) \ .$$

It is immediate to check that

$$\Phi_p(\cos_q(t)) + \Phi_q(\sin_q(t)) = \frac{1}{p} \quad \text{for all} \quad t \in \mathbb{R} \ .$$

To the (u, v) coordinates of the phase plane, we assign *generalized polar coordinates* (ρ, θ) by writing

$$\begin{cases} u = \rho^{\frac{1}{p}} \cos_q(\theta) \\ \\ v = \rho^{\frac{1}{q}} \sin_q(\theta) \end{cases} \tag{18}$$

where

$$\rho = p\ [\Phi_p(u) + \Phi_q(v)] \ .$$

Notice that in case $p = q = 2$, $(\sqrt{\rho}, \theta)$ are the usual polar coordinates of (u, v), and $\cos_q$ and $\sin_q$ are the usual cos and sin functions.

Now, if $(u(r), v(r))$ denotes a solution to (17) and if we define the corresponding polar functions $r \mapsto \rho(r)$ and $r \mapsto \theta(r)$, then it turns out by direct computation that

(ρ, θ) satisfies the following system of equations :

$$\begin{cases} \rho' = p\,\phi_q(v)\,\left[\phi_p(u) - f(u) - \frac{N-1}{r}\,v\right]\,, \\ \theta' = -\frac{1}{\rho}\left[p\,\Phi_q(v) + u\,f(u) + \frac{N-1}{r}\,u\,v\right]\,, \\ \rho(0) = \lambda^p\,, \quad \theta(0) = 0\,. \end{cases} \tag{19}$$

We will denote by $(\rho_\lambda, \theta_\lambda)$ the solution of (19).

The following lemma is a key step for our main result. We establish a lower bound on the angular velocity $|\theta'|$ around the origin, which will later allow us to estimate the number of sign changes of u by counting the number of rotations of the solutions around the origin, in the phase plane. In order to formulate the lemma, we begin by noticing that from (H5), given $\omega \in (0, 1/8)$ there is $s_0 > 0$ such that

$$|f(s)| \geq 4\,\omega\,|s|^{p-1} \quad \text{for all } |s| \geq s_0\ .$$

Lemma 4.1 (Rotation Lemma)**.** *With the previous notation, let assumptions* (H1) *through* (H5) *be satisfied and let* $(\rho_\lambda, \theta_\lambda)$ *be the generalized polar coordinates of a solution* (u_λ, v_λ) *to (17). Set*

$$r_0 := \frac{2\,(N-1)}{\omega\,(p-1)^{1/q}}\,, \quad \sigma_0 \geq \max\left\{2^{1/p} s_0\,,\ \Big(4 \sup_{x\in[-s_0,s_0]} |f(x)|\Big)^{1/(p-1)}\right\}\ .$$

Then, if $r \geq r_0$ *and* $\rho_\lambda \geq \sigma_0^p$*, it holds that*

$$\theta'_\lambda(r) < -\,\omega\ .$$

Proof. We start by observing that with the above notation, *i.e.* $x = \cos_q(\theta)$,

$$\begin{aligned} -\,\theta' &= \left(1 - |x|^p + \frac{x\,f(\sigma\,x)}{\sigma^{p-1}}\right) + \frac{N-1}{r}\frac{u\,v}{\sigma^p} \\ &\geq \left(1 - |x|^p + \frac{x\,f(\sigma\,x)}{\sigma^{p-1}}\right) - \frac{N-1}{r}\,|x|\left(\frac{1-|x|^p}{p-1}\right)^{1/q} \\ &\geq \left(1 - |x|^p + \frac{x\,f(\sigma\,x)}{\sigma^{p-1}}\right) - \frac{N-1}{r}\frac{1}{(p-1)^{1/q}} \end{aligned}$$

where $\sigma = \rho^{1/p}$. It is clear that

$$-\frac{N-1}{r}\frac{1}{(p-1)^{1/q}} > -\,\omega$$

for any $r > r_0$. Hence in order to prove our result we need to estimate the minimum

$$\min_{|x|\leq 1}\left[1 - |x|^p + \frac{x\,f(\sigma\,x)}{\sigma^{p-1}}\right]\ .$$

First assume that $|x|^p \le 1/2$. If $\sigma\,|x| \ge s_0$, then

$$\begin{aligned} 1-|x|^p+\frac{x\,f(\sigma\,x)}{\sigma^{p-1}} &= 1-|x|^p+\frac{|x|^p\,f(\sigma\,x)}{\sigma^{p-1}\,|x|^{p-2}\,x} \\ &\ge 1-|x|^p+4\,\omega\,|x|^p \\ &= 1+(4\,\omega\,-1)\,|x|^p \ge \frac{1}{2}+2\,\omega \ge 2\,\omega\;. \end{aligned}$$

Otherwise, if $\sigma\,|x| \le s_0$, then we have

$$1-|x|^p+\frac{x\,f(\sigma\,x)}{\sigma^{p-1}} \ge 1-|x|^p-\frac{|x|}{\sigma^{p-1}}\sup_{s\in[-s_0,s_0]}|f(s)| > \frac{1}{2}-\frac{1}{\sigma^{p-1}}\sup_{s\in[-s_0,s_0]}|f(s)| > \frac{1}{4}$$

if $\sigma \ge \sigma_0$ and we already know that $\frac{1}{4} > 2\,\omega$.

On the other hand, if $|x|^p \ge 1/2$, then $\sigma\,|x| \ge 2^{-1/p}\,\sigma$, hence for $\sigma \ge \sigma_0$, we have $\sigma\,|x| \ge s_0$ and

$$\begin{aligned} 1-|x|^p+\frac{x\,f(\sigma\,x)}{\sigma^{p-1}} &= 1-|x|^p+\frac{|x|^p\,f(\sigma\,x)}{\sigma^{p-1}\,|x|^{p-2}\,x} \\ &\ge 1-|x|^p+4\,\omega\,|x|^p \ge 2\,\omega\;. \end{aligned}$$

This concludes the proof. □

In preparation for Section 5 we finally relate the energy associated to the flow with the quantity ρ.

Proposition 4.2. *Consider $E(u,v)=F(u)+\Phi_{p'}(v)$ and $\rho(u,v)=p\,[\Phi_p(u)+\Phi_{p'}(v)]$. Under assumptions* (H1) *through* (H5), *it holds that*

$$E(u,v)\to\infty \quad \text{if and only if} \quad \rho(u,v)\to\infty\;,$$

for each (u,v) in $\mathbb{R}^2$.

Proof. The properties $E(u,v)\to\infty$, $\sup(|u|,|v|)\to\infty$, and $\rho(u,v)\to\infty$ are equivalent. □

5. Existence result

We may now state our main result.

Theorem 5.1. *Let $N >$, $p>1$ and suppose that assumptions* (H1)-(H6) *are satisfied. Then there exists an unbounded increasing sequence $\{\lambda_k\}$ of initial data such that for any $k\in\mathbb{N}$, (4) with $\lambda=\lambda_k$, has a compactly supported solution with exactly k nodes.*

The proof is based upon some preliminary results that we state and prove next.

For given $\lambda > A$, let (u_λ, v_λ) be a solution to (17). Recall that the energy function E_λ has been defined by (5). For any $a\in[0,F(\lambda)]$, let us set

$$r_\lambda(a) := \inf\{r\ge 0\,:\,E_\lambda(r)=a\}\;.$$

We first observe that $r_\lambda(0)$ is finite. Indeed, if for some $\lambda \ge A$ (as defined in Section 1) we have that $r_\lambda(0)=\infty$, then $E_\lambda(r)\ge 0$ for all $r\ge 0$, and thus, from

Corollary 3.3, there exists $r_0 > 0$ such that r_0 is a double zero of u_λ implying by the definition that $r_\lambda(0) \le r_0 < \infty$.

We will denote by $N_{[0,R)}(\lambda)$ the number of nodes of u_λ in $[0, R)$. For simplicity of notation, we will denote

$$N(\lambda) := N_{[0,r_\lambda(0))}(\lambda) \ .$$

Notice that all the possible zeros of u_λ in $[0, r_\lambda(0))$ must be simple zeros.

The following proposition was proved in [GHMZ97].

Proposition 5.2. *Under assumptions* (H1) *through* (H6)*, given* $R > 0$*,*

$$\lim_{\lambda\to\infty} E_\lambda(r) = \infty$$

uniformly for $r \in [0, R]$*.*

Now we start to make use of the variables introduced in Section 4.

Proposition 5.3. *If* $N(\lambda) > 1$*, then for any* $r \in (0, r_\lambda(0))$*, the number of nodes of* u_λ *in* $(0, r)$ *is given by*

$$\left[\left(\frac{\pi_p}{2} - \theta_\lambda(r)\right)\frac{1}{\pi_p}\right]$$

where $[x]$ *denotes the integer part of* x*.*

Proof. Follows directly from the change of variables (18). □

Propositions 5.2 and 5.3 combined with the *Rotation Lemma* 4.1 on the angular velocity, yields the following result.

Lemma 5.4. *Under assumptions* (H1) *through* (H6)*,*

$$\lim_{\lambda\to+\infty} N(\lambda) = +\infty \ .$$

Proof. Let $M > 0$. We will show that there exists $\lambda_M > 0$ such that for $\lambda > \lambda_M$, we have $N(\lambda) > M$. We prove this by finding an interval $[0, R]$ with $R = R(M)$, such that the number of nodes in $[0, R]$ is greater than M. To do this we set

$$R = \frac{\pi_p}{\omega}\left(M + \frac{1}{2}\right) + r_0 \ .$$

Using Propositions 4.2 and 5.2, we know that there exists λ_M such that for any $\lambda \ge \lambda_M$, $\rho_\lambda(r) > \rho_0 := \sigma_0^p$ in $[0, R]$. Next we apply the rotation Lemma 4.1 which ensures that

$$-\,\theta_\lambda(R) \ge \omega\, R - \omega\, r_0 - \ \theta_\lambda(r_0) \ge \omega\, R - \omega\, r_0 = \left(M + \frac{1}{2}\right)\pi_p \ ,$$

by the choice of R. Applying Corollary 5.3, it follows that

$$N(\lambda) \ge [M + 1] > M \ .$$

□

Lemma 5.5. *Under assumptions* (H1) *through* (H5), *when $\lambda > 0$ varies, the number of nodes of the solution u_λ can locally change by at most one. Moreover, to the value of λ at which the number of nodes changes corresponds a solution with compact support.*

Proof. This lemma is proved by defining for any $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ the sets

$$A_k := \{\lambda \geq A \ : \ (u_\lambda(r), v_\lambda(r)) \neq (0,0) \quad \text{for all } r \geq 0\,,\ \text{and } N(\lambda) = k\}\,,$$

$$I_k := \{\lambda \geq A \ : \ (u_\lambda(r_\lambda(0)), v_\lambda(r_\lambda(0))) = (0,0) \quad \text{and } N_{[0,r_\lambda(0))}(\lambda) = k\}\,.$$

Recall that $r_\lambda(a) := \inf\{r \geq 0 \ : \ E_\lambda(r) = a\}$, and E_λ has been defined by (5). Notice that we have

$$[A, \infty) = \big(\cup_{k\in\mathbb{N}_0} I_k \big) \cup \big(\cup_{k\in\mathbb{N}_0} A_k \big)\,.$$

Indeed, let $\lambda \geq A$. Then $N(\lambda) = j$ for some $j \in \mathbb{N}_0$. If $u_\lambda(r_\lambda(0)) \neq 0$, then u_λ does not have any double zero in $[0, \infty)$. Indeed, assume by contradiction that $r_1 > r_\lambda(0)$ is a double zero of u_λ. Then by the monotonicity of E_λ, $E_\lambda(r) \equiv 0$ in $[r_\lambda(0), r_1]$. But then also $E'_\lambda(r) \equiv 0$ in $(r_\lambda(0), r_1)$ implying that $u'_\lambda(r) \equiv 0$ in $(r_\lambda(0), r_1)$ and thus $u_\lambda(r_\lambda(0)) = u_\lambda(r_1) = 0$, a contradiction. Hence $\lambda \in A_j$. If $u_\lambda(r_\lambda(0)) = 0$, then by the definition of $r_\lambda(0)$ we also have $u'_\lambda(r_\lambda(0)) = 0$ hence $\lambda \in I_j$. Also, observe that the sets A_i, I_j are disjoint for any i, j, and for $i \neq j$, $A_i \cap A_j = \emptyset$ and $I_i \cap I_j = \emptyset$.

We also observe that if $\lambda \in A_j$, then necessarily $\lim_{r\to\infty} E_\lambda(r) < 0$ (see Corollary 3.3), and if $\lambda \in I_j$, then two cases may occur:

$$\text{either } \lim_{r\to\infty} E_\lambda(r) < 0 \quad \text{or} \quad \lim_{r\to\infty} E_\lambda(r) = 0\,.$$

This due to the possible non-uniqueness of solutions to the initial value problem (4), a solution could reach a double zero but not remain identically zero after that.

The proof of Lemma 5.5 is a consequence of the following technical result.

Proposition 5.6. *With the above notation, we have:*

- (i) A_k *is open in* $[A, \infty)$,
- (ii) $A_k \cup I_k$ *is bounded,*
- (iii) *if* $\lambda_0 \in I_k$, *then there exists* $\delta > 0$ *such that* $(\lambda_0 - \delta, \lambda_0 + \delta) \subset A_k \cup A_{k+1} \cup I_k$,
- (iv) $\sup A_k \in I_{k-1} \cup I_k$, *where we set* $I_{-1} = \emptyset$ *and,*
- (v) $\sup I_k \in I_k$.

Proof. (i) A_k is open in $[A, \infty)$: Indeed, if $\bar\lambda \in A_k$, then in particular $(u_{\bar\lambda}(\bar r), v_{\bar\lambda}(\bar r)) \neq (0,0)$, where $\bar r = r_{\bar\lambda}(0)$. Then there exists $\varepsilon_0 > 0$ such that the solution of (4) is unique in $[0, r_{\bar\lambda}(0) + \varepsilon]$ and $E_{\bar\lambda}(r_{\bar\lambda}(0) + \varepsilon/2) < 0$ for all $\varepsilon \in (0, \varepsilon_0]$, and thus there exists $\delta > 0$ such that

$$E_\lambda(r_{\bar\lambda}(0) + \varepsilon/2) < 0$$

for all $\lambda \in (\bar\lambda - \delta, \bar\lambda + \delta)$ implying that $r_\lambda(0) \leq r_{\bar\lambda}(0) + \varepsilon/2$. On the other hand, for the same reason, there exists $\delta' > 0$ such that

$$E_\lambda(r_{\bar\lambda}(0) - \varepsilon/2) > 0$$

for all $\lambda \in (\bar\lambda - \delta', \bar\lambda + \delta')$ implying that $r_\lambda(0) \geq r_{\bar\lambda}(0) - \varepsilon/2$. We conclude then that $r_\lambda(0) \to r_{\bar\lambda}(0)$. Hence the openness of A_k follows from the continuous dependence of solutions in the initial value λ.

(ii) The boundedness of $A_k \cup I_k$ is a consequence of Lemma 5.4.

(iii) The proof of this statement follows that of [CGY12, Lemma 2.3].

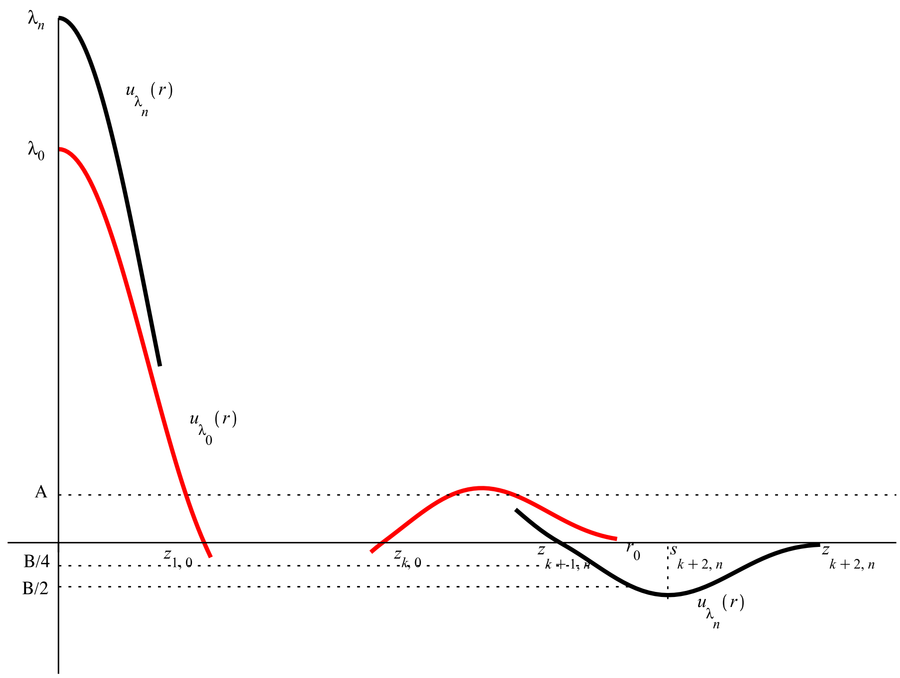

FIGURE 5. Situation for (iii).

Let $\lambda_0 \in I_k$, set $r_0 = r_{\lambda_0}(0)$ and let

$$0 < z_{1,0} < z_{2,0} < \ldots < z_{k,0} < r_0$$

denote the k zeros of u_{λ_0} in $(0, r_0)$.

Assume first that u_{λ_0} is decreasing in $(r_0 - 2\,\varepsilon_0, r_0)$ for some $\varepsilon_0 > 0$, so that it reaches a last maximum point at some $s_{k,0} \in (z_{k,0}, r_0)$. Let

$$H_\lambda(r) := r^{p'(N-1)}\, E_\lambda(r)\;. \tag{20}$$

As $u_{\lambda_0}(r_0) = 0$ and $H_{\lambda_0}(r_0) = 0$, given $\varepsilon > 0$, there exists $\bar r < r_0$ such that

$$0 < u_{\lambda_0}(\bar r) < \frac{A}{2}\;, \quad \text{and} \quad H_{\lambda_0}(\bar r) < \varepsilon\;.$$

Hence by continuous dependence of solutions to (4) in the initial data in any compact subset of $[0, r_0)$, there exists $\delta_0 > 0$ such that for $\lambda \in (\lambda_0 - \delta_0, \lambda_0 + \delta_0)$, the solution u_λ satisfies

$$0 < u_\lambda(\bar r) < A\;, \quad H_\lambda(\bar r) < 2\,\varepsilon \quad \text{and } u_\lambda \text{ has at least } k \text{ simple zeros in } [0, r_0)\;, \tag{21}$$

that is,

$$(\lambda_0 - \delta_0, \lambda_0 + \delta_0) \subset \Big(\cup_{j \geq k} A_j \Big) \cup \Big(\cup_{j \geq k} I_j \Big) \ . \tag{22}$$

Now we argue by contradiction and assume that there is a sequence $\{\lambda_n\}$ converging to λ_0 as $n \to \infty$ such that

$$\lambda_n \notin A_k \cup A_{k+1} \cup I_k \ .$$

From (22),

$$\lambda_n \in \Big(\cup_{j \geq k+2} A_j \Big) \cup \Big(\cup_{j \geq k+1} I_j \Big) \ ,$$

that is, the solution u_{λ_n} has at least $k+2$ zeros and at least the first $k+1$ zeros are simple. Let us denote these zeros by

$$0 < z_{1,n} < z_{2,n} < \ldots < z_{k,n} < z_{k+1,n} < z_{k+2,n} \ .$$

See Figure 5. By the choice of $\bar{r}$ and (21), u_{λ_n} decreases in $[\bar{r}, z_{k+1,n}]$. Let us denote by $s_{k+1,n}$ the point in $(z_{k+1,n}, z_{k+2,n})$ where u_{λ_n} reaches its minimum value. As $E_{\lambda_n}(z_{k+2,n}) \geq 0$, we must have that

$$u_{\lambda_n}(s_{k+1,n}) < B \ .$$

Let us denote by $r_{1,n} < r_{2,n}$ the unique points in $(z_{k+1,n}, s_{k+1,n})$ where

$$u_{\lambda_n}(r_{1,n}) = \frac{B}{4} \ , \quad u_{\lambda_n}(r_{2,n}) = \frac{B}{2} \ .$$

From (20), we have that

$$H'_{\lambda_n}(r) = p'\,(N-1)\, r^{p'(N-1)-1}\, F(u_{\lambda_n}(r)) \ . \tag{23}$$

Therefore, using the first estimate in (21), we have that for n large enough, $H'_{\lambda_n}(r) < 0$ for $r \in [\bar{r}, z_{k+1,n}]$ and thus by the second in (21), $H_{\lambda_n}(z_{k+1,n}) < 2\,\varepsilon$.

Integrating now (23) over $[z_{k+1,n}, r_{2,n}]$, and using that $F(u_{\lambda_n}(t)) < 0$ in this range and $p'(N-1) - 1 = \frac{p}{p-1}(N-1) - 1 \geq p - 1 > 0$, we find that

$$\begin{aligned}
H_{\lambda_n}(r_{2,n}) - H_{\lambda_n}(z_{k+1,n}) \ &= \ -\,p'\,(N-1) \int_{z_{k+1,n}}^{r_{2,n}} t^{p'(N-1)-1} |F(u_{\lambda_n}(t))| \, dt \\
&\leq \ -\,p'\,(N-1)(z_{k+1,n})^{p'(N-1)-1} \int_{z_{k+1,n}}^{r_{2,n}} |F(u_{\lambda_n}(t))| \, dt \\
&\leq \ -\,p'\,(N-1)(z_{k+1,n})^{p'(N-1)-1} \int_{r_{1,n}}^{r_{2,n}} |F(u_{\lambda_n}(t))| \, dt \\
&\leq \ -\,C\, p'\,(N-1)(z_{k+1,n})^{p'(N-1)-1} (r_{2,n} - r_{1,n}) \ ,
\end{aligned}$$

where

$$C := \inf_{s \in [\frac{B}{2}, \frac{B}{4}]} |F(s)| \ .$$

But from the mean value theorem, and for n large enough, we have that

$$\frac{|B|}{4} = |u_{\lambda_n}(r_{2,n}) - u_{\lambda_n}(r_{1,n})| \leq C_{\lambda_0+1}\,(r_{2,n} - r_{1,n}) \ ,$$

and
$$\frac{\lambda_0}{2} \le \lambda_n = |u_{\lambda_n}(0) - u_{\lambda_n}(z_{1,n})| \le C_{\lambda_0+1}\, z_{1,n} \le C_{\lambda_0+1}\, z_{k+1,n} \ ,$$
hence
$$\begin{aligned} H_{\lambda_n}(r_{2,n}) &\le 2\,\varepsilon - C\,p'\,(N-1)\,(z_{k+1,n})^{p'(N-1)-1}\,\frac{|B|}{4\,C_{\lambda_0+1}} \\ &\le 2\,\varepsilon - C\,p'\,(N-1)\left(\frac{\lambda_0}{2\,C_{\lambda_0+1}}\right)^{p'(N-1)-1}\frac{|B|}{4\,C_{\lambda_0+1}} \ . \end{aligned}$$
By choosing from the beginning $\varepsilon \in \left(0, C\,p'\,(N-1)\left(\frac{\lambda_0}{2\,C_{\lambda_0+1}}\right)^{p'(N-1)-1}\frac{|B|}{8C_{\lambda_0+1}}\right)$ we obtain that
$$H_{\lambda_n}(r_{2,n}) = r_{2,n}^{p'(N-1)} E_{\lambda_n}(r_{2,n}) < 0 \ ,$$
contradicting the fact that
$$E_{\lambda_n}(r_{2,n}) \ge E_{\lambda_n}(z_{k+2,n}) \ge 0 \ .$$

A similar computation provides a contradiction if we assume that u_{λ_0} is increasing in $(r_0 - 2\,\varepsilon_0, r_0)$ for some $\varepsilon_0 > 0$. Altogether (iii) is established.

(iv) Assume next that $A_k \neq \emptyset$, let $\lambda_0 = \sup A_k$ and set $r_0 = r_{\lambda_0}(0)$. As A_j is open for every $j \in \mathbb{N}_0$, $\lambda_0 \notin A_j$ for any j hence $\lambda_0 \in I_j$ for some j, and by continuous dependence of the solutions in the initial data in $[0, r_0 - \varepsilon]$ for $\varepsilon > 0$ small enough, $j \le k$. By (iii), there is $\delta > 0$ such that $(\lambda_0 - \delta, \lambda_0] \subset A_j \cup A_{j+1} \cup I_j$, and since $A_k \cap (\lambda_0 - \delta, \lambda_0] \neq \emptyset$, it must be that
$$A_k \cap (A_j \cup A_{j+1} \cup I_j) \neq \emptyset \ ,$$
hence $j = k$ or $j = k-1$.

(v) $\sup I_k \in I_k$: It follows directly from (iii). □

Proof of Theorem 5.1. With the notation of the previous lemma one shows by induction that there exists an increasing sequence $\{\lambda_k\}$, $\lambda_k \to +\infty$, such that $\lambda_k \in I_k$.

As $A \in A_0$, by (ii) we can set $\lambda_0 = \sup A_0$, and by (iv) and (v), $\lambda_0 \in I_0$ and $\lambda_0 \le \sup I_0 \in I_0$. We use now (iii) and find $\delta > 0$ such that
$$(\sup I_0 - \delta, \sup I_0 + \delta) \subset A_0 \cup A_1 \cup I_0 \ .$$
Since $(\sup I_0, \sup I_0 + \delta) \cap A_0 = \emptyset$ by the definition of λ_0 and $(\sup I_0, \sup I_0 + \delta) \cap I_0 = \emptyset$ by the definition of $\sup I_0$, it must be that
$$(\sup I_0, \sup I_0 + \delta) \subset A_1$$
implying that
$$A_1 \neq \emptyset \quad \text{and} \quad \lambda_0 \le \sup I_0 < \lambda_1 := \sup A_1 \ .$$
By (iv), $\lambda_1 \in I_0 \cup I_1$, but as $\sup I_0 < \lambda_1$, it must be that $\sup A_1 \in I_1$. Then
$$I_1 \quad \text{is not empty and} \quad \lambda_1 \le \sup I_1 \ .$$

We use again (iii) to find $\delta > 0$ such that

$$(\sup I_1 - \delta, \sup I_1 + \delta) \subset A_1 \cup A_2 \cup I_1 \ ,$$

and again deduce that

$$(\sup I_1, \sup I_1 + \delta) \subset A_2 \ ,$$

hence $A_2 \neq \emptyset$ and thanks to (ii) we can set $\lambda_2 = \sup A_2$, $\lambda_0 < \lambda_1 \leq \sup I_1 < \lambda_2$ and $\lambda_2 \in I_2$. We continue this procedure to obtain the infinite strictly increasing sequence $\{\lambda_k\}$, defined by $\lambda_k = \sup A_k$ with $\lambda_k \in I_k$. □

6. Qualitative properties of the solutions

Several qualitative properties can be deduced from our intermediate results and from their proofs. Without entering the details let us summarize the most striking ones.

When λ varies, the number of nodes changes of at most one. The energy of any solution decreases as r increases and converges to a finite limit as $r \to \infty$. More precisely, solutions are of two types: either the limit of their energy is negative or the limit of the energy is zero, and the corresponding solutions are compactly supported.

Solutions which have a double zero can be compactly supported or not, as in this case uniqueness may be lost.

For solutions with compact support, the size of the support increases with the number of nodes, and diverges as the number of nodes goes to infinity. This is a consequence of Lemma 4.1 and Proposition 5.2, as can be easily proved arguing by contradiction. With the generality of Theorem 5.1, it is not easy to give quantitative results but one can estimate the size of the support of the solutions and the number of nodes for large values of λ, as the following proposition shows.

Proposition 6.1. *Let $1 < p < N$. Under the assumptions of Theorem 5.1, we have that $r_\lambda(0)$ and $N(\lambda)$ are bounded below by*

$$C\Big(\frac{\lambda^{\frac{N(p-1)}{N-p}}}{f(\lambda)}\Big)^{\frac{N-p}{p(N-1)}}$$

where C is a positive constant independent of λ.

In particular, if $\displaystyle\lim_{\lambda\to\infty} \frac{\lambda^{\frac{N(p-1)}{N-p}}}{f(\lambda)} = \infty$*, then $r_\lambda(0) \to \infty$ and $N(\lambda) \to \infty$ as $\lambda \to \infty$.*

Proof. Let $\theta \in (0,1)$ be as in (H6), and for $\lambda > 0$, let $S_{\theta,\lambda} := \inf\{r > 0 \,:\, u_\lambda(r) = \theta\,\lambda\}$. It can be easily shown that a solution of (4) satisfies

$$(1-\theta)\,\lambda = \lambda - u_\lambda(S_{\theta,\lambda}) = \int_0^{S_{\theta,\lambda}} \left(\frac{1}{r^{N-1}}\int_0^r s^{N-1} f(u_\lambda(s))\,ds\right)^{1/(p-1)} dr\ .$$

As a consequence of the monotonicity of u_λ in $[0, S_{\theta,\lambda}]$ and (H5), for $\theta\lambda$ large enough we obtain

$$N^{1/(p-1)}\,p'\,\frac{(1-\theta)\,\lambda}{[f(\lambda)]^{1/(p-1)}} \leq (S_{\theta,\lambda})^{p'} \leq N^{1/(p-1)}\,p'\,\frac{(1-\theta)\,\lambda}{[f(\theta\,\lambda)]^{1/(p-1)}}\ . \tag{24}$$

On the other hand, since

$$\frac{d}{dr}(E_\lambda+\bar{F}) = -\frac{N-1}{r}\,|v|^{p'} \geq -\frac{N-1}{r}\,p'\,(E_\lambda+\bar{F})\;,$$

for $S_{\theta,\lambda} \leq r \leq r_\lambda(0)$, we obtain

$$\frac{E_\lambda(r)+\bar{F}}{F(\theta\,\lambda)+\bar{F}} \geq \left(\frac{S_{\theta,\lambda}}{r}\right)^{(N-1)\,p'}.$$

Hence

$$\begin{array}{rcl} r_\lambda(0)^{p'} & \geq & S_{\theta,\lambda}^{p'}\left(\dfrac{F(\theta\,\lambda)}{\bar{F}}\right)^{\frac{1}{N-1}} \\ & \geq & \dfrac{1}{\bar{F}^{1/(N-1)}}S_{\theta,\lambda}^{p'}\left(\dfrac{F(\theta\,\lambda)}{\lambda f(\lambda)}\right)^{\frac{1}{N-1}}(\lambda f(\lambda))^{\frac{1}{N-1}} \\ \text{from (H6)} & \geq & C S_{\theta,\lambda}^{p'}(\lambda f(\lambda))^{\frac{1}{N-1}} \\ \text{from the first in (24)} & \geq & C\Big(\dfrac{\lambda^{\frac{N(p-1)}{N-p}}}{f(\lambda)}\Big)^{\frac{N-p}{(p-1)(N-1)}}. \end{array}$$

Hence,

$$r_\lambda(0) \geq C\Big(\frac{\lambda^{\frac{N(p-1)}{N-p}}}{f(\lambda)}\Big)^{\frac{N-p}{p(N-1)}}.$$

This proves the assertion on the size of the support. The conclusion on the number of nodes follows from Lemma 4.1. □

Appendix A.

In this Appendix, for sake of completeness, we state some basic results concerning the initial value problem (4). We begin with a result on the existence of solutions.

Proposition A.1. *Suppose that assumption* (H1) *holds. If* $\lim_{u\to\pm\infty} f(u) = \pm\infty$, *for any fixed* $\lambda \in \mathbb{R}$, *then (4) has a solution defined in* $[0,\infty)$.

Proof. As before let $F(u) = \int_0^u f(s)ds$, then $F(u) \to \infty$ as $u \to \pm\infty$. Recall that

$$-\bar{F} = \inf_{u\in\mathbb{R}} F(u)\;,$$

and suppose that u is a solution to (3) such that $u(0) = \lambda$. Then u satisfies

$$\frac{|u'(r)|^p}{p'} \leq \frac{|u'(r)|^p}{p'} + F(u(r)) + \bar{F} \leq F(\lambda) + \bar{F}\;.$$

Hence $|u'| \leq (p'\,C_\lambda)^{1/p}$ with $C_\lambda = F(\lambda) + \bar{F}$ and $|u| \leq |\lambda| + (p'\,C_\lambda)^{1/p}\,r$ for $r > 0$ in the domain of definition of u. These estimates tell us that if u can be defined in an interval of the form $[0,\delta]$ for $\delta > 0$ and small, then this solution can be extended to $[0,\infty)$.

Consider therefore the Banach space $\mathcal{C} := C([0,\delta];\mathbb{R})$ of continuous functions on $[0,\delta]$, endowed with the sup norm $\|\cdot\|_\infty$. A solution in $\mathcal{C}$ will exist if and only if the operator T defined on $\mathcal{C}$ by

$$T(u)(r) := \lambda - \int_0^r \phi_{p'}\left(\int_0^\tau \frac{s^{N-1}}{\tau^{N-1}}\, f(u(s))\, ds\right)\, d\tau \tag{25}$$

has a fixed point. For $\varepsilon > 0$ given, let $B(\lambda,\varepsilon)$ be the ball in $\mathcal{C}$ with center λ and radius ε. Then if $u \in \overline{B(\lambda,\varepsilon)}$ we have that for all $r \in [0,\delta]$ it holds that $-\varepsilon + \lambda \le u(r) \le \varepsilon + \lambda$. Let us set $m := \max_{|u-\lambda|\le\varepsilon} |f(u)|$. Then from (25), we find the estimate

$$|T(u)(r) - \lambda| \le \frac{\delta^{p'}\, m^{p'-1}}{p'\, N^{p'-1}}\ .$$

If δ is small so that $\frac{\delta^{p'} m^{p'-1}}{p' N^{p'-1}} \le \varepsilon$, we have that $T(\overline{B(\lambda,\varepsilon)}) \subset \overline{B(\lambda,\varepsilon)}$.

To show that T is completely continuous, let $\{u_k\}$ be a sequence in $\overline{B(\lambda,\varepsilon)}$ and consider s, $t \in [0,\delta]$. Then

$$|T(u_k)(t) - T(u_k)(s)| \le \frac{\delta^{p'-1}\, m^{p'-1}}{N^{p'-1}}\, |t-s|\ .$$

From Ascoli-Arzela theorem it follow that T is compact on $\overline{B(\lambda,\varepsilon)}$. To show that T is continuous let $\{u_k\}$ be a sequence in $\overline{B(\lambda,\varepsilon)}$ such that $u_k \to u \in \overline{B(\lambda,\varepsilon)}$, as $k \to \infty$. An application of Lebesgue's dominated theorem to

$$T(u_k)(r) := \lambda - \int_0^r \phi_{p'}\left(\int_0^\tau \frac{s^{N-1}}{\tau^{N-1}}\, f(u_k(s))\, ds\right)\, d\tau\ ,$$

shows that $T(u_k) \to T(u)$ in $\mathcal{C}$ as $k \to \infty$.

Then by the Schauder fixed point theorem, T possesses a fixed point in $\overline{B(\lambda,\varepsilon)}$ which is what we wanted to prove. $\square$

The last proposition states a unique extendibility result for solutions to the initial value problem (4); this result has been proved in [CGY12].

Proposition A.2. *Let f satisfy* (H1)-(H2)*. Then solutions to* (4) *are unique at least until they reach a double zero or a point $u_0 = u_\lambda(r_0)$, where $u'_\lambda(r_0) = 0$ and u_0 is a local maximum of F.*

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E-mail address: J. Dolbeault: dolbeaul@ceremade.dauphine.fr,
M. García-Huidobro: mgarcia@mat.puc.cl, R. Manásevich: manasevi@dim.uchile.cl