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A mixing effect induced by sources concentrated in a soft junction
and the gradient concentration phenomenon

Anne-Laure Bessoud ∗, Pongpol Juntharee † ‡, Christian Licht †, and Gérard Michaille ‡

Abstract

We show that the variational limit of a $\epsilon$-soft and thin junction problem ($\mathcal{P}_\epsilon$) with sources con-
centrated in the junction gives rise to a surface energy mixing the internal energy and sources. The
surface energy functional possesses an integral representation with respect to the Gradient Young-
Concentration measures generated by sequences $(\bar{u}_\epsilon)_{\epsilon>0}$ of minimizers of ($\mathcal{P}_\epsilon$).

AMS subject classifications: 35B40, 49Q20, 28A33

Keywords: $\Gamma$-convergence, Gradient Young measures, concentration measures, minimization problems.

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1 Introduction

This paper concerns a soft thin junction subjected to concentrated sources. More precisely, let \(\Omega\) be a domain in \(\mathbb{R}^N\) and let \(B_\varepsilon := \Sigma \times (-\frac{1}{2}, \frac{1}{2}) \subset \Omega\), \(\Sigma \subset \mathbb{R}^{N-1}\), be the layer occupied by the soft thin junction (cf Figure 1). We consider the minimization problem

\[
\min_{u \in W^{1,2}_0(\Omega) \setminus B_\varepsilon} \left\{ \int_{\Omega \setminus B_\varepsilon} f(\nabla u) \, dx + \varepsilon \int_{B_\varepsilon} g(\nabla u) \, dx - \langle S^\varepsilon, u \rangle_\varepsilon \right\}
\]

where \(W^{1,2}_0(\Omega)\) denotes the space of Sobolev functions with null trace on a part \(\Gamma_0\) of the boundary of \(\Omega\), and the linear form \(\langle S^\varepsilon, \cdot \rangle_\varepsilon\) represents the work of the source (or the loading). Let \(B := \Sigma \times (-\frac{1}{2}, \frac{1}{2})\). A suitably rescaled \(S^\varepsilon\) of \(S\) is assumed to strongly converge to some \(S\) in the dual of the space \(V(B) := \left\{ u \in L^2(\Omega) : \frac{\partial u}{\partial x_N} \in L^2(\Omega) \right\}\) when \(\varepsilon\) tends to zero. A general example of such sources which are measures on \(B_\varepsilon\) is given in Section 4 of the paper. Sources of the form \(c \frac{1}{\varepsilon^2} 1_{B_\varepsilon}\) where \(c\) is any constant and \(L(\varepsilon) \sim \varepsilon\), is a trivial example of measures satisfying this condition with \(S = 1_{B}\). Note that in this paper the source (or the loading) \(S^\varepsilon\) is a non \(L^2\)-continuous perturbation of the energy functional \(\int_{\Omega \setminus B_\varepsilon} f(\nabla u) \, dx + \varepsilon \int_{B_\varepsilon} g(\nabla u) \, dx\).

Among the physical motivations of \((P_\varepsilon)\) one may mention various applications to heat conduction or electrostatic problems involving sources concentrated in the layer \(B_\varepsilon\) with conductivity or permittivity of order the size of \(B_\varepsilon\). One may also think of membrane problems with an exterior loading concentrated in \(B_\varepsilon\) occupied by a material with stiffness of order the small size of \(B_\varepsilon\). Such a problem with a source concentrated in the junction was considered in [3] in a one dimensional case in order to highlight and illustrate a gradient concentration phenomenon, but the authors were not able to express the variational limit problem.

This paper illustrates the same gradient concentration phenomenon with a complete description of the limit problem in the sense of \(\Gamma\)-convergence (Theorem 3.3). When the size \(\varepsilon\) of the layer goes to zero, fields \(u_\varepsilon\) of bounded energy develop a discontinuity through \(\Sigma\). More precisely, at the variational limit, the internal energy functional of the junction \(\varepsilon \int_{B_\varepsilon} g(\nabla u) \, dx\) and the work of the loading \(\langle S^\varepsilon, u \rangle_\varepsilon\) are combined into a functional of the type

\[
\bar{H}(u) = \int_{\Sigma} \bar{h}(\hat{x}, u^+, u^-, \frac{u^+ + u^-}{2}) \, d\hat{x}
\]

and the limit problem reads as

\[
\min_{u \in W^{1,2}_0(\Omega \setminus \Sigma)} \left\{ \int_{\Omega} f(\nabla u) \, dx + \bar{H}(u) \right\}
\]

where \(u^\pm\) denote the traces on \(\Sigma\). When regarding the various studies devoted to the asymptotic modeling of junction problems (see [2, 9, 7, 10] and references therein) the main novelty is that the density \(\bar{h}\) depends also of the mean \(\frac{u^+ + u^-}{2}\). Furthermore we show that the sequence of minimizers of \((P_\varepsilon)\) (which converges to a minimizer \(\bar{u}\) of the limit problem \((P)\)) generates a gradient Young-concentration measure \(\bar{\mu}\) in the sense defined in [3]. Then we can give an integral representation of the internal part of \(\bar{H}\) with respect to the measure \(\bar{\mu}\) (Theorem 5.5) so that it can be localized in \(\Sigma \times \{ \pm 1 \}\). Finally this provides new bounds on the measure \(\bar{\mu}\) (Corollary 5.6).

The paper is organized as follows: in Section 2 we fix notation and give a detailed description of the problem \((P_\varepsilon)\). Section 3 is devoted to the asymptotic analysis of \((P_\varepsilon)\) in the sense of the \(\Gamma\)-convergence of the functional energy extended to \(L^2(\Omega)\) equipped with its strong topology. In Section 4 we describe a large class of suitable sources \(S^\varepsilon\). Finally Section 5 is concerned with the analysis of the gradient concentration phenomenon generated by sequences of minimizers of \((P_\varepsilon)\). We stress the fact that one could treat the problem in \(L^p(\Omega), 1 < p < +\infty\) in the same way without additional difficulties.
2 Description of the minimization problem

Let $\varepsilon > 0$ be a small parameter intended to go to zero, more precisely taking values in a countable subset of $(0, \varepsilon_0]$ whose 0 is the only cluster point. The reference configuration of the assembly of the two adherents and the adhesive is a cylinder $\Omega := \Sigma \times (-r, r)$ (with $r > \varepsilon$), where $\Sigma$ is a bounded domain in $\mathbb{R}^{N-1}$, $N \geq 2$, with Lipschitz boundary. For $x \in \mathbb{R}^N$ we sometimes write $x = (\hat{x}, x_N)$ where $\hat{x} \in \mathbb{R}^{N-1}$.

In all the paper, $C$ denotes a non negative constant which does not depend on $\varepsilon$ and may vary from line to line. We do not relabel the various considered subsequences and the symbols $\rightharpoonup$ and $\rightharpoonup$ denote various strong convergences and weak convergences respectively. We define the following sets:

- $B_\varepsilon := \Sigma \times (-\varepsilon^2, \varepsilon^2)$;
- $B := \Sigma \times (-\frac{1}{2}, \frac{1}{2})$;
- $\Omega_\varepsilon = \Omega \setminus B_\varepsilon$;
- $\Gamma_0$ is a subset of the boundary $\partial \Omega$ of $\Omega$ such that $\text{dist}(\Gamma_0, \partial B_\varepsilon \cap \partial \Omega) > 0$ for all $\varepsilon < \varepsilon_0$;
- we write $\Omega_\varepsilon^-$, $\Omega_\varepsilon^+$, $\Omega_\varepsilon^-$, $\Omega_\varepsilon^+$ and $B_\varepsilon^+$ and $B_\varepsilon^-$ for the sets $\Omega_\varepsilon \cap [x_N < 0]$ and $\Omega_\varepsilon \cap [x_N > 0]$, $\Omega \cap [x_N < 0]$, $\Omega \cap [x_N > 0]$ and $B_\varepsilon \cap [x_N > 0]$, $B_\varepsilon \cap [x_N < 0]$ respectively.

We will be concerned with the following spaces:

- $W^{1,2}_{T_0}(\Omega_\varepsilon) := \{u \in W^{1,2}(\Omega_\varepsilon) : u = 0 \text{ on } \Gamma_0\}$;
- $W^{1,2}_{T_0}(\Omega) := \{u \in W^{1,2}(\Omega) : u = 0 \text{ on } \Gamma_0\}$;
- $W^{1,2}_{T_0}(\Omega \setminus \Sigma) := \{u \in W^{1,2}(\Omega \setminus \Sigma) : u = 0 \text{ on } \Gamma_0\}$, and for every $z \in W^{1,2}_{T_0}(\Omega \setminus \Sigma)$, $z^\pm$ will stand for the traces of $z$ on $\Sigma$ considered as a Sobolev function on $\Omega^+$ and $\Omega^-$ respectively.

We say that a function $h : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ satisfies a growth condition of order 2 if there exist $\alpha$ and $\beta$ in $\mathbb{R}^+$ such that

$$\alpha |\xi|^2 \leq h(\xi) \leq \beta(1 + |\xi|^2) \quad \text{for all } \xi \in \mathbb{R}^N.$$  

We consider two convex functions $f, g : \mathbb{R}^N \rightarrow \mathbb{R}$ satisfying a growth condition of order 2, and we assume that there exists a positively 2-homogeneous function $g^{\infty,2}$ satisfying

$$|g(\xi) - g^{\infty,2}(\xi)| \leq \beta(1 + |\xi|^{2-2}) \quad \text{for all } \xi \in \mathbb{R}^N,$$  

(2.1)
for some \( \delta, 0 < \delta < 2 \). Note that \( g^{\infty,2} \) is the positively 2-homogeneous recession function of \( g \), i.e.,

\[
g^{\infty,2}(\xi) = \lim_{t \to +\infty} \frac{g(t\xi)}{t^2},
\]

is convex and satisfies the same growth condition of order 2. We define the space

\[
V(B_\varepsilon) := \left\{ u \in L^2(B_\varepsilon) : \frac{\partial u}{\partial x_N} \in L^2(B_\varepsilon) \right\}
\]
equipped with the norm

\[
\|u\|_{V(B_\varepsilon)} := \left( \int_{B_\varepsilon} |u|^2 \, dx + \int_{B_\varepsilon} \left| \frac{\partial u}{\partial x_N} \right|^2 \, dx \right)^{\frac{1}{2}}
\]
and we denote the duality bracket between the topological dual space \( V'(B_\varepsilon) \) and \( V(B_\varepsilon) \) by \( \langle \cdot, \cdot \rangle_\varepsilon \). The considered total energy functional \( F_\varepsilon : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\} \) is defined by

\[
F_\varepsilon(u) = \begin{cases} 
\int_{\Omega} f(\nabla u) \, dx + \varepsilon \int_{B_\varepsilon} g(\nabla u) \, dx - \langle S_\varepsilon, u \rangle_\varepsilon & \text{if } u \in W^{1,2}_0(\Omega) \\
+\infty & \text{otherwise,}
\end{cases}
\]

where \( S_\varepsilon \) is given in \( V'(B_\varepsilon) \). Our aim is to describe the asymptotic behavior of the minimization problem

\[
(P_\varepsilon) \quad \min \left\{ F_\varepsilon(u) : u \in L^2(\Omega) \right\},
\]

namely, the limit of \( \min \left\{ F_\varepsilon(u) : u \in L^2(\Omega) \right\} \) together with the limit of the minimizer \( \bar{u}_\varepsilon \), and to identify the limit problem in the framework of \( \Gamma \)-convergence.

Let us consider the space \( V(B) := \left\{ u \in L^2(B) : \frac{\partial u}{\partial x_N} \in L^2(B) \right\} \) equipped with the norm

\[
\|u\|_{V(B)} := \left( \int_B |u|^2 \, dx + \int_B \left| \frac{\partial u}{\partial x_N} \right|^2 \, dx \right)^{\frac{1}{2}},
\]

and denote the duality bracket between \( V'(B) \) and \( V(B) \) by \( \langle \cdot, \cdot \rangle \). The linear continuous operator

\[
\tau_\varepsilon : V(B_\varepsilon) \to V(B)
\]
is defined for every \( x = (\hat{x}, x_N) \in B \) by \( \tau_\varepsilon u(\hat{x}, x_N) := u(\hat{x}, \varepsilon x_N) \) and we denote its transposed operator by \( T \tau_\varepsilon : V'(B) \to V'(B_\varepsilon) \) such that

\[
\langle T \tau_\varepsilon \theta, u \rangle_\varepsilon = \langle \theta, \tau_\varepsilon u \rangle, \quad \forall (\theta, u) \in V'(B) \times V(B_\varepsilon).
\]

We make the following assumption on the source \( S_\varepsilon \): there exists \( S \) in \( V'(B) \) such that

\[
S_\varepsilon := (T \tau_\varepsilon)^{-1} S \text{ strongly converges to } S \text{ in } V'(B).
\]

Then, in order to identify the \( \Gamma \)-limit of the functional \( F_\varepsilon \), it will be more convenient to write the functional \( F_\varepsilon \) as

\[
F_\varepsilon(u) = \begin{cases} 
\int_{\Omega} f(\nabla u) \, dx + \varepsilon^2 \int_B g(\nabla \tau_\varepsilon u, \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon u}{\partial x_N}) \, dx - \langle S_\varepsilon, \tau_\varepsilon u \rangle \text{ if } u \in W^{1,2}_0(\Omega) \\
+\infty & \text{otherwise.}
\end{cases}
\]

## 3 The variational asymptotic model

Let \( H : V(B) \to \mathbb{R} \) be the functional defined by

\[
H(\theta) := \int_B g^{\infty,2}(\hat{\theta}, \frac{\partial \theta}{\partial x_N}) \, dx - \langle S, \theta \rangle = H_{in}(\theta) - \langle S, \theta \rangle.
\]

(3.1)
We refer the functional $H_{in}$ as the *internal part* of $H$. We claim that, when $L^2(\Omega)$ is equipped with its strong topology, the functional $F_\epsilon$ $\Gamma$-converges to the functional $F_0 : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ given by

$$F_0(u) = \begin{cases} \int \Omega f(\nabla u) \, dx + \inf_{\theta \in X(u)} H(\theta) & \text{if } u \in W^{1,2}_0(\Omega \setminus \Sigma), \\ +\infty & \text{otherwise}, \end{cases}$$

where $X(u) := \{ \theta \in V(B) : \theta(\cdot, \pm \frac{1}{2}) = u^\pm \}$.

Before addressing the variational convergence process, we begin by establishing some compactness properties for sequences with bounded energy. Let us introduce the $\varepsilon$-translate operator $T_\varepsilon$ from $W^{1,2}(\Omega)$ into $W^{1,2}(\Omega \setminus \Sigma)$. For any function $w \in W^{1,2}(\Omega)$, $\hat{w}$ stands for its extension by reflexion on $\Sigma \times (-2r, -r) \cup (r, 2r)$ and we define the $\varepsilon$-translate $T_\varepsilon w$ of $w$ by

$$T_\varepsilon w(\hat{x}, x_N) = \begin{cases} \hat{w}(\hat{x}, x_N + \frac{\varepsilon}{2}) & \text{if } x \in \Omega^+; \\
\hat{w}(\hat{x}, x_N - \frac{\varepsilon}{2}) & \text{if } x \in \Omega^-. \end{cases}$$

**Lemma 3.1** (compactness). Let $(u_\varepsilon)_{\varepsilon > 0}$ be a sequence in $L^2(\Omega)$ such that $\sup_{\varepsilon > 0} F_\varepsilon(u_\varepsilon) < +\infty$. Then

(i) $$\int_{B_\varepsilon} |u_\varepsilon|^2 \, dx \leq C\varepsilon \left( \int_{\Omega} |\nabla u_\varepsilon|^2 \, dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 \, dx \right);$$ (3.2)

(ii) $$\sup_{\varepsilon > 0} \left( \int_{\Omega} |\nabla u_\varepsilon|^2 \, dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 \, dx \right) < +\infty;$$ (3.3)

(iii) there exist $u \in W^{1,2}_0(\Omega \setminus \Sigma)$ and a subsequence of $(u_\varepsilon)_{\varepsilon > 0}$ such that $u_\varepsilon \rightharpoonup u$ in $L^2(\Omega)$ and $u_\varepsilon \rightharpoonup u$ in $W^{1,2}_0(\Omega_\eta)$ for all $\eta > 0$;

(iv) there exist $\theta \in V(B)$ and a subsequence such that $\tau_\varepsilon u_\varepsilon \rightharpoonup \theta$ in $V(B)$, i.e.

$$\frac{\partial \tau_\varepsilon u_\varepsilon}{\partial x_N} \rightharpoonup \frac{\partial \theta}{\partial x_N} \quad \text{in } L^2(B);$$

moreover, $\varepsilon \nabla \tau_\varepsilon u_\varepsilon \to 0$ in $L^2(B, \mathbb{R}^{N-1})$;

(v) $\theta(\cdot, \pm \frac{1}{2}) = u^\pm$.

**Proof.** Proof of (i). Without loss of generality, we may assume that the $N - 1$-dimensional Hausdorff measure of the intersection of $\Gamma_0$ with $[x_N > 0]$ is positive so that (i) is a mere consequence of the following Poincaré-like inequality:

$$\exists C > 0, \int_{B_\varepsilon} |\varphi|^2 \, dx \leq C\varepsilon \left( \int_{\Omega^+} |\nabla \varphi|^2 \, dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial \varphi}{\partial x_N} \right|^2 \, dx \right) \forall \varphi \in W^{1,2}_0(\Omega).$$ (3.4)

Indeed, because

$$\varphi(\hat{x}, x_N) = T_\varepsilon \varphi(\hat{x}, 0) + \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \frac{\partial}{\partial x_N} \varphi(\hat{x}, t) \, dt \quad \forall x \in B_\varepsilon,$$

for all smooth function $\varphi \in W^{1,2}_0(\Omega)$, we get

$$|\varphi(\hat{x}, x_N)|^2 \leq 2 \left( |T_\varepsilon \varphi(\hat{x}, 0)|^2 + \varepsilon \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \left| \frac{\partial}{\partial x_N} \varphi(\hat{x}, t) \right|^2 \, dt \right).$$

Hence, integrating on $B_\varepsilon$ and using trace inequality and Poincaré inequality in $\Omega^+$ give the desired inequality (3.4) for smooth $\varphi$, thus for all $\varphi$ in $W^{1,2}_0(\Omega)$ by a density argument.
Proof of (ii). From the coercivity conditions satisfied by $f$ and $g$, estimate (3.2), and the strong convergence of $S_\varepsilon$ in $V'(B)$, one has
\[
\alpha \left( \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 dx \right) \leq C + |(S_\varepsilon, u_\varepsilon)_\varepsilon| \\
= C + |(S_\varepsilon, \tau_\varepsilon u_\varepsilon)| \\
\leq C + \|S_\varepsilon\|_{V'(B)} \|\tau_\varepsilon u_\varepsilon\|_{V(B)} \\
= C + \|S_\varepsilon\|_{V'(B)} \left( \frac{1}{\varepsilon} \int_{B_\varepsilon} |u_\varepsilon|^2 dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 dx \right)^{1/2} \\
\leq C + C \left( \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 dx \right)^{1/2}.
\]

Then, setting $X_\varepsilon := \left( \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N} \right|^2 dx \right)^{1/2}$, (3.3) follows from the estimate $\alpha X_\varepsilon^2 \leq C + CX_\varepsilon$.

Proof of (iii).

Step 1. We claim that there exist $z \in W^{1,2}(\Omega, \Sigma)$ and a subsequence of $(u_\varepsilon)_{\varepsilon > 0}$ such that $T_\varepsilon u_\varepsilon \rightharpoonup z$ in $W^{1,2}(\Omega \setminus \Sigma)$ and strongly in $L^2(\Omega \setminus \Sigma)$. Clearly,
\[
T_\varepsilon u_\varepsilon \in W^{1,2}(\Omega, \Sigma) \text{ and } \frac{\partial}{\partial x_i} T_\varepsilon u_\varepsilon = T_\varepsilon \frac{\partial}{\partial x_i} u_\varepsilon \text{ for all } \varepsilon > 0.
\]

Combining the Poincaré inequality, (3.3) and (3.5), we deduce
\[
\sup_{\varepsilon > 0} \|T_\varepsilon u_\varepsilon\|^2_{W^{1,2}(\Omega, \Sigma)} \leq C \sup_{\varepsilon > 0} \left( \int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x)|^2 dx + \varepsilon \int_{B_\varepsilon} \left| \frac{\partial u_\varepsilon}{\partial x_N}(x) \right|^2 dx \right) < +\infty.
\]

Therefore, $(T_\varepsilon u_\varepsilon)_{\varepsilon > 0}$ is bounded in $W^{1,2}(\Omega \setminus \Sigma)$ and the claim follows immediately.

Step 2. We establish that there exists $u$ in $L^2(\Omega)$ such that we can extract from the previous subsequence $(u_\varepsilon)_{\varepsilon > 0}$ a subsequence strongly converging to $u$ in $L^2(\Omega)$. We can write
\[
\int_{\Omega_\varepsilon} |u_\varepsilon(x)|^2 dx = \int_{\Omega_+ \cup \Omega_-} |T_\varepsilon u_\varepsilon(x)|^2 dx + \int_{\Sigma \times ((r, r - r + \frac{1}{2}) \cup (r - r + \frac{1}{2}, -r))} |T_\varepsilon u_\varepsilon(x)|^2 dx,
\]
so that
\[
\|u_\varepsilon\|^2_{L^2(\Omega)} = \int_{\Omega_+ \cup \Omega_-} |T_\varepsilon u_\varepsilon(x)|^2 dx + \int_{B_\varepsilon} |u_\varepsilon(x)|^2 dx - \int_{\Sigma \times ((r, r - r + \frac{1}{2}) \cup (r - r + \frac{1}{2}, -r))} |T_\varepsilon u_\varepsilon(x)|^2 dx.
\]

From step 1 and (3.2), we deduce that $\sup_{\varepsilon > 0} \|u_\varepsilon\|_{L^2(\Omega)} < +\infty$. Thus there exist $u \in L^2(\Omega)$ and a not relabelled subsequence such that $u_\varepsilon \rightharpoonup u$ in $L^2(\Omega)$. Let us prove that $u = z$. Since $u_\varepsilon \rightharpoonup u$ in $L^2(\Omega)$ and $T_\varepsilon u_\varepsilon \rightharpoonup z$ in $W^{1,2}(\Omega \setminus \Sigma)$, we have for any $\varphi \in C^\infty(\Omega)$,
\[
\int_{\Omega} u(x) \varphi(x) dx = \lim_{\varepsilon \to 0} \int_{\Omega} u_\varepsilon(x) \varphi(x, x_N - \frac{\varepsilon}{2}) dx \\
= \lim_{\varepsilon \to 0} \int_{\Omega} T_\varepsilon u_\varepsilon(x) \varphi(x) dx \\
= \int_{\Omega} z(x) \varphi(x) dx.
\]

Thus $u = z$ almost everywhere in $\Omega$ and we deduce that $u \in W^{1,2}(\Omega \setminus \Sigma)$.

Moreover, from (3.2) we have that $\int_{B_\varepsilon} |u_\varepsilon(x)|^2 dx \to 0$ as $\varepsilon \to 0$. On the other hand, since $T_\varepsilon u_\varepsilon \rightharpoonup z$
in $L^2(\Omega)$, we infer $\int_{\Omega^+ \cup \Omega^-} |T_\varepsilon u_\varepsilon(x)|^2 dx \to \int_{\Omega} |z(x)|^2 dx$ and $\int_{\Sigma \times ((-r, -\varepsilon) \cup (-\varepsilon, r + \varepsilon))} |T_\varepsilon u_\varepsilon(x)|^2 dx \to 0$.

Then we deduce that $\|u_\varepsilon\|_{L^2(\Omega)} \to \|z\|_{L^2(\Omega)} = \|u\|_{L^2(\Omega)}$ and thus that $(u_\varepsilon)_{\varepsilon >0}$ strongly converges to $u$ in $L^2(\Omega)$.

**Step 3.** It remains to establish that for any $\eta > 0$, there exists a subsequence of $(u_\varepsilon)_{\varepsilon >0}$ such that $u_{\varepsilon(\Omega^\varepsilon)} \to u_{\varepsilon(\Omega)}$ in $W^{1,2}_{\varepsilon(\Omega^\varepsilon)}(\Omega)$.

Let $\eta > 0$. Clearly, there exists $0 < \varepsilon_1 < \eta$ such that $\Omega^\varepsilon \subseteq \Omega_{\varepsilon_1}$ for all $\varepsilon \leq \varepsilon_1$. By the Poincaré inequality we have

$$\sup_{\varepsilon >0} \|u_\varepsilon\|_{W^{1,2}(\Omega_{\varepsilon})}^2 \leq C \sup_{\varepsilon >0} \left( \int_{\Omega_{\varepsilon}} |\nabla u_\varepsilon(x)|^2 dx + \varepsilon \int_{B_{\varepsilon}} \left| \frac{\partial u_\varepsilon}{\partial x_1}(x) \right|^2 dx \right) < +\infty.$$ 

Thus, $(u_\varepsilon)_{\varepsilon >0}$ is bounded in $W^{1,2}_{\varepsilon(\Omega)}$, and there exist $w \in W^{1,2}_{\varepsilon(\Omega)}$ and a not relabelled subsequence of $(u_\varepsilon)_{\varepsilon >0}$ satisfying $u_\varepsilon \to w$ in $L^2(\Omega)$ and $u_\varepsilon \to w$ in $W^{1,2}_{\varepsilon(\Omega)}$. It is easily seen that in fact $w = u_{\varepsilon(\Omega)}$.

**Proof of (iv).** The weak convergence of $\tau_{\varepsilon} u_\varepsilon$ to some $\theta$ in $V(B)$ follows from (3.2) and (3.3). Indeed

$$\sup_{\varepsilon >0} \|\tau_{\varepsilon} u_\varepsilon\|_{V(B)} = \sup_{\varepsilon >0} \left( \frac{1}{\varepsilon} \int_{B_{\varepsilon}} |u_\varepsilon|^2 dx + \varepsilon \int_{B_{\varepsilon}} \left| \frac{\partial u_\varepsilon}{\partial x_1}(x) \right|^2 dx \right)^\frac{1}{2} \leq C \sup_{\varepsilon >0} X_{\varepsilon} < +\infty.$$ 

Now we deduce that $\nabla \tau_{\varepsilon} u_\varepsilon \to \nabla \theta$ in the distributional sense so that $\varepsilon \nabla \tau_{\varepsilon} u_\varepsilon \to 0$ in the distributional sense. On the other hand, from the coercivity of $g$, $\varepsilon \nabla \tau_{\varepsilon} u_\varepsilon$ weakly converges to some $L^2(B, \mathbb{R}^{N-1})$ function. Hence, $\varepsilon \nabla \tau_{\varepsilon} u_\varepsilon \to 0$ in $L^2(B, \mathbb{R}^{N-1})$.

**Proof of (v).** Note that $\theta(\cdot, \pm \frac{1}{2})$ is well defined. Indeed, one has

$$V(B) \subset W^{1,2}((-\frac{1}{2}, \frac{1}{2}), L^2(\Sigma)) \subset C([-\frac{1}{2}, \frac{1}{2}], L^2(\Sigma)).$$

Clearly, $\tau_{\varepsilon} u_\varepsilon(\hat{x}, \pm \frac{1}{2}) = (T_{\varepsilon} u_\varepsilon)^\pm(\hat{x})$ (in the sense of traces on $\Sigma$ of $W^{1,2}_{\varepsilon(\Omega \setminus \Sigma)}$-functions) so that $\tau_{\varepsilon} u_\varepsilon(\hat{x}, \pm \frac{1}{2}) \to u^\pm$ in $L^2(\Sigma)$. On the other hand, since

$$\tau_{\varepsilon} u_\varepsilon(\hat{x}, x_N) = \tau_{\varepsilon} u_\varepsilon(\hat{x}, \pm \frac{1}{2}) + \int_{\pm \frac{1}{2}}^{x_N} \frac{\partial \tau_{\varepsilon} u_\varepsilon}{\partial x_N}(\hat{x}, s) \, ds$$

for a.e. $x$ in $B$, we infer that for all $\varphi \in C_c(\Sigma)$,

$$\int_{\pm \frac{1}{2}}^{\frac{1}{2}} \int_{\Sigma} \tau_{\varepsilon} u_\varepsilon(\hat{x}, x_N) \varphi(\hat{x}) \, dx = \int_{\Sigma} (T_{\varepsilon} u_\varepsilon)^\pm(\hat{x}) \varphi(\hat{x}) \, dx + \int_{\pm \frac{1}{2}}^{\frac{1}{2}} \int_{\Sigma} \int_{\pm \frac{1}{2}}^{x_N} \frac{\partial \tau_{\varepsilon} u_\varepsilon}{\partial x_N}(\hat{x}, s) \varphi(\hat{x}) \, ds \, dx.$$ 

By passing to the limit in (3.7), we obtain

$$\int_{\pm \frac{1}{2}}^{\frac{1}{2}} \int_{\Sigma} \theta(\hat{x}, x_N) \varphi(\hat{x}) \, dx = \int_{\Sigma} u^\pm(\hat{x}) \varphi(\hat{x}) \, dx + \int_{\pm \frac{1}{2}}^{\frac{1}{2}} \int_{\Sigma} \int_{\pm \frac{1}{2}}^{x_N} \frac{\partial \theta}{\partial x_N}(\hat{x}, s) \varphi(\hat{x}) \, ds \, dx$$

from which we deduce

$$\int_{\Sigma} u^\pm(\hat{x}) \varphi(\hat{x}) \, dx = \int_{\Sigma} \theta(\hat{x}, \pm \frac{1}{2}) \varphi(\hat{x}) \, dx.$$ 

Thus $\theta(\cdot, \pm \frac{1}{2}) = u^\pm$ almost everywhere in $\Sigma$. \(\square\)

**Lemma 3.2.** For every $u \in W^{1,2}_{\varepsilon(\Omega \setminus \Sigma)}$, $\inf_{\theta \in X(u)} H(\theta) > -\infty$ and there exists $\theta(u) \in X(u)$ such that

$$\inf_{\theta \in X(u)} H(\theta) = H(\theta(u)).$$
Proof. The proof follows from standard arguments used in the direct method of the Calculus of Variation.

As a consequence of Lemma 3.2, in its domain $W^{1,2}_{10}(\Omega \setminus \Sigma)$, the functional $F_0$ may be written

$$F_0(u) = \int_{\Omega} f(\nabla u) \, dx + H(\theta(u)).$$

Theorem 3.3 is the main result of this section.

**Theorem 3.3.** The sequence $(F_\varepsilon)_{\varepsilon > 0}$ $\Gamma$–converges to the functional $F_0$ when $L^2(\Omega)$ is equipped with its strong topology.

The proof results from the following two propositions.

**Proposition 3.4.** For every $u \in L^2(\Omega)$ and every $(u_\varepsilon)_{\varepsilon > 0}$ strongly converging to $u$ in $L^2(\Omega)$ one has

$$F_0(u) \leq \liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon).$$

**Proposition 3.5.** For every $u \in L^2(\Omega)$ there exists $(u_\varepsilon)_{\varepsilon > 0}$ strongly converging to $u$ in $L^2(\Omega)$ satisfying

$$F_0(u) \geq \limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon).$$

Proof of Proposition 3.4. We may assume $\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) < +\infty$. From Lemma 3.1 $u \in W^{1,2}_{10}(\Omega \setminus \Sigma)$ and there exists $\theta \in X(u)$ such that $\tau_\varepsilon u_\varepsilon \to \theta$ in $V(B)$. Since $S_\varepsilon \to S$ in $V'(B)$, one has

$$\lim_{\varepsilon \to 0} \langle S_\varepsilon, \tau_\varepsilon u_\varepsilon \rangle = \langle S, \theta \rangle. \quad (3.8)$$

On the other hand, since from Lemma 3.1, $u_\varepsilon \to u$ in $W^{1,2}_{10}(\Omega_\eta)$ for all $\eta > 0$, one has

$$\liminf_{\varepsilon \to 0} \int_{\Omega_\varepsilon} f(\nabla u_\varepsilon) \, dx \geq \int_{\Omega} f(\nabla u) \, dx. \quad (3.9)$$

Finally from (iv) of Lemma 3.1 and a standard lower semicontinuity argument

$$\liminf_{\varepsilon \to 0} \varepsilon^2 \int_B g(\nabla \tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon} \frac{\partial (\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \, dx$$

$$\geq \liminf_{\varepsilon \to 0} \left( \varepsilon^2 \int_B g(\nabla \tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon} \frac{\partial (\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \, dx - \int_B g^{\infty,2}(\varepsilon \nabla \tau_\varepsilon u_\varepsilon, \frac{\partial (\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \, dx \right)$$

$$+ \liminf_{\varepsilon \to 0} \int_B g^{\infty,2}(\varepsilon \nabla \tau_\varepsilon u_\varepsilon, \frac{\partial (\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \, dx$$

$$\geq \liminf_{\varepsilon \to 0} \left( \varepsilon^2 \int_B g(\nabla \tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon} \frac{\partial (\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \, dx - \int_B g^{\infty,2}(\varepsilon \nabla \tau_\varepsilon u_\varepsilon, \frac{\partial (\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \, dx \right)$$

$$+ \int_B g^{\infty,2}(0, \frac{\partial \theta}{\partial x_N}) \, dx$$

$$= \int_B g^{\infty,2}(0, \frac{\partial \theta}{\partial x_N}) \, dx \quad (3.10)$$

provided that we establish

$$\lim_{\varepsilon \to 0} \left( \varepsilon^2 \int_B g(\nabla \tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon} \frac{\partial (\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \, dx - \int_B g^{\infty,2}(\varepsilon \nabla \tau_\varepsilon u_\varepsilon, \frac{\partial (\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \, dx \right) = 0. \quad (3.11)$$

Since $g^{\infty,2}$ is positively homogeneous of degree 2, and from (2.1), we have

$$\int_B \left[ \varepsilon^2 g(\nabla \tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon} \frac{\partial (\tau_\varepsilon u_\varepsilon)}{\partial x_N}) - g^{\infty,2}(\varepsilon \nabla \tau_\varepsilon - 1 u_\varepsilon, \frac{\partial (\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \right] \, dx$$

$$= \varepsilon^2 \int_B g(\nabla \tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon} \frac{\partial (\tau_\varepsilon u_\varepsilon)}{\partial x_N}) - g^{\infty,2}(\nabla \tau_\varepsilon u_\varepsilon, \frac{1}{\varepsilon} \frac{\partial (\tau_\varepsilon u_\varepsilon)}{\partial x_N}) \, dx$$

$$\leq C \varepsilon^2 \int_B \left[ 1 + \left| \nabla \tau_\varepsilon u_\varepsilon \right|^{2-\delta} + \frac{1}{\varepsilon} \left| \frac{\partial (\tau_\varepsilon u_\varepsilon)}{\partial x_N} \right|^{2-\delta} \right] \, dx.$$
Thus, by using Hölder’s inequality (take $p = \frac{2}{2-\delta}$, $q = \frac{2}{\delta}$) we deduce
\[
\int_B \left| g\left(\nabla \tau_x u_x, \frac{1}{\varepsilon} \frac{\partial \tau_x u_x}{\partial x_N}\right) - g^{\infty, 2}(\frac{\partial \tau_x u_x}{\partial x_N}) \right| dx \leq C\varepsilon^\delta
\]
which proves (3.11). The conclusion of Proposition 3.4 follows by collecting (3.8), (3.9) and (3.10). \qed

**Proof of Proposition 3.5.** Let $u \in L^2(\Omega)$. We have to construct a sequence $(v_\varepsilon)_{\varepsilon > 0}$ strongly converging to $u$ in $L^2(\Omega)$ such that $\limsup_{\varepsilon \to 0} F_\varepsilon(v_\varepsilon) \leq F_0(u)$. If $F_0(u) = +\infty$, then $u \in L^2(\Omega) \setminus W^{1,2}(\Omega \setminus \Sigma)$, and clearly, for any sequence $(v_\varepsilon)_{\varepsilon > 0}$ converging to $u$, $\limsup_{\varepsilon \to 0} F_\varepsilon(v_\varepsilon) \leq F_0(u)$ is true. Now, for the harder part, we assume $F_0(u) < +\infty$. Then $u \in W^{1,2}_0(\Omega \setminus \Sigma)$ and
\[
F_0(u) = \int_\Omega f(\nabla u(x))dx + \inf_{\theta \in X(u)} H(\theta).
\]
To complete the proof, from $\bar{\theta} := \theta(u)$, i.e. $H(\bar{\theta}) = \inf_{\theta \in X(u)} H(\theta)$, we construct a sequence $(v_\varepsilon)_{\varepsilon > 0}$ strongly converging to $u$ in $L^2(\Omega)$ and satisfying
\[
F_0(u) \geq \limsup_{\varepsilon \to 0} F_\varepsilon(v_\varepsilon).
\]
The proof is divided into four steps:

**Step 1.** Let us extend $u$ and $\bar{\theta}$ by 0 into $(\mathbb{R}^{N-1} \setminus \Sigma) \times (-r, r)$ and write these extensions $\bar{u}$ and $\bar{\theta}$. For a sequence $\delta$ of positive numbers intended to go to 0, consider a standard sequence of mollifier $(\rho_\delta)_\delta$ and set
\[
\begin{align*}
\theta_\delta &:= \rho_\delta * \bar{\theta} \\
u_\delta &:= \rho_\delta * \bar{u}(\hat{x}, x_N) = \int_{\mathbb{R}^{N-1}} \rho_\delta(\hat{x} - \hat{y})\bar{u}(\hat{y}, x_N)d\hat{y} \\
\end{align*}
\]
for all $(\hat{x}, x_N) \in \Omega$;
\[
\begin{align*}
\theta_\delta &:= \rho_\delta * \bar{\theta}(\hat{x}, x_N) = \int_{\mathbb{R}^{N-1}} \rho_\delta(\hat{x} - \hat{y})\bar{\theta}(\hat{y}, x_N)d\hat{y} \\
\end{align*}
\]
for all $(\hat{x}, x_N) \in \Omega$.

Clearly,
\[
\begin{align*}
\begin{cases}
\theta_\delta(\hat{x}, \pm \frac{\varepsilon}{2}) = u_\delta(\hat{x}, 0) & \text{for all } \hat{x} \in \Sigma, \\
u_\delta \in W^{1,2}(\Omega \setminus \Sigma), \theta_\delta \in W^{1,2}(B), \\
u_\delta \to u & \text{in } W^{1,2}(\Omega \setminus \Sigma), \theta_\delta \to \bar{\theta} & \text{in } V(B).
\end{cases}
\end{align*}
\]

Next, for each $\delta > 0$, we define the sequence $(v_{\delta, \varepsilon})_{\varepsilon > 0}$ as follows:
\[
\begin{align*}
v_{\delta, \varepsilon}(\hat{x}, x_N) &= \begin{cases}
\begin{array}{ll}
u_\delta(\hat{x}, x_N \pm \varepsilon) & \text{on } \Omega ^{\varepsilon}_x \\
\theta_\delta(\hat{x}, x_N \varepsilon) & \text{on } B_\varepsilon.
\end{array}
\end{cases}
\end{align*}
\]

Obviously $v_{\delta, \varepsilon}(\hat{x}, x_N)$ belongs to $W^{1,2}(\Omega)$ and strongly converges to $u_\delta$ in $L^2(\Omega)$.

**Step 2.** We claim that
\[
\begin{align*}
\lim_{\varepsilon \to 0} \int_{\Omega^+} f(\nabla v_{\delta, \varepsilon})(x)dx &= \int_{\Omega^+} f(\nabla u_\delta)(x)dx \tag{3.14} \\
\lim_{\varepsilon \to 0} \left( \varepsilon^2 \int_{B} g(\nabla \tau_x v_{\delta, \varepsilon}, \frac{1}{\varepsilon} \frac{\partial \tau_x v_{\delta, \varepsilon}}{\partial x_N})(x)dx - \langle S_x, \tau_x v_{\delta, \varepsilon} \rangle \right) &= H(\theta_\delta). \tag{3.15}
\end{align*}
\]

Proof of (3.14): one has
\[
\begin{align*}
\lim_{\varepsilon \to 0} \int_{\Omega^+} f(\nabla v_{\delta, \varepsilon})(x)dx &= \lim_{\varepsilon \to 0} \left( \int_{\Omega^+} f(\nabla u_\delta)(\hat{x}, x_N - \frac{\varepsilon}{2})dx + \int_{\Omega^+} f(\nabla u_\delta)(\hat{x}, x_N + \frac{\varepsilon}{2})dx \right) \\
&= \int_{\Omega^+} f(\nabla u_\delta)(x)dx + \int_{\Omega^+} f(\nabla u_\delta)(x)dx \\
&= \int_{\Omega^+} f(\nabla u_\delta)(x)dx.
\end{align*}
\]
The result is a straightforward consequence of (3.12).

Step 3. We establish that \( \lim_{\delta \to 0} \int_{\Omega} f(\nabla u_\delta) dx + H(\theta_\delta) = F_0(u) \).

Step 4. By using a standard diagonalization argument, from step 2 and step 3, there exists a mapping \( \varepsilon \mapsto \delta(\varepsilon) \) such that \( v_\delta(\varepsilon) \to u \) in \( L^2(\Omega) \) and

\[
\lim_{\varepsilon \to 0} \left( \int_{\Omega} f(\nabla v_\delta(\varepsilon))(x) dx + \varepsilon^2 \int_{B} g(\nabla \tau_\varepsilon v_\delta(\varepsilon), \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon v_\delta(\varepsilon)}{\partial x_N})(x) dx - \langle S, \tau_\varepsilon v_\delta(\varepsilon) \rangle \right) = F_0(u).
\]

The sequence \( (v_\varepsilon)_{\varepsilon > 0} \) where \( v_\varepsilon := v_\delta(\varepsilon) \) fulfills all the conditions except the boundary condition on \( \Gamma_0 \).

From assumption \( \text{dist}(\Gamma_0, \partial B_\varepsilon \cap \Omega) > 0 \), and by using a standard slicing method due to De Giorgi in a neighborhood of \( \Gamma_0 \) (see [4]), one can modify \( v_\varepsilon \) in \( \Omega_\varepsilon \) into a function \( \tilde{v}_\varepsilon \) equal to \( v_\varepsilon \) in \( B_\varepsilon \), satisfying the boundary condition on \( \Gamma_0 \), and \( \limsup_{\varepsilon \to 0} \int_{\Omega_\varepsilon} f(\nabla v_\varepsilon) dx = \limsup_{\varepsilon \to 0} \int_{\Omega_\varepsilon} f(\nabla \tilde{v}_\varepsilon) dx \). Still denoting by \( v_\varepsilon \) this new function, we have \( \lim_{\varepsilon \to 0} F_\varepsilon(v_\varepsilon) = F_0(u) \) and the proof is complete. \( \square \)

Remark 3.6. In order to give an interpretation of the limit energy functional, it is worthwhile to write

\[
\inf_{\theta \in X(u)} H(\theta) = \inf_{\theta \in V_0(B)} \left\{ \int_B g^{\infty,2}(\hat{0}, \frac{\partial \theta}{\partial x_N}) dx + [u]_{\Sigma} \right\} - \langle S, \tilde{u} \rangle
\]

(3.16)

where \( [u] = u^+ - u^- \), \( V_0(B) = \{ \theta \in V(B) : \theta = 0 \text{ on } \Sigma \times \{ \pm \frac{1}{2} \} \} \) and \( \tilde{u}(x) = x_N[u](\hat{x}) + \frac{u^+(\hat{x}) + u^-(\hat{x})}{2} \).

Therefore when the limit source \( S \) vanishes on \( V(B) \), by using Jensen’s inequality, \( \inf_{\theta \in X(u)} H(\theta) \) reduces to

\[
\bar{H}(u) = \int_{\Sigma} g^{\infty,2}(\hat{0}, [u](\hat{x})) d\hat{x}
\]

which is nothing but the surface energy of the model obtained in [9]. When the limit source is not trivial, by using the Euler equation associated with (3.16), it is easily seen that \( \bar{H} \) is a surface energy on \( \Sigma \) of the form

\[
\bar{H}(u) = \int_{\Sigma} \bar{h}(\hat{x}, [u](\hat{x}), \frac{u^+ + u^-}{2}(\hat{x})) d\hat{x}.
\]

In this case we note that the energy density depends explicitly of the mean of the traces and that the surface energy \( \bar{H} \) mixes the internal energy and the work of the loading.

4 Examples of measure sources \( S_\varepsilon \) concentrated in \( B_\varepsilon \)

The general form of elements of \( V'(B) \) is given for every \( \theta \in V(B) \) by \( \langle S, \theta \rangle = \int_B s_0 \theta dx + \int_B s_1 \frac{\partial \theta}{\partial x_N} dx \)

where \( (s_0, s_1) \in L^2(B) \times L^2(B) \). The limit sources \( S \) considered in this section are generated by measures \( S_\varepsilon \) in \( \mathcal{M}(B_\varepsilon) \) whose slicing structure \( H^{N-1} \Sigma \otimes S_\varepsilon \) is such that their slicing components \( S_\varepsilon \) do not present a diffuse singular part in their Lebesgue-Nikodym decomposition in \( \mathcal{M}(\hat{x}, \varepsilon) \), i.e., are of the general form

\[
S_\varepsilon = \frac{1}{\varepsilon} a_\varepsilon(\hat{x}, \varepsilon t) dt + \sum_{n=-\infty}^{+\infty} b_{\varepsilon,n}(\hat{x}) \delta_{\varepsilon t_n(\hat{x})}
\]
where
\[
\begin{cases}
a_c \in L^2(B), \ b_{c,n} \in L^2(\Sigma), \\
t_n : \Sigma \rightarrow \left(-\frac{1}{2}, \frac{1}{2}\right) \text{ is a Borel measurable map.}
\end{cases}
\]

Roughly, such sources \(S^c\) are sums of a function in \(L^2(B)\) and a countable sum of surface sources, each of them being concentrated in the \(N - 1\)-dimensional surface included in \(B_c\) whose graph is \(\varepsilon t_n\). We make the following additional assumptions:

(H1) there exists \(a \in L^2(B)\) such that \(a_c \rightarrow a\) in \(L^2(B)\);

(H2) there exists \(b_n \in L^2(\Sigma)\) such that \(b_{c,n} \rightarrow b_n\) in \(L^2(\Sigma)\) when \(\varepsilon \rightarrow 0\);

(H3) there exists \(c_n \in \mathbb{R}^+\) such that \(\|b_{c,n}\|_{L^2(\Sigma)} \leq c_n\) and \(\sum_{n=-\infty}^{+\infty} c_n < +\infty\);

It is easy to check that the measure \(S_\varepsilon =^T \tau_\varepsilon S^c\) of \(M(B)\) is given by: \(S_\varepsilon = \mathcal{H}^{N-1} |\Sigma \otimes (S_\varepsilon)_\varepsilon\) where
\[
(S_\varepsilon)_\varepsilon = a_c(\hat{x}, t) \, dt + \sum_{n=-\infty}^{+\infty} b_{c,n}(\hat{x}) \delta_{n}(\hat{x}).
\]

**Proposition 4.1.** The measure \(S_\varepsilon\) strongly converges in \(V'(B)\) to the measure \(S\) defined for every \(\theta \in V(B)\) by
\[
\langle S, \theta \rangle = \int_B a(x)\theta(x) \, dx + \sum_{n=-\infty}^{+\infty} \int_\Sigma b_n(\hat{x})\theta(\hat{x}, t_n(\hat{x})) \, d\hat{x}.
\]

Therefore, the functional \(F_\varepsilon\) \(\Gamma\)-converges to the functional \(F_0 : L^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}\) given by
\[
F_0(u) = \begin{cases}
\int_\Omega f(\nabla u) \, dx + \inf_{\theta \in X(u)} \left\{ \int_B g^{\infty,2}(\hat{0}, \frac{\partial \theta}{\partial x_N}) \, dx - \int_B a\theta \, dx - \sum_{n=-\infty}^{+\infty} \int_\Sigma b_n(\hat{x})\theta(\hat{x}, t_n(\hat{x})) \, d\hat{x} \right\} \\
+\infty \text{ otherwise.}
\end{cases}
\]

**Proof.** The second assertion is a straightforward consequence of Theorem 3.3 provided that we establish the strong convergence of \(S_\varepsilon\) to \(S\) in \(V'(B)\). For every \(\theta \in V(B)\) we have
\[
\langle S_\varepsilon - S, \theta \rangle = \int_B (a_c - a)\theta \, dx + \sum_{n=-\infty}^{+\infty} (b_{c,n} - b_n)\theta(\hat{x}, t_n(\hat{x})) \, d\hat{x},
\]

thus
\[
\|S_\varepsilon - S, \theta\| \leq \|\theta\|_{L^2(B)} \|a_c - a\|_{L^2(B)} + \sum_{n=-\infty}^{+\infty} \|b_{c,n} - b_n\|_{L^2(\Sigma)} \left( \int_\Sigma |\theta(\hat{x}, t_n(\hat{x}))|^2 \, d\hat{x} \right)^{1/2}.
\]

But it is easy to establish that there exists a non negative constant \(C\) such that
\[
\left( \int_\Sigma |\theta(\hat{x}, t_n(\hat{x}))|^2 \, d\hat{x} \right)^{1/2} \leq C \|\theta\|_{V(B)}
\]
so that (4.1) yields
\[
\|S_\varepsilon - S\|_{V'(B)} \leq \|a_c - a\|_{L^2(B)} + C \sum_{n=-\infty}^{+\infty} \|b_{c,n} - b_n\|_{L^2(\Sigma)}.
\]

The conclusion follows from assumptions (H1), (H2) and (H3). □
5 The gradient concentration phenomenon

We first recall the notion of gradient Young-concentration measure introduced in [3]. Let us denote the unit sphere \{-1, 1\} of \(\mathbb{R}\) by \(S^0\), and consider \(\Sigma' \subset \subset \Sigma\), \(B^\prime := \Sigma' \times (-\frac{1}{2}, \frac{1}{2})\).

**Definition 5.1.** A pair \((v, \mu_\Sigma) \in L^2(\Omega) \times \mathcal{M}^+(\Omega \times S^0)\) is a gradient Young-concentration measure (localized on \(\Sigma'\)) if there exists a sequence \((v_\varepsilon)_{\varepsilon > 0}\) in \(W^{1,2}_{\Gamma_0}\)(\(\Omega\)) satisfying

\[
\begin{align*}
\sup_{\varepsilon > 0} \int_{\Omega \setminus B^\prime_\varepsilon} |\nabla v_\varepsilon|^2 \, dx &< +\infty, \\
v_\varepsilon &\rightharpoonup v \text{ in } L^2(\Omega), \\
\mu_\varepsilon := &\, \delta \frac{\varepsilon}{\partial N} \left| \frac{\varepsilon}{\partial N} \right| (x) \otimes \varepsilon \mathbb{1}_{B^\prime_\varepsilon} \left| \frac{\varepsilon}{\partial N} \right|^2 \, dx \rightharpoonup \mu_\Sigma'.
\end{align*}
\]

We say that the sequence \((v_\varepsilon)_{\varepsilon > 0}\) generates the gradient Young-concentration measure \((v, \mu_\Sigma')\). We denote the set of gradient Young-concentration measures localized on \(\Sigma'\) by \(YC(\Sigma')\).

Recall that the weak convergence \(\rightharpoonup\) above is defined by

\[
\int_{\Omega} \int_{S^0} \theta(x) \varphi(\zeta) d\mu_\varepsilon = \int_{B^\prime_\varepsilon} \theta(x) \hat{\varphi} \left( \frac{\varepsilon}{\partial N} \right) \, dx \rightarrow \int_{\Omega} \int_{S^0} \theta(x) \varphi(\zeta) \, d\mu_\Sigma'
\]

for all \(\theta \in \mathcal{C}(\Omega)\) and all \(\varphi \in \mathcal{C}(S^0)\), where the 2-homogeneous extension \(\hat{\varphi} : \mathbb{R} \rightarrow \mathbb{R}\) of \(\varphi \in \mathcal{C}(S^0)\) is defined for all \(\zeta \in \mathbb{R}^m\) by

\[
\hat{\varphi}(\zeta) = \begin{cases} 
|\zeta|^2 \varphi\left( \frac{\zeta}{|\zeta|} \right), & \text{if } \zeta \neq 0, \\
0 & \text{otherwise}.
\end{cases}
\]

In [3], Theorem 3.1, the gradient Young-concentration measures was characterized as follows.

**Theorem 5.2** (Characterization). A pair \((v, \mu_\Sigma') = \mu_\Sigma \otimes \pi\) belongs to \(YC(\Sigma')\) if and only if \(v \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma)\), \(\pi\) is concentrated on \(\Sigma'\) and, for every \(\varphi \in \mathcal{C}(S^0)\) such that \(\varphi^{**} > -\infty\),

\[
\begin{align*}
\frac{d\pi}{d\mathcal{H}^{N-1}|\Sigma'}(x) &\int_{S^0} \varphi(\zeta) \, d\mu_\varepsilon \geq \varphi^{**}(|v|(x)) \quad \text{for } \mathcal{H}^{N-1} \text{ a.e. } x \in \Sigma' \\
&\int_{S^0} \varphi(\zeta) \, d\mu_\varepsilon \geq 0 \quad \text{for } \pi_\varepsilon \text{ a.e. } x \in \Sigma'
\end{align*}
\]

where \(\pi = \frac{d\pi}{d\mathcal{H}^{N-1}|\Sigma'}\mathcal{H}^{N-1}|\Sigma' + \pi_\varepsilon\) is the Radon-Nikodym decomposition of \(\pi\) with respect to the measure \(\mathcal{H}^{N-1}|\Sigma'\).

**Remark 5.3.** Although from (3.3), \(\delta \frac{\varepsilon}{\partial N} \left| \frac{\varepsilon}{\partial N} \right| (x) \otimes \varepsilon \mathbb{1}_{B^\prime_\varepsilon} \left| \frac{\varepsilon}{\partial N} \right|^2 \, dx\) possesses weak cluster points in the sense of the weak convergence \(\rightharpoonup\) made precise above, for technical reason (proof of the sufficient conditions in Proposition 3.5 in [3]), it was not possible to state such a characterization for these cluster points because of possible concentration effects on the boundary of \(\Sigma\). This is the reason why we deal with gradient Young-concentration measures localized on \(\Sigma' \subset \subset \Sigma\).

Taking into account that the 2-homogeneous extension \(\hat{\varphi} : \mathbb{R} \rightarrow \mathbb{R}\) of \(\varphi \in \mathcal{C}(S^0)\) satisfying \(\varphi^{**} > -\infty\) is of the form

\[
\varphi(\zeta) = \begin{cases} 
\epsilon \zeta^2 & \text{if } \zeta \geq 0 \\
\delta \zeta^2 & \text{if } \zeta \leq 0,
\end{cases}
\]

with \((c, d) \in \mathbb{R}_+ \times \mathbb{R}_+\), the above characterization theorem can be reduced to the following (cf Corollary 3.6 in [3])

**Corollary 5.4.** A measure \((v, \mu_\Sigma') = (a(x) \delta_1 + b(x) \delta_{-1}) \otimes \pi\) belongs to \(YC(\Sigma')\) if and only if \(v \in W^{1,2}_{\Gamma_0}(\Omega \setminus \Sigma)\), \(\pi\) is concentrated on \(\Sigma'\) and

\[
\begin{align*}
\frac{d\pi}{d\mathcal{H}^{N-1}|\Sigma'}(x)(a(x)c + b(x)d) &\geq \varphi(|v|(x)) \quad \text{for } \mathcal{H}^{N-1}|\Sigma' \text{ a.e. } x \text{ and for all } (c, d) \in \mathbb{R}_+ \times \mathbb{R}_+
\end{align*}
\]
where $\varphi(\zeta) = \begin{cases} c\zeta^2 & \text{if } \zeta \geq 0 \\ d\zeta^2 & \text{if } \zeta \leq 0 \end{cases}$.

As stated in [3] Remark 2.5, every sequence $(u_\varepsilon)_{\varepsilon > 0}$ satisfying (3.3) generates a gradient Young-concentration measure. Therefore every sequence $(\bar{u}_\varepsilon)_{\varepsilon > 0}$, $\bar{u}_\varepsilon \in \text{argmin} F_\varepsilon$, generates a measure $\bar{\mu}_{\Sigma'} \in \mathcal{Y}(\Sigma')$. Let $\bar{u}$ be a strong limit of $(\bar{u}_\varepsilon)_{\varepsilon > 0}$ in $L^2(\Omega)$, then, under the condition $g^{\infty,2}(\hat{\theta}, \xi_3) \geq g^{\infty,2}(\hat{0}, \xi_3)$, the next theorem states that the internal term $H_{in}(\bar{\theta})$ (cf (3.1)) where $\bar{\theta}$ is the solution of $\inf_{\theta \in X(\bar{u})} H(\theta)$, possesses an integral representation with respect to the Young-concentration measure $\bar{\mu}_{\Sigma'}$. In some sense we localize $H_{in}$ on $S \times \{ \pm 1 \}$. Moreover, by using Theorem 5.2 we will deduce some bounds on $\bar{\mu}_{\Sigma'}$.

**Theorem 5.5.** Let $\bar{u}_\varepsilon$ be a minimizer of $\min \left\{ F_\varepsilon(v) : v \in L^2(\Omega) \right\}$ and, for every $\Sigma' \subset \subset \Sigma$, $(\bar{u}, \bar{\mu}_{\Sigma'})$ be a gradient Young-concentration measure localized on $\Sigma'$ generated by the sequence $(\bar{u}_\varepsilon)_{\varepsilon > 0}$. Then the two following assertions hold:

i) $\bar{u}_\varepsilon \to \bar{u}$ in $L^2(\Omega)$, $F_\varepsilon(\bar{u}_\varepsilon) \to F_0(\bar{u}) = \min \left\{ F_0(u) : u \in L^2(\Omega) \right\}$;

ii) Let $\mathcal{F}$ be a countable family of $\Sigma' \subset \subset \Sigma$, then there exists $\bar{\mu} \in \mathcal{M}(\hat{\Omega} \times \mathcal{S}^0)$, $\bar{\mu} = \bar{\mu}_{\Sigma'} \otimes \bar{\pi}$ with $\bar{\pi}$ concentrated on $\hat{\Sigma}$ such that for all $\Sigma' \in \mathcal{F}$, $\bar{\mu}_{\Sigma'} \otimes \mathcal{S}^0 = \bar{\mu}_{\Sigma'}$. Assume furthermore that $g^{\infty,2}$ satisfies the condition

$$\forall \xi \in \mathbb{R}^3, \ g^{\infty,2}(\hat{\xi}, \xi_3) \geq g^{\infty,2}(\hat{0}, \xi_3). \quad (5.2)$$

Then, every weak cluster point $\bar{\theta}$ of the sequence $(\tau_\varepsilon \bar{u}_\varepsilon)_{\varepsilon > 0}$ in $V(B)$ satisfies $H(\bar{\theta}) = \inf_{\theta \in X(\bar{u})} H(\theta)$ and

$$\int_{\Omega} g^{\infty,2}(\hat{0}, \frac{\partial \bar{\theta}}{\partial x_N})(\hat{x}, s) \, dx = \int_{\Omega} \frac{d\bar{\pi}}{dx}(\hat{x}) \int_{\mathcal{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\Sigma'},$$

$$H_{in}(\bar{\theta}) = \int_{\Sigma} \left[ \frac{d\bar{\pi}}{dx}(\hat{x}) \int_{\mathcal{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\Sigma'} \right] d\hat{x} \quad \text{for a.e. } \hat{x} \in \Sigma'. \quad (5.3)$$

**Proof.** According to the variational nature of the $\Gamma$-convergence, for a subsequence one has

$$\bar{u}_\varepsilon \to \bar{u} \text{ in } L^2(\Omega),$$

$$\lim_{\varepsilon \to 0} F_\varepsilon(\bar{u}_\varepsilon) = F_0(\bar{u}) = \min \left\{ F_0(v) : v \in L^2(\Omega) \right\} = \int_{\Omega} f(\nabla \bar{u}) \, dx + \inf_{\theta \in X(\bar{u})} H(\theta). \quad (5.4)$$

Fix $\Sigma' \subset \subset \Sigma$. From (3.3), for the subsequence (possibly dependent on $\Sigma'$) associated with the gradient Young-concentration measure $(\bar{u}, \bar{\mu}_{\Sigma'})$, there exist a subsequence and a measure $\bar{\mu} = \bar{\mu}_{\Sigma} \otimes \bar{\pi}$ in $\mathcal{M}(\hat{\Omega} \times \mathcal{S}^0)$ with $\bar{\pi}$ concentrated in $\hat{\Sigma}$, such that

$$\delta_{\bar{\mu}_{\Sigma'} / \bar{\mu}_{\Sigma}} \bigg| \begin{pmatrix} 1 \\ \bar{\pi} \end{pmatrix} \bigg|_B \otimes \varepsilon \frac{\partial \bar{u}_\varepsilon}{\partial x_N} \, dx \to \bar{\mu}.$$

Thus, from (3.11) and (5.2) we infer

$$\lim_{\varepsilon \to 0} \varepsilon^2 \int_B g(\nabla \tau_\varepsilon \bar{u}_\varepsilon, \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon}{\partial x_N} \bar{u}_\varepsilon) \, dx = \lim_{\varepsilon \to 0} \int_B g^{\infty,2}(\varepsilon \nabla \tau_\varepsilon \bar{u}_\varepsilon, \frac{\partial \tau_\varepsilon}{\partial x_N} \bar{u}_\varepsilon) \, dx$$

$$\geq \lim_{\varepsilon \to 0} \int_{B_\varepsilon} g^{\infty,2}(\hat{0}, \frac{\partial \bar{u}_\varepsilon}{\partial x_N}) \, dx = \int_{\Sigma} \left( \int_{\mathcal{S}^0} g^{\infty,2}(\hat{0}, \xi_3) \, d\bar{\mu}_{\Sigma'} \right) d\bar{\pi}. \quad (5.5)$$

Let $\bar{\theta}$ be the weak limit of $\tau_\varepsilon \bar{u}_\varepsilon$ in $V(B)$ for the considered subsequence. Then, from (5.5) and since

$$\liminf_{\varepsilon \to 0} \int_{\Omega} f(\nabla \bar{u}_\varepsilon) \, dx \geq \int_{\Omega} f(\nabla \bar{u}) \, dx \text{ and } \lim_{\varepsilon \to 0} \langle \mathcal{S}_\varepsilon, \tau_\varepsilon \bar{u}_\varepsilon \rangle = \langle \mathcal{S}, \bar{\theta} \rangle,$$
we infer
\[\lim_{\varepsilon \to 0} F_{\varepsilon}(\bar{u}_\varepsilon) \geq \int_{\Omega} f(\nabla \bar{u}) \, dx + \int_{\bar{\Omega}} \left( \int_{\mathbb{S}^0} g^{\infty,2}(\bar{0}, \xi_3) \, d\bar{\mu}_{\bar{x}} \right) \, d\bar{\pi} - \langle \mathcal{S}, \bar{\theta} \rangle. \]  
(5.6)

Collecting (5.4) and (5.6) we obtain
\[\int_{\Omega} f(\nabla \bar{u}) \, dx + \inf_{\bar{\theta} \in \mathcal{X}(u)} H(\bar{\theta}) \geq \int_{\Omega} f(\nabla \bar{u}) \, dx + \int_{\bar{\Omega}} \left( \int_{\mathbb{S}^0} g^{\infty,2}(\bar{0}, \xi_3) \, d\bar{\mu}_{\bar{x}} \right) \, d\bar{\pi} - \langle \mathcal{S}, \bar{\theta} \rangle,\]
in particular
\[\int_{\Omega} f(\nabla \bar{u}) \, dx + H(\bar{\theta}) \geq \int_{\Omega} f(\nabla \bar{u}) \, dx + \int_{\bar{\Omega}} \left( \int_{\mathbb{S}^0} g^{\infty,2}(\bar{0}, \xi_3) \, d\bar{\mu}_{\bar{x}} \right) \, d\bar{\pi} - \langle \mathcal{S}, \bar{\theta} \rangle,\]
thus
\[\int_{B} g^{\infty,2}(\bar{0}, \frac{\partial \bar{\theta}}{\partial x_1}) \, dx \geq \int_{\bar{\Omega}} \left( \int_{\mathbb{S}^0} g^{\infty,2}(\bar{0}, \xi_3) \, d\bar{\mu}_{\bar{x}} \right) \, d\bar{\pi} \geq \int_{\bar{\Omega}} \varphi(\bar{x}) g^{\infty,2}(\bar{0}, \xi_3) \, d\bar{\mu}_{\bar{x}} \]
for a.e. \(\bar{x} \in \bar{\Sigma}\).  
(5.7)

On the other hand, by a standard lower semicontinuity argument, for every \(\varphi \in C_c(\Sigma)\), \(\varphi \geq 0\),
\[\liminf_{\varepsilon \to 0} \int_{B} \varphi(\bar{x}) g^{\infty,2}(\bar{0}, \frac{\partial \tau_\varepsilon \bar{u}_\varepsilon}{\partial x_N}) \, dx = \int_{\bar{\Omega}} \varphi(\bar{x}) \left( \int_{\mathbb{S}^0} g^{\infty,2}(\bar{0}, \xi_3) \, d\bar{\mu}_{\bar{x}} \right) \, d\bar{\pi} \]
so that
\[\int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\bar{0}, \frac{\partial \bar{\theta}}{\partial x_1}) (\bar{x}, s) \, ds \leq \frac{d\bar{\pi}}{d\bar{x}}(\bar{x}) \int_{\mathbb{S}^0} g^{\infty,2}(\bar{0}, \xi_3) \, d\bar{\mu}_{\bar{x}} \quad \text{for a.e. } \bar{x} \in \bar{\Sigma}. \quad (5.8)\]

Combining (5.7) and (5.8) we deduce
\[\int_{-\frac{1}{2}}^{\frac{1}{2}} g^{\infty,2}(\bar{0}, \frac{\partial \bar{\theta}}{\partial x_1}) (\bar{x}, s) \, ds = \frac{d\bar{\pi}}{d\bar{x}}(\bar{x}) \int_{\mathbb{S}^0} g^{\infty,2}(\bar{0}, \xi_3) \, d\bar{\mu}_{\bar{x}} \quad \text{for a.e. } \bar{x} \in \bar{\Sigma}. \]

Clearly, \(\bar{\mu}|\bar{\Sigma}' \times \mathbb{S}^0 = \bar{\mu}_{\bar{\Sigma}'}\). Now, by using a standard Cantor’s diagonal process, the same equality holds for all \(\Sigma'\) of the countable familly \(\mathcal{F}\). It remains to show that \(H(\bar{\theta}) = \inf_{\bar{\theta} \in \mathcal{X}(u)} H(\bar{\theta})\). It’s enough to notice that
\[\lim_{\varepsilon \to 0} F_{\varepsilon}(\bar{u}_\varepsilon) = \int_{\Omega} f(\nabla \bar{u}) \, dx + \inf_{\bar{\theta} \in \mathcal{X}(u)} H(\bar{\theta}) \]
\[\geq \lim_{\varepsilon \to 0} \int_{\Omega} f(\nabla \bar{u}_\varepsilon) \, dx + \liminf_{\varepsilon \to 0} \left( \varepsilon^2 \int_{B} g(\nabla \tau_\varepsilon \bar{u}_\varepsilon, \frac{1}{\varepsilon} \frac{\partial \tau_\varepsilon \bar{u}_\varepsilon}{\partial x_N}) \, dx - \langle \mathcal{S}, \tau_\varepsilon \bar{u}_\varepsilon \rangle \right) \]
\[\geq \int_{\Omega} f(\nabla \bar{u}) \, dx + \int_{B} g^{\infty,2}(\bar{0}, \frac{\partial \bar{\theta}}{\partial x_N}) \, dx - \langle \mathcal{S}, \bar{\theta} \rangle \]
\[= \int_{\Omega} f(\nabla \bar{u}) \, dx + H(\bar{\theta}) \]
which completes the proof. \(\square\)

We define the following two constants associated with the function \(g\):
\[c(g) := \min \left( \frac{g^{\infty,2}(\bar{0}, -1)}{g^{\infty,2}(0, 1)}, \frac{g^{\infty,2}(\bar{0}, 1)}{g^{\infty,2}(0, -1)} \right), \quad C(g) = \frac{1}{c(g)} = \max \left( \frac{g^{\infty,2}(\bar{0}, 1)}{g^{\infty,2}(0, -1)}, \frac{g^{\infty,2}(\bar{0}, 1)}{g^{\infty,2}(0, -1)} \right). \]
Recall that
\[
g_{\infty,2}(0, \xi) = \begin{cases} 
g_{\infty,2}(0, -1) |\xi|^2 & \text{if } \xi \leq 0 
g_{\infty,2}(0, 1) |\xi|^2 & \text{if } \xi > 0 \end{cases}.
\]

Moreover, from the assumption on the function \(g\), clearly,
\[
g_{\infty,2}(0, 1) > 0 \text{ and } g_{\infty,2}(0, -1) > 0.
\]

We make precise the probability measure \(\mu_{\hat{x}}\) localized on \(\Sigma' \subset \subset \Sigma\) as follows:
\[
\hat{\mu}_{\hat{x}} := p(\hat{x})\delta_1 + q(\hat{x})\delta_{-1} \quad \text{with} \quad p(\hat{x}) + q(\hat{x}) = 1 \text{ a.e. } \hat{x} \in \Sigma'.
\]

**Corollary 5.6.** Under the assumptions of Theorem 5.5, the three following estimates hold:

(i) for a.e. \(\hat{x}\) in \(\Sigma'\)
\[
c(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \theta}{\partial x_N}(\hat{x}, s) \right|^2 ds \leq \frac{d\pi}{dx_N}(\hat{x}) \leq C(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \theta}{\partial x_N}(\hat{x}, s) \right|^2 ds,
\]

and \(\frac{d\pi}{dx_N}(\hat{x}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \theta}{\partial x_N}(\hat{x}, s) \right|^2 ds\) when \(g_{\infty,2}(0, -1) = g_{\infty,2}(0, 1)\);

(ii) \[
\frac{c(g) \left| [\tilde{u}](\hat{x}) \right|^2}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \theta}{\partial x_N}(\hat{x}, s) \right|^2 ds} \leq p(\hat{x}) \leq 1 \text{ for a.e. } \hat{x} \text{ such that } [\tilde{u}](\hat{x}) > 0;
\]

(iii) \[
\frac{c(g) \left| [\tilde{u}](\hat{x}) \right|^2}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \theta}{\partial x_N}(\hat{x}, s) \right|^2 ds} \leq q(\hat{x}) \leq 1 \text{ for a.e. } \hat{x} \text{ such that } [\tilde{u}](\hat{x}) < 0.
\]

**Proof:** Since \(\mu_{\hat{x}} = p(\hat{x})\delta_1 + q(\hat{x})\delta_{-1}\), we have \(\int_{\mathbb{R}^d} g_{\infty,2}(\xi)d\mu_{\hat{x}} = p(\hat{x})g_{\infty,2}(0, 1) + q(\hat{x})g_{\infty,2}(0, -1)\) with \(p(\hat{x}) + q(\hat{x}) = 1\) a.e. \(\hat{x}\) in \(\Sigma'\) so that from (5.3), one has
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} g_{\infty,2}(0, \frac{\partial \theta}{\partial x_N}(\hat{x}, s))ds = \left( \int_{\mathbb{R}^d} g_{\infty,2}(\xi)d\mu_{\hat{x}} \right) \frac{d\pi}{dx}(\hat{x})
\]
\[
= \frac{d\pi}{dx}(\hat{x}) \left\{ p(\hat{x})g_{\infty,2}(0, 1) + q(\hat{x})g_{\infty,2}(0, -1) \right\} \text{ a.e. } \hat{x} \in \Sigma'.
\]

We are going to establish
\[
c(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \theta}{\partial x_N}(\hat{x}, s) \right|^2 ds \leq \frac{d\pi}{dx}(\hat{x}) \leq C(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \theta}{\partial x_N}(\hat{x}, s) \right|^2 ds.
\]

From (5.10) we deduce that
\[
\min \left\{ g_{\infty,2}(0, -1), g_{\infty,2}(0, 1) \right\} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \theta}{\partial x_N}(\hat{x}, s) \right|^2 ds \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} g_{\infty,2}(0, \frac{\partial \theta}{\partial x_N}(\hat{x}, s))ds
\]
\[
= \left( \int_{\mathbb{R}^d} g_{\infty,2}(\xi)d\mu_{\hat{x}} \right) \frac{d\pi}{dx}(\hat{x})
\]
\[
= \left\{ p(\hat{x})g_{\infty,2}(0, 1) + q(\hat{x})g_{\infty,2}(0, -1) \right\} \frac{d\pi}{dx}(\hat{x})
\]
\[
\leq \max \left\{ g_{\infty,2}(0, -1), g_{\infty,2}(0, 1) \right\} \frac{d\pi}{dx}(\hat{x}).
\]
and
\[
\min \left\{ g^{∞,2}(0, -1), g^{∞,2}(0, 1) \right\} \frac{d\pi}{dx}(\hat{x}) = \min \left\{ g^{∞,2}(0, -1), g^{∞,2}(0, 1) \right\} \{p(\hat{x}) + q(\hat{x})\} \frac{d\pi}{dx}(\hat{x})
\]
\[
\leq \left\{ p(\hat{x})g^{∞,2}(0, 1) + q(\hat{x})g^{∞,2}(0, -1) \right\} \frac{d\pi}{dx}(\hat{x})
\]
\[
= \left( \int_{g_0} g^{∞,2}(\xi) d\mu_\pi(\xi) \right) \frac{d\pi}{dx}(\hat{x})
\]
\[
= \int_{-\frac{1}{2}}^{\frac{1}{2}} g^{∞,2}(\hat{x}, s) \frac{\partial \overline{\theta}}{\partial x_N}(\hat{x}, s) ds
\]
\[
\leq \max \left\{ g^{∞,2}(0, -1), g^{∞,2}(0, 1) \right\} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds
\]

Then, from (5.11) and (5.12) we have
\[
c(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds = \min \left\{ g^{∞,2}(0, -1), g^{∞,2}(0, 1) \right\} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds \leq \frac{d\pi}{dx}(\hat{x})
\]
and
\[
\frac{d\pi}{dx}(\hat{x}) \leq \max \left\{ g^{∞,2}(0, -1), g^{∞,2}(0, 1) \right\} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds = C(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds
\]
from which we deduce,
\[
c(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds \leq \frac{d\pi}{dx}(\hat{x}) \leq C(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds.
\]

Let us prove (ii) and (iii). According to Theorem 5.2, for every $\varphi \in C(S^0)$ such that $\varphi^{**} > -\infty$,
\[
\frac{d\pi}{d\mathcal{H}^N_{\Sigma'}}(x) \int_{S^0} \varphi(\xi) \ d\mu_x(\xi) \geq \varphi^{**}(v(x)) \quad \text{for } \mathcal{H}^N_{\Sigma'} \text{-a.e. } x \in \Sigma',
\]
where $\pi = \frac{d\pi}{d\mathcal{H}^N_{\Sigma'}} \mathcal{H}^N_{\Sigma'} + \pi_s$ is the Radon-Nikodym decomposition of $\pi$ with respect to the measure $\mathcal{H}^N_{\Sigma'}$. We assume that $[\hat{u}](\hat{x}) > 0$ and show that
\[
\frac{c(g) \left| [\hat{u}](\hat{x}) \right|^2}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds} \leq p(\hat{x}) \leq 1.
\]

Let
\[
\varphi(\xi) = \begin{cases} \varphi(1) |\xi|^2 & \text{if } \xi \geq 0 \\ 0 & \text{if } \xi < 0 \end{cases}
\]
Clearly, $\varphi^{**}([\hat{u}](\hat{x})) = \varphi([\hat{u}](\hat{x})) = \varphi(1) \left| [\hat{u}](\hat{x}) \right|^2$. From the inequality (5.13), it follows that
\[
\varphi(1) \left| [\hat{u}](\hat{x}) \right|^2 = \varphi^{**}([\hat{u}](\hat{x}))
\]
\[
\leq \frac{d\pi}{dx}(\hat{x}) \left( \int_{S^0} \varphi(\xi) d\mu_x(\xi) \right)
\]
\[
\leq C(g) \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds \right) \left( \int_{S^0} \varphi(\xi) d\mu_x(\xi) \right)
\]
\[
= C(g)p(\hat{x}) \varphi(1) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \overline{\theta}}{\partial x_N}(\hat{x}, s) \right|^2 ds.
\]
Then, we obtain
\[
\frac{[\bar{u}](\hat{x})^2}{C(g) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{v}}{\partial x_N}(\hat{x}, s) \right|^2 ds} = \frac{c(g) [\bar{u}](\hat{x})^2}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\partial \bar{v}}{\partial x_N}(\hat{x}, s) \right|^2 ds} \leq p(\hat{x}) \leq 1.
\]

The proof of (iii) is similar. \qed

6 References


