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► To cite this version:

Daniela Jarušková, Vladimir I. Piterbarg. Log-likelihood ratio test for detecting transient change. Statistics and Probability Letters, 2011, 81 (5), pp.552. 10.1016/j.spl.2011.01.006 . hal-00725105

HAL Id: hal-00725105

<https://hal.science/hal-00725105>

Submitted on 24 Aug 2012

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Accepted Manuscript

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PII: S0167-7152(11)00013-7
DOI: [10.1016/j.spl.2011.01.006](https://doi.org/10.1016/j.spl.2011.01.006)
Reference: STAPRO 5881

To appear in: *Statistics and Probability Letters*

Received date: 5 October 2009

Revised date: 11 January 2011

Accepted date: 11 January 2011

Please cite this article as: Jarušková, D., Piterbarg, V.I., Log – likelihood ratio test for detecting transient change. *Statistics and Probability Letters* (2011), doi:10.1016/j.spl.2011.01.006

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Log-likelihood ratio test for detecting transient change

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Abstract

The limit behaviour of scan statistics in a form of maxima of moving sum processes with changing window size for detecting a transient change is studied. Approximate critical values obtained from tail behaviour of the limit variables that are maxima of some locally homogeneous Gaussian fields are compared with critical values obtained by simulation.

Keywords: change points detection, scan statistics, asymptotics, maxima of locally homogeneous Gaussian fields

1. Introduction

In many situations we observe at equidistant time points a sequence of random vectors $\{\mathbf{X}_i, i = 1, \dots, n\}$, or on a grid in a square $\{(i, j), i = 1, \dots, n; j = 1, \dots, n\}$ an array of vectors $\{\mathbf{X}_{ij}, i = 1, \dots, n; j = 1, \dots, n\}$. The question may arise whether all observed vectors have the same mean or whether for some $i \in [t_1, t_2]$, resp. for $(i, j) \in A \subset [x_1, x_2] \times [y_1, y_2]$, at least one coordinate of the mean vector is different. For example, components of vectors $\{\mathbf{X}_i\}$ may represent signals that are transmitted through several channels and we are interested to detect the transient change in signal values. In medicine the one-dimensional variables $\{X_i\}$ may represent numbers per day of new cases of a disease and an increase of their mean

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value for a certain time period means an epidemic outbreak. In the control of a homogeneity of a textile fabric the variables $\{X_{ij}\}$ may represent values of a pixel with the coordinates (i, j) in a digital black and white photo. The goal is to detect a speck (a region of a fabric) where a mean gray level is different. The useful tools in statistical inference are scan statistics based on moving sum process $\{\mathbf{S}([t, t+a]) = \sum_{i=t}^{t+a} (\mathbf{X}_i - \bar{\mathbf{X}}), t = 1, \dots, n-a\}$, resp. $\{\mathbf{S}([t, t+a] \times [s, s+b]) = \sum_{i=t}^{t+a} \sum_{j=s}^{s+b} (\mathbf{X}_{ij} - \bar{\mathbf{X}}), t = 1, \dots, n-a; s = 1, \dots, n-b\}$, where $\bar{\mathbf{X}}$ is the overall mean, for more details see Glaz et al. (2001). Sometimes, it is difficult to say how long an epidemic may last or how large a speck of a different grey level may be. In such a case the window width is not specified in advance but scan statistics with varying window width are considered. This corresponds to the situation studied in our paper.

The problem may be generalized to a search for a transient change in parameters of a regular distribution or for a transient change in an intensity of a Poisson process.

In the scope of mathematical statistics a decision whether parameters of interest changed is based on hypotheses testing where the null hypothesis claims that there is no change while the alternative claims that during some time interval or in some place the observed variables or vectors are distributed according to a distribution with different parameters. The described problem belongs to the change point analysis and maximum type test statistics based on standardized partial moving sum process may be applied. For applying them we need to know corresponding critical values. As an exact distribution under the null hypothesis is very complex we usually apply an asymptotic distribution. As always, the maximum type test statistics tend to infinity and to get a limit distribution we have to consider their “trimmed versions”. In such a case the limit variables correspond to maxima of Gaussian fields on compact sets.

The exact distributions of limit variables are unknown, but approximate critical values may be obtained from their tail behaviour. The exceedance probability of Gaussian fields of this type (as the exceedance level tends to infinity) was studied by Siegmund (1988) and Siegmund and Yakir (2000). Yao (1993) obtained an approximation for detecting a transient change in mean of normally distributed random variables while Loader (1991) obtained an approximation for a special case of detecting a transient change in an intensity of a one and two-dimensional Poisson process. We obtained tail approximations of the limit variables by the double-sum method for locally homoge-

neous Gaussian fields suggested by Piterbarg (1996). More precisely, we have shown that the conditions of Theorem 7.1, Chapter 2 of Piterbarg (1996) are satisfied. A recently published paper by Chan and Lai (2006) presents similar ideas of tail approximation for nonhomogeneous Gaussian fields. As a consequence of their Theorem 2.1, see Example 2.2, they obtained a high-level exceedance probability for a maximum of standardized increments of a Wiener process. It is clear that Theorem 2.1. of Chan and Lai (2006) could be also applied to obtain a high-level exceedance probability for a maximum of standardized increments of a Brownian bridge, i.e., the limit variable of our test statistic in one-dimensional case. We would like to thank the referee to bring our attention to the paper by Chan and Lai (2006).

Our paper shows how the new probabilistic results concerning Gaussian fields may be applied in statistics. To show how accurate the approximations are, we included a small simulation study where we compare critical values obtained by simulation with critical values obtained by the suggested approximations.

2. Mathematical formulation of problem

One - dimensional case

Suppose that we observe a sequence of d -dimensional random vectors $\mathbf{X}_1, \dots, \mathbf{X}_n$. We would like to test the null hypothesis H_{01} against the alternative A_1 :

$$H_{01} : \mathbf{X}_i = \boldsymbol{\mu} + \mathbf{e}_i, \quad i = 1, \dots, n, \quad (1)$$

$$A_1 : \exists \quad 0 \leq k_1^* < k_2^* \leq n \quad \text{satisfying} \quad \alpha n \leq k_2^* - k_1^* \leq (1 - \beta) n$$

such that

$$\mathbf{X}_i = \boldsymbol{\mu} + \mathbf{e}_i, \quad i = 1, \dots, k_1^*;$$

$$\mathbf{X}_i = \boldsymbol{\mu} + \boldsymbol{\delta} + \mathbf{e}_i, \quad i = k_1^* + 1, \dots, k_2^*;$$

$$\mathbf{X}_i = \boldsymbol{\mu} + \mathbf{e}_i, \quad i = k_2^* + 1, \dots, n.$$

We suppose that $\boldsymbol{\mu}$ and $\boldsymbol{\delta} \neq \mathbf{0}$ are unknown real vectors. The error terms $\{\mathbf{e}_i\}$ are i.i.d. and $E \mathbf{e}_i = \mathbf{0}$, $Var \mathbf{e}_i = \boldsymbol{\Sigma}$ (known) and $E \|\mathbf{e}_i\|^{2+\Delta} < \infty$. The problem was introduced by Levin and Kline (1985) and studied in a general setting by Antoch and Hušková (1996).

For $0 \leq k_1 < k_2 \leq n$ we introduce

$$\mathbf{S}(k_1, k_2) = \sum_{i=k_1+1}^{k_2} (\mathbf{X}_i - \bar{\mathbf{X}}) = \sum_{i=1}^{k_2} (\mathbf{X}_i - \bar{\mathbf{X}}) - \sum_{i=1}^{k_1} (\mathbf{X}_i - \bar{\mathbf{X}}),$$

where $\bar{\mathbf{X}}$ denotes an average of all observed vectors. The test statistic has a form:

$$T_1^2 = \max_{\substack{0 \leq k_1 < k_2 \leq n \\ [\alpha n] \leq k_2 - k_1 \leq [(1-\beta)n]}} \frac{(\mathbf{S}(k_1, k_2))^T \boldsymbol{\Sigma}^{-1} \mathbf{S}(k_1, k_2)}{(k_2 - k_1)(1 - (k_2 - k_1)/n)} \quad (2)$$

For $l = 1, \dots, d$ we introduce zero mean unit variance Gaussian fields

$$\{Y_l(x_1, x_2) = \frac{B_l(x_2) - B_l(x_1)}{\sqrt{(x_2 - x_1)(1 - (x_2 - x_1))}}, 0 < x_1 < x_2 < 1\},$$

where $\{(B_1(x), \dots, B_d(x))^T, 0 \leq x \leq 1\}$ is a d -dimensional Gaussian process whose coordinates are independent Brownian bridges.

Then, under H_0 it holds:

$$T_1^2 \xrightarrow{D} \max_{\substack{0 \leq x_1 < x_2 \leq 1 \\ \alpha \leq x_2 - x_1 \leq (1-\beta)}} \sum_{l=1}^d (Y_l(x_1, x_2))^2 \quad \text{as } n \rightarrow \infty. \quad (3)$$

The assertion (3) may be proved using Theorem 3 of Bickel and Wichura (1971).

The exact distribution of (2) is unknown. In one-dimensional case Yao (1993) derived an accurate tail approximation for normally distributed random variables $\{X_i\}$. The exact distribution of the limit variable given by the righthand side of (3) is also unknown but it is again possible to derive its “tail behaviour”.

In what follows the sign “ $\sim$ ” means an asymptotic equivalence.

Theorem 1. *Under the above set up*

$$P\left(\max_{\substack{0 \leq x_1 < x_2 \leq 1 \\ \alpha \leq x_2 - x_1 \leq (1-\beta)}} \sum_{l=1}^d (Y_l(x_1, x_2))^2 > u^2\right) \sim \frac{1}{2^{d/2-1} \Gamma(d/2)} C_1(\alpha, \beta) u^{2+d} e^{-u^2/2} \quad (4)$$

as $u \rightarrow \infty$, where

$$C_1(\alpha, \beta) = \frac{1}{4} \left(\log \frac{(1-\beta)(1-\alpha)}{\beta\alpha} + \frac{1}{\alpha} - \frac{1}{1-\beta} \right). \quad (5)$$

PROOF. We introduce a locally homogeneous Gaussian field

$$\{YP(x_1, x_2, \theta_1, \dots, \theta_{d-1}), 0 < x_1 < x_2 < 1, -\frac{\pi}{2} \leq \theta_k \leq \frac{\pi}{2}, k = 1, \dots, d-2, \\ -\pi \leq \theta_{d-1} \leq \pi\} \quad \text{by}$$

$$YP(x_1, x_2, \theta_1, \dots, \theta_{d-1}) = Y_1(x_1, x_2) u_1(\theta_1, \dots, \theta_{d-1}) + \dots \\ + Y_d(x_1, x_2) u_d(\theta_1, \dots, \theta_{d-1}),$$

where $u_1(\theta_1, \dots, \theta_{d-1}) = \sin \theta_1, \dots, u_d(\theta_1, \dots, \theta_{d-1}) = \cos \theta_{d-1} \cdots \cos \theta_1$ are spherical coordinates. Clearly,

$$\max_{\substack{0 \leq x_1 < x_2 \leq 1 \\ \alpha \leq x_2 - x_1 \leq (1-\beta)}} \sqrt{\sum_{l=1}^d (Y_l(x_1, x_2))^2} = \max_Q YP(x_1, x_2, \theta_1, \dots, \theta_{d-1}),$$

where $Q = \{(x_1, x_2, \theta_1, \dots, \theta_{d-1}), 0 < x_1 < x_2 < 1, \alpha \leq x_2 - x_1 \leq (1-\beta), -\frac{\pi}{2} \leq \theta_k \leq \frac{\pi}{2}, k = 1, \dots, d-2, -\pi \leq \theta_{d-1} \leq \pi\}$. The correlation function of $\{YP(x_1, x_2, \theta_1, \dots, \theta_{d-1})\}$ satisfies

$$E \left(YP(x_1 + h_1, x_2 + h_2, \theta_1 + \psi_1, \dots, \theta_{d-1} + \psi_{d-1}) YP(x_1, x_2, \theta_1, \dots, \theta_{d-1}) \right) = \\ 1 - \frac{1}{2(x_2 - x_1)(1 - (x_2 - x_1))} |h_1| - \frac{1}{2(x_2 - x_1)(1 - (x_2 - x_1))} |h_2| - \\ - \frac{1}{2} \psi_1^2 - \frac{1}{2} (\cos^2 \theta_1) \psi_2^2 - \dots - \frac{1}{2} (\cos^2 \theta_1 \cdots \cos^2 \theta_{d-2}) \psi_{d-1}^2 + o(|h_1| + |h_2| + \\ + \psi_1^2 + \dots + \psi_{d-1}^2) \quad \text{as } h_1 \rightarrow 0, h_2 \rightarrow 0, \psi_1 \rightarrow 0, \dots, \psi_{d-1} \rightarrow 0.$$

Using Theorem 7.1 (see also Corollary 7.5) of Piterbarg (1996) it holds (as $u \rightarrow \infty$)

$$P \left(\max_Q YP(x_1, x_2, \theta_1, \dots, \theta_{d-1}) > u \right) \sim H_{1,1,2,\dots,2} u^{2+d} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \times \\ \int_Q \frac{1}{4(x_2 - x_1)^2 (1 - (x_2 - x_1))^2} \prod_{i=1}^{d-2} \sqrt{\frac{1}{2} \cos^2 \theta_{d-1-i}} d\theta_1 \dots d\theta_{d-1} dx_1 dx_2.$$

We get (4) as

$$H_{1,1,2,\dots,2} = 1/\pi^{(d-1)/2},$$

$$\begin{aligned} C_1(\alpha, \beta) &= \iint_{\substack{0 \leq x_1 < x_2 \leq 1 \\ \alpha \leq x_2 - x_1 \leq (1-\beta)}} \frac{1}{4(x_2 - x_1)^2(1 - (x_2 - x_1))^2} dx_1 dx_2 \\ &= \int_{\alpha}^{1-\beta} \int_0^{1-x} \frac{1}{4x^2(1-x)^2} d\xi dx = \int_{\alpha}^{1-\beta} \frac{1}{4x^2(1-x)} dx \\ &= \frac{1}{4} \left(\log \frac{(1-\beta)(1-\alpha)}{\beta\alpha} + \frac{1}{\alpha} - \frac{1}{1-\beta} \right) \end{aligned}$$

and

$$\frac{1}{2^{(d-1)/2}} \int_{-\pi}^{\pi} \dots \int_{-\pi/2}^{\pi/2} \prod_{i=1}^{d-2} \cos^i \theta_{d-1-i} d\theta_1 \dots d\theta_{d-1} = \frac{1}{2^{(d-1)/2}} \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

□

For n large the approximation (4) may serve to find approximate critical values of the test statistic (2).

Remark 1. In case the variance-covariance matrix Σ is unknown, it can be replaced by any consistent estimator.

Remark 2. For $d = 1$ we can also consider a one-sided alternative (e.g. $\delta > 0$). The test statistic

$$\tilde{T}_1 = \max_{\substack{0 \leq k_1 < k_2 \leq n \\ [\alpha n] \leq k_2 - k_1 \leq [(1-\beta)n]}} \frac{S(k_1, k_2)}{\sigma \sqrt{(k_2 - k_1)(1 - (k_2 - k_1)/n)}} \quad (6)$$

converges (as $n \rightarrow \infty$) in distribution to

$$\max_{\substack{0 \leq x_1 < x_2 \leq 1 \\ \alpha \leq x_2 - x_1 \leq (1-\beta)}} \frac{B_1(x_2) - B_1(x_1)}{\sqrt{(x_2 - x_1)(1 - (x_2 - x_1))}}.$$

To find approximate critical values we may apply the approximation (valid for u large) that follows from the limit behaviour as $y \rightarrow \infty$:

$$P\left(\max_{\substack{0 \leq x_1 < x_2 \leq 1 \\ \alpha \leq x_2 - x_1 \leq (1-\beta)}} \frac{B_1(x_2) - B_1(x_1)}{\sqrt{(x_2 - x_1)(1 - (x_2 - x_1))}} > u\right) \sim C_1(\alpha, \beta) u^3 \phi(u), \quad (7)$$

where $\phi(u)$ is the density of a standard normal distribution. As it has been already mentioned, (7) can be also obtained using Theorem 2.1 and Lemma 2.3 of Chan and Lai (2006).

Remark 3. We can consider a more general alternative supposing that a shift in mean may occur not only in one interval but in a union of a fixed number of intervals. For such a case a test statistic as well as its limit behaviour that enables to calculate approximate critical values can be derived analogously. However, with an increase number of change points the convergence to a limit distribution will be slower and approximate critical values less accurate.

Two-dimensional case

Suppose that on two-dimensional grid $\{(i, j), i = 1, \dots, n; j = 1, \dots, n\}$ we observe a d -dimensional array $\{\mathbf{X}_{ij}, i = 1, \dots, n, j = 1, \dots, n\}$. We would like to test the following hypotheses testing problem

$$H_{02} : \mathbf{X}_{ij} = \boldsymbol{\mu} + \mathbf{e}_{ij}, \quad i = 1, \dots, n, j = 1, \dots, n \quad (8)$$

$$\begin{aligned} A_2 : \exists \quad & 0 \leq k_1^* < k_2^* \leq n \text{ and } 0 \leq m_1^* < m_2^* < n \\ & \text{satisfying } \alpha n^2 \leq (k_2^* - k_1^*)(m_2^* - m_1^*) \leq (1 - \beta) n^2 \quad \text{such that} \\ & \mathbf{X}_{ij} = \boldsymbol{\mu} + \mathbf{e}_{ij}, \quad (i, j) \notin [k_1^*, k_2^*] \times [m_1^*, m_2^*], \\ & \mathbf{X}_{ij} = \boldsymbol{\mu} + \boldsymbol{\delta} + \mathbf{e}_{ij}, \quad (i, j) \in [k_1^*, k_2^*] \times [m_1^*, m_2^*]. \end{aligned}$$

We suppose that $\boldsymbol{\mu}$ and $\boldsymbol{\delta} \neq \mathbf{0}$ are unknown real vectors. The error terms $\{\mathbf{e}_{ij}\}$ are i.i.d. and $E \mathbf{e}_{ij} = \mathbf{0}$, $Var \mathbf{e}_{ij} = \boldsymbol{\Sigma}$ (known) and $E \|\mathbf{e}_{ij}\|^{2+\Delta} < \infty$.

Denote $\mathbf{S}(k_1, m_1, k_2, m_2) = \sum_{i=k_1}^{k_2} \sum_{j=m_1}^{m_2} (\mathbf{X}_{ij} - \bar{\mathbf{X}})$ where

$\bar{\mathbf{X}} = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \mathbf{X}_{ij}$. The test statistic has a form

$$T_2^2 = \max_{\substack{0 \leq k_1 < k_2 \leq n; 0 \leq m_1 < m_2 \leq n \\ [\alpha n^2] \leq (m_2 - m_1)(k_2 - k_1) \leq [(1-\beta)n^2]}} \frac{\frac{1}{n} (\mathbf{S}(k_1, m_1, k_2, m_2))^T \boldsymbol{\Sigma}^{-1} \mathbf{S}(k_1, k_2, m_1, m_2)}{\sqrt{\frac{(k_2 - k_1)}{n} \frac{(m_2 - m_1)}{n} \left(1 - \frac{(k_2 - k_1)(m_2 - m_1)}{n^2}\right)}}. \quad (9)$$

For $l = 1, \dots, d$ we introduce zero mean Gaussian fields $\{Z_l(x_1, y_1, x_2, y_2), 0 < x_1 < x_2 < 1, 0 < y_1 < y_2 < 1\}$ defined by

$$Z_l(x_1, y_1, x_2, y_2) = \frac{W_l(x_2, y_2) - W_l(x_2, y_1) - W_l(x_1, y_2) + W_l(x_1, y_1) - (x_2 - x_1)(y_2 - y_1)W_l(1, 1)}{\sqrt{(x_2 - x_1)(y_2 - y_1)(1 - (x_2 - x_1)(y_2 - y_1))}}$$

where $\{(W_1(x, y), \dots, W_d(x, y))^T, x \geq 0, y \geq 0\}$ is a zero mean Gaussian field whose coordinates are independent Gaussian fields with a covariance structure

$$E(W_l(x_1, y_1) W_l(x_2, y_2)) = \min(x_1, x_2) \min(y_1, y_2), \quad l = 1, \dots, d.$$

We also introduce a subset A of R^4 by $A = \{0 \leq x_1 < x_2 \leq 1, 0 \leq y_1 < y_2 \leq 1, \alpha \leq (x_2 - x_1)(y_2 - y_1) \leq (1 - \beta)\}$. Applying again results by Bickel and Wichura (1971) we get

$$T_2^2 \xrightarrow{D} \max_{(x_1, y_1, x_2, y_2) \in A} \sum_{l=1}^d (Z_l(x_1, y_1, x_2, y_2))^2 \quad \text{as } n \rightarrow \infty. \quad (10)$$

Theorem 2. *Under the above set up*

$$P\left(\max_{(x_1, y_1, x_2, y_2) \in A} \sum_{l=1}^d (Z_l(x_1, y_1, x_2, y_2))^2 > u^2\right) \sim C_2(\alpha, \beta) \frac{1}{2^{d/2-1} \Gamma(d/2)} u^{6+d} e^{-u^2/2}, \quad (11)$$

as $u \rightarrow \infty$, where

$$C_2(\alpha, \beta) = \int_{\alpha}^{1-\beta} \frac{-2(1-\xi) - (1+\xi) \log \xi}{16 \xi^2 (1-\xi)^4} d\xi. \quad (12)$$

PROOF. The zero mean Gaussian field

$$ZP(x_1, y_1, x_2, y_2, \theta_1, \dots, \theta_{d-1}) = Z_1(x_1, y_1, x_2, y_2)u_1(\theta_1, \dots, \theta_{d-1}) + Z_d(x_1, y_1, x_2, y_2)u_d(\theta_1, \dots, \theta_{d-1})$$

is locally homogeneous. Denote $x = x_2 - x_1$, $y = y_2 - y_1$. The correlation function of $\{ZP(x_1, y_1, x_2, y_2, \theta_1, \dots, \theta_{d-1})\}$ satisfies

$$\begin{aligned} E\left(ZP(x_1 + h_1, y_1 + f_1, x_2 + h_2, y_2 + f_2, \theta_1 + \psi_1, \dots, \theta_{d-1} + \psi_{d-1}) \right. \\ \left. ZP(x_1, y_1, x_2, y_2, \theta_1, \dots, \theta_{d-1})\right) = \\ 1 - \frac{1}{2x(1-xy)}|h_1| - \frac{1}{2x(1-xy)}|h_2| - \frac{1}{2y(1-xy)}|f_1| - \frac{1}{2y(1-xy)}|f_2| - \\ - \frac{1}{2}\psi_1^2 - \frac{1}{2}(\cos^2 \theta_1)\psi_2^2 - \dots - \frac{1}{2}(\cos^2 \theta_1 \dots \cos^2 \theta_{d-1})\psi_{d-1}^2 + \\ + o(|h_1| + |h_2| + |f_1| + |f_2| + \psi_1^2 + \dots + \psi_{d-1}^2), \end{aligned}$$

as $h_1 \rightarrow 0, h_2 \rightarrow 0, f_1 \rightarrow 0, f_2 \rightarrow 0, \psi_1 \rightarrow 0, \dots, \psi_{d-1} \rightarrow 0$. The assertion follows again from Theorem 7.1 of Piterbarg (1996), where

$$\begin{aligned} C_2(\alpha, \beta) &= \\ &= \iiint \int_A \frac{1}{16(x_2 - x_1)^2(y_2 - y_1)^2(1 - (y_2 - y_1)(x_2 - x_1))^4} dx_1 dx_2 dy_1 dy_2 \\ &= \iint_{\alpha \leq x, y \leq (1-\beta)} \frac{(1-x)(1-y)}{16x^2y^2(1-xy)^4} dx dy \\ &= \int_{\alpha}^{1-\beta} \frac{1}{16\xi^2(1-\xi)^4} d\xi \int_{\xi}^1 \frac{(\eta - \xi)(1 - \eta)}{\eta^2} d\eta = (12). \end{aligned}$$

□

For n large the approximation (11) may serve for finding approximate critical values of the test statistic (9).

Remark 4. In case the variance-covariance matrix Σ is unknown it can be replaced by any consistent estimator.

Remark 5. For $d = 1$ we can also consider a one-sided alternative (e.g. $\delta > 0$). The test statistic

$$\tilde{T}_2 = \max_{\substack{0 \leq k_1 < k_2 \leq n; 0 \leq m_1 < m_2 \leq n \\ [\alpha n^2] \leq (m_2 - m_1)(k_2 - k_1) \leq [(1 - \beta)n^2]}} \frac{\frac{1}{\sigma} \frac{1}{n} S(k_1, m_1, k_2, m_2)}{\sqrt{\frac{(k_2 - k_1)}{n} \frac{(m_2 - m_1)}{n} \left(1 - \frac{(k_2 - k_1)(m_2 - m_1)}{n^2}\right)}}, \quad (13)$$

converges as $(n \rightarrow \infty)$ in distribution to

$$\max_{(x_1, y_1, x_2, y_2) \in A} Z_1(x_1, y_1, x_2, y_2)$$

and

$$P\left(\max_{(x_1, y_1, x_2, y_2) \in A} Z_1(x_1, y_1, x_2, y_2) > u\right) \sim C_2(\alpha, \beta) u^7 \phi(u) \quad \text{as } u \rightarrow \infty. \quad (14)$$

The limit behaviour (14) may be also obtained as a consequence of Theorem 2.1 and Lemma 2.3 of Chan and Lai (2006).

Remark 6. We can again consider a more general alternative supposing that a shift in mean can occur on a more complex set A . For applications a limit distribution of a corresponding test statistic, if the set A has a form of an ellipsoid, may be of interest. This may be a goal for our future research.

3. Transient change in a parameter of a distribution

The problem (1) can be called a test for an “epidemic type of change” in mean. However, it can be formulated in more general way as a transient change in a multidimensional parameter θ of a distribution. More precisely, we suppose that $\mathbf{X}_1, \dots, \mathbf{X}_n$ are independent vectors distributed according to distributions with densities $f(\mathbf{x}_1; \theta_1, \eta_1), \dots, f(\mathbf{x}_n; \theta_n, \eta_n)$ where $\theta_1, \dots, \theta_n \in \Theta^{(1)} \subset R^d$ and $\eta_1, \dots, \eta_n \in \Theta^{(2)} \subset R^c$ are parameters. We would like to test the null hypothesis H_{03} against the alternative A_3 :

$$H_{03} : \theta_1 = \dots = \theta_n \quad \text{and} \quad \eta_1 = \dots = \eta_n = \eta \quad (15)$$

$$A_3 : \exists \quad 0 \leq k_1^* < k_2^* \leq n \quad \text{satisfying} \quad \alpha n \leq k_2^* - k_1^* \leq (1 - \beta) n$$

such that

$$\theta_1 = \dots = \theta_{k_1^*} \neq \theta_{k_1^*+1} = \dots = \theta_{k_2^*} \neq \theta_{k_2^*+1} = \dots = \theta_n,$$

$$\theta_1 = \dots = \theta_{k_1^*} = \theta_{k_2^*+1} = \dots = \theta_n,$$

$$\eta_1 = \dots = \eta_n = \eta,$$

where $\theta_1, \dots, \theta_n$ are parameters of interest while $\eta_1, \dots, \eta_n$ are nuisance parameters, see Csörgő and Horváth (1997). If we consider the log-likelihood ratio test statistic in the form

$$L^2 = \max_{\substack{0 \leq k_1 < k_2 \leq n \\ [\alpha n] \leq k_2 - k_1 \leq [(1-\beta)n]}} -2 \log \Lambda_{k_1, k_2} \quad (16)$$

with

$$\Lambda_{k_1, k_2} = \frac{\sup_{\theta, \eta} \prod_{1 \leq i \leq n} f(\mathbf{X}_i; \theta, \eta)}{\sup_{\theta_1, \theta_2, \eta} \prod_{\substack{1 \leq i \leq k_1 \\ k_2 < i \leq n}} f(\mathbf{X}_i; \theta_1, \eta) \prod_{k_1 < i \leq k_2} f(\mathbf{X}_i; \theta_2, \eta)},$$

then under the condition C.1–C.9 (or their analogues) of Csörgő and Horváth (1997) we get again

$$L^2 \xrightarrow{D} \max_{\substack{0 \leq x_1 < x_2 \leq 1 \\ \alpha \leq x_2 - x_1 \leq (1-\beta)}} \sum_{l=1}^d (Y_l(x_1, x_2))^2 \quad \text{as } n \rightarrow \infty. \quad (17)$$

For instance, if we test for a transient change in the one-dimensional parameter θ of an exponential distribution (two-sided alternative) the test statistic (16) is equivalent to the test statistic

$$|T_e| = \max_{\substack{0 \leq k_1 < k_2 \leq n \\ [\alpha n] \leq k_2 - k_1 \leq [(1-\beta)n]}} \frac{|S(k_1, k_2)|}{\bar{X} \sqrt{(k_2 - k_1)(1 - (k_2 - k_1)/n)}}. \quad (18)$$

For one-sided alternative, the test statistic has a form

$$\tilde{T}_e = \max_{\substack{0 \leq k_1 < k_2 \leq n \\ [\alpha n] \leq k_2 - k_1 \leq [(1-\beta)n]}} \frac{S(k_1, k_2)}{\bar{X} \sqrt{(k_2 - k_1)(1 - (k_2 - k_1)/n)}}. \quad (19)$$

The test statistic (18), resp. (19), may be applied for testing a transient change in an intensity λ of a one-dimensional Poisson process because intervals between successive events have an exponential distribution. Instead of (18) we may use for a two-sided alternative a test statistic

$$|P_1| = \max_{\substack{0 \leq t_1 < t_2 \leq T \\ \alpha T \leq t_2 - t_1 \leq (1-\beta)T}} \frac{|N((t_1, t_2)) - (t_2 - t_1)N((0, T))/T|}{\sqrt{N((0, T)) \frac{t_2 - t_1}{T} (1 - \frac{t_2 - t_1}{T})}}, \quad (20)$$

and for a one-sided alternative a test statistic

$$\tilde{P}_1 = \max_{\substack{0 \leq t_1 < t_2 \leq T \\ \alpha T \leq t_2 - t_1 \leq (1-\beta)T}} \frac{N((t_1, t_2)) - (t_2 - t_1)N((0, T))/T}{\sqrt{N((0, T)) \frac{t_2 - t_1}{T} (1 - \frac{t_2 - t_1}{T})}}, \quad (21)$$

where $N(A)$ denotes a number of events in a set A . Under the null hypothesis of no change in an intensity it holds (by applying Bickel and Wichura (1971))

$$|P_1| \xrightarrow{D} \max_{\substack{0 \leq x_1 < x_2 \leq 1 \\ \alpha \leq x_2 - x_1 \leq (1-\beta)}} |Y_1(x_1, x_2)|, \quad \tilde{P}_1 \xrightarrow{D} \max_{\substack{0 \leq x_1 < x_2 \leq 1 \\ \alpha \leq x_2 - x_1 \leq (1-\beta)}} Y_1(x_1, x_2) \quad (22)$$

as $T \rightarrow \infty$.

The same asymptotic results were obtained by Loader (1991) together with a more accurate approximation (for smaller value of T) based on large deviations theory. The analogue test statistics may be derived for testing a transient change in an intensity of a Poisson process in a plane, more precisely for a two-sided alternative

$$|P_2| = \max_{\substack{1 \leq t_1 < t_2 \leq T \\ 1 \leq s_1 < s_2 \leq T \\ \alpha T^2 \leq (t_2 - t_1)(s_2 - s_1) \leq (1-\beta)T^2}} \frac{|N([t_1, t_2] \times [s_1, s_2]) - \frac{(t_2 - t_1)(s_2 - s_1)}{T^2} N([0, T]^2)|}{\sqrt{N([0, T]^2) \frac{(t_2 - t_1)(s_2 - s_1)}{T^2} (1 - \frac{(t_2 - t_1)(s_2 - s_1)}{T^2})}}, \quad (23)$$

resp. for a one-sided alternative

$$\tilde{P}_2 = \max_{\substack{1 \leq t_1 < t_2 \leq T \\ 1 \leq s_1 < s_2 \leq T \\ \alpha T^2 \leq (t_2 - t_1)(s_2 - s_1) \leq (1-\beta)T^2}} \frac{N([t_1, t_2] \times [s_1, s_2]) - \frac{(t_2 - t_1)(s_2 - s_1)}{T^2} N([0, T]^2)}{\sqrt{N([0, T]^2) \frac{(t_2 - t_1)(s_2 - s_1)}{T^2} (1 - \frac{(t_2 - t_1)(s_2 - s_1)}{T^2})}}. \quad (24)$$

Under the null hypothesis of no change in a intensity it holds

$$|P_2| \xrightarrow{D} \max_{(x_1, y_1, x_2, y_2) \in A} |Z_1(x_1, y_1, x_2, y_2)|, \quad (25)$$

and

$$\tilde{P}_2 \xrightarrow{D} \max_{(x_1, y_1, x_2, y_2) \in A} Z_1(x_1, y_1, x_2, y_2) \quad (26)$$

as $T \rightarrow \infty$.

The results (25) and (26) were again obtained by Loader (1991) together with a better approximation for smaller value of T .

4. Simulation study

The quality of the suggested approximations is affected by distribution (under H_0) of observations, and by a choice of α and β . The agreement is worse for smaller value of α , resp. β . For comparison Table 1 and 2 present the 5 % critical values of $\tilde{T}_1$ and $\tilde{T}_e$, (i.e. $d = 1$ and one sided-alternative) for normally and exponential distributed variables for several choice of α and β , obtained by (7) and by simulations.

trimmed portions	(7)	$n = 1000$	$n = 2000$	$n = 4000$
$\alpha = 0.01, \beta = 0.01$	4.44	4.13	4.24	4.27
$\alpha = 0.20, \beta = 0.20$	3.57	3.46	3.50	3.51
$\alpha = 0.01, \beta = 0.95$	4.02	3.85	3.89	3.92

Table 1. The 5% critical values of $\tilde{T}_1$ for normally distributed observations.

trimmed portions	(7)	$n = 1000$	$n = 2000$	$n = 4000$
$\alpha = 0.01, \beta = 0.01$	4.44	5.20	4.96	4.84
$\alpha = 0.20, \beta = 0.20$	3.57	3.54	3.56	3.61
$\alpha = 0.01, \beta = 0.95$	4.02	5.14	4.94	4.80

Table 2. The 5% critical values of $\tilde{T}_e$ for exponentially distributed observations.

Table 3 shows the agreement between the 5 % critical values of $|T_1|$ obtained by (4) and by simulations with normally distributed random vectors for $d = 1, 2, 3$ and $\alpha = \beta = 0.01$.

trimmed portions	(4)	$n = 1000$	$n = 2000$	$n = 4000$
$d = 1$	4.62	4.31	4.41	4.46
$d = 2$	5.05	4.76	4.82	4.87
$d = 3$	5.38	5.06	5.11	5.19

Table 3. The 5% critical values of $|T_1|$ for normally distributed observations.

Table 4 shows the 5% critical values of $\tilde{T}_2$ obtained by (14) and by a simulation study when the variables are normally distributed.

trimmed portions	(14)	$n^2 = 1600$	$n^2 = 3600$
$\alpha = 0.20, \beta = 0.20$	4.72	4.09	4.12
$\alpha = 0.01, \beta = 0.95$	5.80	4.67	4.81

Table 4. The 5% critical values of $\tilde{T}_2$ for normally distributed observations.

All simulated critical values were obtained from 5000 repetitions.

Acknowledgment: We would like to thank our referees for all their recommendations and comments. Especially, we would like to thank one of the referees to pointing us to the paper by Chan and Lai (2006). The work of the first author was supported by grant GAČR 201/09/0755.

References

- [1] Antoch J., Hušková M., 1996. *Tests and estimators for epidemic alternatives*. Tatra Mountains Math. Publ. **7**, 311–329.
- [2] Antoch J., Jarušková D., 2007. *Testing a homogeneity of stochastic processes*. Kybernetika **43**, 415–430.
- [3] Bickel P. J., Wichura M. J., 1971. *Convergence criteria for multiparameter stochastic processes and some applications*. The Annals of Mathematical Statistics **42**, 1656–1670.
- [4] Chan H. P., Lai T. L., 2006. *Maxima of asymptotically Gaussian random fields and moderate deviation approximations to boundary crossing probabilities of sums of random variables with multidimensional indices*. The Annals of Probability **34**, 80–121.
- [5] Csörgő M., Horváth L., 1997. *Limit Theorems in Change Point Analysis*. J. Wiley, New York.

- [6] Glaz J., Naus J., Wallenstein S., 2001. *Scan statistics*. Springer, New York – Berlin – Heidelberg.
- [7] Levin B., Kline J. , 1985. *The cusum test of homogeneity with an application to spontaneous abortion epidemiology*. Statist. in Medicine **4**, 469–488.
- [8] Loader C. R., 1991. *Large – deviation approximations to the distribution of scan statistics*. Adv. Appl. Prob. **23**, 751–771.
- [9] Piterbarg V.I., 1996. *Asymptotic methods in the theory of Gaussian processes and fields*. American Mathematical Society. Providence.
- [10] Siegmund D., 1988. *Approximate tail probabilities for the maxima of some random fields*. The Annals of Probability **16**, 487–501.
- [11] Siegmund D., Yakir B., 2000 *Tail probabilities for the null distribution of scanning statistics*. Bernoulli **6**, 191–213.
- [12] Yao Q., 1993. *Boundary-crossing probabilities of some random fields related to likelihood ratio tests for epidemic alternatives*. J. Appl. Prob. **30**, 52–65.