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A hybridized discontinuous Galerkin method for 3d time-harmonic Maxwell’s equations

Stéphane Lanterti1, Liang Li1,2 and Ronan Perrussel3

1 Equipe NACHOS, INRIA Sophia Antipolis-Méditerranée, Sophia Antipolis, France
2 School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, P. R. China
3 Université de Toulouse; CNRS; INPT; UPS; Laboratoire Plasma et Conversion d’Énergie (LAPLACE), Toulouse, France
E-mails: stephane.lanteri@inria.fr, plum@liang@uestc.edu.cn, perrussel@laplace.univ-tlse.fr

Abstract — Hybridized discontinuous Galerkin methods preserve the advantages of classical discontinuous Galerkin methods and in addition enable to circumvent the issue of the number of degrees of freedom. The principles of these numerical methods are summed up for 3d time-harmonic Maxwell’s equations and basic examples are proposed to assess their efficiency.

I. INTRODUCTION

Discontinuous Galerkin (DG) methods are currently widespread for the discretization of time-transient Maxwell’s equations [1, 2]. This success can be explained by several advantages compared to other approaches such as the flexibility for hp-adaptivity or a natural parallelism [3]. Nonetheless for stationary problems as time-harmonic Maxwell’s equations, the number of degrees of freedom is several times larger when using a classical conforming finite element method and this aspect is cumbersome. Hybridized DG (HDG) methods have been proposed recently to circumvent this problem [4]: preserving the discontinuous approximations of both electric and magnetic fields, the only global linear system to solve determines the degrees of freedom of a new hybrid variable “living” on the interface of each element of the mesh. Thus, a HDG method leads to the solution of a reduced-size linear system without altering the nice properties of a DG method. We consider here such a HDG method for solving the 3d time-harmonic Maxwell’s equations and illustrate the accuracy and the computing cost of the approach.

II. HDG DISCRETIZATION

Electric E and magnetic H fields are approximated by discontinuous vector fields EH and HH whose components are polynomials in each element, i.e. EH and HH belong to (Pp(K))3 in each element K of the mesh with Pp(K) the space of polynomials of degree p on K. The fields EH and HH are computed by weakly verifying time-harmonic Maxwell’s equations in each element K of the mesh; it means that for all V in (Pp(K))3; EH and HH satisfy

\[
\begin{align*}
(\omega \varepsilon \text{E}_\text{h}, \text{V}) - (\text{H}_\text{h}, \text{curl} \text{V}) + (\text{H}_\text{h} \times \text{n}, \text{V}) &= 0, \\
(\omega \mu \text{H}_\text{h}, \text{V}) + (\text{E}_\text{h}, \text{curl} \text{V}) - (\text{E}_\text{h} \times \text{n}, \text{V}) &= 0.
\end{align*}
\]

(1)

Here i is the imaginary unit, \( \omega \) the angular frequency, \( \varepsilon \) and \( \mu \) the relative permittivity and permeability. Symbols (\cdot, \cdot) and (\cdot, \cdot) denote the scalar product of complex-valued vectors integrated over K and \( \partial K \); the boundary of K; n is the outward unit normal vector on \( \partial K \) with \( \times \) the cross-product. Quantities EH and EH are only defined on \( \partial K \) and

assumed to be approximations of the tangential component of the magnetic and electric fields on \( \partial K \).

In the classical DG approach, both quantities HH and EH are defined as linear combinations of the electric and magnetic fields of the element K and of its neighboring elements. This strategy leads to the solution of a linear system which can be several times larger than the system obtained by a more classical conforming finite element method.

In the HDG approach, HH is considered as a new hybrid variable, which is single-valued on each interface of the mesh. On the boundary of each K, the variable EH is then defined from the hybrid variable and the local field by

\[
\text{E}_\text{h} = \text{E}_\text{h} + \tau \text{n} \times (\text{H}_\text{h} - \text{H}_\text{h}^\prime) \text{on } \partial K,
\]

(2)

where \( \tau \) is (often strictly) positive parameter. Contrary to HH, EH has a priori two distinct values on each side of an interface between two elements. We thus need to weakly enforce that HH is single-valued on each interface

\[
(\text{E}_\text{h}^\prime) \cdot \text{n} = 0, \forall F \in F_\text{h},
\]

(3a)

and that the scheme is consistent with the boundary conditions (Silver-Müller conditions for instance)

\[
(\text{E}_\text{h}^\prime \times \text{n} + \text{H}_\text{h}^\prime - \text{G}^\prime, \eta) \text{ on } \partial F = 0, \forall F \in F_\text{h},
\]

(3b)

Here Fh and Fh are the set of the interior and boundary faces of the mesh (Fh = \( F_\text{h} \cup F_\text{h} \)), [V] denotes the jump of V on any face, G is a source term provided by an incident electromagnetic wave on the boundary and the meaning of (\cdot, \cdot) comes from (1) with F replacing \( \partial K \). The test functions \( \eta \) belongs to the tangential trace space Mh:

\[
M_\text{h} = \{ \eta \in (L^2(F_\text{h}))^3 | \eta \times \text{n} \in (P_{p-1}(F))^3, (\eta \cdot \text{n}) \times \text{n} = 0, \forall F \in F_\text{h} \},
\]

(4)

with \( L^2(F_\text{h}) \) the space of square-integrable functions on Fh. The hybrid variable HH belongs to this space Mh.

Thus, from (1), we obtain disconnected local problems in each element and a reduced-size linear system can be assembled only for the hybrid variable in a classical finite element way starting from the interface conditions (3); see for instance [5] for more details. Once the linear system for the unknowns associated to the hybrid variable has been solved, the local electromagnetic field within each K can be recovered locally by solving (1).

III. PRELIMINARY NUMERICAL RESULTS

A. Accuracy of the method

We consider the propagation of a plane wave in vacuum. The computational domain is chosen to be the unit cube
\( \Omega = (-0.5:0.5)^3 \), and a Silver-Müller absorbing boundary condition is imposed on the whole boundary. Parameters \( \epsilon \), \( \mu \), and \( \gamma \) are set to be 1, and \( \omega \) to be \( 2\pi \).

A sequence of regular tetrahedral meshes is employed. Table I displays the error in \( L^2 \)-norm and the convergence orders of the approximate solutions for both the hybrid variable. We observe that the asymptotic convergence rate \( h^p \), for the HDG method and \( p=1 \). Table II yields the same characteristics for \( p=2 \). In Table I and II, "mesh size" denotes the edge length of the tetrahedrons on the edge of the unit cube and \( N_{\text{dof}} \) the number of degrees of freedom for the hybrid variable. We observe that the asymptotic convergence orders of the approximate solutions for both \( \mathbf{E} \) and \( \mathbf{H} \) are optimal, i.e., of order \( p+1 \) for both \( \mathbf{E} \) and \( \mathbf{H} \) when using polynomial order \( p \). This convergence rate has proved to be the Helmholtz equation in 2d [6] but no theoretical result is currently available for Maxwell’s equations in 3d.

### Table I. Convergence of the HDG method with \( p = 1 \)

<table>
<thead>
<tr>
<th>Mesh</th>
<th>( N_{\text{dof}} )</th>
<th>( E_\infty )</th>
<th>value</th>
<th>order</th>
<th>( E_{L^2} )</th>
<th>value</th>
<th>order</th>
<th>( H_2 )</th>
<th>value</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>780</td>
<td>2.36E-1</td>
<td>-</td>
<td></td>
<td>2.36E-1</td>
<td>-</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/8</td>
<td>39168</td>
<td>1.54E-2</td>
<td>2.0</td>
<td>1.78E-2</td>
<td>1.9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table II. Convergence of the HDG method with \( p = 2 \)

<table>
<thead>
<tr>
<th>Mesh</th>
<th>( N_{\text{dof}} )</th>
<th>( E_\infty )</th>
<th>value</th>
<th>order</th>
<th>( E_{L^2} )</th>
<th>value</th>
<th>order</th>
<th>( H_2 )</th>
<th>value</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>1440</td>
<td>3.13E-2</td>
<td>-</td>
<td></td>
<td>3.36E-2</td>
<td>-</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/8</td>
<td>78336</td>
<td>4.93E-4</td>
<td>3.0</td>
<td>4.46E-3</td>
<td>2.9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### B. Computational costs

We add a perfectly conducting cube of side length 1/3 centered at the origin in the previous problem and \( \omega \) is set to be \( 4\pi \). We can link the HDG method with a UPwind Flux (UF) DG discretization [7] and, because both approaches provide us the same solution, we can “fairly” compare their numerical costs. For each method the solution of the problem is based on a Schwarz-type domain decomposition method as described in [8]. A global iterative solver is coupled with local solvers in each subdomain of the decomposition. Several kinds of local solvers are studied but they all benefit of a single-precision LU factorization of the local problems. The timing measures for this factorization phase are given in Table IV where \( N_{\text{UF}} \) denotes the number of subdomains (one per processor). Three local solvers are considered: two Krylov subspace solvers preconditioned by the LU factorization, GMRes (DD-gmres) or BiCGStab(l) (DD-bicgl), and one direct solver using iterative refinement (DD-itref) to get the double-precision accuracy. The corresponding CPU times are summarized in Table V where \#it denotes the number of iterations for the Schwarz algorithm.

### Table V. Computing times (Schwarz algorithm).

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Strategy</th>
<th>#it</th>
<th>CPU time (min/max)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>DD-bicgl</td>
<td>10</td>
<td>85.0/200.21s</td>
</tr>
<tr>
<td>2</td>
<td>DD-gmres</td>
<td>10</td>
<td>43.15/46.22s</td>
</tr>
<tr>
<td>3</td>
<td>DD-itref</td>
<td>10</td>
<td>32.26/34.64s</td>
</tr>
<tr>
<td>4</td>
<td>DD-bicgl</td>
<td>13</td>
<td>144.7/158.89s</td>
</tr>
<tr>
<td>5</td>
<td>DD-gmres</td>
<td>13</td>
<td>77.15/82.05s</td>
</tr>
<tr>
<td>6</td>
<td>DD-itref</td>
<td>13</td>
<td>54.16/59.55s</td>
</tr>
</tbody>
</table>

## IV. CONCLUSION AND PERSPECTIVES

HDG methods lead to the same optimal accuracy as some classical DG methods but greatly reduce the computational burden in comparison. They make DG methods more attractive for stationary problems. We are currently working on efficient strategies for solving the linear systems for the hybrid variable unknowns, as well as on the further development and application of the proposed HDG method to more challenging propagation problems. Other questions need to be addressed, especially the use of a variable order \( p \) and non-conforming meshes, as it can be naturally dealt with the HDG strategy [4]. Moreover, it should also facilitate the rigorous coupling with other discretizations because the HDG approach clearly separates the local volume problems (1) and the interface conditions (3).

### REFERENCES


