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A mixing effect induced by sources concentrated in a soft junction and the gradient concentration phenomenon

Anne-Laure Bessoud ^{*}, Pongpol Juntharee ^{†‡}, Christian Licht [†], and Gérard Michaille [‡]

Abstract

We show that the variational limit of a ε -soft and thin junction problem $(\mathcal{P}_\varepsilon)$ with sources concentrated in the junction gives rise to a surface energy mixing the internal energy and sources. The surface energy functional possesses an integral representation with respect to the Gradient Young-Concentration measures generated by sequences $(\bar{u}_\varepsilon)_{\varepsilon>0}$ of minimizers of $(\mathcal{P}_\varepsilon)$.

AMS *subject classifications*: 35B40, 49Q20, 28A33

Keywords: Γ -convergence, Gradient Young measures, concentration measures, minimization problems.

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1 Introduction

This paper concerns a soft thin junction subjected to concentrated sources. More precisely, let Ω be a domain in $\mathbb{R}^N$ and let $B_\varepsilon := \Sigma \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \subset \Omega$, $\Sigma \subset \mathbb{R}^{N-1}$, be the layer occupied by the soft thin junction (cf Figure 1). We consider the minimization problem

$$\min_{u \in W_{\Gamma_0}^{1,2}(\Omega)} \left\{ \int_{\Omega \setminus B_\varepsilon} f(\nabla u) \, dx + \varepsilon \int_{B_\varepsilon} g(\nabla u) \, dx - \langle \mathcal{S}^\varepsilon, u \rangle_\varepsilon \right\} \quad (\mathcal{P}_\varepsilon)$$

where $W_{\Gamma_0}^{1,2}(\Omega)$ denotes the space of Sobolev functions with null trace on a part Γ_0 of the boundary of Ω , and the linear form $\langle \mathcal{S}^\varepsilon, \cdot \rangle_\varepsilon$ represents the work of the source (or the loading). Let $B := \Sigma \times (-\frac{1}{2}, \frac{1}{2})$. A suitably rescaled $\mathcal{S}_\varepsilon$ of $\mathcal{S}^\varepsilon$ is assumed to strongly converge to some $\mathcal{S}$ in the dual of the space $V(B) := \left\{ u \in L^2(\Omega) : \frac{\partial u}{\partial x_N} \in L^2(\Omega) \right\}$ when ε tends to zero. A general example of such sources which are measures on B_ε is given in Section 4 of the paper. Sources of the form $c \frac{1}{L(\varepsilon)} \mathbb{1}_{B_\varepsilon}$ where c is any constant and $L(\varepsilon) \sim \varepsilon$, is a trivial example of measures satisfying this condition with $\mathcal{S} = \mathbb{1}_B$. Note that in this paper the source (or the loading) $\mathcal{S}_\varepsilon$ is a non L^2 -continuous perturbation of the energy functional $\int_{\Omega \setminus B_\varepsilon} f(\nabla u) \, dx + \varepsilon \int_{B_\varepsilon} g(\nabla u) \, dx$.

Among the physical motivations of $(\mathcal{P}_\varepsilon)$ one may mention various applications to heat conduction or electrostatic problems involving sources concentrated in the layer B_ε with conductivity or permittivity of order the size of B_ε . One may also think of membrane problems with an exterior loading concentrated in B_ε occupied by a material with stiffness of order the small size of B_ε . Such a problem with a source concentrated in the junction was considered in [3] in a one dimensional case in order to highlight and illustrate a gradient concentration phenomenon, but the authors were not able to express the variational limit problem.

This paper illustrates the same gradient concentration phenomenon with a complete description of the limit problem in the sense of Γ -convergence (Theorem 3.3). When the size ε of the layer goes to zero, fields u_ε of bounded energy develop a discontinuity through Σ . More precisely, at the variational limit, the internal energy functional of the junction $\varepsilon \int_{B_\varepsilon} g(\nabla u) \, dx$ and the work of the loading $\langle \mathcal{S}^\varepsilon, u \rangle_\varepsilon$ are combined into a functional of the type

$$\bar{H}(u) = \int_{\Sigma} \bar{h}(\hat{x}, u^+ - u^-, \frac{u^+ + u^-}{2}) \, d\hat{x}$$

and the limit problem reads as

$$\min_{u \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)} \left\{ \int_{\Omega} f(\nabla u) \, dx + \bar{H}(u) \right\} \quad (\mathcal{P})$$

where $u^\pm$ denote the traces on Σ . When regarding the various studies devoted to the asymptotic modeling of junction problems (see [2, 9, 7, 10] and references therein) the main novelty is that the density $\bar{h}$ depends also of the mean $\frac{u^+ + u^-}{2}$. Furthermore we show that the sequence of minimizers of $(\mathcal{P}_\varepsilon)$ (which converges to a minimizer $\bar{u}$ of the limit problem $(\mathcal{P})$) generates a *gradient Young-concentration measure* $\bar{\mu}$ in the sense defined in [3]. Then we can give an *integral representation* of the internal part of $\bar{H}$ with respect to the measure $\bar{\mu}$ (Theorem 5.5) so that it can be localized in $\Sigma \times \{\pm 1\}$. Finally this provides new bounds on the measure $\bar{\mu}$ (Corollary 5.6).

The paper is organized as follows: in Section 2 we fix notation and give a detailed description of the problem $(\mathcal{P}_\varepsilon)$. Section 3 is devoted to the asymptotic analysis of $(\mathcal{P}_\varepsilon)$ in the sense of the Γ -convergence of the functional energy extended to $L^2(\Omega)$ equipped with its strong topology. In Section 4 we describe a large class of suitable sources $\mathcal{S}_\varepsilon$. Finally Section 5 is concerned with the analysis of the gradient concentration phenomenon generated by sequences of minimizers of $(\mathcal{P}_\varepsilon)$. We stress the fact that one could treat the problem in $L^p(\Omega)$, $1 < p < +\infty$ in the same way without additional difficulties.

2 Description of the minimization problem

Let $\varepsilon > 0$ be a small parameter intended to go to zero, more precisely taking values in a countable subset of $(0, \varepsilon_0]$ whose 0 is the only cluster point. The reference configuration of the assembly of the two adherents and the adhesive is a cylinder $\Omega := \Sigma \times (-r, r)$ (with $r > \varepsilon$), where Σ is a bounded domain in $\mathbb{R}^{N-1}$, $N \geq 2$, with Lipschitz boundary. For $x \in \mathbb{R}^N$ we sometimes write $x = (\hat{x}, x_N)$ where $\hat{x} \in \mathbb{R}^{N-1}$. In all the paper, C denotes a non negative constant which does not depend on ε and may vary from line to line. We do not relabel the various considered subsequences and the symbols $\rightarrow$ and $\rightharpoonup$ denote various strong convergences and weak convergences respectively. We define the following sets:

Figure 1: physical domain

- . $B_\varepsilon := \Sigma \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$;
- . $B := \Sigma \times (-\frac{1}{2}, \frac{1}{2})$;
- . $\Omega_\varepsilon = \Omega \setminus \overline{B_\varepsilon}$;
- . Γ_0 is a subset of the boundary $\partial\Omega$ of Ω such that $\text{dist}(\overline{\Gamma_0}, \overline{\partial B_\varepsilon \cap \partial\Omega}) > 0$ for all $\varepsilon < \varepsilon_0$;
- . we write $\Omega_\varepsilon^-, \Omega_\varepsilon^+, \Omega^-, \Omega^+, B_\varepsilon^+$ and B_ε^- for the sets $\Omega_\varepsilon \cap [x_N < 0]$ and $\Omega_\varepsilon \cap [x_N > 0]$, $\Omega \cap [x_N < 0]$, $\Omega \cap [x_N > 0]$ and $B_\varepsilon \cap [x_N > 0]$, $B_\varepsilon \cap [x_N < 0]$ respectively.

We will be concerned with the following spaces:

- . $W_{\Gamma_0}^{1,2}(\Omega_\varepsilon) := \{u \in W^{1,2}(\Omega_\varepsilon) : u = 0 \text{ on } \Gamma_0\}$;
- . $W_{\Gamma_0}^{1,2}(\Omega) := \{u \in W^{1,2}(\Omega) : u = 0 \text{ on } \Gamma_0\}$;
- . $W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma) := \{u \in W^{1,2}(\Omega \setminus \Sigma) : u = 0 \text{ on } \Gamma_0\}$, and for every $z \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$, $z^\pm$ will stand for the traces of z on Σ considered as a Sobolev function on Ω^+ and Ω^- respectively.

We say that a function $h : \mathbb{R}^N \longrightarrow \mathbb{R} \cup \{+\infty\}$ satisfies a growth condition of order 2 if there exist α and β in $\mathbb{R}^+$ such that

$$\alpha |\xi|^2 \leq h(\xi) \leq \beta(1 + |\xi|^2) \text{ for all } \xi \in \mathbb{R}^N.$$

We consider two convex functions $f, g : \mathbb{R}^N \longrightarrow \mathbb{R}$ satisfying a growth condition of order 2, and we assume that there exists a positively 2-homogeneous function $g^{\infty,2}$ satisfying

$$|g(\xi) - g^{\infty,2}(\xi)| \leq \beta(1 + |\xi|^{2-\delta}) \text{ for all } \xi \in \mathbb{R}^N, \quad (2.1)$$

for some δ , $0 < \delta < 2$. Note that $g^{\infty,2}$ is the positively 2-homogeneous recession function of g , i.e.,

$$g^{\infty,2}(\xi) = \lim_{t \rightarrow +\infty} \frac{g(t\xi)}{t^2},$$

is convex and satisfies the same growth condition of order 2. We define the space

$$V(B_\varepsilon) := \left\{ u \in L^2(B_\varepsilon) : \frac{\partial u}{\partial x_N} \in L^2(B_\varepsilon) \right\}$$

equipped with the norm

$$\|u\|_{V(B_\varepsilon)} := \left(\int_{B_\varepsilon} |u|^2 dx + \int_{B_\varepsilon} \left| \frac{\partial u}{\partial x_N} \right|^2 dx \right)^{\frac{1}{2}}$$

and we denote the duality bracket between the topological dual space $V'(B_\varepsilon)$ and $V(B_\varepsilon)$ by $\langle \cdot, \cdot \rangle_\varepsilon$. The considered total energy functional $F_\varepsilon : L^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$F_\varepsilon(u) = \begin{cases} \int_{\Omega_\varepsilon} f(\nabla u) dx + \varepsilon \int_{B_\varepsilon} g(\nabla u) dx - \langle \mathcal{S}^\varepsilon, u \rangle_\varepsilon & \text{if } u \in W_{\Gamma_0}^{1,2}(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathcal{S}^\varepsilon$ is given in $V'(B_\varepsilon)$. Our aim is to describe the asymptotic behavior of the minimization problem

$$(\mathcal{P}_\varepsilon) \quad \min \{ F_\varepsilon(u) : u \in L^2(\Omega) \},$$

namely, the limit of $\min \{ F_\varepsilon(u) : u \in L^2(\Omega) \}$ together with the limit of the minimizer $\bar{u}_\varepsilon$, and to identify the limit problem in the framework of Γ -convergence.

Let us consider the space $V(B) := \left\{ u \in L^2(B) : \frac{\partial u}{\partial x_N} \in L^2(B) \right\}$ equipped with the norm

$$\|u\|_{V(B)} := \left(\int_B |u|^2 dx + \int_B \left| \frac{\partial u}{\partial x_N} \right|^2 dx \right)^{\frac{1}{2}},$$

and denote the duality bracket between $V'(B)$ and $V(B)$ by $\langle \cdot, \cdot \rangle$. The linear continuous operator

$$\tau_\varepsilon : V(B_\varepsilon) \rightarrow V(B)$$

is defined for every $x = (\hat{x}, x_N) \in B$ by $\tau_\varepsilon u(\hat{x}, x_N) := u(\hat{x}, \varepsilon x_N)$ and we denote its transposed operator by ${}^T \tau_\varepsilon$:

$$\langle {}^T \tau_\varepsilon \theta, u \rangle_\varepsilon = \langle \theta, \tau_\varepsilon u \rangle, \quad \forall (\theta, u) \in V'(B) \times V(B_\varepsilon).$$

We make the following assumption on the source $\mathcal{S}^\varepsilon$: there exists $\mathcal{S}$ in $V'(B)$ such that

$$\mathcal{S}_\varepsilon := ({}^T \tau_\varepsilon)^{-1} \mathcal{S}^\varepsilon \text{ strongly converges to } \mathcal{S} \text{ in } V'(B).$$

Then, in order to identify the Γ -limit of the functional F_ε , it will be more convenient to write the functional F_ε as

$$F_\varepsilon(u) = \begin{cases} \int_{\Omega_\varepsilon} f(\nabla u) dx + \varepsilon^2 \int_B g(\widehat{\nabla} \tau_\varepsilon u, \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon u}{\partial x_N}) dx - \langle \mathcal{S}_\varepsilon, \tau_\varepsilon u \rangle & \text{if } u \in W_{\Gamma_0}^{1,2}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

3 The variational asymptotic model

Let $H : V(B) \rightarrow \mathbb{R}$ be the functional defined by

$$\begin{aligned} H(\theta) &:= \int_B g^{\infty,2}(\hat{0}, \frac{\partial \theta}{\partial x_N}) dx - \langle \mathcal{S}, \theta \rangle \\ &= H_{in}(\theta) - \langle \mathcal{S}, \theta \rangle. \end{aligned} \tag{3.1}$$

We refer the functional H_{in} as the *internal part* of H . We claim that, when $L^2(\Omega)$ is equipped with its strong topology, the functional F_ε Γ -converges to the functional $F_0 : L^2(\Omega) \longrightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$F_0(u) = \begin{cases} \int_{\Omega} f(\nabla u) \, dx + \inf_{\theta \in X(u)} H(\theta) & \text{if } u \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma), \\ +\infty & \text{otherwise,} \end{cases}$$

where $X(u) := \{\theta \in V(B) : \theta(\cdot, \pm \frac{1}{2}) = u^\pm\}$.

Before addressing the variational convergence process, we begin by establishing some compactness properties for sequences with bounded energy. Let us introduce the ε -translate operator T_ε from $W^{1,2}(\Omega)$ into $W^{1,2}(\Omega \setminus \Sigma)$. For any function $w \in W^{1,2}(\Omega)$, $\tilde{w}$ stands for its extension by reflexion on $\Sigma \times (-2r, -r) \cup (r, 2r)$ and we define the ε -translate $T_\varepsilon w$ of w by

$$T_\varepsilon w(\hat{x}, x_N) = \begin{cases} \tilde{w}(\hat{x}, x_N + \frac{\varepsilon}{2}) & \text{if } x \in \Omega^+; \\ \tilde{w}(\hat{x}, x_N - \frac{\varepsilon}{2}) & \text{if } x \in \Omega^-. \end{cases}$$

Lemma 3.1 (compactness). *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^2(\Omega)$ such that $\sup_{\varepsilon>0} F_\varepsilon(u_\varepsilon) < +\infty$. Then*

(i)

$$\int_{B_\varepsilon} |u_\varepsilon|^2 \, dx \leq C\varepsilon \left(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 \, dx \right); \quad (3.2)$$

(ii)

$$\sup_{\varepsilon>0} \left(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 \, dx \right) < +\infty; \quad (3.3)$$

(iii) *there exist $u \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$ and a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ such that $u_\varepsilon \rightharpoonup u$ in $L^2(\Omega)$ and $u_\varepsilon \rightharpoonup u$ in $W_{\Gamma_0}^{1,2}(\Omega_\eta)$ for all $\eta > 0$;*

(iv) *there exist $\theta \in V(B)$ and a subsequence such that $\tau_\varepsilon u_\varepsilon \rightharpoonup \theta$ in $V(B)$, i.e.*

$$\begin{aligned} \tau_\varepsilon u_\varepsilon &\rightharpoonup \theta \text{ in } L^2(B), \\ \frac{\partial \tau_\varepsilon u_\varepsilon}{\partial x_N} &\rightharpoonup \frac{\partial \theta}{\partial x_N} \text{ in } L^2(B); \end{aligned}$$

moreover, $\varepsilon \widehat{\nabla} \tau_\varepsilon u_\varepsilon \rightharpoonup 0$ in $L^2(B, \mathbb{R}^{N-1})$;

(v) $\theta(\cdot, \pm \frac{1}{2}) = u^\pm$.

Proof. Proof of (i). Without loss of generality, we may assume that the $N - 1$ -dimensional Hausdorff measure of the intersection of Γ_0 with $[x_N > 0]$ is positive so that (i) is a mere consequence of the following Poincaré-like inequality:

$$\exists C > 0, \quad \int_{B_\varepsilon} |\varphi|^2 \, dx \leq C\varepsilon \left(\int_{\Omega_\varepsilon^+} |\nabla \varphi|^2 \, dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial \varphi}{\partial x_N} \right|^2 \, dx \right) \quad \forall \varphi \in W_{\Gamma_0}^{1,2}(\Omega). \quad (3.4)$$

Indeed, because

$$\varphi(\hat{x}, x_N) = T_\varepsilon \varphi(\hat{x}, 0) + \int_{\frac{\varepsilon}{2}}^{x_N} \frac{\partial}{\partial x_N} \varphi(\hat{x}, t) \, dt \quad \forall x \in B_\varepsilon,$$

for all smooth function $\varphi \in W_{\Gamma_0}^{1,2}(\Omega)$, we get

$$|\varphi(\hat{x}, x_N)|^2 \leq 2 \left(|T_\varepsilon \varphi(\hat{x}, 0)|^2 + \varepsilon \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \left| \frac{\partial}{\partial x_N} \varphi(\hat{x}, t) \right|^2 \, dt \right).$$

Hence, integrating on B_ε and using trace inequality and Poincaré inequality in Ω^+ give the desired inequality (3.4) for smooth φ , thus for all φ in $W_{\Gamma_0}^{1,2}(\Omega)$ by a density argument.

Proof of (ii). From the coercivity conditions satisfied by f and g , estimate (3.2), and the strong convergence of $\mathcal{S}_\varepsilon$ in $V'(B)$, one has

$$\begin{aligned}
\alpha \left(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 dx \right) &\leq C + |\langle \mathcal{S}_\varepsilon, u_\varepsilon \rangle_\varepsilon| \\
&= C + |\langle \mathcal{S}_\varepsilon, \tau_\varepsilon u_\varepsilon \rangle| \\
&\leq C + \|\mathcal{S}_\varepsilon\|_{V'(B)} \|\tau_\varepsilon u_\varepsilon\|_{V(B)} \\
&= C + \|\mathcal{S}_\varepsilon\|_{V'(B)} \left(\frac{1}{\varepsilon} \int_{B_\varepsilon} |u_\varepsilon|^2 dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 dx \right)^{\frac{1}{2}} \\
&\leq C + C \left(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 dx \right)^{1/2}.
\end{aligned}$$

Then, setting $X_\varepsilon := \left(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 dx \right)^{1/2}$, (3.3) follows from the estimate $\alpha X_\varepsilon^2 \leq C + CX_\varepsilon$.

Proof of (iii).

Step 1. We claim that there exist $z \in W^{1,2}(\Omega \setminus \Sigma)$ and a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ such that $T_\varepsilon u_\varepsilon \rightharpoonup z$ in $W^{1,2}(\Omega \setminus \Sigma)$ and strongly in $L^2(\Omega \setminus \Sigma)$. Clearly,

$$T_\varepsilon u_\varepsilon \in W^{1,2}(\Omega \setminus \Sigma) \text{ and } \frac{\partial}{\partial x_i} T_\varepsilon u_\varepsilon = T_\varepsilon \frac{\partial}{\partial x_i} u_\varepsilon \text{ for all } \varepsilon > 0. \quad (3.5)$$

Combining the Poincaré inequality, (3.3) and (3.5), we deduce

$$\sup_{\varepsilon>0} \|T_\varepsilon u_\varepsilon\|_{W^{1,2}(\Omega \setminus \Sigma)}^2 \leq C \sup_{\varepsilon>0} \left(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x)|^2 dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N}(x) \right|^2 dx \right) < +\infty.$$

Therefore, $(T_\varepsilon u_\varepsilon)_{\varepsilon>0}$ is bounded in $W^{1,2}(\Omega \setminus \Sigma)$ and the claim follows immediately.

Step 2. We establish that there exists u in $L^2(\Omega)$ such that we can extract from the previous subsequence $(u_\varepsilon)_{\varepsilon>0}$ a subsequence strongly converging to u in $L^2(\Omega)$. We can write

$$\int_{\Omega_\varepsilon} |u_\varepsilon(x)|^2 dx = \int_{\Omega^+ \cup \Omega^-} |T_\varepsilon u_\varepsilon(x)|^2 dx - \int_{\Sigma \times ((r-\frac{\varepsilon}{2}, r) \cup (-r, -r+\frac{\varepsilon}{2}))} |T_\varepsilon u_\varepsilon(x)|^2 dx,$$

so that

$$\|u_\varepsilon\|_{L^2(\Omega)}^2 = \int_{\Omega^+ \cup \Omega^-} |T_\varepsilon u_\varepsilon(x)|^2 dx + \int_{B_\varepsilon} |u_\varepsilon(x)|^2 dx - \int_{\Sigma \times ((r-\frac{\varepsilon}{2}, r) \cup (-r, -r+\frac{\varepsilon}{2}))} |T_\varepsilon u_\varepsilon(x)|^2 dx. \quad (3.6)$$

From step 1 and (3.2), we deduce that $\sup_{\varepsilon>0} \|u_\varepsilon\|_{L^2(\Omega)} < +\infty$. Thus there exist $u \in L^2(\Omega)$ and a not relabelled subsequence such that $u_\varepsilon \rightharpoonup u$ in $L^2(\Omega)$. Let us prove that $u = z$. Since $u_\varepsilon \rightharpoonup u$ in $L^2(\Omega)$ and $T_\varepsilon u_\varepsilon \rightharpoonup z$ in $W^{1,2}(\Omega \setminus \Sigma)$, we have for any $\varphi \in C_c^\infty(\Omega)$,

$$\begin{aligned}
\int_{\Omega} u(x) \varphi(x) dx &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_\varepsilon(x) \varphi(\hat{x}, x_N - \frac{\varepsilon}{2}) dx \\
&= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} T_\varepsilon u_\varepsilon(x) \varphi(x) dx \\
&= \int_{\Omega} z(x) \varphi(x) dx.
\end{aligned}$$

Thus $u = z$ almost everywhere in Ω and we deduce that $u \in W^{1,2}(\Omega \setminus \Sigma)$.

Moreover, from (3.2) we have that $\int_{B_\varepsilon} |u_\varepsilon(x)|^2 dx \rightarrow 0$ as $\varepsilon \rightarrow 0$. On the other hand, since $T_\varepsilon u_\varepsilon \rightarrow z$

in $L^2(\Omega)$, we infer $\int_{\Omega \cup \cup \Omega^-} |T_\varepsilon u_\varepsilon(x)|^2 dx \rightarrow \int_{\Omega} |z(x)|^2 dx$ and $\int_{\Sigma \times ((r-\frac{\varepsilon}{2}, r) \cup (-r, -r+\frac{\varepsilon}{2}))} |T_\varepsilon u_\varepsilon(x)|^2 dx \rightarrow 0$. Then we deduce that $\|u_\varepsilon\|_{L^2(\Omega)} \rightarrow \|z\|_{L^2(\Omega)} = \|u\|_{L^2(\Omega)}$ and thus that $(u_\varepsilon)_{\varepsilon>0}$ strongly converges to u in $L^2(\Omega)$.

Step 3. It remains to establish that for any $\eta > 0$, there exists a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ such that $u_\varepsilon|_{\Omega_\eta} \rightharpoonup u|_{\Omega_\eta}$ in $W_{\Gamma_0}^{1,2}(\Omega_\eta)$. It will immediately result that $u \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$. Let $\eta > 0$. Clearly, there exists $0 < \varepsilon_1 < \eta$ such that $\Omega_\eta \subseteq \Omega_\varepsilon$ for all $\varepsilon \leq \varepsilon_1$. By the Poincaré inequality we have

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{W^{1,2}(\Omega_\eta)}^2 \leq C \sup_{\varepsilon>0} \left(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x)|^2 dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N}(x) \right|^2 dx \right) < +\infty.$$

Thus, $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W_{\Gamma_0}^{1,2}(\Omega_\eta)$, and there exist $w \in W_{\Gamma_0}^{1,2}(\Omega_\eta)$ and a not relabelled subsequence of $(u_\varepsilon)_{\varepsilon>0}$ satisfying $u_\varepsilon \rightharpoonup w$ in $L^2(\Omega_\eta)$ and $u_\varepsilon \rightharpoonup w$ in $W_{\Gamma_0}^{1,2}(\Omega_\eta)$. It is easily seen that in fact $w = u|_{\Omega_\eta}$.

Proof of (iv). The weak convergence of $\tau_\varepsilon u_\varepsilon$ to some θ in $V(B)$ follows from (3.2) and (3.3). Indeed

$$\begin{aligned} \sup_{\varepsilon>0} \|\tau_\varepsilon u_\varepsilon\|_{V(B)} &= \sup_{\varepsilon>0} \left(\frac{1}{\varepsilon} \int_{B_\varepsilon} |u_\varepsilon|^2 dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 dx \right)^{\frac{1}{2}} \\ &\leq C \sup_{\varepsilon>0} X_\varepsilon < +\infty. \end{aligned}$$

Now we deduce that $\widehat{\nabla} \tau_\varepsilon u_\varepsilon \rightharpoonup \widehat{\nabla} \theta$ in the distributional sense so that $\varepsilon \widehat{\nabla} \tau_\varepsilon u_\varepsilon \rightharpoonup 0$ in the distributional sense. On the other hand, from the coercivity of g , $\varepsilon \widehat{\nabla} \tau_\varepsilon u_\varepsilon$ weakly converges to some $L^2(B, \mathbb{R}^{N-1})$ function. Hence, $\varepsilon \widehat{\nabla} \tau_\varepsilon u_\varepsilon \rightharpoonup 0$ in $L^2(B, \mathbb{R}^{N-1})$.

Proof of (v). Note that $\theta(\cdot, \pm \frac{1}{2})$ is well defined. Indeed, one has

$$V(B) \subset W^{1,2}((-\frac{1}{2}, \frac{1}{2}), L^2(\Sigma)) \subset \mathcal{C}([-\frac{1}{2}, \frac{1}{2}], L^2(\Sigma)).$$

Clearly, $\tau_\varepsilon u_\varepsilon(\hat{x}, \pm \frac{1}{2}) = (T_\varepsilon u_\varepsilon)^\pm(\hat{x})$ (in the sense of traces on Σ of $W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$ -functions) so that $\tau_\varepsilon u_\varepsilon(\hat{x}, \pm \frac{1}{2}) \rightarrow u^\pm$ in $L^2(\Sigma)$. On the other hand, since

$$\tau_\varepsilon u_\varepsilon(\hat{x}, x_N) = \tau_\varepsilon u_\varepsilon(\hat{x}, \pm \frac{1}{2}) + \int_{\pm \frac{1}{2}}^{x_N} \frac{\partial \tau_\varepsilon u_\varepsilon}{\partial x_N}(\hat{x}, s) ds$$

for a.e. x in B , we infer that for all $\varphi \in \mathcal{C}_c(\Sigma)$,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\Sigma} \tau_\varepsilon u_\varepsilon(\hat{x}, x_N) \varphi(\hat{x}) dx = \int_{\Sigma} (T_\varepsilon u_\varepsilon)^\pm(\hat{x}) \varphi(\hat{x}) d\hat{x} + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\Sigma} \int_{\pm \frac{1}{2}}^{x_N} \frac{\partial \tau_\varepsilon u_\varepsilon}{\partial x_N}(\hat{x}, s) \varphi(\hat{x}) ds dx. \quad (3.7)$$

By passing to the limit in (3.7), we obtain

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\Sigma} \theta(\hat{x}, x_N) \varphi(\hat{x}) dx = \int_{\Sigma} u^\pm(\hat{x}) \varphi(\hat{x}) d\hat{x} + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\Sigma} \int_{\pm \frac{1}{2}}^{x_N} \frac{\partial \theta}{\partial x_N}(\hat{x}, s) \varphi(\hat{x}) ds dx$$

from which we deduce

$$\int_{\Sigma} u^\pm(\hat{x}) \varphi(\hat{x}) d\hat{x} = \int_{\Sigma} \theta(\hat{x}, \pm \frac{1}{2}) \varphi(\hat{x}) d\hat{x}.$$

Thus $\theta(\cdot, \pm \frac{1}{2}) = u^\pm$ almost everywhere in Σ . □

Lemma 3.2. *For every $u \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$, $\inf_{\theta \in X(u)} H(\theta) > -\infty$ and there exists $\theta(u) \in X(u)$ such that $\inf_{\theta \in X(u)} H(\theta) = H(\theta(u))$.*

Proof. The proof follows from standard arguments used in the direct method of the Calculus of Variation. $\square$

As a consequence of Lemma 3.2, in its domain $W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$, the functional F_0 may be written

$$F_0(u) = \int_{\Omega} f(\nabla u) \, dx + H(\theta(u)).$$

Theorem 3.3 is the main result of this section.

Theorem 3.3. *The sequence $(F_{\varepsilon})_{\varepsilon>0}$ Γ -converges to the functional F_0 when $L^2(\Omega)$ is equipped with its strong topology.*

The proof results from the following two propositions.

Proposition 3.4. *For every $u \in L^2(\Omega)$ and every $(u_{\varepsilon})_{\varepsilon>0}$ strongly converging to u in $L^2(\Omega)$ one has*

$$F_0(u) \leq \liminf_{\varepsilon \rightarrow 0} F_{\varepsilon}(u_{\varepsilon}).$$

Proposition 3.5. *For every $u \in L^2(\Omega)$ there exists $(v_{\varepsilon})_{\varepsilon>0}$ strongly converging to u in $L^2(\Omega)$ satisfying*

$$F_0(u) \geq \limsup_{\varepsilon \rightarrow 0} F_{\varepsilon}(v_{\varepsilon}).$$

Proof of Proposition 3.4. We may assume $\liminf_{\varepsilon \rightarrow 0} F_{\varepsilon}(u_{\varepsilon}) < +\infty$. From Lemma 3.1 $u \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$ and there exists $\theta \in X(u)$ such that $\tau_{\varepsilon} u_{\varepsilon} \rightharpoonup \theta$ in $V(B)$. Since $\mathcal{S}_{\varepsilon} \rightarrow \mathcal{S}$ in $V'(B)$, one has

$$\lim_{\varepsilon \rightarrow 0} \langle \mathcal{S}_{\varepsilon}, \tau_{\varepsilon} u_{\varepsilon} \rangle = \langle \mathcal{S}, \theta \rangle. \quad (3.8)$$

On the other hand, since from Lemma 3.1, $u_{\varepsilon} \rightharpoonup u$ in $W_{\Gamma_0}^{1,2}(\Omega_{\eta})$ for all $\eta > 0$, one has

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega_{\varepsilon}} f(\nabla u_{\varepsilon}) \, dx \geq \int_{\Omega} f(\nabla u) \, dx. \quad (3.9)$$

Finally from (iv) of Lemma 3.1 and a standard lower semicontinuity argument

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \int_B g(\widehat{\nabla} \tau_{\varepsilon} u_{\varepsilon}, \frac{1}{\varepsilon} \frac{\partial(\tau_{\varepsilon} u_{\varepsilon})}{\partial x_N}) \, dx \\ & \geq \liminf_{\varepsilon \rightarrow 0} \left(\varepsilon^2 \int_B g(\widehat{\nabla} \tau_{\varepsilon} u_{\varepsilon}, \frac{1}{\varepsilon} \frac{\partial(\tau_{\varepsilon} u_{\varepsilon})}{\partial x_N}) \, dx - \int_B g^{\infty,2}(\varepsilon \widehat{\nabla} \tau_{\varepsilon} u_{\varepsilon}, \frac{\partial(\tau_{\varepsilon} u_{\varepsilon})}{\partial x_N}) \, dx \right) \\ & \quad + \liminf_{\varepsilon \rightarrow 0} \int_B g^{\infty,2}(\varepsilon \widehat{\nabla} \tau_{\varepsilon} u_{\varepsilon}, \frac{\partial(\tau_{\varepsilon} u_{\varepsilon})}{\partial x_N}) \, dx \\ & \geq \liminf_{\varepsilon \rightarrow 0} \left(\varepsilon^2 \int_B g(\widehat{\nabla} \tau_{\varepsilon} u_{\varepsilon}, \frac{1}{\varepsilon} \frac{\partial(\tau_{\varepsilon} u_{\varepsilon})}{\partial x_N}) \, dx - \int_B g^{\infty,2}(\varepsilon \widehat{\nabla} \tau_{\varepsilon} u_{\varepsilon}, \frac{\partial(\tau_{\varepsilon} u_{\varepsilon})}{\partial x_N}) \, dx \right) \\ & \quad + \int_B g^{\infty,2}(\hat{0}, \frac{\partial \theta}{\partial x_N}) \, dx \\ & = \int_B g^{\infty,2}(\hat{0}, \frac{\partial \theta}{\partial x_N}) \, dx \end{aligned} \quad (3.10)$$

provided that we establish

$$\lim_{\varepsilon \rightarrow 0} \left(\varepsilon^2 \int_B g(\widehat{\nabla} \tau_{\varepsilon} u_{\varepsilon}, \frac{1}{\varepsilon} \frac{\partial(\tau_{\varepsilon} u_{\varepsilon})}{\partial x_N}) \, dx - \int_B g^{\infty,2}(\varepsilon \widehat{\nabla} \tau_{\varepsilon} u_{\varepsilon}, \frac{\partial(\tau_{\varepsilon} u_{\varepsilon})}{\partial x_N}) \, dx \right) = 0. \quad (3.11)$$

Since $g^{\infty,2}$ is positively homogeneous of degree 2, and from (2.1), we have

$$\begin{aligned} & \int_B \left| \varepsilon^2 g(\widehat{\nabla} \tau_{\varepsilon} u_{\varepsilon}, \frac{1}{\varepsilon} \frac{\partial(\tau_{\varepsilon} u_{\varepsilon})}{\partial x_N}) - g^{\infty,2}(\varepsilon \widehat{\nabla} \tau_{\varepsilon}^{-1} u_{\varepsilon}, \frac{\partial(\tau_{\varepsilon} u_{\varepsilon})}{\partial x_N}) \right| \, dx \\ & = \varepsilon^2 \int_B \left| g(\widehat{\nabla} \tau_{\varepsilon} u_{\varepsilon}, \frac{1}{\varepsilon} \frac{\partial(\tau_{\varepsilon} u_{\varepsilon})}{\partial x_N}) - g^{\infty,2}(\widehat{\nabla} \tau_{\varepsilon} u_{\varepsilon}, \frac{1}{\varepsilon} \frac{\partial(\tau_{\varepsilon} u_{\varepsilon})}{\partial x_N}) \right| \, dx \\ & \leq C \varepsilon^2 \int_B \left[1 + \left| \widehat{\nabla} \tau_{\varepsilon} u_{\varepsilon} \right|^{2-\delta} + \left| \frac{1}{\varepsilon} \frac{\partial(\tau_{\varepsilon} u_{\varepsilon})}{\partial x_N} \right|^{2-\delta} \right] \, dx. \end{aligned}$$

Thus, by using Hölder's inequality (take $p = \frac{2}{2-\delta}$, $q = \frac{2}{\delta}$) we deduce

$$\int_B \left| \varepsilon^2 g(\widehat{\nabla} \tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon u_\varepsilon}{\partial x_N}) - g^{\infty,2}(\varepsilon \widehat{\nabla} \tau_\varepsilon u_\varepsilon, \frac{\partial(\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \right| dx \leq C \varepsilon^\delta$$

which proves (3.11). The conclusion of Proposition 3.4 follows by collecting (3.8), (3.9) and (3.10). $\square$

Proof of Proposition 3.5. Let $u \in L^2(\Omega)$. We have to construct a sequence $(v_\varepsilon)_{\varepsilon>0}$ strongly converging to u in $L^2(\Omega)$ such that $\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon) \leq F_0(u)$. If $F_0(u) = +\infty$, then $u \in L^2(\Omega) \setminus W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$, and clearly, for any sequence $(v_\varepsilon)_{\varepsilon>0}$ converging to u , $\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon) \leq F_0(u)$ is true. Now, for the harder part, we assume $F_0(u) < +\infty$. Then $u \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$ and

$$F_0(u) = \int_\Omega f(\nabla u(x)) dx + \inf_{\theta \in X(u)} H(\theta).$$

To complete the proof, from $\bar{\theta} := \theta(u)$, i.e. $H(\bar{\theta}) = \inf_{\theta \in X(u)} H(\theta)$, we construct a sequence $(v_\varepsilon)_{\varepsilon>0}$ strongly converging to u in $L^2(\Omega)$ and satisfying

$$F_0(u) \geq \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon).$$

The proof is divided into four steps:

Step 1. Let us extend u and $\bar{\theta}$ by 0 into $(\mathbb{R}^{N-1} \setminus \Sigma) \times (-r, r)$ and write these extensions $\tilde{u}$ and $\tilde{\theta}$. For a sequence δ of positive numbers intended to go to 0, consider a standard sequence of mollifier $(\rho_\delta)_\delta$ and set

$$\begin{aligned} u_\delta &:= \rho_\delta * \tilde{u} \text{ defined by } \rho_\delta * \tilde{u}(\hat{x}, x_N) = \int_{\mathbb{R}^{N-1}} \rho_\delta(\hat{x} - \hat{y}) \tilde{u}(\hat{y}, x_N) d\hat{y} \quad \text{for all } (\hat{x}, x_N) \in \Omega; \\ \theta_\delta &:= \rho_\delta * \tilde{\theta} \text{ defined by } \rho_\delta * \tilde{\theta}(\hat{x}, x_N) = \int_{\mathbb{R}^{N-1}} \rho_\delta(\hat{x} - \hat{y}) \tilde{\theta}(\hat{y}, x_N) d\hat{y} \quad \text{for all } (\hat{x}, x_N) \in \Omega. \end{aligned}$$

Clearly,

$$\begin{cases} \theta_\delta(\hat{x}, \pm \frac{1}{2}) = u_\delta(\hat{x}, 0) \text{ for all } \hat{x} \in \Sigma, \\ u_\delta \in W^{1,2}(\Omega \setminus \Sigma), \theta_\delta \in W^{1,2}(B), \\ u_\delta \rightarrow u \text{ in } W^{1,2}(\Omega \setminus \Sigma), \theta_\delta \rightarrow \bar{\theta} \text{ in } V(B). \end{cases} \quad (3.12)$$

Next, for each $\delta > 0$, we define the sequence $(v_{\delta,\varepsilon})_{\varepsilon>0}$ as follows:

$$v_{\delta,\varepsilon}(\hat{x}, x_N) = \begin{cases} u_\delta(\hat{x}, x_N \pm \frac{\varepsilon}{2}) & \text{on } \Omega_\varepsilon^\mp \\ \theta_\delta(\hat{x}, \frac{x_N}{\varepsilon}) & \text{on } B_\varepsilon. \end{cases} \quad (3.13)$$

Obviously $v_{\delta,\varepsilon}(\hat{x}, x_N)$ belongs to $W^{1,2}(\Omega)$ and strongly converges to u_δ in $L^2(\Omega)$.

Step 2. We we claim that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} f(\nabla v_{\delta,\varepsilon})(x) dx = \int_\Omega f(\nabla u_\delta)(x) dx \quad (3.14)$$

$$\lim_{\varepsilon \rightarrow 0} \left(\varepsilon^2 \int_B g(\widehat{\nabla} \tau_\varepsilon v_{\delta,\varepsilon}, \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon v_{\delta,\varepsilon}}{\partial x_N})(x) dx - \langle \mathcal{S}_\varepsilon, \tau_\varepsilon v_{\delta,\varepsilon} \rangle \right) = H(\theta_\delta). \quad (3.15)$$

Proof of (3.14): one has

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} f(\nabla v_{\delta,\varepsilon})(x) dx &= \lim_{\varepsilon \rightarrow 0} \left(\int_{\Omega_\varepsilon^+} f(\nabla u_\delta)(\hat{x}, x_N - \frac{\varepsilon}{2}) dx + \int_{\Omega_\varepsilon^-} f(\nabla u_\delta)(\hat{x}, x_N + \frac{\varepsilon}{2}) dx \right) \\ &= \int_{\Omega^+} f(\nabla u_\delta)(x) dx + \int_{\Omega^-} f(\nabla u_\delta)(x) dx \\ &= \int_\Omega f(\nabla u_\delta)(x) dx. \end{aligned}$$

Proof of (3.15): Since $g^{\infty,2}$ is positively homogeneous of degree 2 and $\mathcal{S}_\varepsilon$ strongly converges to $\mathcal{S}$ in $V'(B)$, one has

$$\lim_{\varepsilon \rightarrow 0} \left(\varepsilon^2 \int_B g(\widehat{\nabla} \theta_\delta, \frac{1}{\varepsilon} \frac{\partial \theta_\delta}{\partial x_N})(x) dx - \langle \mathcal{S}_\varepsilon, \theta_\delta \rangle \right) = \int_B g^{\infty,2}(\widehat{0}, \frac{\partial \theta_\delta}{\partial x_N}) dx - \langle \mathcal{S}, \theta_\delta \rangle = H(\theta_\delta).$$

Step 3. We establish that $\lim_{\delta \rightarrow 0} \int_\Omega f(\nabla u_\delta) dx + H(\theta_\delta) = F_0(u)$. Since

$$\int_\Omega f(\nabla u_\delta) dx + H(\theta_\delta) = \int_\Omega f(\nabla u_\delta) dx + \int_B g^{\infty,2}(\widehat{0}, \frac{\partial \theta_\delta}{\partial x_N}) dx - \langle \mathcal{S}, \theta_\delta \rangle,$$

the result is a straightforward consequence of (3.12).

Step 4. By using a standard diagonalization argument, from step 2 and step 3, there exists a mapping $\varepsilon \mapsto \delta(\varepsilon)$ such that $v_{\delta(\varepsilon)} \rightarrow u$ in $L^2(\Omega)$ and

$$\lim_{\varepsilon \rightarrow 0} \left(\int_\Omega f(\nabla v_{\delta(\varepsilon)})(x) dx + \varepsilon^2 \int_B g(\widehat{\nabla} \tau_\varepsilon v_{\delta(\varepsilon)}, \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon v_{\delta(\varepsilon)}}{\partial x_N})(x) dx - \langle \mathcal{S}_\varepsilon, \tau_\varepsilon v_{\delta(\varepsilon)} \rangle \right) = F_0(u).$$

The sequence $(v_\varepsilon)_{\varepsilon > 0}$ where $v_\varepsilon := v_{\delta(\varepsilon)}$ fulfils all the conditions except the boundary condition on Γ_0 . From assumption $\text{dist}(\overline{\Gamma_0}, \overline{\partial B_\varepsilon} \cap \partial \Omega) > 0$, and by using a standard slicing method due to De Giorgi in a neighborhood of Γ_0 (see [4]), one can modify v_ε in Ω_ε into a function $\tilde{v}_\varepsilon$ equal to v_ε in B_ε , satisfying the boundary condition on Γ_0 , and $\limsup_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} f(\nabla v_\varepsilon) dx = \limsup_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} f(\nabla \tilde{v}_\varepsilon) dx$. Still denoting by v_ε this new function, we have $\lim_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon) = F_0(u)$ and the proof is complete. $\square$

Remark 3.6. In order to give an interpretation of the limit energy functional, it is worthwhile to write

$$\inf_{\theta \in X(u)} H(\theta) = \inf_{\theta \in V_0(B)} \left\{ \int_B g^{\infty,2}(\widehat{0}, \frac{\partial \theta}{\partial x_N}(x) + [u](\hat{x})) dx - \langle \mathcal{S}, \theta \rangle \right\} - \langle \mathcal{S}, \tilde{u} \rangle \quad (3.16)$$

where $[u] = u^+ - u^-$, $V_0(B) = \{\theta \in V(B) : \theta = 0 \text{ on } \Sigma \times \{\pm \frac{1}{2}\}\}$ and $\tilde{u}(x) = x_N[u](\hat{x}) + \frac{u^+(\hat{x}) + u^-(\hat{x})}{2}$. Therefore when the limit source $\mathcal{S}$ vanishes on $V(B)$, by using Jensen's inequality, $\inf_{\theta \in X(u)} H(\theta)$ reduces to

$$\bar{H}(u) = \int_\Sigma g^{\infty,2}(\widehat{0}, [u](\hat{x})) d\hat{x}$$

which is nothing but the surface energy of the model obtained in [9]. When the limit source is not trivial, by using the Euler equation associated with (3.16), it is easily seen that $\bar{H}$ is a surface energy on Σ of the form

$$\bar{H}(u) = \int_\Sigma \bar{h}(\hat{x}, [u](\hat{x}), \frac{u^+ + u^-}{2}(\hat{x})) d\hat{x}.$$

In this case we note that the energy density depends explicitly of the mean of the traces and that the surface energy $\bar{H}$ mixes the internal energy and the work of the loading.

4 Examples of measure sources $\mathcal{S}_\varepsilon$ concentrated in B_ε

The general form of elements of $V'(B)$ is given for every θ in $V(B)$ by $\langle \mathcal{S}, \theta \rangle = \int_B s_0 \theta dx + \int_B s_1 \frac{\partial \theta}{\partial x_N} dx$ where $(s_0, s_1) \in L^2(B) \times L^2(B)$. The limit sources $\mathcal{S}$ considered in this section are generated by measures $\mathcal{S}^\varepsilon$ in $\mathbb{M}(B_\varepsilon)$ whose slicing structure $\mathcal{H}^{N-1}[\Sigma \otimes \mathcal{S}_x^\varepsilon]$ is such that their slicing components $\mathcal{S}_x^\varepsilon$ do not present a diffuse singular part in their Lebesgue-Nikodym decomposition in $\mathbb{M}(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, i.e., are of the general form

$$\mathcal{S}_x^\varepsilon = \frac{1}{\varepsilon} a_\varepsilon(\hat{x}, \varepsilon t) dt + \sum_{n=-\infty}^{+\infty} b_{\varepsilon,n}(\hat{x}) \delta_{\varepsilon t_n}(\hat{x})$$

where

$$\begin{cases} a_\varepsilon \in L^2(B), \quad b_{\varepsilon,n} \in L^2(\Sigma), \\ t_n : \Sigma \longrightarrow (-\frac{1}{2}, \frac{1}{2}) \text{ is a Borel measurable map.} \end{cases}$$

Roughly, such sources $\mathcal{S}^\varepsilon$ are sums of a function in $L^2(B)$ and a countable sum of surface sources, each of them being concentrated in the $N - 1$ -dimensional surface included in B_ε whose graph is εt_n . We make the following additional assumptions:

- (H1) there exists $a \in L^2(B)$ such that $a_\varepsilon \rightarrow a$ in $L^2(B)$;
- (H2) there exists $b_n \in L^2(\Sigma)$ such that $b_{\varepsilon,n} \rightarrow b_n$ in $L^2(\Sigma)$ when $\varepsilon \rightarrow 0$;
- (H3) there exists $c_n \in \mathbb{R}^+$ such that $\|b_{\varepsilon,n}\|_{L^2(\Sigma)} \leq c_n$ and $\sum_{n=-\infty}^{+\infty} c_n < +\infty$;

It is easy to check that the measure $\mathcal{S}_\varepsilon = {}^T \tau_\varepsilon \mathcal{S}^\varepsilon$ of $\mathbb{M}(B)$ is given by: $\mathcal{S}_\varepsilon = \mathcal{H}^{N-1} \llcorner [\Sigma \otimes (\mathcal{S}_\varepsilon)_{\hat{x}}]$ where

$$(\mathcal{S}_\varepsilon)_{\hat{x}} = a_\varepsilon(\hat{x}, t) dt + \sum_{n=-\infty}^{+\infty} b_{\varepsilon,n}(\hat{x}) \delta_{t_n(\hat{x})}.$$

Proposition 4.1. *The measure $\mathcal{S}_\varepsilon$ strongly converges in $V'(B)$ to the measure $\mathcal{S}$ defined for every $\theta \in V(B)$ by*

$$\langle \mathcal{S}, \theta \rangle = \int_B a(x) \theta(x) dx + \sum_{n=-\infty}^{+\infty} \int_\Sigma b_n(\hat{x}) \theta(\hat{x}, t_n(\hat{x})) d\hat{x}.$$

Therefore, the functional F_ε Γ -converges to the functional $F_0 : L^2(\Omega) \longrightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$F_0(u) = \begin{cases} \int_\Omega f(\nabla u) dx + \inf_{\theta \in X(u)} \left\{ \int_B g^{\infty,2}(\hat{0}, \frac{\partial \theta}{\partial x_N}) dx - \int_B a \theta dx - \sum_{n=-\infty}^{+\infty} \int_\Sigma b_n(\hat{x}) \theta(\hat{x}, t_n(\hat{x})) d\hat{x} \right\} \\ +\infty \text{ otherwise.} \end{cases}$$

if $u \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$

Proof. The second assertion is a straightforward consequence of Theorem 3.3 provided that we establish the strong convergence of $\mathcal{S}_\varepsilon$ to $\mathcal{S}$ in $V'(B)$. For every $\theta \in V(B)$ we have

$$\langle \mathcal{S}_\varepsilon - \mathcal{S}, \theta \rangle = \int_B (a_\varepsilon - a) \theta dx + \int_\Sigma \sum_{n=-\infty}^{+\infty} (b_{\varepsilon,n} - b_n) \theta(\hat{x}, t_n(\hat{x})) d\hat{x},$$

thus

$$|\langle \mathcal{S}_\varepsilon - \mathcal{S}, \theta \rangle| \leq \|\theta\|_{L^2(B)} \|a_\varepsilon - a\|_{L^2(B)} + \sum_{n=-\infty}^{+\infty} \left[\|b_{\varepsilon,n} - b_n\|_{L^2(\Sigma)} \left(\int_\Sigma |\theta(\hat{x}, t_n(\hat{x}))|^2 d\hat{x} \right)^{\frac{1}{2}} \right]. \quad (4.1)$$

But it is easy to establish that there exists a non negative constant C such that

$$\left(\int_\Sigma |\theta(\hat{x}, t_n(\hat{x}))|^2 d\hat{x} \right)^{\frac{1}{2}} \leq C \|\theta\|_{V(B)}$$

so that (4.1) yields

$$\|\mathcal{S}_\varepsilon - \mathcal{S}\|_{V'(B)} \leq \|a_\varepsilon - a\|_{L^2(B)} + C \sum_{n=-\infty}^{+\infty} \|b_{\varepsilon,n} - b_n\|_{L^2(\Sigma)}.$$

The conclusion follows from assumptions (H1), (H2) and (H3). $\square$

5 The gradient concentration phenomenon

We first recall the notion of gradient Young-concentration measure introduced in [3]. Let us denote the unit sphere $\{-1, 1\}$ of $\mathbb{R}$ by $\mathbb{S}^0$, and consider $\Sigma' \subset \subset \Sigma$, $B'_\varepsilon := \Sigma' \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$.

Definition 5.1. A pair $(v, \mu_{\Sigma'}) \in L^2(\Omega) \times \mathbb{M}^+(\bar{\Omega} \times \mathbb{S}^0)$ is a gradient Young-concentration measure (localized on Σ') iff there exists a sequence $(v_\varepsilon)_{\varepsilon>0}$ in $W_{\Gamma_0}^{1,2}(\Omega)$ satisfying

$$\begin{cases} \sup_{\varepsilon>0} \int_{\Omega \setminus B_\varepsilon} |\nabla v_\varepsilon|^2 dx < +\infty, \\ v_\varepsilon \rightarrow v \text{ in } L^2(\Omega), \\ \mu_\varepsilon := \delta_{\frac{\partial v_\varepsilon}{\partial x_N} / \left| \frac{\partial v_\varepsilon}{\partial x_N} \right|}(x) \otimes \varepsilon \mathbb{1}_{B'_\varepsilon} \left| \frac{\partial v_\varepsilon}{\partial x_N} \right|^2 dx \xrightarrow{*} \mu_{\Sigma'}. \end{cases}$$

We say that the sequence $(v_\varepsilon)_{\varepsilon>0}$ generates the gradient Young-concentration measure $(v, \mu_{\Sigma'})$. We denote the set of gradient Young-concentration measures localized on Σ' by $\mathcal{YC}(\Sigma')$.

Recall that the weak convergence $\xrightarrow{*}$ above is defined by

$$\int_{\Omega} \int_{\mathbb{S}^0} \theta(x) \varphi(\zeta) d\mu_\varepsilon = \int_{B'_\varepsilon} \varepsilon \theta(x) \tilde{\varphi} \left(\frac{\partial v_\varepsilon}{\partial x_N} \right) dx \rightarrow \int_{\Omega} \int_{\mathbb{S}^0} \theta(x) \varphi(\zeta) d\mu_{\Sigma'}$$

for all $\theta \in \mathcal{C}(\bar{\Omega})$ and all $\varphi \in \mathcal{C}(\mathbb{S}^0)$, where the 2-homogeneous extension $\tilde{\varphi} : \mathbb{R} \rightarrow \mathbb{R}$ of $\varphi \in \mathcal{C}(\mathbb{S}^0)$ is defined for all $\zeta \in \mathbb{R}^m$ by

$$\tilde{\varphi}(\zeta) = \begin{cases} |\zeta|^2 \varphi\left(\frac{\zeta}{|\zeta|}\right), & \text{if } \zeta \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In [3], Theorem 3.1, the gradient Young-concentration measures was characterized as follows.

Theorem 5.2 (Characterization). A pair $(v, \mu_{\Sigma'} = \mu_x \otimes \pi)$ belongs to $\mathcal{YC}(\Sigma')$ if and only if $v \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$, π is concentrated on $\bar{\Sigma}'$ and, for every $\varphi \in \mathcal{C}(\mathbb{S}^0)$ such that $\varphi^{**} > -\infty$,

$$\begin{aligned} \frac{d\pi}{d\mathcal{H}^{N-1}|_{\Sigma'}}(x) \int_{\mathbb{S}^0} \varphi(\zeta) d\mu_x &\geq \varphi^{**}([v](x)) \quad \text{for } \mathcal{H}^{N-1} \text{ a. e. } x \in \Sigma' \\ \int_{\mathbb{S}^0} \varphi(\zeta) d\mu_x &\geq 0 \quad \text{for } \pi_s \text{ a. e. } x \in \bar{\Sigma}' \end{aligned} \quad (5.1)$$

where $\pi = \frac{d\pi}{d\mathcal{H}^{N-1}|_{\Sigma'}} \mathcal{H}^{N-1}|_{\Sigma'} + \pi_s$ is the Radon-Nikodym decomposition of π with respect to the measure $\mathcal{H}^{N-1}|_{\Sigma'}$.

Remark 5.3. Although from (3.3), $\delta_{\frac{\partial v_\varepsilon}{\partial x_N} / \left| \frac{\partial v_\varepsilon}{\partial x_N} \right|}(x) \otimes \varepsilon \mathbb{1}_{B'_\varepsilon} \left| \frac{\partial v_\varepsilon}{\partial x_N} \right|^2 dx$ possesses weak cluster points in the sense of the weak convergence $\xrightarrow{*}$ made precise above, for technical reason (proof of the sufficient conditions in Proposition 3.5 in [3]), it was not possible to state such a characterization for these cluster points because of possible concentration effects on the boundary of Σ . This is the reason why we deal with gradient Young-concentration measures localized on $\Sigma' \subset \subset \Sigma$.

Taking into account that the 2-homogeneous extension $\tilde{\varphi} : \mathbb{R} \rightarrow \mathbb{R}$ of $\varphi \in \mathcal{C}(\mathbb{S}^0)$ satisfying $\varphi^{**} > -\infty$ is of the form

$$\varphi(\zeta) = \begin{cases} c\zeta^2 & \text{if } \zeta \geq 0 \\ d\zeta^2 & \text{if } \zeta \leq 0, \end{cases}$$

with $(c, d) \in \mathbb{R}^+ \times \mathbb{R}^+$, the above characterization theorem can be reduced to the following (cf Corollary 3.6 in [3])

Corollary 5.4. A measure $(v, \mu_{\Sigma'} = (a(x)\delta_1 + b(x)\delta_{-1}) \otimes \pi)$ belongs to $\mathcal{YC}(\Sigma')$ if and only if $v \in W_{\Gamma_0}^{1,2}(\Omega \setminus \Sigma)$, π is concentrated on $\bar{\Sigma}'$ and

$$\frac{d\pi}{d\mathcal{H}^{N-1}|_{\Sigma'}}(x) (a(x)c + b(x)d) \geq \varphi([v](x)) \text{ for } \mathcal{H}^{N-1}|_{\Sigma'} \text{ a.e. } x \text{ and for all } (c, d) \in \mathbb{R}^+ \times \mathbb{R}^+$$

$$\text{where } \varphi(\zeta) = \begin{cases} c\zeta^2 & \text{if } \zeta \geq 0 \\ d\zeta^2 & \text{if } \zeta \leq 0 \end{cases}.$$

As stated in [3] Remark 2.5, every sequence $(u_\varepsilon)_{\varepsilon>0}$ satisfying (3.3) generates a gradient Young-concentration measure. Therefore every sequence $(\bar{u}_\varepsilon)_{\varepsilon>0}$, $\bar{u}_\varepsilon \in \operatorname{argmin} F_\varepsilon$, generates a measure $\bar{\mu}_{\Sigma'} \in \mathcal{YC}(\Sigma')$. Let $\bar{u}$ be a strong limit of $(\bar{u}_\varepsilon)_{\varepsilon>0}$ in $L^2(\Omega)$, then, under the condition $g^{\infty,2}(\hat{\xi}, \xi_3) \geq g^{\infty,2}(\hat{0}, \xi_3)$, the next theorem states that the internal term $H_{in}(\bar{\theta})$ (cf (3.1)) where $\bar{\theta}$ is the solution of $\inf_{\theta \in X(\bar{u})} H(\theta)$, possesses an integral representation with respect to the Young-concentration measure $\bar{\mu}_{\Sigma'}$. In some sense we localize H_{in} on $S \times \{\pm 1\}$. Moreover, by using Theorem 5.2 we will deduce some bounds on $\bar{\mu}_{\Sigma'}$.

Theorem 5.5. *Let $\bar{u}_\varepsilon$ be a minimizer of $\min \{F_\varepsilon(v) : v \in L^2(\Omega)\}$ and, for every $\Sigma' \subset \subset \Sigma$, $(\bar{u}, \bar{\mu}_{\Sigma'})$ be a gradient Young-concentration measure localized on Σ' generated by the sequence $(\bar{u}_\varepsilon)_{\varepsilon>0}$. Then the two following assertions hold:*

$$i) \quad \bar{u}_\varepsilon \rightarrow \bar{u} \text{ in } L^2(\Omega), \quad F_\varepsilon(\bar{u}_\varepsilon) \rightarrow F_0(\bar{u}) = \min \{F_0(u) : u \in L^2(\Omega)\};$$

$$ii) \quad \text{Let } \mathcal{F} \text{ be a countable family of } \Sigma' \subset \subset \Sigma, \text{ then there exists } \bar{\mu} \in \mathbb{M}(\bar{\Omega} \times \mathbb{S}^0), \bar{\mu} = \bar{\mu}_{\hat{x}} \otimes \bar{\pi} \text{ with } \bar{\pi} \text{ concentrated on } \bar{\Sigma} \text{ such that for all } \Sigma' \in \mathcal{F}, \bar{\mu}|_{\bar{\Sigma}' \times \mathbb{S}^0} = \bar{\mu}_{\Sigma'}. \text{ Assume furthermore that } g^{\infty,2} \text{ satisfies the condition}$$

$$\forall \xi \in \mathbb{R}^3, \quad g^{\infty,2}(\hat{\xi}, \xi_3) \geq g^{\infty,2}(\hat{0}, \xi_3). \quad (5.2)$$

Then, every weak cluster point $\bar{\theta}$ of the sequence $(\tau_\varepsilon \bar{u}_\varepsilon)_{\varepsilon>0}$ in $V(B)$ satisfies $H(\bar{\theta}) = \inf_{\theta \in X(\bar{u})} H(\theta)$ and

$$\begin{aligned} \int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\hat{0}, \frac{\partial \bar{\theta}}{\partial x_N})(\hat{x}, s) \, ds &= \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \text{ for a.e. } \hat{x} \text{ in } \Sigma'; \\ H_{in}(\bar{\theta}) &= \int_{\Sigma} \left[\frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \right] d\hat{x}. \end{aligned} \quad (5.3)$$

Proof. According to the variational nature of the Γ -convergence, for a subsequence one has

$$\begin{aligned} \bar{u}_\varepsilon &\rightarrow \bar{u} \text{ in } L^2(\Omega) \\ \lim_{\varepsilon \rightarrow 0} F_\varepsilon(\bar{u}_\varepsilon) &= F_0(\bar{u}) = \min \{F_0(v) : v \in L^2(\Omega)\} \\ &= \int_{\Omega} f(\nabla \bar{u}) \, dx + \inf_{\theta \in X(\bar{u})} H(\theta). \end{aligned} \quad (5.4)$$

Fix $\Sigma' \subset \subset \Sigma$. From (3.3), for the subsequence (possibly dependent on Σ') associated with the gradient Young-concentration measure $(\bar{u}, \bar{\mu}_{\Sigma'})$, there exist a subsequence and a measure $\bar{\mu} = \bar{\mu}_{\hat{x}} \otimes \bar{\pi}$ in $\mathbb{M}(\bar{\Omega} \times \mathbb{S}^0)$ with $\bar{\pi}$ concentrated in $\bar{\Sigma}$, such that

$$\delta \frac{\partial \bar{u}_\varepsilon}{\partial x_N} / \left| \frac{\partial \bar{u}_\varepsilon}{\partial x_N} \right| (x) \otimes \varepsilon \mathbf{1}_{B_\varepsilon} \left| \frac{\partial \bar{u}_\varepsilon}{\partial x_N} \right|^2 \, dx \rightharpoonup \bar{\mu}.$$

Thus, from (3.11) and (5.2) we infer

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \int_B g(\widehat{\nabla} \tau_\varepsilon \bar{u}_\varepsilon, \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon \bar{u}_\varepsilon}{\partial x_N}) \, dx &= \lim_{\varepsilon \rightarrow 0} \int_B g^{\infty,2}(\varepsilon \widehat{\nabla} \tau_\varepsilon \bar{u}_\varepsilon, \frac{\partial \tau_\varepsilon \bar{u}_\varepsilon}{\partial x_N}) \, dx \\ &\geq \lim_{\varepsilon \rightarrow 0} \int_{B_\varepsilon} g^{\infty,2}(\hat{0}, \frac{\partial \bar{u}_\varepsilon}{\partial x_N}) \, dx \\ &= \int_{\Sigma} \left(\int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \right) d\bar{\pi}. \end{aligned} \quad (5.5)$$

Let $\bar{\theta}$ be the weak limit of $\tau_\varepsilon \bar{u}_\varepsilon$ in $V(B)$ for the considered subsequence. Then, from (5.5) and since

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} f(\nabla \bar{u}_\varepsilon) \, dx \geq \int_{\Omega} f(\nabla \bar{u}) \, dx \text{ and } \lim_{\varepsilon \rightarrow 0} \langle \mathcal{S}_\varepsilon, \tau_\varepsilon \bar{u}_\varepsilon \rangle = \langle \mathcal{S}, \bar{\theta} \rangle,$$

we infer

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon(\bar{u}_\varepsilon) \geq \int_{\Omega} f(\nabla \bar{u}) \, dx + \int_{\Sigma} \left(\int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \right) d\bar{\pi} - \langle \mathcal{S}, \bar{\theta} \rangle. \quad (5.6)$$

Collecting (5.4) and (5.6) we obtain

$$\int_{\Omega} f(\nabla \bar{u}) \, dx + \inf_{\theta \in X(\bar{u})} H(\theta) \geq \int_{\Omega} f(\nabla \bar{u}) \, dx + \int_{\Sigma} \left(\int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \right) d\bar{\pi} - \langle \mathcal{S}, \bar{\theta} \rangle,$$

in particular

$$\int_{\Omega} f(\nabla \bar{u}) \, dx + H(\bar{\theta}) \geq \int_{\Omega} f(\nabla \bar{u}) \, dx + \int_{\Sigma} \left(\int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \right) d\bar{\pi} - \langle \mathcal{S}, \bar{\theta} \rangle,$$

thus

$$\begin{aligned} \int_B g^{\infty,2}(\hat{0}, \frac{\partial \bar{\theta}}{\partial x_N}) \, dx &\geq \int_{\Sigma} \left(\int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \right) d\bar{\pi} \\ &\geq \int_{\Sigma} \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \left(\int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \right) d\hat{x}. \end{aligned} \quad (5.7)$$

On the other hand, by a standard lower semicontinuity argument, for every $\varphi \in \mathcal{C}_c(\Sigma)$, $\varphi \geq 0$,

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \int_B \varphi(\hat{x}) g^{\infty,2}(\hat{0}, \frac{\partial \tau_\varepsilon \bar{u}_\varepsilon}{\partial x_N}) \, dx &= \int_{\Sigma} \varphi(\hat{x}) \left(\int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \right) d\bar{\pi} \\ &\geq \int_B \varphi(\hat{x}) g^{\infty,2}(\hat{0}, \frac{\partial \bar{\theta}}{\partial x_N}) \, dx \end{aligned}$$

so that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\hat{0}, \frac{\partial \bar{\theta}}{\partial x_N})(\hat{x}, s) \, ds \leq \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \quad \text{for a.e. } \hat{x} \in \Sigma. \quad (5.8)$$

Combining (5.7) and (5.8) we deduce

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\hat{0}, \frac{\partial \bar{\theta}}{\partial x_N})(\hat{x}, s) \, ds = \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \int_{\mathbb{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\hat{x}} \quad \text{for a.e. } \hat{x} \in \Sigma.$$

Clearly, $\bar{\mu}[\bar{\Sigma}' \times \mathbb{S}^0] = \bar{\mu}_{\Sigma'}$. Now, by using a standard Cantor's diagonal process, the same equality holds for all Σ' of the countable family $\mathcal{F}$. It remains to show that $H(\bar{\theta}) = \inf_{\theta \in X(\bar{u})} H(\theta)$. It's enough to notice that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} F_\varepsilon(\bar{u}_\varepsilon) &= \int_{\Omega} f(\nabla \bar{u}) \, dx + \inf_{\theta \in X(\bar{u})} H(\theta) \\ &\geq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} f(\nabla \bar{u}_\varepsilon) \, dx + \liminf_{\varepsilon \rightarrow 0} \left(\varepsilon^2 \int_B g(\widehat{\nabla} \tau_\varepsilon \bar{u}_\varepsilon, \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon \bar{u}_\varepsilon}{\partial x_N}) \, dx - \langle \mathcal{S}_\varepsilon, \tau_\varepsilon \bar{u}_\varepsilon \rangle \right) \\ &\geq \int_{\Omega} f(\nabla \bar{u}) \, dx + \int_B g^{\infty,2}(\hat{0}, \frac{\partial \bar{\theta}}{\partial x_N}) \, dx - \langle \mathcal{S}, \bar{\theta} \rangle \\ &= \int_{\Omega} f(\nabla \bar{u}) \, dx + H(\bar{\theta}) \end{aligned}$$

which completes the proof. $\square$

We define the following two constants associated with the function g :

$$c(g) := \min \left(\frac{g^{\infty,2}(\hat{0}, -1)}{g^{\infty,2}(\hat{0}, 1)}, \frac{g^{\infty,2}(\hat{0}, 1)}{g^{\infty,2}(\hat{0}, -1)} \right), \quad C(g) = \frac{1}{c(g)} = \max \left(\frac{g^{\infty,2}(\hat{0}, -1)}{g^{\infty,2}(\hat{0}, 1)}, \frac{g^{\infty,2}(\hat{0}, 1)}{g^{\infty,2}(\hat{0}, -1)} \right).$$

Recall that

$$g^{\infty,2}(\widehat{0}, \xi) = \begin{cases} g^{\infty,2}(\widehat{0}, -1) |\xi|^2 & \text{if } \xi \leq 0 \\ g^{\infty,2}(\widehat{0}, 1) |\xi|^2 & \text{if } \xi > 0 \end{cases}.$$

Moreover, from the assumption on the function g , clearly,

$$g^{\infty,2}(\widehat{0}, 1) > 0 \text{ and } g^{\infty,2}(\widehat{0}, -1) > 0.$$

We make precise the probability measure $\bar{\mu}_{\hat{x}}$ localized on $\Sigma' \subset \Sigma$ as follows:

$$\bar{\mu}_{\hat{x}} := p(\hat{x})\delta_1 + q(\hat{x})\delta_{-1} \quad \text{with} \quad p(\hat{x}) + q(\hat{x}) = 1 \text{ a.e. } \hat{x} \in \Sigma'.$$

Corollary 5.6. *Under the assumptions of Theorem 5.5, the three following estimates hold:*

(i) *for a.e. $\hat{x}$ in Σ'*

$$c(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds \leq \frac{d\bar{\pi}}{dx_N}(\hat{x}) \leq C(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds, \quad (5.9)$$

$$\text{and } \frac{d\bar{\pi}}{dx_N}(\hat{x}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds \quad \text{when } g^{\infty,2}(\widehat{0}, -1) = g^{\infty,2}(\widehat{0}, 1);$$

$$(ii) \quad \frac{c(g) \left| [\bar{u}](\hat{x}) \right|^2}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds} \leq p(\hat{x}) \leq 1 \text{ for a.e. } \hat{x} \text{ such that } [\bar{u}](\hat{x}) > 0;$$

$$(iii) \quad \frac{c(g) \left| [\bar{u}](\hat{x}) \right|^2}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds} \leq q(\hat{x}) \leq 1 \text{ for a.e. } \hat{x} \text{ such that } [\bar{u}](\hat{x}) < 0.$$

Proof. Since $\bar{\mu}_{\hat{x}} = p(\hat{x})\delta_1 + q(\hat{x})\delta_{-1}$, we have $\int_{\mathbb{S}^0} g^{\infty,2}(\xi) d\bar{\mu}_{\hat{x}} = p(\hat{x})g^{\infty,2}(\widehat{0}, 1) + q(\hat{x})g^{\infty,2}(\widehat{0}, -1)$ with $p(\hat{x}) + q(\hat{x}) = 1$ a.e. $\hat{x}$ in Σ' so that from (5.3), one has

$$\begin{aligned} \int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\widehat{0}, \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s)) ds &= \left(\int_{\mathbb{S}^0} g^{\infty,2}(\xi) d\bar{\mu}_{\hat{x}} \right) \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \\ &= \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \left\{ p(\hat{x})g^{\infty,2}(\widehat{0}, 1) + q(\hat{x})g^{\infty,2}(\widehat{0}, -1) \right\} \text{ a.e. } \hat{x} \in \Sigma'. \end{aligned} \quad (5.10)$$

We are going to establish

$$c(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds \leq \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \leq C(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds.$$

From (5.10) we deduce that

$$\begin{aligned} \min \left\{ g^{\infty,2}(\widehat{0}, -1), g^{\infty,2}(\widehat{0}, 1) \right\} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds &\leq \int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\widehat{0}, \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s)) ds \\ &= \left(\int_{\mathbb{S}^0} g^{\infty,2}(\xi) d\bar{\mu}_x \right) \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \\ &= \left\{ p(\hat{x})g^{\infty,2}(\widehat{0}, 1) + q(\hat{x})g^{\infty,2}(\widehat{0}, -1) \right\} \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \\ &\leq \max \left\{ g^{\infty,2}(\widehat{0}, -1), g^{\infty,2}(\widehat{0}, 1) \right\} \frac{d\bar{\pi}}{d\hat{x}}(\hat{x}) \end{aligned} \quad (5.11)$$

and

$$\begin{aligned}
\min \left\{ g^{\infty,2}(\widehat{0}, -1), g^{\infty,2}(\widehat{0}, 1) \right\} \frac{d\bar{\pi}}{d\widehat{x}}(\widehat{x}) &= \min \left\{ g^{\infty,2}(\widehat{0}, -1), g^{\infty,2}(\widehat{0}, 1) \right\} \{p(\widehat{x}) + q(\widehat{x})\} \frac{d\bar{\pi}}{d\widehat{x}}(\widehat{x}) \\
&\leq \left\{ p(\widehat{x})g^{\infty,2}(\widehat{0}, 1) + q(\widehat{x})g^{\infty,2}(\widehat{0}, -1) \right\} \frac{d\bar{\pi}}{d\widehat{x}}(\widehat{x}) \\
&= \left(\int_{\mathbb{S}^0} g^{\infty,2}(\xi) d\bar{\mu}_x \right) \frac{d\bar{\pi}}{d\widehat{x}}(\widehat{x}) \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\widehat{0}, \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s)) ds \\
&\leq \max \left\{ g^{\infty,2}(\widehat{0}, -1), g^{\infty,2}(\widehat{0}, 1) \right\} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds \quad (5.12)
\end{aligned}$$

Then, from (5.11) and (5.12) we have

$$c(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds = \frac{\min \left\{ g^{\infty,2}(\widehat{0}, -1), g^{\infty,2}(\widehat{0}, 1) \right\}}{\max \left\{ g^{\infty,2}(\widehat{0}, -1), g^{\infty,2}(\widehat{0}, 1) \right\}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds \leq \frac{d\bar{\pi}}{d\widehat{x}}(\widehat{x})$$

and

$$\frac{d\bar{\pi}}{d\widehat{x}}(\widehat{x}) \leq \frac{\max \left\{ g^{\infty,2}(\widehat{0}, -1), g^{\infty,2}(\widehat{0}, 1) \right\}}{\min \left\{ g^{\infty,2}(\widehat{0}, -1), g^{\infty,2}(\widehat{0}, 1) \right\}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds = C(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds$$

from which we deduce,

$$c(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds \leq \frac{d\bar{\pi}}{d\widehat{x}}(\widehat{x}) \leq C(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds.$$

Let us prove (ii) and (iii). According to Theorem 5.2, for every $\varphi \in \mathcal{C}(\mathbb{S}^0)$ such that $\varphi^{**} > -\infty$,

$$\frac{d\bar{\pi}}{d\mathcal{H}_{[\Sigma']}^{N-1}}(x) \int_{\mathbb{S}^0} \varphi(\zeta) d\bar{\mu}_x \geq \varphi^{**}([v](x)) \quad \text{for } \mathcal{H}^{N-1} \text{ a.e. } x \in \Sigma', \quad (5.13)$$

where $\bar{\pi} = \frac{d\bar{\pi}}{d\mathcal{H}_{[\Sigma']}^{N-1}} \mathcal{H}_{[\Sigma']}^{N-1} + \bar{\pi}_s$ is the Radon-Nikodym decomposition of $\bar{\pi}$ with respect to the measure $\mathcal{H}_{[\Sigma']}^{N-1}$. We assume that $[\bar{u}](\widehat{x}) > 0$ and show that

$$\frac{c(g) |[\bar{u}](\widehat{x})|^2}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds} \leq p(\widehat{x}) \leq 1.$$

Let

$$\varphi(\xi) = \begin{cases} \varphi(1) |\xi|^2 & \text{if } \xi \geq 0 \\ 0 & \text{if } \xi < 0 \end{cases}. \quad (5.14)$$

Clearly, $\varphi^{**}([\bar{u}](\widehat{x})) = \varphi([\bar{u}](\widehat{x})) = \varphi(1) |[\bar{u}](\widehat{x})|^2$. From the inequality (5.13), it follows that

$$\begin{aligned}
\varphi(1) |[\bar{u}](\widehat{x})|^2 &= \varphi^{**}([\bar{u}](\widehat{x})) \\
&\leq \frac{d\bar{\pi}}{d\widehat{x}}(\widehat{x}) \left(\int_{\mathbb{S}^0} \varphi(\xi) d\bar{\mu}_x \right) \\
&\leq C(g) \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds \right) \left(\int_{\mathbb{S}^0} \varphi(\xi) d\bar{\mu}_x \right) \\
&= C(g) p(\widehat{x}) \varphi(1) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\widehat{x}, s) \right|^2 ds.
\end{aligned}$$

Then, we obtain

$$\frac{|[\bar{u}](\hat{x})|^2}{C(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds} = \frac{c(g) |[\bar{u}](\hat{x})|^2}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds} \leq p(\hat{x}) \leq 1.$$

The proof of (iii) is similar. □

6 References

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