On the fast Khintchine spectrum in continued fractions

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ON THE FAST KHINTCHINE SPECTRUM IN CONTINUED FRACTIONS

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ABSTRACT. For $x \in [0,1)$, let $x = [a_1(x), a_2(x), \cdots]$ be its continued fraction expansion with partial quotients $\{a_n(x), n \geq 1\}$. Let $\psi : \mathbb{N} \to \mathbb{N}$ be a function with $\psi(n)/n \to \infty$ as $n \to \infty$. In this note, the fast Khintchine spectrum, i.e., the Hausdorff dimension of the set

$$E(\psi) := \Big\{ x \in [0,1) : \lim_{n\to\infty} \frac{1}{\psi(n)} \sum_{j=1}^{n} \log a_j(x) = 1 \Big\}$$

is completely determined without any extra condition on ψ.

1. Introduction

Continued fraction expansions are induced by the Gauss transformation $T : [0,1) \to [0,1)$ given by

$$T(0) := 0, \ T(x) = \frac{1}{x} \ (\text{mod } 1), \ \text{for } x \in (0,1).$$

Let $a_1(x) = \lfloor x^{-1} \rfloor$ ($\lfloor \cdot \rfloor$ stands for the integral part) and $a_n(x) = a_1(T^{n-1}(x))$ for $n \geq 2$. Each irrational number $x \in [0,1)$ admits a unique infinite continued fraction expansion of the form

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}}. \tag{1.1}$$

Sometimes, (1.1) is written as $x = [a_1, a_2, \cdots]$. The integers a_n are called the partial quotients of x. The n-th convergent $p_n(x)/q_n(x)$ of x is given by $p_n(x)/q_n(x) = [a_1, \cdots, a_n]$.

The continued fraction is tightly connected with the classic Diophantine approximation. For example, for any $v \geq 2$, the well-known Jarník set

$$\Big\{ x : |x - p/q| < q^{-v}, \ \text{ for infinitely many } \ (p,q) \in \mathbb{Z}^2 \Big\}$$

is equal to J_{v-2}, where for any $\beta > 0$, the set J_β is defined by continued fractions as

$$J_\beta := \Big\{ x : a_{n+1}(x) \geq q_n(x)^\beta, \ \text{ for infinitely many } \ n \in \mathbb{N} \Big\}.$$

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The Gauss transformation is identified with an infinite symbolic dynamical system if we consider the partial quotients as symbols. The appearence of infinite symbols brings us new phenomena in relative to the case of finite symbols. For example, consider the set

$$\left\{x \in [0,1) : \mathbb{A}\left\{ \frac{1}{n}\sharp\{1 \le j \le n : a_j(x) = 1\}\right\}_{n\ge 1} = [0,1]\right\}$$

where $\mathbb{A}(E)$ denotes the set of the accumulation points of a set E. The Hausdorff dimension of this set is $1/2$ (see [9]), while in b-adic expansion a similar set is of Haudorff dimension 0 (see [12]). Another example is that the multifractal spectrum of the level sets of the Khintchine constant

$$\left\{x \in [0,1) : \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log a_j(x) = \xi\right\}$$

is neither concave nor convex [3]. Because of the difference from the finite symbolic dynamical systems and of the observed new phenomena, continued fractions attracted much attention. One can find rich properties of the continued fraction dynamical system in [2, 3, 5, 6, 7, 10, 11, 14] and related works therein.

Let $\psi : \mathbb{N} \to \mathbb{N}$. Define

$$E(\psi) = \left\{x \in [0,1) : \lim_{n\to\infty} \frac{\log a_1(x) + \cdots + \log a_n(x)}{\psi(n)} = 1\right\}.$$

When $\psi(n) = \lambda n$ for some $\lambda > 0$, the set $E(\psi)$ is a level set of the classic Khintchine constant. Besides a detailed spectrum analysis of the classic Khintchine constant in [3], the authors also studied the fast Khintchine spectrum, i.e. the Hausdorff dimension of $E(\psi)$ when $\psi(n)/n \to \infty$ as $n \to \infty$. But the result for the latter case is incomplete. Only under the conditions that $\lim_{n\to\infty} \frac{\psi(n+1)}{\psi(n)} = b$ and $\lim_{n\to\infty}(\psi(n) - \psi(n-1)) = \infty$, the dimension of $E(\psi)$ was given [3]. In this note, we show that these extra conditions are unnecessary for determining the dimension of $E(\psi)$ in the case of fast Khintchine spectrum.

Two functions ψ and $\tilde{\psi}$ defined on $\mathbb{N}$ are said to be *equivalent* if $\frac{\psi(n)}{\tilde{\psi}(n)} \to 1$ as $n \to \infty$.

Theorem 1.1. *Let $\psi : \mathbb{N} \to \mathbb{N}$ with $\psi(n)/n \to \infty$ as $n \to \infty$. If ψ is equivalent to an increasing function, then $E(\psi) \neq \emptyset$ and*

$$\dim_H E(\psi) = \frac{1}{1+b}, \quad \text{with } b = \limsup_{n\to\infty} \frac{\psi(n+1)}{\psi(n)}.$$

Otherwise, $E(\psi) = \emptyset$.

Remark 1. The method used in [3] does not apply to general ψ. This is explained in Section 3 below.

Remark 2. The upper bound of $\dim_H E(\psi)$ is the difficult part of the proof of Theorem 1.1. As a byproduct of the proof, we get that for any $\beta > 0$, the Hausdorff dimension of the set

$$J_\beta^* := \Big\{x \in J_\beta : \lim_{n\to\infty} \frac{\log q_n(x)}{n} = \infty\Big\}$$

is $1/(2+\beta)$, i.e. one half of the dimension of the Jarník set J_β. A detailed explanation is given at the end of this paper.

2. Preliminary

This section is devoted to fixing some notation, recalling some elementary properties enjoyed by continued fractions and citing some technical lemmas in dimension estimation.

Throughout this paper, we use $\lfloor\cdot\rfloor$ to denote the integral part of a real number, $|A|$ the diameter of a set $A \subset \mathbb{R}$, $\mathcal{H}^s$ the s-dimensional Hausdorff measure, and $\dim_H$ the Hausdorff dimension of a subset of $[0,1)$.

Recall that for any irrational number $x \in [0,1)$, $p_n(x)$ and $q_n(x)$ are the numerator and denominator of the n-th convergent of x. It is known that $p_n = p_n(x)$ and $q_n = q_n(x)$ can be obtained recursively by the following relations.

$$p_n = a_n(x)p_{n-1} + p_{n-2}, \quad q_n = a_n(x)q_{n-1} + q_{n-2} \tag{2.1}$$

with the conventions $p_0 = q_{-1} = 0$ and $p_{-1} = q_0 = 1$. For each $n \geq 1$,

$$p_{n-1}q_n - p_nq_{n-1} = (-1)^n. \tag{2.2}$$

For any $n \geq 1$ and $(a_1, a_2, \cdots, a_n) \in \mathbb{N}^n$, define

$$I_n(a_1, a_2, \cdots, a_n) = \{x \in [0,1) : \ a_1(x) = a_1, \cdots, a_n(x) = a_n\},$$

which is the set of points beginning with $(a_1, \cdots, a_n)$ in their continued fraction expansions, and is called a *cylinder* of order n.

Note that p_n and q_n are determined by the first n partial quotients of x. So all points in $I_n(a_1, \cdots, a_n)$ determine the same p_n and q_n. Hence sometimes, we write $p_n = p_n(a_1, \cdots, a_n)$ and $q_n = q_n(a_1, \cdots, a_n)$ to denote $p_n(x)$ and $q_n(x)$ for $x \in I_n(a_1, \cdots, a_n)$.

Proposition 2.1 ([8])**.** *For any $n \geq 1$ and $(a_1, \cdots, a_n) \in \mathbb{N}^n$, let q_n be given recursively by (2.1). The cylinder $I_n(a_1, \cdots, a_n)$ is an interval with the endpoints p_n/q_n and $(p_n + p_{n-1})/(q_n + q_{n-1})$. Then*

$$\frac{1}{2q_n^2} \leq \Big|I_n(a_1, \cdots, a_n)\Big| = \frac{1}{q_n(q_n + q_{n-1})} \leq \frac{1}{q_n^2}. \tag{2.3}$$

For each $n \geq 1$, $q_n(a_1, \cdots, a_n) \geq 2^{(n-1)/2}$ and

$$\prod_{k=1}^n a_k \leq q_n(a_1, \cdots, a_n) \leq 2^n \prod_{k=1}^n a_k. \tag{2.4}$$

Now we mention some known results concerning the dimension of sets in continued fractions. Let $\{s_n\}_{n\geq 1}$ be a sequence of integers and $\ell \geq 2$ be some fixed integer. Set

$$F(\{s_n\}_{n=1}^{\infty};\ell) = \{x \in [0,1) : s_n \leq a_n(x) < \ell s_n, \text{ for all } n \geq 1\}.$$

Lemma 2.2 ([3]). *Under the assumption that* $\frac{1}{n}\sum_{k=1}^{n} s_k \to \infty$ *as* $n \to \infty$, *one has*

$$\dim_H F(\{s_n\}_{n=1}^{\infty};\ell) = \liminf_{n\to\infty} \frac{\log(s_1 s_2 \cdots s_n)}{2\log(s_1 s_2 \cdots s_n) + \log s_{n+1}}.$$

Lemma 2.3 ([3]).

$$\dim_H \left\{x \in [0,1) : \limsup_{n\to\infty} \frac{\log q_n(x)}{n} = \infty\right\} = \frac{1}{2}.$$

3. Proof of Theorem 1.1

Notice that $E(\psi) = E(\tilde{\psi})$ if ψ and $\tilde{\psi}$ are equivalent. We can assume that ψ is increasing because of the following simple lemma.

Lemma 3.1. *The set* $E(\psi) \neq \emptyset$ *if and only if* ψ *is equivalent to an increasing function.*

Proof. If $E(\psi)$ is nonempty, take an $x_0 \in E(\psi)$. Then put

$$\tilde{\psi}(n) = \Big\lfloor \log a_1(x_0) + \cdots + \log a_n(x_0) \Big\rfloor$$

for all $n \geq 1$. Clearly $\tilde{\psi}$ is increasing. The functions ψ and $\tilde{\psi}$ are equivalent.

On the other hand, if ψ is increasing, we have a point $x \in E(\psi)$ such that for each $n \geq 1$

$$a_n(x) = \lfloor e^{\psi(n)-\psi(n-1)+1} \rfloor.$$

□

Now we can proceed the proof of Theorem 1.1 with the assumption that ψ is increasing.

• Lower bound. Apply Lemma 2.2 to $s_n = \lfloor e^{\psi(n)-\psi(n-1)} \rfloor$ and $\ell = 2$. Let

$$F = \left\{x \in [0,1) : \left\lfloor e^{\psi(n)-\psi(n-1)} \right\rfloor \leq a_n(x) < 2\left\lfloor e^{\psi(n)-\psi(n-1)} \right\rfloor, \text{ for all } n \geq 1\right\}$$

which is subset of $E(\psi)$. We get immediately that

$$\dim_H E(\psi) \geq \frac{1}{1+b}.$$

• Upper bound. This is the main part of the proof.

Let us first recall the method used in [3] under the extra condition that $\lim_{n\to\infty} \frac{\psi(n+1)}{\psi(n)} = b \geq 1$. Especially when $b > 1$, we constructed a set containing $E(\psi)$ by posing precise restrictions on each partial quotients, namely

$$\left\{x \in [0,1) : e^{L_n} \leq a_n(x) \leq e^{M_n}, \text{ when } n \gg 1\right\}, \tag{3.1}$$

where (with a small $\epsilon > 0$)

$$L_n = \frac{\psi(n)}{1+\epsilon} - \frac{\psi(n-1)}{1-\epsilon} \quad \text{and} \quad M_n = \frac{\psi(n)}{1-\epsilon} - \frac{\psi(n-1)}{1+\epsilon}.$$

By a standard covering argument, together with $\lim_{n\to\infty} \frac{\psi(n+1)}{\psi(n)} = b$, we get the exact upper bound of the dimension of $E(\psi)$. But as far as a general function ψ is concerned, the above argument fails. For example, take

$$\psi(n) = (k+2)!, \text{ when } k! \le n < (k+1)!.$$

Then the set in (3.1) reads as

$$\left\{ x \in [0,1) : \begin{cases} e^{c_1(k+2)!} \le a_n(x) \le e^{c_2(k+2)!}, & \text{when } n = k!; \\ 1 \le a_n(x) \le e^{c_3(k+2)!}, & \text{when } k! < n < (k+1)!. \end{cases} \right\}$$

for suitably chosen constants c_1, c_2, c_3. According to Lemma 2.2, this set has Hausdorff dimension $\ge 1/2$. However, the dimension of $E(\psi)$ is equal to zero by Theorem 1.1.

Now we are going to prove the upper bound of $\dim_H E(\psi)$ for a general function ψ. Since ψ is increasing, we always have $b \ge 1$. We distinguish two cases: $b = 1$ and $b > 1$.

Case 1. $b = 1$. Lemma 2.3 serves for this case. According to the estimation (2.4), since $\psi(n)/n \to \infty$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{\log q_n(x)}{\psi(n)} = \lim_{n\to\infty} \frac{\log a_1(x) + \cdots + \log a_n(x)}{\psi(n)}.$$

Thus Lemma 2.3 gives us

$$\dim_H E(\psi) \le \frac{1}{2} = \frac{1}{1+b}.$$

Case 2. $b > 1$. Fix an $\epsilon > 0$. Choose a sequence of integers $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ with n_1 large enough and for each $k \ge 1$ one has

$$\psi(n_k + 1) \ge \psi(n_k) b(1-\epsilon), \quad n_k \le \epsilon\psi(n_k). \tag{3.2}$$

For each $N \ge 1$, let

$$E_N(\psi) = \Big\{ x \in [0,1) : (1-\epsilon) < \frac{1}{\psi(n)} \sum_{j=1}^{n} \log a_j(x) < (1+\epsilon), \ \forall\, n \ge N \Big\}.$$

Then

$$E(\psi) \subset \bigcup_{N\ge1} E_N(\psi).$$

To estimate the dimension of $E_N(\psi)$ for $N \ge 1$, we proceed in three steps.

Step i. **Find a cover of** $E_N(\psi)$. For any $n \ge N$, set

$$D_n(\epsilon) = \Big\{ (a_1, \cdots, a_n) \in \mathbb{N}^n : (1-\epsilon) < \frac{1}{\psi(n)} \sum_{j=1}^{n} \log a_j < (1+\epsilon) \Big\}. \tag{3.3}$$

For every $(a_1, \cdots, a_n) \in D_n(\epsilon)$, we define

$$D_{n+1}\Big(\epsilon; (a_1, \cdots, a_n)\Big) = \Big\{ a_{n+1} \in \mathbb{N} : (a_1, \cdots, a_n, a_{n+1}) \in D_{n+1}(\epsilon) \Big\}.$$

Clearly, by the definition of $D_n(\epsilon)$, we have

$$E_N(\psi) \subset \bigcap_{n=N}^{\infty} \mathfrak{D}_n(\epsilon), \text{ with } \mathfrak{D}_n(\epsilon) = \bigcup_{(a_1,\cdots,a_n)\in D_n(\epsilon)} I_n(a_1,\cdots,a_n). \tag{3.4}$$

Now instead of considering the intersections in (3.4) from $n = N$ until $n = \infty$, we only consider the intersection of two consecutive terms. Namely, for any $n \geq N$,

$$E_N(\psi) \subset \Big(\mathfrak{D}_n(\epsilon) \cap \mathfrak{D}_{n+1}(\epsilon)\Big) = \bigcup_{(a_1,\cdots,a_n)\in D_n(\epsilon)} J_n(a_1,\cdots,a_n),$$

where

$$J_n(a_1,\cdots,a_n) = \bigcup_{a_{n+1}\in D_{n+1}(\epsilon;(a_1,\cdots,a_n))} I_{n+1}(a_1,\cdots,a_n,a_{n+1}).$$

Hence, for each $n \geq N$, we get a cover of $E_N(\psi)$:

$$\Big\{J_n(a_1,\cdots,a_n) : (a_1,\cdots,a_n) \in D_n(\epsilon)\Big\}. \tag{3.5}$$

Thus the s-dimensional Hausdorff measure of $E_N(\psi)$ can be estimated as

$$\mathcal{H}^s(E_N(\psi)) \leq \liminf_{n\to\infty} \sum_{(a_1,\cdots,a_n)\in D_n(\epsilon)} \Big|J_n(a_1,\cdots,a_n)\Big|^s. \tag{3.6}$$

As we shall see, $J_n(a_1,\cdots,a_n)$ is a union of cylinders of order $(n+1)$, say $I_{n+1}(a_1,\cdots,a_n,a_{n+1})$ with a taking large values (Lemma 3.3). Using this fact, the length of $J_n(a_1,\cdots,a_n)$ will be well estimated.

Step ii. **Lengths of** $J_n(a_1,\cdots,a_n)$. We begin with a fact on $D_{n+1}(\epsilon; a_1,\cdots,a_n)$.

Lemma 3.2. *For each* $(a_1,\cdots,a_n) \in D_n(\epsilon)$,

$$D_{n+1}\Big(\epsilon;(a_1,\cdots,a_n)\Big) \neq \emptyset.$$

Proof. This follows from the following simple constructions.

(a) If $\sum_{j=1}^{n} \log a_j > (1-\epsilon)\psi(n+1)$, we choose $a_{n+1} = 1$.

(b) If $\sum_{j=1}^{n} \log a_j \leq (1-\epsilon)\psi(n+1)$, we can choose

$$a_{n+1} = \Big\lfloor \frac{e^{\psi(n+1)}}{a_1\cdots a_n} \Big\rfloor.$$

□

Recall that the sequence of integers $\{n_k\}_{k\geq 1}$ is given in (3.2).

Lemma 3.3. *For any* $(a_1,\cdots,a_{n_k}) \in D_{n_k}(\epsilon)$ *and* $a_{n_k+1} \in D_{n_k+1}(\epsilon,(a_1,\cdots,a_{n_k}))$, *we have*

$$\log a_{n_k+1} \geq (1-\epsilon)\Big(\frac{b(1-\epsilon)^2}{1+\epsilon} - 1\Big)\log q_{n_k} =: \beta \log q_{n_k}, \tag{3.7}$$

Proof. By the definitions of $D_n(\epsilon)$ and the first inequality in (3.2), for any $(a_1, \cdots, a_{n_k}) \in D_{n_k}(\epsilon)$ and $a_{n_k+1} \in D_{n_k+1}(\epsilon, (a_1, \cdots, a_{n_k}))$, one has

$$\sum_{j=1}^{n_k+1} \log a_j \geq \psi(n_k+1)(1-\epsilon) \geq \psi(n_k) b (1-\epsilon)^2$$

$$\geq \frac{b(1-\epsilon)^2}{1+\epsilon} \sum_{j=1}^{n_k} \log a_j. \tag{3.8}$$

On the other hand, by (2.4) and the second inequality in (3.2), we get

$$q_{n_k}(a_1, \cdots, a_{n_k}) \leq 2^{n_k} \prod_{j=1}^{n_k} a_j \leq \left(\prod_{j=1}^{n_k} a_j \right)^{\frac{1}{1-\epsilon}}. \tag{3.9}$$

Combining (3.8) and (3.9), we obtain the desired result. □

Now return back to the cover of $E_N(\psi)$ given in (3.5) especially when $n = n_k$. We estimate the length of $J_{n_k}(a_1, \cdots, a_{n_k})$ for every $(a_1, \cdots, a_{n_k}) \in D_{n_k}(\epsilon)$. For $n = n_k$, by (3.7) and Proposition 2.1, we have

$$\left| J_n(a_1, \cdots, a_n) \right| \leq \sum_{a : a \geq q_n^\beta} \left| \frac{a \cdot p_n + p_{n-1}}{a \cdot q_n + q_{n-1}} - \frac{(a+1)p_n + p_{n-1}}{(a+1)q_n + q_{n-1}} \right|.$$

By (2.2), for all $a \in \mathbb{N}$, the differences appearing in the series have the same sign depending only the parity of n. Thus the series is telescopic. Since $\frac{(a+1)p_n+p_{n-1}}{(a+1)q_n+q_{n-1}}$ tends to p_n/q_n as $a \to \infty$, we get

$$\left| J_n(a_1, \cdots, a_n) \right| \leq \left| \frac{q_n^\beta p_n + p_{n-1}}{q_n^\beta q_n + q_{n-1}} - \frac{p_n}{q_n} \right| = \frac{1}{(q_n^\beta q_n + q_{n-1}) q_n} \leq \frac{1}{q_n^{2+\beta}}.$$

Consider the liminf in (3.6) along the subsequence $\{n_k\}_{k \geq 1}$, then we obtain

$$\mathcal{H}^s(E_N(\psi)) \leq \liminf_{k \to \infty} \sum_{(a_1, \cdots, a_{n_k}) \in D_{n_k}(\epsilon)} \left(\frac{1}{q_{n_k}} \right)^{s(2+\beta)}. \tag{3.10}$$

The last step is devoted to estimating the summation in (3.10) under a suitable choice of s.

Step iii. **Bernoulli measures**. A family of measures μ_t defined on cylinders is constructed firstly. For each $t > 1$ and for any $(a_1, \cdots, a_n) \in \mathbb{N}^n$, set

$$\mu_t(I_n(a_1, \cdots, a_n)) = e^{-nP(t) - t \sum_{j=1}^n \log a_j}, \tag{3.11}$$

where $e^{P(t)} = \zeta(t) = \sum_{k=1}^{\infty} k^{-t}$. By Kolmogorov's consistency theorem, μ_t can be extended into a probability measure on $[0, 1)$.

Fix $\epsilon > 0$. By the assumption that $\lim_{n \to \infty} \psi(n)/n = \infty$, one can choose some integer $N(\epsilon) \in \mathbb{N}$ such that for all $n \geq N(\epsilon)$,

$$nP(1 + \frac{\epsilon}{2}) \leq \frac{\epsilon}{2}(1-\epsilon)\psi(n). \tag{3.12}$$

We claim that for each $n \geq N(\epsilon)$ and $(a_1, \cdots, a_n) \in D_n$,

$$q_n^{-(1+\epsilon)} \leq \mu_{(1+\epsilon/2)}(I_n(a_1, \cdots, a_n)). \tag{3.13}$$

More precisely, for any $(a_1, \cdots, a_n) \in D_n$, by (3.3) and (3.12), we have

$$\frac{\epsilon}{2} \sum_{j=1}^{n} \log a_j \geq nP(1 + \frac{\epsilon}{2}). \tag{3.14}$$

Thus by (2.4) and then (3.14), we get

$$q_n^{-(1+\epsilon)} \leq e^{-(1+\epsilon) \sum_{j=1}^n \log a_j} \leq e^{-nP(1+\frac{\epsilon}{2})-(1+\frac{\epsilon}{2}) \sum_{j=1}^n \log a_j}.$$

Choose $s = \frac{1+\epsilon}{2+\beta}$ in (3.10). By (3.13), we have

$$\mathcal{H}^{\frac{1+\epsilon}{2+\beta}}(E_N(\psi)) \leq \liminf_{k\to\infty} \sum_{(a_1,\cdots,a_{n_k})\in D_{n_k}(\epsilon)} \mu_{(1+\epsilon/2)}(I_{n_k}(a_1, \cdots, a_{n_k})) \leq 1.$$

Hence

$$\dim_H E(\psi) \leq \sup_{N\geq 1} \Big\{ \dim_H E_N(\psi) \Big\} \leq \frac{1+\epsilon}{2+\beta}.$$

Then the desired result follows by letting $\epsilon \to 0$. □

Final remark: Now we give a remark on the dimension of J_β^* and that of J_β. Recall that J_β^* and J_β are defined in Section 1. For any $(a_1, \cdots, a_n) \in \mathbb{N}^n$, we define

$$\tilde{J}_n(a_1, \cdots, a_n) = \bigcup_{a_{n+1}\geq q_n^\beta} I_{n+1}(a_1, \cdots, a_n, a_{n+1}).$$

Then it is clear that

$$J_\beta = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup \tilde{J}_n(a_2, \cdots, a_n),$$

where the last union is taken over all $(a_1, \cdots, a_n) \in \mathbb{N}^n$. While

$$J_\beta^* \subset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup \tilde{J}_n(a_2, \cdots, a_n),$$

where the last union is taken over all $(a_1, \cdots, a_n) \in \mathbb{N}^n$ with $\frac{\log q_n}{n}$ being sufficiently large. As a result,

$$\mathcal{H}^s(J_\beta) \leq \liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{(a_1,\cdots,a_n)\in\mathbb{N}^n} \left(\frac{1}{q_n}\right)^{s(2+\beta)},$$

while

$$\mathcal{H}^s(J_\beta^*) \leq \liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{(a_1,\cdots,a_n)\in\mathbb{N}^n,(\log q_n)/n \text{ large}} \left(\frac{1}{q_n}\right)^{s(2+\beta)}.$$

By (2.3), we know that

$$1 \leq \sum_{a_1,\cdots,a_n\in\mathbb{N}^n} q_n^{-2} \leq 2. \tag{3.15}$$

While, by (3.13), we get

$$\sum_{a_1,\cdots,a_n:\log q_n/n \text{ large}} {q_n}^{-(1+\epsilon)} \le 1. \tag{3.16}$$

Comparing of (3.15) and (3.16) reveals that

$$\dim_H J_\beta \le \frac{2}{2+\beta}, \quad \dim_H J_\beta^* \le \frac{1}{2+\beta}.$$

Actually we have proven that $\dim_H J_\beta^* = \frac{1}{2+\beta}$ since $E(\psi)$ can serve as a subset of J_β^*.

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