Explicit computation of the electrostatic energy for an elliptical charged disc.

Sophie Laurens, Sébastien Tordeux

Explicit computation of the electrostatic energy for an elliptical charged disc

S. Laurens[a,b], S. Tordeux[c]

[a] *Mathematical Institute of Toulouse, 118 route de Narbonne, F-31400 Toulouse*
[b] *CERFACS, 42 Avenue Gaspard Coriolis F-31100 Toulouse*
[c] *INRIA Bordeaux Sud-Ouest-LMA, avenue de l'Université, F-64013 Pau*

Abstract

This letter describes a method for obtaining an explicit expression for the electrostatic energy of a charged elliptical infinitely thin disc. The charge distribution is assumed to be polynomial. Such explicit values for this energy are fundamental for assessing the accuracy of boundary element codes. The main tools used are an extension of Copson's method and a diagonalization, given by Leppington and Levine, of the single-layer potential operator associated with the electrostatic potential created by a distribution of charges on the elliptical disc.

1. Introduction

In recent years, integral equations have become an essential tool for solving both industrial and scientific problems in electromagnetism and acoustics. The assessment of the accuracy delivered by such codes, in particular in their handling of the singular integrals involved, is a major issue. Here, we present a method for deriving an analytical expression for the electrostatic energy of a charged elliptical infinitely thin plate, providing a means for the validation of these codes.

Let us denote by $\mathsf{A} = \big\{(x_1, x_2) \in \mathbb{R}^2 \text{ with } x_1^2/a^2 + x_2^2/b^2 - 1 < 0 \text{ and } a > b\big\}$ the ellipse with major and minor semi-axes a and b. Let f be the electrostatic potential generated by a density of charges σ distributed over A:

$$f(\mathbf{x}) = \frac{1}{4\pi}\int_{\mathsf{A}} \frac{\sigma(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} ds_{\mathbf{y}} \qquad \text{for all } \mathbf{x} \text{ in } \mathsf{A}, \tag{1}$$

with $\mathbf{x} = (x_1, x_2)$. The units have been chosen such that the electric permittivity of air is 1. The electrostatic energy I can be expressed in either the two following forms:

$$I_\sigma = \int_{\mathsf{A}} f(\mathbf{x})\,\overline{\sigma(\mathbf{x})} ds_{\mathbf{x}} = \int_{\mathsf{A}} \frac{1}{4\pi}\int_{\mathsf{A}} \frac{\overline{\sigma(\mathbf{x})}\,\sigma(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\, ds_{\mathbf{x}} ds_{\mathbf{y}}. \tag{2}$$

Email address: sophie.laurens@insa-toulouse.fr (S. Laurens)

We aim in this letter at proving and numerically illustrating the following theorem, where ε is the eccentricity of the ellipse A given by $\varepsilon = \sqrt{1 - b^2/a^2}$.

Theorem 1.1. *Let $\sigma(\mathbf{x}) = \alpha_0 + \alpha_1 x_1/a + \alpha_2 x_2/b$, with $\alpha \in \mathbb{R}^3$, be the distribution of charges over A. The corresponding electrostatic energy is given by*

$$I_\sigma = \frac{8ab^2}{15\pi}\left[\left(5\alpha_0^2 + \alpha_2^2\right) K(\varepsilon) + \left(\alpha_1^2 - \alpha_2^2\right) \frac{K(\varepsilon) - E(\varepsilon)}{\varepsilon^2}\right],$$

with $K(\varepsilon)$ and $E(\varepsilon)$ the complete elliptic integrals of the first and second kind

$$K(\varepsilon) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \varepsilon^2 \sin^2 \phi}} \quad \text{and} \quad E(\varepsilon) = \int_0^{\pi/2} \sqrt{1 - \varepsilon^2 \sin^2 \phi}\, d\phi. \tag{3}$$

2. Diagonalization of the electrostatic energy

Following [1], we consider the spheroidal coordinate system (θ, φ) giving a parametrization of A in terms of the unit half-sphere

$$x_1 = a \sin\theta \cos\varphi \;\text{ and }\; x_2 = b \sin\theta \sin\varphi, \;\text{ with }\; \theta \in [0, \pi/2],\ \varphi \in [0, 2\pi]. \tag{4}$$

The elemental area associated with the new variables is $ab \cos\theta \sin\theta\, d\theta d\varphi$. In these spheroidal coordinates, the electrostatic potential f defined in (1) and the electrostatic energy can also be written in terms of θ and φ as

$$\begin{cases} f(\theta, \varphi) = \dfrac{ab}{4\pi} \displaystyle\int_0^{\pi/2} \int_0^{2\pi} \frac{g(\theta', \varphi')}{d(\theta, \varphi, \theta', \varphi')} \sin\theta'\, d\theta' d\varphi', \\ I_\sigma = ab \displaystyle\int_0^{\pi/2} \int_0^{2\pi} \overline{f(\theta, \varphi)} g(\theta, \varphi) \sin\theta\, d\theta d\varphi, \end{cases}$$

with $g(\theta, \varphi) = \sigma(\theta, \varphi)\ \cos\theta$ and $d(\theta, \varphi, \theta', \varphi')$ the distance separating $\mathbf{x}$ from $\mathbf{y}$ $d = |\mathbf{x} - \mathbf{y}|$ expressed in the spheroidal coordinates (4).
The next step consists in introducing a well chosen spectral basis for the half-sphere involving the even Legendre functions Q_n^m normalized by

$$\int_0^{\pi} Q_n^m(\cos\theta)\, Q_{n'}^m(\cos\theta) \sin\theta d\theta = \delta_{n,n'}. \tag{5}$$

This basis yields a block diagonalization of the convolution operator (see [1])

$$\frac{1}{d} = \frac{1}{\sqrt{ab}} \sum_{n=0}^{\infty} \sum_{\substack{m=-n \\ n-m \text{ even}}}^{n} \sum_{\substack{m'=-n \\ n-m' \text{ even}}}^{n} d_{mm'}^n Q_n^m(\cos\theta)\, Q_n^{m'}(\cos\theta')\, e^{i\left(m\varphi - m'\varphi'\right)},$$

$$\text{with } d_{mm'}^n = \frac{Q_n^m(0)\, Q_n^{m'}(0)}{2n+1} \int_0^{2\pi} \frac{e^{i(m-m')\varphi}}{\sqrt{\frac{b}{a}\cos^2\varphi + \frac{a}{b}\sin^2\varphi}} d\varphi. \tag{6}$$

The functions f and g can be expanded in this basis as

$$u(\theta,\varphi) = \sum_{n=0}^{\infty} \sum_{\substack{m=-n \\ n-m \text{ even}}}^{n} u_n^m Q_n^m(\cos\theta)\, e^{im\varphi} \quad \text{with } u = f \text{ or } g \tag{7}$$

with

$$u_n^m = \frac{1}{\pi}\int_0^{\pi/2}\int_0^{2\pi} u(\theta,\varphi)\, Q_n^m(\cos\theta)\, e^{-im\varphi} \sin(\theta) d\theta d\varphi. \tag{8}$$

Due to (6), coefficients f_n^m are related to g_n^m by

$$f_n^m = \frac{\sqrt{ab}}{4} \sum_{\substack{m'=-n \\ n-m' \text{ even}}}^{n} d_{mm'}^n g_{n'}^{m'}. \tag{9}$$

Moreover, the orthogonal properties of the spectral basis yield

$$I_\sigma = \frac{\pi}{4}(ab)^{3/2} \sum_{n,m,m'} d_{mm'}^n g_n^m \overline{g_n^{m'}},$$

where we have lightened the notation by making the range of the summation index implicit. Indices n, n' are varying from 0 to ∞, and m, m' are such that $|m| \leq n$, $|m'| \leq n'$, with $n-m$ and $n-m'$ even. Substituting expression (6) for $d_{mm'}^n$ and introducing the eccentricity of the ellipse ε, we get

$$I_\sigma = \pi\, ab^2 \sum_{n,m,m'} g_n^m\, \overline{g_n^{m'}}\, \frac{Q_n^m(0)\, Q_n^{m'}(0)}{2n+1} \int_0^{\pi/2} \frac{\cos(m-m')\varphi}{\sqrt{1-\varepsilon^2\cos^2\varphi}} d\varphi. \tag{10}$$

3. Electrostatic energy for an affine distribution of charges

This section is dedicated to the calculation of the electrostatic energy generated by an affine density of charges.

Proof of Theorem 1.1. Let $\sigma_0(\mathbf{x}) = 1$, $\sigma_1(\mathbf{x}) = x_1/a$ and $\sigma_2(\mathbf{x}) = x_2/b$. In view of the symmetry of A with respect to x_1 and x_2, we have

$$\int_{\mathsf{A}} \frac{1}{4\pi} \int_{\mathsf{A}} \frac{\sigma_i(\mathbf{x})\sigma_j(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\, ds_{\mathbf{x}} ds_{\mathbf{y}} = 0 \qquad \text{for } i \neq j.$$

Consequently, the electrostatic energy I_σ can be expanded as

$$I_\sigma = \alpha_0^2\, I_{\sigma_0} + \alpha_1^2\, I_{\sigma_1} + \alpha_2^2\, I_{\sigma_2}$$

The result will follow from the computation of I_{σ_0}, I_{σ_1} and I_{σ_2}.

3.1. Computation of I_{σ_0}

For $\sigma(\mathbf{x}) = \sigma_0(\mathbf{x}) = 1$, the function $g(\theta,\varphi) = \cos\theta$ does not depend on φ. The g_n^m coefficients are independent from a and b, and since $g_n^m = 0$ for all

$m \neq 0$, the function g can be expanded as $g(\theta) = \sum_{n=0}^{+\infty} g_n^0 Q_n^0 (\cos\theta)$. Due to (10), the electrostatic energy I_{σ_0} depends only on a and b and is given by

$$I_{\sigma_0}(a,b) = \pi\, ab^2 \sum_{n=0}^{+\infty} \frac{|g_n^0|^2 \, \left(Q_n^0\,(0)\right)^2}{2n+1} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-\varepsilon^2\cos^2\varphi}} = \kappa\, ab^2\, K(\varepsilon), \quad (11)$$

with $\varepsilon = \sqrt{1-b^2/a^2}$ and κ a constant depending neither on a nor on b. The constant κ is deduced from the classical case of an unit circle which has been detailed for example in [2]

$$I_{\sigma_0}(1,1) \;=\; \int_{\mathsf{C}} \frac{1}{4\pi} \int_{\mathsf{C}} \frac{1}{|\mathbf{x}-\mathbf{y}|}\, ds_{\mathbf{x}} ds_{\mathbf{y}} = 4/3. \quad (12)$$

Comparing (11) and (12), this yields to $\kappa = 8/3\pi$ since $K(0) = \pi/2$. Therefore,

$$I_{\sigma_0} = \frac{8}{3\pi}\, ab^2\, K(\varepsilon). \quad (13)$$

3.2. Computation of I_{σ_1}

For $\sigma(\mathbf{x}) = \sigma_1(\mathbf{x}) = x_1/a$, the function g is given by $g(\theta,\varphi) = \sin\theta\cos\theta\cos\varphi$. In that case, the g_n^m are zero except for $|m| = 1$. By definition of the Legendre functions, we have $Q_n^{-1} = -Q_n^1$. As the function g is even, it emerges that $g_n^{-1} = -g_n^1$, and thus

$$I_{\sigma_1} = \pi ab^2 \sum_n \frac{\left(g_n^1 Q_n^1(0)\right)^2}{2n+1} \left[2\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-\varepsilon^2\cos^2\varphi}} + 2\int_0^{\pi/2} \frac{\cos 2\varphi\;d\varphi}{\sqrt{1-\varepsilon^2\cos^2\varphi}} \right].$$

Due to (3), it emerges that

$$I_{\sigma_1}(a,b) = \kappa\, ab^2\, \frac{K(\varepsilon)-E(\varepsilon)}{\varepsilon^2}. \quad (14)$$

To determine the constant κ, we consider again the case of an unit circle. In this case, I_{σ_1} can be explicitly computed (see the Appendix A), and is given by $I_{\sigma_1}(1,1) = 2/15$. Evaluating (14) at $a = b = 1$ we get

$$I_{\sigma_1}(1,1) = \kappa\frac{\pi}{4} \quad \text{since } \lim_{\varepsilon\to 0} \frac{K(\varepsilon)-E(\varepsilon)}{\varepsilon^2} = \frac{\pi}{4}. \quad (15)$$

It follows that $\kappa = 8/15\pi$ and therefore we have

$$I_{\sigma_1} = \frac{8}{15\pi}\, ab^2\, \frac{K(\varepsilon)-E(\varepsilon)}{\varepsilon^2}. \quad (16)$$

3.3. Computation of I_{σ_2}

For I_{σ_2}, we consider $\sigma(\mathbf{x}) = \sigma_2(\mathbf{x}) = x_2/b$, meaning that $g(\theta,\varphi) = \sin\theta\cos\theta\sin\varphi$. We still have $g_n^m = 0$ except for $|m| = 1$, but in that case, $g_n^{-1} = g_n^1$. Thus

$$I_{\sigma_2} = \pi a b^2 \sum_{n=0}^{+\infty} \frac{\left(g_n^1 Q_n^1(0)\right)^2}{2n+1} \left[2\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-\varepsilon^2\cos^2\varphi}} - 2\int_0^{\pi/2} \frac{\cos 2\varphi \ d\varphi}{\sqrt{1-\varepsilon^2\cos^2\varphi}} \right]$$

Moreover, both integrals I_{σ_1} and I_{σ_2} are equal on C by symmetry. We obtain

$$I_{\sigma_2} = \frac{8}{15\pi}\, a b^2 \left(K(\varepsilon) - \frac{K(\varepsilon) - E(\varepsilon)}{\varepsilon^2} \right). \tag{17}$$

4. Numerical tests and conclusion

Tables 1 and 2 give a comparison of the exact values given by an analytical expression with numerical approximate values obtained by the boundary element code CESC of CERFACS with P^1 continuous elements. It can be observed that the two values coincide at least up to the fourth decimal digit. Table 3 shows the maximum relative error for each of the cases of Tables 1 and 2 cases, which is less than 0.35 per mil.

a	0.5	0.7	0.9	1.1	1.3	1.5
$I_{\sigma_0}^{\text{comp}}$	0.1666	0.2741	0.3939	0.5234	0.6608	0.8048
$I_{\sigma_0}^{\text{exact}}$	0.1666	0.2741	0.3939	0.5234	0.6608	0.8048
$I_{\sigma_1}^{\text{comp}} \times 10^{-1}$	0.0417	0.1455	0.3651	0.7535	1.3715	2.2781
$I_{\sigma_1}^{\text{exact}} \times 10^{-1}$	0.0417	0.1456	0.3656	0.7543	1.3717	2.2800
$I_{\sigma_2}^{\text{comp}} \times 10^{-2}$	0.4167	0.6280	0.8426	1.0585	1.2748	1.4910
$I_{\sigma_2}^{\text{exact}} \times 10^{-2}$	0.4167	0.6280	0.8427	1.0586	1.2748	1.4911

Table 1: Exact and computed values of the electrostatic energy, $I_{\sigma_i}^{\text{exact}}$ and $I_{\sigma_i}^{\text{comp}}$, for an elliptical disc with minor axis $b = 0.5$, given by (13), (16), (17) .

a	0.75	0.9	1.05	1.2	1.35	1.5
I_{σ}^{comp}	2.7734	3.6157	4.5162	5.4706	6.4761	7.5313
$I_{\sigma}^{\text{exact}}$	2.7736	3.6159	4.5165	5.4708	6.4763	7.5316

Table 2: Affine density of charges $\sigma = x_1 + 2x_2 + 3$ for an elliptical disc with $b = 0.5$.

	I_{σ_0}	I_{σ_1}	I_{σ_2}	I_{σ}
ε_{rel}	$0.055\,{}^0\!/_{00}$	$0.195\,{}^0\!/_{00}$	$9.1 \times 10^{-4}\,{}^0\!/_{00}$	$0.334\,{}^0\!/_{00}$

Table 3: Maximum value of the relative error $\varepsilon_{\text{rel}} = \max |I_{\sigma_i}^{\text{exact}} - I_{\sigma_i}^{\text{comp}}|$ in per mil for an elliptical disc for $b = 0.5$ and $a = 0.5 : 0.05 : 1.5$.

Acknowledgements

The authors would like to express their thanks to A. Bendali (INSA) for fruitful discussions and M. Fares (CERFACS) for the numerical computations achieved with the CERFACS code CESC. Part of this work was supported by the French National Research Agency under grant no. ANR−08−SYSC−001.

Appendix A. The case of an unit circle disc

Let C be the circle with radius 1. We aim here at computing the integral

$$I_{\mathsf{C}} = \frac{1}{4\pi}\int_{\mathsf{C}} \frac{x_1\, y_1}{|\mathbf{x}-\mathbf{y}|}\, ds_{\mathbf{x}} ds_{\mathbf{y}}$$

This integral is rewritten in polar coordinates (r, ϕ for $\mathbf{x}$ and ρ, ϕ' for $\mathbf{y}$) as

$$I_{\mathsf{C}} = \frac{1}{4\pi}\int_0^1\int_0^{2\pi}\int_0^1\int_0^{2\pi} \frac{r\cos\phi\ \rho\cos\phi'}{\sqrt{r^2+\rho^2-2r\rho\cos(\phi-\phi')}}\, rdrd\phi\ \rho d\rho d\phi'$$

and evaluated using to the formula (3.4.5) of [3, p70]

$$\int_0^{2\pi} \frac{\cos\phi\, d\phi}{\sqrt{\rho^2+r^2-2r\rho\cos(\phi-\phi')}} = \frac{4\cos\phi'}{\rho r}\int_0^{\min(\rho,r)} \frac{t^2\, dt}{\sqrt{\rho^2-t^2}\sqrt{r^2-t^2}}.$$

This leads to

$$I_{\mathsf{C}} = \frac{1}{\pi}\int_0^1\int_0^1\int_0^{\min(\rho,r)} \frac{r\rho\, t^2}{\sqrt{\rho^2-t^2}\sqrt{r^2-t^2}}\, drd\rho dt \int_0^{2\pi}\cos^2\phi' d\phi'.$$

This integral is symmetric in ρ and r, and since $\displaystyle\int_0^{2\pi}\cos^2\phi' d\phi' = \pi$, we have

$$I_{\mathsf{C}} = 2\int_{t=0}^1 t^2 \int_{\rho=t}^1 \frac{\rho}{\sqrt{\rho^2-t^2}}\int_{r=\rho}^1 \frac{r}{\sqrt{r^2-t^2}}\, dr\, d\rho\, dt = \frac{2}{15}.$$

References

[1] F. Leppington, H. Levine, Reflexion and transmission at a plane screen with periodically arranged circular or elliptical apertures, J. Fluid Mech 61 (1973) 109–127.

[2] S. Laurens, S. Tordeux, A. Bendali, M. Fares, R. Kotiuga, Lower and upper bounds for the Rayleigh conductivity of a perforated plate, submitted, http://hal.archives-ouvertes.fr/hal-00686438.

[3] E. Copson, On the problem of the electrified disc, Proceedings of the Edinburgh Mathematical Society (Series 2) 8 (01) (1947) 14–19.