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**Abstract**

We solve a Mean Variance Hedging problem in an incomplete market where multiple defaults can appear. For this, we use a default-density modeling approach. The global market information is formulated as progressive enlargement of a default-free Brownian filtration and the dependence of default times is modeled by a conditional density hypothesis. We prove the quadratic form of each value process between consecutive defaults times and solve recursively systems of quadratic backward stochastic differential equations. Moreover, we obtain an explicit formula of the optimal trading strategy. We illustrate our results with some specific cases.

**Keywords:** Mean variance hedging; default-density modeling; Quadratic backward stochastic differential equation (BSDE); Dynamic programming.

**MSC Classification (2010):** 60J75, 91B28, 93E20.

## Introduction

In this paper, we study the problem of mean variance hedging in a financial market model subject to defaults and contagion risk. We consider multiple defaults events corresponding for example of a succession of crisis periods for a country or a succession of bad annual financial results for a firm. These defaults could induce loss or gain on the asset price. A classic approach to model this is to use an Itô process governed by some Brownian motion $W$ for the asset price $S$ and jumps appearing at random default times, associated to a marked point process $\mu$. Hence the mean variance hedging problem in this incomplete market framework may be then studied by stochastic control and dynamic programming methods in the global filtration $\mathbb{G}$ generated by $W$ and $\mu$. This leads in principle to Hamilton-Jacobi-Bellman integro-differential equations in a Markovian framework, and more generally to Backward Stochastic Differential Equations (BSDEs) with jumps, and the derivation relies on a martingale representation under $\mathbb{G}$ with respect to $W$ and $\mu$, which holds under intensity hypothesis on the defaults, and the so-called immersion

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property (or $(\mathcal{H})$-hypothesis). Such an approach was used in [4] for the multiple defaults case or in [3] for the mean variance hedging problem under $\mathbb{G}$ for defaultable claims.

The mean variance hedging problem was introduced in [2] and many papers have followed and developed this approach. In most of these papers, this problem was solved with continuous filtration [11], [12]. The authors use the dual's approach to show the existence of the variance optimal measure (VOM). Moreover, they can write the solution of the primal problem using Backward Stochastic Differential Equations (BSDEs) whose existence of solutions are deduced by the existence of the VOM. In the case of discontinuous filtration, the VOM is not always a probability measure (see [1] for conditions), so we cannot use the previous approach to solve our problem. That is why, in general, in the case of discontinuous filtration, the authors make the assumption the VOM is a true probability measure as [9] and then deduce the solution of the primal problem using BSDEs. They so prove the existence of the solution of each BSDE using the VOM. Indeed, without the fact that the VOM is a true probability, it is difficult to show the existence of solution of the corresponding BSDEs with jumps since these BSDEs coefficients are not standard.

In a general model with discontinuous filtration generated by a continuous process and a discontinuous process, the author in [10] proved the existence of the solution of the BSDEs for the mean variance problem assuming that the coefficients of its asset are adapted with respect to the continuous filtration $\mathbb{F}$. This strong assumption allows him not to assume that the VOM is a true probability and leads him to solve directly the main BSDE without any assumption on the VOM.

In this paper we work also in the case of a discontinuous filtration $\mathbb{G}$. In our model, jumps are generated by default times. So, we cannot use the same technics as [10], since his strong assumption is not well satisfied in our framework. Indeed, our assets coefficients depend on the jumps (defaults) . Therefore, we use a different approach than the one mentioned previously. Indeed, we use an approach initiated and studied in [5]. By viewing the global filtration $\mathbb{G}$ as a progressive enlargement of filtrations of the default-free filtration $\mathbb{F}$ generated by the Brownian motion $W$, with the default filtration generated by the random times, the basic idea is to split the global mean variance problem, into sub-control problems in the reference filtration $\mathbb{F}$ and corresponding to mean variance problems in default-free markets between two default times. More precisely, we derive a backward recursive decomposition by starting from the mean variance problem when all defaults occurred, and then going back to the initial mean variance problem before any default. The main point is to connect this family of stochastic control problems in the $\mathbb{F}$-filtration, and this is achieved by assuming the existence of a conditional density on the default times given the default-free information $\mathbb{F}$. So we will use the approach of [5] to show that between each default time, using dynamic programming method, we can first characterize each dynamic version of the mean variance hedging problem in a quadratic decomposition form. These decompositions will depend explicitly on the parameters and default times of our model. Secondly, we will express the three terms appearing in this quadratic decomposition form as solution of three explicits backward stochastic differential equations (BSDEs). Then, starting after the last default event and then going back to the initial mean variance problem we will obtain for this each subset a system of recursive BSDEs. We will prove explicitly the existence and uniqueness of the solution of theses systems of quadratic BSDEs which is not trivial and we will find the optimal mean variance hedging strategy.

The paper is so structured as follows: In section 1, we will introduce our model and the corresponding mean variance hedging problem. We will give the systems of BSDEs. Then, in section 2, we will give the solution to the mean variance hedging problem. For this, firstly, we will begin by giving a proof of the existence of a solution of the recursive system of BSDEs. Secondly, we will give the BSDEs characterization by a verification theorem. Finally, in section 3, we will give some numerical illustrations.

# 1 Multiple defaults model

## 1.1 Market information

We adopt in this paper the same model and notations as in [5]. Let $\boldsymbol{\tau} = (\tau_1, ..., \tau_n)$ be now a vector of the $n$ random times and $\boldsymbol{L} = (L_1, ..., L_n)$ be a vector of the $n$ marks associated to $\tau$, $L_i$ being a $\mathcal{G}$-measurable random variable taking values in $E \subset \mathbb{R}$ and representing for example the loss given default at time $\tau_i$. We denote, for $k = \{1, \dots, n\}$, $\mathbb{D}^k = (\mathcal{D}_t^k)_{t\in[0,T]}$ where $\mathcal{D}_t^k = \tilde{\mathcal{D}}_{t+}^k$ and $\tilde{\mathcal{D}}_t^k = \sigma(1_{\tau_k \le s}, s \le t) \vee \sigma(L_k 1_{\tau_k \le s}, s \le t)$ the filtrations generated by the associated jump processes. Then $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ will be the enlarged progressive filtration $\mathbb{F} \vee \mathbb{D}^1 \vee ... \vee \mathbb{D}^n$, representing the structure of the global information available for the investors over $[0, T]$. In other words, $\mathbb{G}$ is the smallest right-continuous filtration containing $\mathbb{F}$ such that for any $1 \le k \le n$, $\tau_k$ is a $\mathbb{G}$-stopping time and $L_k$ is $\mathcal{G}_{\tau_k}$-measurable. We shall assume that the default times are ordered (i.e. $\tau_1 \le ... \le \tau_n$) and so valued in $\Delta_n$ on $\{\theta_n \le T\}$ where, for $k = 1, ..., n$, we denote

$$\Delta_k \;:=\; \Big\{(\theta_1, ..., \theta_k) \in (\mathbb{R}_+)^k : \theta_1 \le ... \le \theta_k\Big\}.$$

This means that we do not distinguish specific credit names and only observe the successive default times. For any $(\theta_1, ..., \theta_n) \in \Delta_n$, $(l_1, ..., l_n) \in E^n$, we denote by $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$, $\boldsymbol{l} = (l_1, ..., l_n)$, and $\boldsymbol{\theta}_k = (\theta_1, ..., \theta_k)$, $\boldsymbol{l}_k = (l_1, ..., l_k)$ for $0 \le k \le n$ with the convention $\theta_0 = l_0 = \emptyset$. We also denote $\boldsymbol{\tau}_k = (\tau_1, ..., \tau_k)$ and $\boldsymbol{L}_k = (L_1, ..., L_k)$. Moreover, for $0 \le t \le T$, the set $\Omega_t^k$ denotes the event

$$\Omega_t^k \;:=\; \{\tau_k \le t < \tau_{k+1}\},$$

(with $\Omega_t^0 = \{t < \tau_1\}$ and $\Omega_t^n = \{\tau_n \le t\}$) and represents the scenario where $k$ defaults occur before time $t$. We call $\Omega_t^k$ the $k$-default scenario at time t. We define similarly $\Omega_{t-}^k = \{\tau_k < t \le \tau_{k+1}\}$. We denote by $\mathcal{P}(\mathbb{F})$ the $\sigma$-algebra of $\mathbb{F}$-predictable measurable subsets on $\mathbb{R}_+ \times \Omega$, and by $\mathcal{P}_{\mathbb{F}}(\Delta^k, E^k)$ the set of indexed $\mathbb{F}$-predictable processes $Z^k(.,.)$, i.e. s.t. the map $(t, \omega, \boldsymbol{\theta}_k, \boldsymbol{l}_k) \to Z_t^k(\omega, \boldsymbol{\theta}_k, \boldsymbol{l}_k)$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable. We also denote by $\mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$ the set of indexed $\mathbb{F}$-adapted processes $Z^k(.,.)$, i.e. s.t. for all $0 \le t \le T$, the map $(\omega, \boldsymbol{\theta}_k, \boldsymbol{l}_k) \to Z_t^k(\omega, \boldsymbol{\theta}_k, \boldsymbol{l}_k)$ is $\mathcal{F}_t \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable. Hence we have that any $\mathbb{G}$-predictable process $Z = (Z_t)_{0\le t\le T}$ has a decomposition in the form

$$Z_t \;=\; \sum_{k=0}^{n} 1_{\Omega_{t-}^k} Z_t^k(\tau_k, L_k), \quad 0 \le t \le T$$

where $Z^k$ lies in $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$. We assume also the *density hypothesis* which is given in multiple defaults case by the following statement:

**Assumption 1.1** (*Density hypothesis*)**.** *There exists $\alpha \in \mathcal{O}_{\mathbb{F}}(\Delta_n, E^n)$ such that for any Borel function $f$ on $\Delta_n \times E^n$ and $0 \le t \le T$ :*

$$\mathbb{E}[f(\tau, L)|\mathcal{F}_t] \;=\; \int_{\Delta_n \times E^n} f(\boldsymbol{\theta}, \boldsymbol{l})\alpha_t(\boldsymbol{\theta}, \boldsymbol{l})d\boldsymbol{\theta}\eta(d\boldsymbol{l}) \quad a.s., \tag{1.1}$$

*where $d\boldsymbol{\theta} = d\theta_1...d\theta_n$ is the Lebesgue measure on $\mathbb{R}^n$, and $\eta(d\boldsymbol{l})$ is a Borel measure on $E^n$ in the form $\eta(d\boldsymbol{l}) = \eta_1(dl_1)\prod_{k=1}^{n-1}\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})$, with $\eta_1$ a nonnegative Borel measure on $E$ and $\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})$ a nonnegative transition kernel on $E^k \times E$.*

**Remark 1.1.** *The condition* (1.1) *implies that in the case that* $\alpha$ *is separable in the form* $\alpha_t(\boldsymbol{\theta}, \boldsymbol{l}) = \alpha_t^{\boldsymbol{\tau}}(\boldsymbol{\theta})\alpha_t^{\boldsymbol{L}}(\boldsymbol{l})$ *that the random times and marks are independent given* $\mathcal{F}_t$.

### 1.2 Asset price model under default risk

The trading asset $S$ is a $\mathbb{G}$-adapted process which admits (as in [5]) the following decomposed form

$$S_t = \sum_{k=0}^{n} 1_{\Omega_t^k} S_t^k(\boldsymbol{\tau}_k, \boldsymbol{L}_k), \tag{1.2}$$

where $S^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$, $\boldsymbol{\theta}_k = (\theta_1, ..., \theta_k) \in \Delta_k$, $\boldsymbol{l}_k = (l_1, ..., l_k) \in E^k$, is an indexed process in $\mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$, valued in $\mathbb{R}_+$, representing the asset value in the k-default scenario, given the past default events $\boldsymbol{\tau}_k = \boldsymbol{\theta}_k$, and the marks at default $\boldsymbol{L}_k = \boldsymbol{l}_k$. Notice that $S_t$ is equal to the value $S_t^k$ only on the set $\Omega_t^k$, that is, only for $\tau_k \leq t < \tau_{k+1}$. The dynamic of the indexed process $S^k$ is given by

$$dS_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) \;=\; S_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)(\mu_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)dt + \sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)dW_t), \quad \theta_k \leq t \leq T \tag{1.3}$$

where $W$ is a one-dimensional $(P, \mathbb{F})$-Brownian motion, $\mu^k$ and $\sigma^k$ are indexed processes in $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$, valued in $\mathbb{R}$. We make, as in the one default case, the usual no-arbitrage assumption that there exists an indexed risk premium process $\lambda^k \in \mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$ s.t. for all $(\boldsymbol{\theta}_k, \boldsymbol{l}_k) \in \Delta_k \times E^k$,

$$\sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) = \mu_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k), \quad 0 \leq t \leq T. \tag{1.4}$$

Moreover, in this contagion risk model, each default time may induce a jump in the assets portfolio. This is formalized by considering a family of indexed processes $\gamma^k$, $0 \leq k \leq n-1$, in $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k, E)$, and valued in $[-1, \infty)$. For $(\boldsymbol{\theta}_k, \boldsymbol{l}_k) \in \Delta_k \times E^k$, and $l_{k+1} \in E$, $\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})$ represents the relative vector jump size on the asset at time $t = \theta_{k+1} \geq \theta_k$ with a mark $l_{k+1}$, given the past default events $(\boldsymbol{\tau}_k, \boldsymbol{L}_k) = (\boldsymbol{\theta}_k, \boldsymbol{l}_k)$. In other words, we have :

$$S_{\theta_{k+1}}^{k+1}(\boldsymbol{\theta}_{k+1}, \boldsymbol{l}_{k+1}) = S_{\theta_{k+1}^-}^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\left(1 + \gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})\right) \tag{1.5}$$

### 1.3 Strategy and wealth process

The trading strategy is a $\mathbb{G}$-predictable process $\pi$, hence decomposed in the form

$$\pi_t = \sum_{k=0}^{n} 1_{\Omega_{t-}^k} \pi_t^k(\boldsymbol{\tau}_k, \boldsymbol{L}_k), \quad 0 \leq t \leq T \tag{1.6}$$

where $\pi^k$ is an indexed process in $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$, and $\pi^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ is valued in closed set $A^k$ of $\mathbb{R}$ containing the zero element, and representing the amount invested continuously in the asset in the $k$-default scenario, given the past default events $\boldsymbol{\tau}_k = \boldsymbol{\theta}_k$ and the marks at default $\boldsymbol{L}_k = \boldsymbol{l}_k$, for $(\boldsymbol{\theta}_k, \boldsymbol{l}_k) \in \Delta_k \times E^k$. We shall often identify the strategy $\pi$ with the family $(\pi^k)_{0 \leq k \leq n}$ given in 1.6, and we require the integrability conditions : for all $\boldsymbol{\theta}_k \in \Delta_k$, $\boldsymbol{l}_k \in E^k$,

$$\int_0^T |\pi_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\mu_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)|dt + \int_0^T |\pi_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)|^2 dt < \infty, \quad a.s. \tag{1.7}$$

Given a trading strategy $\pi = (\pi^k)_{0 \leq k \leq n}$, the corresponding wealth process is given by

$$X_t = \sum_{k=0}^{n} 1_{\Omega_t^k} X_t^k(\boldsymbol{\tau}_k, \boldsymbol{L}_k), \quad 0 \leq t \leq T \tag{1.8}$$

where $X^k(\boldsymbol{\tau}_k, \boldsymbol{L}_k)$, $\boldsymbol{\theta}_k \in \Delta_k$, $\boldsymbol{l}_k \in E^k$, is an indexed process in $\mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$, representing the wealth controlled by $\pi^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ in the price process $S^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$, given the past default events $\boldsymbol{\tau}_k = \boldsymbol{\theta}_k$ and the marks at default $\boldsymbol{L}_k = \boldsymbol{l}_k$. From the dynamics (1.3) and under (1.7), it is governed by

$$dX_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) = \pi_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)(\mu_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)dt + \sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)dW_t), \quad \theta_k \le t \le T. \tag{1.9}$$

Moreover, each default time induces a jump in the asset price process, and then also on the wealth process. From (1.5), it is given by

$$X_{\theta_{k+1}}^{k+1}(\boldsymbol{\theta}_{k+1}, \boldsymbol{l}_{k+1}) = X_{\theta_{k+1}^-}^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) + \pi_{\theta_{k+1}}^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}).$$

Finally, the payoff is a bounded $\mathcal{G}_T$-measurable random variable $H_T$ which admits the decomposition form given by

$$H_T = \sum_{k=0}^n 1_{\Omega_T^k} H_t^k(\boldsymbol{\tau}_k, \boldsymbol{L}_k), \tag{1.10}$$

where $H_T^k(.,.)$ is $\mathcal{F}_T \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable and represents the payoff when $k$ defaults occurred before maturity $T$.

**Remark 1.2.** *We have between each default time (i.e. in each time events $\Omega_t^k := \{\tau_k \le t < \tau_{k+1}\}$, $t \in [0,T]$) that the market is complete.*

### 1.4 The mean variance problem

On our problem of *mean variance hedging (MVH)*, the performance of an admissible trading strategy $\pi \in \mathcal{A}_{\mathbb{G}}$ started with an initial capital $x \in \mathbb{R}$ is measured over the finite horizon T by

$$J_0^H(x, \pi) = \mathbb{E}[(H_T - X_T^{x,\pi})^2] \tag{1.11}$$

and the MVH problem is formulated as

$$V_0^H(x) \ = \ \inf_{\pi \in \mathcal{A}_{\mathbb{G}}} J_0^H(x, \pi).$$

#### 1.4.1 Value functions

We define, first, the corresponding multiple defaults admissible trading strategies set:

**Definition 1.1.** *For $0 \le k \le n$, $\mathcal{A}_{\mathbb{F}}^k$ denotes the set of indexed processes $\pi^k$ in $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$, valued in $A^k$ satisfying* (1.7)*, and such that*

$$\mathbb{E}\left[\int_{\theta_k}^T |\pi_s^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)|^2 ds\right] < \infty \tag{1.12}$$

*We then denote by $\mathcal{A}_{\mathbb{G}} = (\mathcal{A}_{\mathbb{F}}^k)_{0 \le k \le n}$ the set of admissible trading strategies $\pi = (\pi^k)_{0 \le k \le n}$.*

Under the density hypothesis 1.1, let us define a family of auxiliary processes $\alpha^k \in \mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$, $0 \le k \le n$, which is related to the survival probability and is defined by recursive induction from $\alpha^n = \alpha$,

$$\alpha_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) = \int_t^\infty \int_E \alpha_t^{k+1}(\boldsymbol{\theta}_k, \theta_{k+1}, \boldsymbol{l}_k, l_{k+1}) d\theta_{k+1} \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}), \tag{1.13}$$

for $0 \leq k \leq n-1$, so that $\mathbb{P}[\tau_{k+1} > t|\mathcal{F}_t] = \int_{\Delta_k \times E^k} \alpha_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) d\boldsymbol{\theta}_k \eta(d\boldsymbol{l}_k)$ and $\mathbb{P}[\tau_1 > t|\mathcal{F}_t] = \alpha_t^0$, where $d\boldsymbol{\theta}_k = d\theta_1 ... d\theta_k$, $\eta(d\boldsymbol{l}_k) = \eta_1(dl_1)...\eta_k(\boldsymbol{l}_{k-1}, dl_k)$. Given $\pi^k \in \mathcal{A}_{\mathbb{F}}^k$, we denote by $X^{k,x}(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ the controlled process solution to (1.9) and starting from $x$ at $\theta_k$. We now give our model hypothesis:

**Assumption 1.2.** *We assume for all $t \in [\theta_k, T]$ and $0 \leq k \leq n$ that $\mu_t^k$, $\sigma_t^k$,$\gamma_t^k$ and the family processes $\alpha^k \in \mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$ are uniformly bounded. Moreover, we assume for $0 \leq k \leq n$ that the measure $\eta_k(dl_k)$ is uniformly bounded too.*

### 1.4.2 The mean variance hedging problem

The value function to the global mean variance $\mathbb{G}$-problem (1.11) is then given, in the multiple defaults case, in a backward induction from the $\mathbb{F}$-problems (see [5] for more details) :

$$V^n(x, \boldsymbol{\theta}, \boldsymbol{l}) = \operatorname*{ess\,inf}_{\pi^n \in \mathcal{A}_{\mathbb{F}}^n} \mathbb{E}\left[(H_T^n - X_T^{n,x}(\boldsymbol{\theta}, \boldsymbol{l}))^2 \alpha_T(\boldsymbol{\theta}, \boldsymbol{l})|\mathcal{F}_{\theta_n}\right] \tag{1.14}$$

$$\begin{aligned} V^k(x, \boldsymbol{\theta}_k, \boldsymbol{l}_k) &= \operatorname*{ess\,inf}_{\pi^k \in \mathcal{A}_{\mathbb{F}}^k} \mathbb{E}[(H_T^k - X_T^{k,x}(\boldsymbol{\theta}_k, \boldsymbol{l}_k))^2 \alpha_T^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) + \\ \int_{\theta_k}^T \int_E V^{k+1}(X_{\theta_{k+1}}^{k,x}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) &+ \pi_{\theta_{k+1}}^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k).\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}), \boldsymbol{\theta}_{k+1}, \boldsymbol{l}_{k+1}) \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) d\theta_{k+1}|\mathcal{F}_{\theta_k}] \end{aligned} \tag{1.15}$$

where we recall that $\boldsymbol{\theta}_n = \boldsymbol{\theta}, \boldsymbol{l}_n = \boldsymbol{l}$, $\boldsymbol{\theta}_0 = \theta_0 = \emptyset$ and $\boldsymbol{l}_0 = l_0 = \emptyset$.

**Remark 1.3.** *If there exists, for all $0 \leq k \leq n$, some $\pi^{k,*} \in \mathcal{A}_{\mathbb{F}}^k$ attaining the essential infimum in the previous equations, then the strategy $\pi^* = (\pi^{k,*})_{0 \leq k \leq n} \in \mathcal{A}_{\mathbb{G}}$ is optimal for the MVH problem.*

## 2 Solution to the mean variance hedging problem

We exploit the quadratic form of the mean variance hedging problem in order to characterize by dynamic programming methods the solutions to the stochastic optimization problems (1.14) and (1.15) in terms of a recursive system of indexed BSDEs with respect to the filtration $\mathbb{F}$. We use a verification approach in the following sense:

1. Firstly, we derive formally the system of BSDEs associated to the $\mathbb{F}$-stochastic control problems (1.14) and (1.15) using dynamic programming principle.

2. Secondly, we obtain the existence of the solutions of the corresponding system of BSDEs (see Theorem 2.1).

3. Finally, in a verification Theorem (see Theorem 2.2), we prove that these BSDEs solutions are unique and provide the solution to our mean variance hedging problem. We prove also that the strategy found in step 1 is optimal and admissible. Moreover, we prove that the quadratic representation form of our value function are true.

So let's begin with point 1: For $t \in [\theta_n, T]$, $\nu^n \in \mathcal{A}_{\mathbb{F}}^n$, let us introduce the set of controls coinciding with strategy $\nu$ until time $t$:

$$\mathcal{A}_{\mathbb{F}}^n(t, \nu^n) = \{\pi^n \in \mathcal{A}_{\mathbb{F}}^n : \pi_{.\wedge t}^n = \nu_{.\wedge t}^n\}$$

We can now define the dynamic version of (1.14) by considering the family of $\mathbb{F}$-adapted processes:

$$V_t^n(x,\boldsymbol{\theta},\boldsymbol{l},\nu^n) = \operatorname*{ess\,inf}_{\pi^n\in\mathcal{A}_{\mathbb{F}}^n(t,\nu^n)} \mathbb{E}\left[(H_T^n - X_T^{n,x}(\boldsymbol{\theta},\boldsymbol{l}))^2\alpha_T(\boldsymbol{\theta},\boldsymbol{l})|\mathcal{F}_t\right], \quad t\geq\theta_n, \tag{2.16}$$

so that $V_{\theta_n}^n(x,\boldsymbol{\theta},\boldsymbol{l},\nu^n) = V^n(x,\boldsymbol{\theta},\boldsymbol{l})$ for any $\nu^n \in \mathcal{A}_{\mathbb{F}}^n$. From the dynamic programing principle, one should have the submartingale property on $\{V_t^n(x,\boldsymbol{\theta},\boldsymbol{l},\nu^n) \ \ ,\theta_n\leq t\leq T\}$, for any $\nu\in\mathcal{A}_{\mathbb{F}}^n$, and if an optimal strategy exists for (2.16), we should have the martingale property of $\{V_t^n(x,\boldsymbol{\theta},\boldsymbol{l},\pi^{*,n}),\theta_n\leq t\leq T\}$ for some $\pi^{*,n}\in\mathcal{A}_{\mathbb{F}}^n$. Moreover, since we work on a quadratic minimization approach, the value process $V_t^n(x,\boldsymbol{\theta},\boldsymbol{l},\nu^n)$ should admit the quadratic form decomposition given by

$$V_t^n(x,\boldsymbol{\theta},\boldsymbol{l},\nu^n) = v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\left(X_t^{n,x}(\boldsymbol{\theta},\boldsymbol{l}) - Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2 + \xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}}, \quad t\in[\theta_n,T]$$

We search a triple $(v^{n,\boldsymbol{\theta},\boldsymbol{l}}, Y^{n,\boldsymbol{\theta},\boldsymbol{l},}, \xi^{n,\boldsymbol{\theta},\boldsymbol{l}})$ in the form

$$(\mathbf{En}) \quad \begin{cases} \dfrac{dv_t^{n,\boldsymbol{\theta},\boldsymbol{l}}}{v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}} = -g_t^{n,\boldsymbol{\theta},\boldsymbol{l},(1)}(v_t^{n,\boldsymbol{\theta},\boldsymbol{l}},\beta_t^{n,\boldsymbol{\theta},\boldsymbol{l}})dt + \beta_t^{n,\boldsymbol{\theta},\boldsymbol{l}}dW_t, \\[2ex] dY_t^{n,\boldsymbol{\theta},\boldsymbol{l}} \quad = -g_t^{n,\boldsymbol{\theta},\boldsymbol{l},(2)}(Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}},Z_t^{n,\boldsymbol{\theta},\boldsymbol{l}})dt + Z_t^{n,\boldsymbol{\theta},\boldsymbol{l}}dW_t \\[2ex] d\xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}} = -g_t^{n,\boldsymbol{\theta},\boldsymbol{l},(3)}(\xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}},R_t^{n,\boldsymbol{\theta},\boldsymbol{l}})dt + R_t^{n,\boldsymbol{\theta},\boldsymbol{l}}dW_t. \end{cases} \tag{2.17}$$

Then, by using the above submartingale and martingale property of the dynamic programming principle and since $V_T^n(x,\boldsymbol{\theta},\boldsymbol{l},\nu^n) = \left(X_T^{n,x}(\boldsymbol{\theta},\boldsymbol{l}) - H_T^n(\boldsymbol{\theta},\boldsymbol{l})\right)^2\alpha_T(\boldsymbol{\theta},\boldsymbol{l})$ by (2.16), we see from Itô calculus (see Proposition 3.5 of Goutte and Ngoupeyou [3] for more details) that the triple $(v^{n,\boldsymbol{\theta},\boldsymbol{l}},Y^{n,\boldsymbol{\theta},\boldsymbol{l}},\xi^{n,\boldsymbol{\theta},\boldsymbol{l}})$ satisfies (2.17) for all $t\in[\theta_n,T]$ with terminal conditions $v_T^{n,\boldsymbol{\theta},\boldsymbol{l}} = \alpha_T(\boldsymbol{\theta},\boldsymbol{l})$, $Y_T^{n,\boldsymbol{\theta},\boldsymbol{l}} = H_T^n(\boldsymbol{\theta},\boldsymbol{l})$ and $\xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}} = 0$. The corresponding coefficients of the BSDEs are given by the following equations:

$$g_t^{n,\boldsymbol{\theta},\boldsymbol{l},(1)} = -\frac{\left(\mu^n(\boldsymbol{\theta},\boldsymbol{l}) + \sigma^n(\boldsymbol{\theta},\boldsymbol{l})\beta_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2}{(\sigma^n(\boldsymbol{\theta},\boldsymbol{l}))^2}, \quad g_t^{n,\boldsymbol{\theta},\boldsymbol{l},(2)} = -\frac{\mu^n(\boldsymbol{\theta},\boldsymbol{l})}{\sigma^n(\boldsymbol{\theta},\boldsymbol{l})}Z_t^{n,\boldsymbol{\theta},\boldsymbol{l}} \quad \text{and} \quad g_t^{n,\boldsymbol{\theta},\boldsymbol{l},(3)} = 0.$$

We have, also, that the optimal strategy $\pi^{n,*}$ (such that $V_t^n(x,\boldsymbol{\theta},\boldsymbol{l},\pi^{n,*})$ is a true martingale) is given for all $t\in[\theta_n,T]$ by

$$\pi_t^{n,*}(\boldsymbol{\theta},\boldsymbol{l}) \quad = \quad f_t^{n,\boldsymbol{\theta},\boldsymbol{l},1}X_t^{n,x}(\boldsymbol{\theta},\boldsymbol{l}) + f_t^{n,\boldsymbol{\theta},\boldsymbol{l},2} \tag{2.18}$$

where

$$f_t^{n,\boldsymbol{\theta},\boldsymbol{l},1} := -\frac{1}{\left(\sigma_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2}\left(\mu_t^{n,\boldsymbol{\theta},\boldsymbol{l}} + \sigma_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\beta_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)$$

and

$$f_t^{n,\boldsymbol{\theta},\boldsymbol{l},2} := \frac{1}{\left(\sigma_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2}\left[\sigma_t^{n,\boldsymbol{\theta},\boldsymbol{l}}Z_t^{n,\boldsymbol{\theta},\boldsymbol{l}} + Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\left(\mu_t^{n,\boldsymbol{\theta},\boldsymbol{l}} + \sigma_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\beta_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)\right]$$

Hence, the optimal strategy is linear in $X$ which is the case in the no default model. We will refer in the sequel to this problem as the **(En)** problem.

Next, consider the problem (1.15) and define similarly the dynamic version by considering the value function process given by:

$$V_t^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k) \;=\; \operatorname*{ess\,inf}_{\pi^k\in\mathcal{A}_{\mathbb{F}}^k(t,\nu^k)} \mathbb{E}[(H_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)-X_T^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2\alpha_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)+ \tag{2.19}$$

$$\int_t^T\int_E V_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\;+\;\pi_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k).\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}),\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})d\theta_{k+1}|\mathcal{F}_t]$$

for $\theta_k\le t\le T$, where $\mathcal{A}_{\mathbb{F}}^k(t,\nu^k)=\{\pi^k\in\mathcal{A}_{\mathbb{F}}^k:\pi_{.\wedge t}^k=\nu_{.\wedge t}^k\}$, for $\nu^k\in\mathcal{A}_{\mathbb{F}}^k$ so that $V_{\theta_k}^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k)=V^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k)$. Similarly, we will refer in the sequel to this problem as **(Ek)** problem for $k=0,\dots,n-1$. The dynamic programming principle for (2.19) formally implies that the process

$$V_t^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k)+\int_0^t\int_E V_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)+\pi_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k).\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}),\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})d\theta_{k+1}$$

for $t\in[\theta_k,T]$ is a submartingale for any $\nu^k\in\mathcal{A}_{\mathbb{F}}^k$ and a true martingale for $\pi^{*,k}$ if it is an optimal strategy for (2.19). Again, since we work on a quadratic minimization approach, the value process $V_t^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k)$ should admit the quadratic form decomposition given by

$$V_t^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k)=v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_t^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)-Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2+\xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\quad,\forall k=0,\dots,n-1$$

We search also a triple $\left(v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k},Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k},\xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)$ for all $k=0,\dots,n-1$, in the form

$$(\mathbf{Ek})\quad\begin{cases}\dfrac{dv_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}{v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}=-g_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}(v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k},\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})dt+\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}dW_t\\[2ex] dY_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}=-g_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(2)}(Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k},Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})dt+Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}dW_t\\[2ex] d\xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}=-g_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(3)}(\xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k},R_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})dt+R_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}dW_t\end{cases}\tag{2.20}$$

Then, by using the above submartingale and martingale property of the dynamic programming principle and since $V_T^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k)=\left(X_T^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)-H_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\right)^2\alpha_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)$ by (2.19), we see from Itô calculus (see again Proposition 3.5 of Goutte and Ngoupeyou [3] for more details) that the triple $\left(v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k},Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k},\xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)$ satisfies (2.20) for all $t\in[\theta_k,T]$ with terminal conditions $v_T^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}=\alpha_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)$, $Y_T^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}=H_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)$ and $\xi_T^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}=0$. And the corresponding coefficients of the BSDEs are given by the following equations:

$$\begin{aligned}g_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}&=\int_E(1+v_t^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})\\&-\frac{\left(\mu_t^k+\sigma_t^k\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}+\int_E(1+v_t^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})\right)^2}{(\sigma_t^k)^2+\int_E(1+v_t^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})(\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}))^2\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})},\end{aligned}$$

$$\begin{aligned}g_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(2)}&=\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}+\int_E U_t^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(1+v_t^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})\\&+\frac{\left(-\int_E U_t^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(1+v_t^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})-\sigma_t^kZ_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)}{(\sigma_t^k)^2+\int_E(1+v_t^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})(\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}))^2\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})}\\&\times\left(\mu_t^k+\sigma_t^k\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}+\int_E(1+v_t^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})\right)\end{aligned}$$

and

$$g_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(3)} = v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\Big[\int_E (U_t^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2(1+v_t^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1}) + (Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2$$
$$- \frac{\Big(-\int_E(1+v_t^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})U_t^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1}) - \sigma_t^k Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\Big)^2}{(\sigma_t^k)^2 + \int_E(1+v_t^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})(\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}))^2\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})}\Big]$$

where

$$1+v^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} = \frac{v^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}}{v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}} \quad \text{and} \quad U^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} = Y^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} - Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.$$

The optimal strategy $\pi^{k,*}$ (such that $V_t^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\pi^{k,*})$ is a true martingale) is given by

$$\pi_t^{k,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) = \frac{1}{(\sigma_t^k)^2 + \int_E(1+v_t^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1})^2\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})}\Big[\sigma_t^k Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} - K_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(\mu_t^k + \sigma_t^k\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})$$
$$+ \frac{\int_E(X_t^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} - Y^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}})\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})}{v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}\Big] \quad (2.21)$$

with $K_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} := X_t^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$. Again, we obtain a linear form of the optimal strategy with respect to X. We will refer in the sequel to this problem as the **(Ek)** problem, $k \in \{0,\dots,n-1\}$.

**Remark 2.4.** *For all* **(Ek)** *problems, $k \in \{0,1,\dots n\}$, we work in the time interval given for all $t \in [\theta_k,T]$. Hence for the particular case where we take the value function for $t = \theta_k$, we obtain that $V_{t=\theta_k}^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k) := V_{\theta_k}^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k) = V^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k)$, where we recall that $x$ is the value of $X^k$ in $\theta_k$, so $X_{\theta_k}^k = x$.*

Hence, **(Ek)** and **(En)** define thus a recursive system of families of BSDEs, indexed by $(\boldsymbol{\theta},\boldsymbol{l}) \in \Delta_n(T)\times E^n$, and the rest of this paper is devoted first to prove the existence of a solution of these system of BSDEs, and then to its uniqueness via verification theorem relating the solution to the value function 2.19 and 2.16.

## 2.1 Existence of a solution of the recursive system of BSDEs

The generators of our recursive system of BSDEs (2.17) and (2.20) are not trivial since the coefficients $g^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, k \in \{0,\dots,n\}$ are not standards. Hence, we give a Theorem to insure that recursive BSDEs solutions exist and stay in their own solution's space for all $k \in \{0,1,\dots,n\}$. Let consider the family $\{Q(\boldsymbol{\theta}),(\boldsymbol{\theta},\boldsymbol{l}) \in [0,T]\times E^n\}$ of probability measures such that the Radon Nikodym density of $Q(\boldsymbol{\theta},\boldsymbol{l})$ with respect to $P$ on $\mathcal{F}_T$ is given by

$$Z_T^Q(\boldsymbol{\theta},\boldsymbol{l}) := \frac{dQ(\boldsymbol{\theta},\boldsymbol{l})}{dP}|_{\mathcal{F}_T} = \exp\left[\int_\theta^T \frac{\mu_s^n(\boldsymbol{\theta},\boldsymbol{l})}{\sigma_s^n(\boldsymbol{\theta},\boldsymbol{l})}dW_s - \frac{1}{2}\int_\theta^T \left|\frac{\mu_s^n(\boldsymbol{\theta},\boldsymbol{l})}{\sigma_s^n(\boldsymbol{\theta},\boldsymbol{l})}\right|^2 ds\right]. \quad (2.22)$$

**Theorem 2.1.** *For all $k \in \{0,1,\dots,n\}$ and $t \in [\theta_k,T]$, we have that*

1. *There exists a couple $\left(v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k},\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right) \in \mathcal{S}^\infty \times \mathrm{BMO}$ of the first BSDE of (2.20) (if $k \neq n$) and (2.17) (if $k = n$) and there exists constants $\delta_1^k$ and $\delta_2^k$ such that*

$$0 < \delta_1^k \le v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \le \delta_2^k.$$

*Moreover, for the case $k = n$, we have an explicit solution which is*

$$v_t^{n,\boldsymbol{\theta},\boldsymbol{l}} = \left( \mathbb{E}\left[ \left( \frac{Z_T^Q(\boldsymbol{\theta},\boldsymbol{l})}{Z_t^Q(\boldsymbol{\theta},\boldsymbol{l})} \right)^2 \frac{1}{\alpha_T(\boldsymbol{\theta},\boldsymbol{l})} |\mathcal{F}_t \right] \right)^{-1} \tag{2.23}$$

2. *There exists a couple $\left(Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right) \in \mathcal{S}^\infty \times \mathrm{BMO}$ solution of the second BSDE of* (2.20) *(if $k \neq n$) and* (2.17) *(if $k = n$). Moreover, for the case $k = n$, we have an explicit solution which is*

$$Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}} = \mathbb{E}\left[ \frac{Z_T^Q(\boldsymbol{\theta},\boldsymbol{l})}{Z_t^Q(\boldsymbol{\theta},\boldsymbol{l})} H_T^n(\boldsymbol{\theta},\boldsymbol{l}) \Big| \mathcal{F}_t \right] = \mathbb{E}^{Q(\boldsymbol{\theta},\boldsymbol{l})}\left[ H_T^n(\boldsymbol{\theta},\boldsymbol{l}) \Big| \mathcal{F}_t \right]. \tag{2.24}$$

3. *There exists a couple $\left(\xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, R_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right) \in \mathcal{S}^\infty \times \mathrm{BMO}$ solution of the third BSDE of* (2.20) *(if $k \neq n$) and* (2.17) *(if $k = n$). Moreover, for the case $k = n$, we have an explicit solution which is $\xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}} = 0$ since the market is complete (i.e. we are after the last default).*

*Proof.* For each BSDE, we will proceed in a backward recursive proof.

<u>**First BSDE:**</u> **(En) problem:** If $k = n$ (i.e. we are after the last default), the market is complete. Following (2.22) and from Itô calculus, we get that $\left[\frac{\left(Z_t^Q(\boldsymbol{\theta},\boldsymbol{l})\right)^2}{v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}}\right]_{t\in[\theta_n,T]}$ is a $\mathbb{P}$-martingale. Using its terminal condition $v_T^{n,\boldsymbol{\theta},\boldsymbol{l}} = \alpha_T(\boldsymbol{\theta},\boldsymbol{l})$ we finally obtain, for all $t \in [\theta_n, T]$, that

$$v_t^{n,\boldsymbol{\theta},\boldsymbol{l}} = \left( \mathbb{E}\left[ \left( \frac{Z_T^Q(\boldsymbol{\theta},\boldsymbol{l})}{Z_t^Q(\boldsymbol{\theta},\boldsymbol{l})} \right)^2 \frac{1}{\alpha_T(\boldsymbol{\theta},\boldsymbol{l})} |\mathcal{F}_t \right] \right)^{-1}.$$

Moreover, under integrability condition 1.7, the martingale $\frac{\mu^n(\boldsymbol{\theta},\boldsymbol{l})}{\sigma^n(\boldsymbol{\theta},\boldsymbol{l})}.W$ is BMO. This implies that the family $\{Q(\boldsymbol{\theta},\boldsymbol{l}), (\boldsymbol{\theta},\boldsymbol{l}) \in \Delta_n(T) \times E^n\}$ of measures of probability, such that the Radon Nikodym density of $Q(\boldsymbol{\theta},\boldsymbol{l})$ with respect to $P$ is given by (2.22), satisfies the reverse Holder inequality $R_2(P)$. Hence there exists a positive constant $c_4$ such that for all stopping time $\tau \leq T$ we have $\frac{\mathbb{E}\left[Z_T^Q(\boldsymbol{\theta},\boldsymbol{l})^2|\mathcal{F}_\tau\right]}{Z_\tau^Q(\boldsymbol{\theta},\boldsymbol{l})^2} \leq c_4$. This result implies in particular that for all $t \in [0,T]$, $\frac{Z_t^Q(\boldsymbol{\theta},\boldsymbol{l})^2}{\mathbb{E}\left[Z_T^Q(\boldsymbol{\theta},\boldsymbol{l})^2|\mathcal{F}_t\right]} \geq \frac{1}{c_4} > 0$. We conclude by Assumption 1.2 there exists a constant $\delta_1^n$ such that $v^{n,\boldsymbol{\theta},\boldsymbol{l}} \geq \delta_1^n$. Moreover using Jensen's inequality and Assumption 1.2, there exists a positive constant $\delta_2^n$ such that for all $t \in [0,T]$: $v_t^{n,\boldsymbol{\theta},\boldsymbol{l}} \leq \delta_2^n$.

**(Ek) problems:** Now, assume that the solution exists for $\tilde{k} := k+1$ with $k \in \{0, 1, \ldots, n-1\}$ (our recursive hypothesis), we have to show that it is still true for $\tilde{k} - 1 := k$. We prove that the problem is equivalent to a problem of BSDE with quadratic growth and bounded terminal condition, therefore using Kobylanski's results in [8], we will get the result. Hence, the proof is divided in two parts. Firstly, we will give results for a modified quadratic BSDE. Secondly, we will use comparison theorem of quadratic BSDE to show that the first component solution of the modified BSDE is non negative and we will conclude the proof. Let define, so, the modified BSDE for $k \in \{0, 1, \dots, n-1\}$ given by:

$$dv_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} = -\overline{g_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}}(v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, \overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}})dt + \overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}dW_t \tag{2.25}$$

with generator given by

$$\begin{aligned}\overline{g_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}} &= \int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) \\ &- \frac{\left(\mu_t^k|v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| + \sigma_t^k\overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}} + \int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})\right)^2}{(\sigma_t^k)^2|v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| + \int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}(\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}))^2\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})}.\end{aligned}$$

Using our recursive hypothesis that there exists constants $\delta_1^{k+1}$ and $\delta_2^{k+1}$ such that

$$0 < \delta_1^{k+1} \le v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} \le \delta_2^{k+1}.$$

and Assumption 1.2, we have that there exists a constant $C > 0$ such that:

$$|\overline{g_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}}| \quad \le \quad C\left[1 + |v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| + |\overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}|^2\right]. \tag{2.26}$$

Therefore this coefficient follows a quadratic growth (with respect to $\overline{\beta^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}$) and linear growth (with respect to $v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$), using Kobylanski Theorem [8], there exists a pair $(v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, \overline{\beta^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}) \in \mathcal{S}^\infty \times \text{BMO}$ solution of this modified BSDE. Let now find a suitable lower bound of the coefficient $\overline{g^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}}$. Let first define:

$$e_t^k = \int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) \qquad , \qquad l_t^k = 2\left(\frac{\mu_t^k}{\sigma_t^k} + \frac{\sigma_t^k e_t^k}{d_t^k}\right) \tag{2.27}$$

$$d_t^k = \int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}(\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}))^2\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) \qquad and \qquad c_t^k = \frac{2\mu_t^k e_t^k}{d_t^k} + \left(\frac{\mu_t^k}{\sigma_t^k}\right)^2 \tag{2.28}$$

Using (2.26), we find $-\overline{g^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}} = K_t^0 + K_t^1 + K_t^2 + K_t^3$ where

$$K_t^0 = -\int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})$$

$$K_t^1 = \frac{\left(\mu_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + e_t^k\right)^2}{(\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2|v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| + d_t^k} \le \left(\frac{\mu_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}{\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}\right)^2|v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| + \frac{2\mu_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}|v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}|e_t^k}{d_t^k} + \frac{(e_t^k)^2}{d_t^k}$$

$$K_t^2 = \frac{(\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}})^2}{(\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2|v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| + d_t^k} \le \frac{|\overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}|^2}{|v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}|}$$

and

$$K_t^3 = \frac{2\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}(\mu_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}|v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| + e_t^k)}{(\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2|v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| + d_t^k} \le 2\frac{\mu_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}{\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}\overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}} + 2\frac{\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}e_t^k}{d_t^k}.$$

Since the processes $\mu^k, \sigma^k, \gamma^k, v^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}$ are bounded from Assumption 1.2 and our recursive hypothesis at step $k+1$, we conclude that the processes $l^k$ and $c^k$ are bounded too. Using the expressions of $K^0$, $K^1$, $K^2$ and $K^3$, we obtain:

$$-\overline{g^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}} \le \frac{|\overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}|^2}{|v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}|} + c_t^k|v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| + l_t^k\overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}} + \frac{(e_t^k)^2}{d_t^k} - \int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})$$

Using Cauchy's inequality on the expression of $e_t^k$, we find:

$$\begin{aligned}
(e_t^k)^2 &= \left(\int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} \gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}) \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})\right)^2 \\
&\le \int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} \left(\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})\right)^2 \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) \int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})
\end{aligned}$$

then we get:

$$\begin{aligned}
&\frac{(e_t^k)^2}{d_t^k} - \int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) \\
&= \frac{\left(\int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} \gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}) \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})\right)^2}{\int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} \left(\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})\right)^2 \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})} - \int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) \le 0.
\end{aligned}$$

Hence we obtain a suitable lower bound $\bar{f}_t^k$ of the generator $\overline{g_t^{k,(1)}}$

$$\overline{g^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}} \ge \bar{f}_t^k := -\left(c_t^k |v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| + l_t^k \overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}} + \frac{\overline{|\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}|}^2}{|v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}|}\right).$$

Hence if we consider now the following BSDE:

$$d\bar{Y}_t = \left(c_t^k \bar{Y}_t + l_t^k \bar{Z}_t^k + \frac{|\bar{Z}_t^k|^2}{\bar{Y}_t}\right) dt + \bar{Z}_t^k dW_t, \quad \bar{Y}_T = \alpha_T^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) \in (0,1).$$

then from Proposition 5.1 of [10], there exists a pair $(\bar{Y}, \bar{Z}) \in \mathcal{S}^\infty \times \mathrm{BMO}$ solution of the BSDE:

$$d\bar{Y}_t = -\bar{f}_t^k dt + \bar{Z}_t dW_t, \quad \bar{Y}_T = \alpha_T^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k).$$

with $\bar{Y} \ge \delta_1^k$ and the coefficient $\bar{f}^k$ satisfies a quadratic growth (with respect to $\bar{Z}$) and linear growth (with respect to $\bar{Y}$). Since $\overline{g^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}} \ge \bar{f}^k$, applying finally comparison theorem of Kobylanski [8], then the first component's solution of the modified BSDE (2.25) gives

$$v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \ge \bar{Y}_t \ge \delta_1^k > 0.$$

Therefore the modified BSDE is equivalent to the first BSDE of the **(Ek)** problem (2.20), then we get the proof of the existence of the solution of this first BSDE.

Moreover, to obtain the upper bound $\delta_2^k$ of $v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$, we take the terminal condition of the corresponding BSDE: $v_T^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} = \alpha_T^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) := \delta_2^k$. These prove that there exist well constants $\delta_1^k$ and $\delta_2^k$ such that

$$0 < \delta_1^k \le v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \le \delta_2^k.$$

**Second BSDE:** **(En) problem:** Following the resolution of the existence of the first BSDE for $k = n$ and (2.22), we obtain an explicit solution of the second BSDE which is given by

$$Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}} = \mathbb{E}\left[\frac{Z_T^Q(\boldsymbol{\theta}, \boldsymbol{l})}{Z_t^Q(\boldsymbol{\theta}, \boldsymbol{l})} H_T^n(\boldsymbol{\theta}, \boldsymbol{l}) \Big| \mathcal{F}_t\right]. \tag{2.29}$$

Since for all $(\boldsymbol{\theta}, \boldsymbol{l}) \in \Delta_n(T) \times E^n$, $H^n(\boldsymbol{\theta}, \boldsymbol{l}) \in L^\infty$ by assumption on the contingent claim, then from (2.29), we find $Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}} \in \mathcal{S}^\infty$. Moreover, we have a representation Theorem

$$Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}} = H_T(\boldsymbol{\theta}, \boldsymbol{l}) - \int_t^T Z_s^{n,\boldsymbol{\theta},\boldsymbol{l}} dW_s^{Q(\boldsymbol{\theta},\boldsymbol{l})}, \quad t \in [\theta_n, T] \tag{2.30}$$

where $W_s^{Q(\boldsymbol{\theta},\boldsymbol{l})} = W_s - \frac{\mu_s^{n,\boldsymbol{\theta},\boldsymbol{l}}}{\sigma_s^{n,\boldsymbol{\theta},\boldsymbol{l}}}$ is a $Q(\boldsymbol{\theta}, \boldsymbol{l})$ Brownian motion. For any stopping times $\theta_n \leq \tau \leq T$ and from (2.30), there exists a constant $d > 0$ such that

$$\mathbb{E}^{Q(\boldsymbol{\theta},\boldsymbol{l})}\left[\int_\tau^T \left(Z_s^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2 ds|\mathcal{F}_\tau\right] \leq \mathbb{E}^{Q(\boldsymbol{\theta},\boldsymbol{l})}\left[\left(H_T^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2 |\mathcal{F}_\tau\right] \leq d.$$

Then $Z^{n,\boldsymbol{\theta},\boldsymbol{l}}.W^{Q(\boldsymbol{\theta},\boldsymbol{l})}$ is a BMO-martingale under the probability measure $Q(\boldsymbol{\theta}, \boldsymbol{l})$, so $Z^{n,\boldsymbol{\theta},\boldsymbol{l}}.W$ is a BMO martingale under the probability measure $P$ from Kazamaki [7] Theorem 3.3. Therefore we conclude $Z^{n,\boldsymbol{\theta},\boldsymbol{l}} \in \text{BMO}$.

**(Ek) problems:** Now, assume that the solution exists for $\tilde{k} := k+1$ with $k \in \{0, 1, \ldots, n-1\}$ (our recursive hypothesis), we have to show that it is still true for $\tilde{k} - 1 := k$. We would like now to prove that $\left(Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right) \in \mathcal{S}^\infty \times \text{BMO}$ for all $k \in \{0, 1, \ldots, n\}$. We can actually prove the existence of the solution of the second BSDE, since the solution of the first one exists. Given the solution of the first BSDE, the coefficient of the second one is linear. Therefore, we can characterize explicitly the solution.

Step 1: Preliminary results.

Given the explicit formula of the coefficient $g^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(2)}$ in (2.20), we get

$$g_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(2)} = a_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \kappa_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \Lambda_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.$$

with

$$a_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} = \beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} - \sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \frac{\left(\mu_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \int_E \gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})(1 + v_t^{J,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})\right)}{\left(\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)^2 + \int_E (1 + v_t^{J,\boldsymbol{\theta}_k,\boldsymbol{l}_k})(\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}))^2 \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})},$$

$$\begin{aligned}
\kappa_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} &= -\int_E (1 + v_t^{J,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) + \int_E (1 + v_t^{J,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) \\
&\times \frac{\left(\mu_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \int_E (1 + v_t^{J,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})\right)}{\left(\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)^2 + \int_E (1 + v_t^{J,\boldsymbol{\theta}_k,\boldsymbol{l}_k})(\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}))^2 \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})}
\end{aligned}$$

and

$$\begin{aligned}
\Lambda_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} &= \int_E (1 + v_t^{J,\boldsymbol{\theta}_k,\boldsymbol{l}_k})Y_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) \\
&- \int_E (1 + v_t^{J,\boldsymbol{\theta}_k,\boldsymbol{l}_k})Y_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) \\
&\times \frac{\left(\mu_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \int_E (1 + v_t^{J,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})\right)}{\left(\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)^2 + \int_E (1 + v_t^{J,\boldsymbol{\theta}_k,\boldsymbol{l}_k})(\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}))^2 \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})}.
\end{aligned}$$

Under Assumption 1.2 and the integrability condition 1.7, coefficients $\sigma^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$, $\mu^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ and $\gamma^k$ are bounded. Moreover from the solution of the first BSDE and the boundness of the processes $v^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}$ and $Y_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}$ (recursive hypothesis), we have that the processes $v^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ are bounded for all $\boldsymbol{l}_k \in E^k$ and $\beta^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W$ is a BMO martingale.
Therefore we deduce that the martingales $\Lambda^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W$, $a^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W$ and $\kappa^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W$ are BMO under the probability measure $P$. Let define the probability measure $Q \sim P$ with Radon Nikodym density on $\mathcal{F}_T$ defined by $Z_T^Q = \mathcal{E}(a^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W)_T$. Since the martingale $a^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W$ is BMO, the process $Z_t^Q = \mathbb{E}\left[Z_T^Q|\mathcal{F}_t\right]$ is uniformly integrable and from Theorem 3.3 of Kazamaki [7], the martingale $\kappa^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W$ is still BMO under the probability measure $Q$. Therefore, there exists a non negative constant $c$ such that $\mathbb{E}^Q\left[\int_t^T |\kappa_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}|^2 ds|\mathcal{F}_t\right] \le c$, for all $\theta_k \le t \le T$ and $k \in \{0,1,\dots,n\}$.

Step 2: Integrability of the adjoint process $\Gamma$:

Let define for all $k \in \{0,1,\dots,n\}$

$$\widetilde{\Gamma}_t := \exp\left(\int_0^t \kappa_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} ds\right).$$

We prove that $\widetilde{\Gamma} \in \mathrm{L}^p(Q)$ for any $p > 1$ and $\delta > 0$:

$$\begin{aligned}\left|\frac{\widetilde{\Gamma}_T}{\widetilde{\Gamma}_t}\right|^p &= \exp\left(p\int_t^T \kappa_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} ds\right) \le \exp\left(\int_t^T \left(\delta(\kappa_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \frac{p^2}{4\delta}\right) ds\right)\\ &\le \exp\left(\frac{p^2}{4\delta}T\right)\exp\left(\delta\int_t^T (\kappa_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 ds\right).\end{aligned}$$

Since there exists a non-negative constant $c$ such that

$$\mathbb{E}^Q\left[\int_t^T |\kappa_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}|^2 ds|\mathcal{F}_t\right] \le c$$

we deduce form Proposition 3.1 in Appendix that there exists $0 \le \delta \le \frac{1}{c^2}$ such that $\mathbb{E}^Q\left[\exp(\int_t^T \delta|(\kappa_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2|ds)|\mathcal{F}_t\right] \le \frac{1}{1-\delta c^2}$. Therefore we conclude there exists a non negative constant $C_1$ such that

$$\mathbb{E}^Q\left[\left|\frac{\widetilde{\Gamma}_T}{\widetilde{\Gamma}_t}\right|^p|\mathcal{F}_t\right] \le C_1. \tag{2.31}$$

Step 3: The solution of the BSDE.

Let define now for $k \in \{0,1,\dots,n-1\}$

$$Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} = \mathbb{E}^Q\left[\frac{1}{\widetilde{\Gamma}_t}\left(\widetilde{\Gamma}_T H_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) + \int_t^T \widetilde{\Gamma}_s \Lambda_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} ds\right)|\mathcal{F}_t\right], \quad \theta_k \le t \le T. \tag{2.32}$$

Since $\Gamma = Z^Q\widetilde{\Gamma}$, using Bayes formula equation (2.32) is equivalent to

$$Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} = \mathbb{E}\left[\frac{1}{\Gamma_t}\left(\Gamma_T H_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) + \int_t^T \Gamma_s\Lambda_s ds\right)|\mathcal{F}_t\right], \quad t \le T. \tag{2.33}$$

Moreover since $\Lambda^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ is bounded and $H_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\in \mathrm{L}^\infty$, there exists a non negative constant $C$ such that

$$|Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}|\leq C\mathbb{E}^Q\left(\left[\left|\frac{\widetilde{\Gamma}_T}{\widetilde{\Gamma}_t}\right|+\int_t^T\left(\left|\frac{\widetilde{\Gamma}_s}{\widetilde{\Gamma}_t}\right|^2+(\Lambda_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2\right)ds\right]\Big|\mathcal{F}_t\right)$$

Since the process $\Lambda^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W^Q$ is a BMO martingale under the probability measure $Q$ and using (2.31),there exists a constant $\bar{C}>0$ such that:

$$|Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}|\leq \bar{C},\quad t\leq T.$$

Let consider $Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ defined by (2.32), then the process

$$\widetilde{\Gamma}_tY_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}+\int_0^t\Lambda_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\widetilde{\Gamma}_sds=\mathbb{E}^Q\left[\widetilde{\Gamma}_TH_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)+\int_0^T\widetilde{\Gamma}_s\Lambda_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}ds|\mathcal{F}_t\right]$$

is a squared integrable $Q$-martingale since $H^k$ is bounded by assumption, $\Lambda^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W$ is BMO and $\widetilde{\Gamma}$ satisfies (2.31). Therefore from representation theorem, there exists a process $\bar{Z}\in\mathcal{H}^2$ such that $d(\widetilde{\Gamma}_tY_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}+\int_0^t\widetilde{\Gamma}_s\Lambda_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}ds)=\bar{Z}_tdW_t^Q$. Setting $Z^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}=\frac{\bar{Z}}{\widetilde{\Gamma}}$, using integration by part formula we find:

$$dY_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}=-(\Lambda_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}+Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}a_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}+\kappa_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})dt+Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}dW_t,\quad Y_T^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}=H_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k).$$

Applying Itô's formula, we find

$$d(Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2=2Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}[-(\Lambda_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}+\kappa_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})dt+Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.dW_t^Q]+(Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2dt,$$

therefore, for any stopping time $\sigma$, we find:

$$\mathbb{E}^Q\left[\int_\sigma^T(Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2dt\Big|\mathcal{F}_\sigma\right]\leq\mathbb{E}^Q\left[(H_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2+2\int_\sigma^T2Y_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(\Lambda_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}+\kappa_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}Y_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})ds\Big|\mathcal{F}_\sigma\right].$$

Since $H^k, Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ are bounded, $\Lambda^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W^Q$ and $\kappa^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W^Q$ are BMO martingales under the probability measure $Q$, we conclude $Z^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W^Q$ is a BMO martingale measure under $Q$ then $Z^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W$ is a BMO martingale under the probability measure $P$ from Kazamaki [7] Theorem 3.3. Therefore we conclude $(Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k},Z^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\in\mathcal{S}^\infty\times$ BMO is a solution of the second BSDE.

**Third BSDE: (En) problem:** Since $g_t^{3,\boldsymbol{\theta},\boldsymbol{l}}\equiv 0$, we have directly $\xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\equiv 0$.

**(Ek) problems:** Now, assume that the solution exists for $\tilde{k}:=k+1$ with $k\in\{0,1,\ldots,n-1\}$ (our recursive hypothesis), we have to show that it is still true for $\tilde{k}-1:=k$. It lets to prove that $\left(\xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k},R_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)\in\mathcal{S}^\infty\times$ BMO. Since, for all $k\in\{0,1,\ldots,n\}$, all the terms appearing in the coefficient $g_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(3)}$ are bounded and $Z^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\in$ BMO by previous step, we conclude using representation Theorem that $(\xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k},R^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\in\mathcal{S}^\infty\times$ BMO, for all $k\in\{0,1,\ldots,n\}$.

□

## 2.2 BSDEs characterization by verification theorem

Now, we show that the triple $\left(v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, \xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)$, appearing in the quadratic decomposition form, solution to the recursive system indexed BSDEs provides actually the solution to the global optimal investment problem in terms of the value functions $V^k$, $k \in \{0,1,\dots,n\}$ in (2.16) and (2.19). As a byproduct, we will obtain the existence of the optimal strategy $\pi^{k,*}$.

**Theorem 2.2.** *The value functions $V^k$ , $k = 0,\dots,n$ defined in* (2.16) *and* (2.19) *are given, for all $t \in [\theta_k, T]$, by*

$$V_t^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k) = v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_t^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \tag{2.34}$$

*for all $x \in \mathbb{R}$, $(\boldsymbol{\theta}_k,\boldsymbol{l}_k) \in \Delta_k \times E^k$, $\nu^k \in \mathcal{A}_{\mathbb{F}}^k$ where $(v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, \xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})$ is the unique solution of the recursive triple BSDEs systems given for all $k = \{0,1,\dots,n\}$ in 2.17 and 2.20.*

*In particular, the solution of the Mean Variance Hedging problem is given by*

$$V_0^H(x) \;=\; \inf_{\pi\in\mathcal{A}_{\mathbb{G}}} \mathbb{E}\left[(H_T - X_T^{x,\pi})^2\right] = v_0^0(x - Y_0^0)^2 + \xi_0^0, \quad x \in \mathbb{R}. \tag{2.35}$$

*where the triple $\left(v^0, Y_0^0, \xi_0^0\right)$ is solution of the recursive system of BSDEs:* **(En)***:* (2.17) *and* **(Ek)***:* (2.20)*, $k \in \{0,1,\dots,n-1\}$.*

*Moreover, there exists an optimal strategy $\pi^* := \left(\pi^{0,*},\pi^{1,*},\dots,\pi^{n,*}\right)$ given by:*

$$\begin{aligned}\pi_t^{k,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) \;&=\; \frac{1}{(\sigma_t^k)^2 + \int_E (1+v_t^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1})^2\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})}\Big[\sigma_t^k Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} - K_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(\mu_t^k + \sigma_t^k\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}) \\ &+\; \frac{\int_E (X_t^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} - Y^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}})\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})}{v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}\Big]\end{aligned} \tag{2.36}$$

*with $K_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} := X_t^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$. And for the after last default problem:*

$$\pi_t^{n,*}(\boldsymbol{\theta},\boldsymbol{l}) = \frac{1}{(\sigma_t^n)^2}\left[\sigma_t^n Z_t^{n,\boldsymbol{\theta},\boldsymbol{l}} - \left(X_t^{n,x}(\boldsymbol{\theta},\boldsymbol{l}) - Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)\left(\mu_t^n + \sigma_t^n\beta_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)\right] \tag{2.37}$$

**Remark 2.5.** *Following* (2.35)*, we can give some financial comments of our quadratic decomposition form:*

- *The process $v^0$ doesn't depend on the payoff $H$. Moreover, we have that*

$$v_0^0 = V_0^0(1) := \inf_{\pi\in\mathcal{A}_{\mathbb{G}}} \mathbb{E}\left[X_T^{1,\pi}\right]^2.$$

  *Therefore $v^0$ is related to the minimal variance of a pure investment on the asset $S$ with an initial wealth $x = 1$.*

- *The process $Y^0$ is the quadratic approximation price of the option H.*

- *The process $\xi^0$ represents the incompleteness of this market. Since if the market is complete (as in the* **(En)** *problem) then this process vanishes.*

*Proof.* **Step1:** We begin by proving for all $k = \{0,1,\dots,n\}$, $t \in [\theta_k, T]$ and $\nu^k \in \mathcal{A}_{\mathbb{F}}^k$, that

$$v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_t^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \le V_t^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k) \tag{2.38}$$

Let denote by $D^k$ the process defined for all $k = \{0, \dots, n-1\}$, $t \in [\theta_k, T]$ and $\nu^k \in \mathcal{A}^k_{\mathbb{F}}$ by

$$D^k_t(x, \boldsymbol{\theta}_k, \boldsymbol{l}_k, \nu^k) \quad := \quad v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t \big(X^{k,x}_t(\boldsymbol{\theta}_k, \boldsymbol{l}_k) - Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t\big)^2 + \xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t \tag{2.39}$$

$$+ \int_{\theta_k}^t \int_E \Big( v^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_s \big(X^{k,x}_s(\boldsymbol{\theta}_k, \boldsymbol{l}_k) + \pi^{k,*}_s \gamma^k_s(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}) - Y^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_s\big)^2 + \xi^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_s \Big) \eta(\boldsymbol{l}_k, dl_{k+1}) ds$$

and $D^n_t(x, \boldsymbol{\theta}, \boldsymbol{l}, \nu^n) := v^{n,\boldsymbol{\theta},\boldsymbol{l}}_t \big(X^{n,x}_t(\boldsymbol{\theta}, \boldsymbol{l}) - Y^{n,\boldsymbol{\theta},\boldsymbol{l}}_t\big)^2 + \xi^{n,\boldsymbol{\theta},\boldsymbol{l}}_t$.

Since $D^k$ is a local submartingale, let $T_i$ be a localizing $\mathbb{F}$-stopping times sequence valued in $[\theta_k, T]$ for $D^k_t$, we have for all $\theta_k \le t \le s \le T$

$$D^k_{t \wedge T_i}(x, \boldsymbol{\theta}_k, \boldsymbol{l}_k, \nu^k) \quad \le \quad \mathbb{E}\Big[D^k_{s \wedge T_i}(x, \boldsymbol{\theta}_k, \boldsymbol{l}_k, \nu^k) | \mathcal{F}_t\Big]$$

Now using Definition 1.1 of the admissibility condition for $\nu^k$, Assumption 1.2, the fact that $Y^{n,\boldsymbol{\theta},\boldsymbol{l}}$, $\xi^{n,\boldsymbol{\theta},\boldsymbol{l}}$ are squared integrable and $v^{n,\boldsymbol{\theta},\boldsymbol{l}}$ is bounded, we obtain that the sequence $\Big(D^k_{s \wedge T_i}(x, \boldsymbol{\theta}_k, \boldsymbol{l}_k, \nu^k)\Big)_i$ is uniformly integrable for $s \in [\theta_k, T]$, and so we obtain the submartingale property for $D^k$. Writing now, this submartingale property between time t and T and recalling terminal conditions of the three BSDEs, we obtain the expected results which are for all $\nu^k \in \mathcal{A}^k_{\mathbb{F}}$ and $k \in \{0, 1, \dots, n-1\}$

$$v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t \big(X^{k,x}_t(\boldsymbol{\theta}_k, \boldsymbol{l}_k) - Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t\big)^2 + \xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t \quad \le \quad \mathbb{E}\Big[(H^k_T(\boldsymbol{\theta}_k, \boldsymbol{l}_k) - X^{k,x}_T(\boldsymbol{\theta}_k, \boldsymbol{l}_k))^2 \alpha^k_T(\boldsymbol{\theta}_k, \boldsymbol{l}_k) | \mathcal{F}_t\Big] \tag{2.40}$$

$$+\mathbb{E}\Bigg[\int_t^T \int_E V^{k+1}_{\theta_{k+1}}(X^{k,x}_{\theta_{k+1}}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) + \pi^k_{\theta_{k+1}}(\boldsymbol{\theta}_k, \boldsymbol{l}_k).\gamma^k_{\theta_{k+1}}(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}), \boldsymbol{\theta}_{k+1}, \boldsymbol{l}_{k+1}) \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) d\theta_{k+1} | \mathcal{F}_t\Bigg]$$

and for $k = n$

$$v^{n,\boldsymbol{\theta}_n,\boldsymbol{l}_n}_t \big(X^{n,x}_t(\boldsymbol{\theta}, \boldsymbol{l}) - Y^{n,\boldsymbol{\theta}_n,\boldsymbol{l}_n}_t\big)^2 + \xi^{n,\boldsymbol{\theta}_n,\boldsymbol{l}_n}_t \quad \le \quad \mathbb{E}\Big[(H^n_T - X^{n,x}_T(\boldsymbol{\theta}, \boldsymbol{l}))^2 \alpha_T(\boldsymbol{\theta}, \boldsymbol{l}) | \mathcal{F}_t\Big] \tag{2.41}$$

**Step2:** We need now to check that the trading strategy $\pi^* = (\pi^{k,*})_{k=0,\dots,n}$ is admissible in the sense of Definition 1.1. For more readability, we forget the dependence parameter $(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ for $\pi^{k,*}_t(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ and we will use the simpler notation $\pi^{k,*}_t$. We recall $(D^k_t)_{t \in [0,T]}$, the local martingale (since we take this quantity with the optimal strategy $\pi^*$) is defined in (2.39) for all $k = \{0, \dots, n-1\}$ and $t \in [\theta_k, T]$ by

$$D^k_t(x, \boldsymbol{\theta}_k, \boldsymbol{l}_k, \pi^{k,*}) \quad = \quad v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t \big(X^{k,x,*}_t(\boldsymbol{\theta}_k, \boldsymbol{l}_k) - Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t\big)^2 + \xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t$$

$$+ \int_{\theta_k}^t \int_E \Big( v^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_s \big(X^{k,x,*}_s(\boldsymbol{\theta}_k, \boldsymbol{l}_k) + \pi^{k,*}_s \gamma^k_s(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}) - Y^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_s\big)^2 + \xi^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_s \Big) \eta(\boldsymbol{l}_k, dl_{k+1}) ds$$

Let $T_i$ be a localizing $\mathbb{F}$-stopping times sequence valued in $[\theta_k, T]$ for the local martingale $D^k_t$, then

$$D^k_{t \wedge T_i}(x, \boldsymbol{\theta}_k, \boldsymbol{l}_k, \pi^{k,*}) \quad = \quad v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{t \wedge T_i} \big(X^{k,x,*}_{t \wedge T_i}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) - Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{t \wedge T_i}\big)^2 + \xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{t \wedge T_i}$$

$$+ \int_{\theta_k}^{t \wedge T_i} \int_E \Big( v^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_s \big(X^{k,x,*}_s(\boldsymbol{\theta}_k, \boldsymbol{l}_k) + \pi^{k,*}_s \gamma^k_s(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}) - Y^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_s\big)^2 + \xi^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_s \Big) \eta(\boldsymbol{l}_k, dl_{k+1}) ds$$

Since $(D^k)$ is a local martingale, taking the expectation, we get

$$\mathbb{E}\left[v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{t\wedge T_i}(X^{k,x,*}_{t\wedge T_i}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)-Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{t\wedge T_i})^2+\xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{t\wedge T_i}|\mathcal{F}_{\theta_k}\right] = v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{\theta_k}(X^{k,x,*}_{\theta_k}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)-Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{\theta_k})^2+\xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{\theta_k} \tag{2.42}$$

$$-\mathbb{E}\left[\int_{\theta_k}^{t\wedge T_i}\int_E\left(v^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_s(X^{k,x,*}_s(\boldsymbol{\theta}_k,\boldsymbol{l}_k)+\pi^{k,*}_s\gamma^k_s(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1})-Y^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_s)^2+\xi^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_s\right)\eta(\boldsymbol{l}_k,dl_{k+1})ds|\mathcal{F}_{\theta_k}\right]$$

By recursive backward induction and using Theorem 3.2, we have for all $k=\{0,\dots,n-1\}$ that $v^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_s(X^{k,x,*}_s(\boldsymbol{\theta}_k,\boldsymbol{l}_k)+\pi^{k,*}_s\gamma^k_s(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1})-Y^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_s)^2+\xi^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_s$ is positive for all $s\in[0,T]$. Hence we obtain for all $t\in[\theta_k,T]$ that

$$\begin{aligned}\mathbb{E}\left[v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{t\wedge T_i}(X^{k,x,*}_{t\wedge T_i}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)-Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{t\wedge T_i})^2+\xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{t\wedge T_i}|\mathcal{F}_{\theta_k}\right] &\le v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{\theta_k}(X^{k,x,*}_{\theta_k}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)-Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{\theta_k})^2+\xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{\theta_k}\\ &\le v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{\theta_k}(x-Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{\theta_k})^2+\xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{\theta_k}<\infty \quad (2.43)\end{aligned}$$

Using Theorem 2.1, we know that there exists a positive constant $\delta$ such that $v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t\ge\delta$ for all $t\in[\theta_k,T]$. Letting now $i\to\infty$, it follows from Fatou's Lemma and similarly as in the proof of Proposition 3.2 in [10] that

$$\mathbb{E}\left[v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{t\wedge T_i}(X^{k,x,*}_{t\wedge T_i}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)-Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{t\wedge T_i})^2+\xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_{t\wedge T_i}|\mathcal{F}_t\right] \ge \tilde{\delta}\left(\mathbb{E}\left[|X^{k,x,*}_t(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|^2\right]+1\right)$$

Hence, we obtain that there exist constants $c_1$ and $c_2$ such that

$$\mathbb{E}\left[|X^{k,x,*}_T(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|^2\right]\le c_1 \quad\text{and}\quad \mathbb{E}\left[\int_{\theta_k}^T|X^{k,x,*}_s(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|^2ds\right]<c_2 \tag{2.44}$$

We need now to prove that this inequality implies Definition 1.1. Indeed, applying Itô formula to $\left(X^{k,x,*}_t(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\right)^2$ gives

$$d\left(X^{k,x,*}_t(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\right)^2=2X^{k,x,*}_{t-}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)dX^{k,x,*}_t(\boldsymbol{\theta}_k,\boldsymbol{l}_k)+d\left[X^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\right]_t$$

Using the dynamic of $X^{k,x,*}_t(\boldsymbol{\theta}_k,\boldsymbol{l}_k)$ and let $(T_i)_{i\in\mathbb{N}}$ be a sequence of localizing time, we get

$$x^2+\mathbb{E}\left[\int_{\theta_k}^{T\wedge T_i}|\pi^{k,*}_s|^2(\sigma^k_s(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2ds\right]\le\mathbb{E}\left[\left(X^{k,x,*}_{T\wedge T_i}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\right)^2\right]-2\mathbb{E}\left[\int_{\theta_k}^{T\wedge T_i}\pi^{k,*}_s\mu^k_s(\boldsymbol{\theta}_k,\boldsymbol{l}_k)X^{k,x,*}_s(\boldsymbol{\theta}_k,\boldsymbol{l}_k)ds\right] \tag{2.45}$$

Since, by assumption, processes $\mu^k_s(\boldsymbol{\theta}_k,\boldsymbol{l}_k)$ and $\sigma^k_s(\boldsymbol{\theta}_k,\boldsymbol{l}_k)$ are bounded. We obtain that there is a constant $K_2\le(\sigma^k_s(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2$ such that for all $s\in[0,T]$

$$-2\pi^{k,*}_s\mu^k_s(\boldsymbol{\theta}_k,\boldsymbol{l}_k)X^{k,x,*}_s(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\le\frac{2}{K_2}|X^{k,x,*}_s(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|^2|\mu^k_s(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|^2+\frac{K_2}{2}|\pi^{k,*}_s|^2 \tag{2.46}$$

Using (2.46) in (2.45) gives

$$\begin{aligned}x^2+\mathbb{E}\left[\int_{\theta_k}^{T\wedge T_i}|\pi^{k,*}_s|^2(\sigma^k_s(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2ds\right] &\le \mathbb{E}\left[\left(X^{k,x,*}_{T\wedge T_i}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\right)^2\right]\\ &+ \mathbb{E}\left[\int_{\theta_k}^{T\wedge T_i}\frac{2}{K_2}|X^{k,x,*}_s(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|^2|\mu^k_s(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|^2ds\right]+\mathbb{E}\left[\int_{\theta_k}^{T\wedge T_i}\frac{K_2}{2}|\pi^{k,*}_s|^2ds\right]\end{aligned}$$

Applying now Fatou's Lemma, when $i$ goes to infinity we get

$$x^2+\mathbb{E}\left[\int_{\theta_k}^T |\pi_s^{k,*}|^2(\sigma_s^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2ds\right] \leq \mathbb{E}\left[\left(X_T^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\right)^2\right] \tag{2.47}$$
$$+\frac{2}{K_2}\mathbb{E}\left[\int_{\theta_k}^T |X_s^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|^2|\mu_s^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|^2ds\right]+\frac{K_2}{2}\mathbb{E}\left[\int_{\theta_k}^T |\pi_s^{k,*}|^2ds\right]$$

Moreover since $K_2\leq(\sigma_s^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2$, we obtain

$$\frac{K_2}{2}\mathbb{E}\left[\int_{\theta_k}^T |\pi_s^{k,*}|^2ds\right]\leq\mathbb{E}\left[\left(X_T^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\right)^2-x^2\right]+\frac{2}{K_2}\mathbb{E}\left[\int_{\theta_k}^T |X_s^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|^2|\mu_s^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|^2ds\right]$$

Therefore, since by assumption $\mu_s^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)$ is bounded and by (2.44), we conclude that (1.12) is satisfied, which is that $\pi^{k,*}$ is admissible in sense of Definition 1.1.

**Step3:** We need to show that the wealth process $X_t^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)$ taken with the strategy $\pi_t^{k,*}$ exists for all $k\in\{0,1,\ldots,n\}$. Firstly, we can remark that the optimal strategy (2.36) admits a linear form with respect to $X_t^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)$ for $\theta_k\leq t\leq T$. Let denote this linear form as

$$\pi_t^{k,*}=a_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)X_t^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)+d_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k),\quad \forall k\in\{0,1,\ldots,n\}$$

Then substituting this expression in (1.9) gives for $\theta_k\leq t\leq T$

$$\begin{aligned} dX_t^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) &= \left(a_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)X_t^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)+d_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\right)\left(\mu_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)dt+\sigma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)dW_t\right)\\ &= X_t^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\left(a_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\mu_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)dt+a_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\sigma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)dW_t\right) \quad (2.48)\\ &+ \left(d_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\mu_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)dt+d_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\sigma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)dW_t\right)\end{aligned}$$

We recall that the solution for $\theta_k\leq t\leq T$ of the SDE given by

$$d\phi_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)=\phi_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\left(a_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\mu_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)dt+a_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\sigma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)dW_t\right)$$

is $\phi_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)=\phi_{\theta_k}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\exp\left\{\int_{\theta_k}^T\left(a_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\mu_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)-\frac{1}{2}\left(a_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\sigma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\right)^2\right)dt\right\}$.
Therefore setting $X_t^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k):=L_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\phi_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)$ with $dL_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k):=\bar{\mu}_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)dt+\bar{\sigma}_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)dW_t$ and $L_{\theta_k}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)=1$ and applying integration by part formula we obtain for all $\theta_k\leq t\leq T$

$$\begin{aligned} dX_t^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) &= X_t^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\left[a_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\mu_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)dt+a_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\sigma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)dW_t\right]\\ &+ \phi_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\left[\left(\bar{\mu}_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)+a_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\sigma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\bar{\sigma}_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\right)dt+\bar{\sigma}_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)dW_t\right]\end{aligned}$$

Hence from (2.48), we get $\bar{\mu}_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)=\frac{d_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\left(\mu_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)-a_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\left(\sigma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\right)^2\right)}{\phi_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)}$ and $\bar{\sigma}_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)=\frac{d_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\sigma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)}{\phi_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)}$. Then we deduce that $X_t^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k):=L_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\phi_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)$ is a solution of the SDE (1.9).

**Step4:** We would like now to prove that the trading strategy $\pi^*=(\pi^{k,*})_{k=0,\ldots,n}$ is optimal. Since the trading strategy $\pi^*=(\pi^{k,*})_{k=0,\ldots,n}$ is admissible in the sense of Definition 1.1

and processes $D^k$ are "true" martingales for $k = \{0, \dots, n\}$, we have for all $(\boldsymbol{\theta}_k, \boldsymbol{l}_k) \in \Delta_k(T) \times E^k$, $x \in \mathbb{R}$, $t \in [\theta_k, T]$ and $k = \{0, 1, \dots, n-1\}$ that

$$v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\left(X_t^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)^2 + \xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} = \mathbb{E}\Big[(H_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - X_T^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2\alpha_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|\mathcal{F}_t\Big] \tag{2.49}$$
$$+\mathbb{E}\left[\int_t^T \int_E V_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) + \pi_{\theta_{k+1}}^{k,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k).\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}),\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})d\theta_{k+1}|\mathcal{F}_t\right]$$

and for $k = n$

$$v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\left(X_t^{n,x,*}(\boldsymbol{\theta},\boldsymbol{l}) - Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2 + \xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}} = \mathbb{E}\Big[(H_T^n(\boldsymbol{\theta},\boldsymbol{l}) - X_T^{n,x,*}(\boldsymbol{\theta},\boldsymbol{l}))^2\alpha_T(\boldsymbol{\theta},\boldsymbol{l})|\mathcal{F}_t\Big] \tag{2.50}$$

where $X^{n,x,*}(\boldsymbol{\theta},\boldsymbol{l})$ means that we take the strategy $\pi^* = (\pi^{k,*})_{k=0,\dots,n}$ to evaluate these wealth processes. Starting with $k = n$, let $F_t^n(\boldsymbol{\theta},\boldsymbol{l})$ be the process given by

$$F_t^n(\boldsymbol{\theta},\boldsymbol{l}) := \operatorname*{ess\,inf}_{\pi^n\in\mathcal{A}_{\mathbb{F}}^n(t,\nu^n)} \mathbb{E}\Big[(H_T^n(\boldsymbol{\theta},\boldsymbol{l}) - X_T^{n,x}(\boldsymbol{\theta},\boldsymbol{l}))^2\alpha_T(\boldsymbol{\theta},\boldsymbol{l}) - v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}((X_t^{n,x}(\boldsymbol{\theta},\boldsymbol{l}))^2 - 2X_t^{n,x}(\boldsymbol{\theta},\boldsymbol{l})Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}})|\mathcal{F}_t\Big]$$

By the submartingale property given in (2.41) we get

$$\begin{aligned}
F_t^n(\boldsymbol{\theta},\boldsymbol{l}) &:= \operatorname*{ess\,inf}_{\pi^n\in\mathcal{A}_{\mathbb{F}}^n(t,\nu^n)} \mathbb{E}\Big[(H_T^n(\boldsymbol{\theta},\boldsymbol{l}) - X_T^{n,x}(\boldsymbol{\theta},\boldsymbol{l}))^2\alpha_T(\boldsymbol{\theta},\boldsymbol{l}) - v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}((X_t^{n,x}(\boldsymbol{\theta},\boldsymbol{l}))^2 - 2X_t^{n,x}(\boldsymbol{\theta},\boldsymbol{l})Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}})|\mathcal{F}_t\Big] \\
&\geq v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\left(X_t^{n,x}(\boldsymbol{\theta},\boldsymbol{l}) - Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2 + \xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}} - v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\Big((X_t^{n,x}(\boldsymbol{\theta},\boldsymbol{l}))^2 - 2X_t^{n,x}(\boldsymbol{\theta},\boldsymbol{l})Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\Big) \\
&= v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\left(Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2 + \xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}} \\
&= v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\left(X_t^{n,x,*}(\boldsymbol{\theta},\boldsymbol{l}) - Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2 + \xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}} - v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\Big((X_t^{n,x,*}(\boldsymbol{\theta},\boldsymbol{l}))^2 - 2X_t^{n,x,*}(\boldsymbol{\theta},\boldsymbol{l})Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\Big)
\end{aligned}$$

Using now the martingale property when we take the strategy (i.e. (2.50)), we obtain

$$\begin{aligned}
F_t^n(\boldsymbol{\theta},\boldsymbol{l}) &\geq \mathbb{E}\Big[(H_T^n(\boldsymbol{\theta},\boldsymbol{l}) - X_T^{n,x,*}(\boldsymbol{\theta},\boldsymbol{l}))^2\alpha_T(\boldsymbol{\theta},\boldsymbol{l}) - v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}((X_t^{n,x,*}(\boldsymbol{\theta},\boldsymbol{l}))^2 - 2X_t^{n,x,*}(\boldsymbol{\theta},\boldsymbol{l})Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}})|\mathcal{F}_t\Big] \\
&\geq \operatorname*{ess\,inf}_{\pi^n\in\mathcal{A}_{\mathbb{F}}^n(t,\nu^n)} \mathbb{E}\Big[(H_T^n(\boldsymbol{\theta},\boldsymbol{l}) - X_T^{n,x}(\boldsymbol{\theta},\boldsymbol{l}))^2\alpha_T(\boldsymbol{\theta},\boldsymbol{l}) - v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}((X_t^{n,x}(\boldsymbol{\theta},\boldsymbol{l}))^2 - 2X_t^{n,x}(\boldsymbol{\theta},\boldsymbol{l})Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}})|\mathcal{F}_t\Big] \\
&= F_t^n(\boldsymbol{\theta},\boldsymbol{l})
\end{aligned}$$

Hence $F_t^n(\boldsymbol{\theta},\boldsymbol{l}) = v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\left(Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2 + \xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}}$. Combining with its definition we finally get the first expected result

$$\begin{aligned}
V_t^n(x,\boldsymbol{\theta},\boldsymbol{l},\nu^n) &:= \operatorname*{ess\,inf}_{\pi^n\in\mathcal{A}_{\mathbb{F}}^n(t,\nu^n)} \mathbb{E}\Big[(H_T^n(\boldsymbol{\theta},\boldsymbol{l}) - X_T^{n,x}(\boldsymbol{\theta},\boldsymbol{l}))^2\alpha_T(\boldsymbol{\theta},\boldsymbol{l})|\mathcal{F}_t\Big] \\
&= v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\left(X_t^{n,x}(\boldsymbol{\theta},\boldsymbol{l}) - Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2 + \xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}}
\end{aligned}$$

Let now $k = \{0, 1, \dots, n-1\}$ and assume that (2.34) holds true at step $k+1$. Then we observe similarly as above that for any $t \in [\theta_k, T]$, $\pi^k \in \mathcal{A}_{\mathbb{F}}^k(t,\nu^k)$

$$\begin{aligned}
F_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) &:= \operatorname*{ess\,inf}_{\pi^k\in\mathcal{A}_{\mathbb{F}}^k(t,\nu^k)} \mathbb{E}\Big[(H_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - X_T^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2\alpha_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) \\
&- v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}((X_t^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2 - 2X_t^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}) \\
&+ \int_t^T \int_E V_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) + \pi_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k).\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}),\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})d\theta_{k+1}|\mathcal{F}_t\Big]
\end{aligned}$$

By again the submartingale property given in (2.40) we get

$$\begin{array}{rcl} F_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) & \geq & v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_t^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \\ & - & v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\Big((X_t^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2 - 2X_t^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\Big) \\ & = & v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\Big(Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\Big)^2 + \xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \\ & = & v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_t^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \\ & - & v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}((X_t^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2 - 2X_t^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}) \end{array}$$

Using the martingale property (2.49), we obtain

$$\begin{array}{rcl} F_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) & \geq & \mathbb{E}\Big[(H_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - X_T^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2\alpha_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) \\ & - & v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}((X_t^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2 - 2X_t^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}) \\ & + & \displaystyle\int_t^T\int_E V_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) + \pi_{\theta_{k+1}}^{k,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k).\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}),\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})d\theta_{k+1}|\mathcal{F}_t\Big] \\ & \geq & F_t^k(\boldsymbol{\theta},\boldsymbol{l}) \end{array}$$

Hence $F_t^n(\boldsymbol{\theta}_k,\boldsymbol{l}_k) = v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\Big(Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\Big)^2 + \xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$. Combining with its definition we finally get the second expected result which is for all $k \in \{0,1,\ldots,n-1\}$

$$\begin{array}{rcl} V_t^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k) & = & \underset{\pi^k\in\mathcal{A}_{\mathbb{F}}^k(t,\nu^k)}{\operatorname{ess\,inf}}\ \mathbb{E}[(H_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - X_T^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2\alpha_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) + \\ \displaystyle\int_t^T\int_E V_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) & + & \pi_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k).\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}),\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})d\theta_{k+1}|\mathcal{F}_t] \\ & = & v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_t^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \end{array}$$

Moreover taking now $t = \theta_k$ and using relations (2.49), (2.50) and (2.34) we get

$$\begin{array}{rcl} V_{\theta_n}^n(x,\boldsymbol{\theta},\boldsymbol{l}) & = & \mathbb{E}\Big[(H_T^n(\boldsymbol{\theta},\boldsymbol{l}) - X_T^{n,x,*}(\boldsymbol{\theta},\boldsymbol{l}))^2\alpha_T(\boldsymbol{\theta},\boldsymbol{l})|\mathcal{F}_{\theta_n}\Big] \\ & = & v_{\theta_n}^{n,\boldsymbol{\theta},\boldsymbol{l}}(X_{\theta_n}^{n,x,*}(\boldsymbol{\theta},\boldsymbol{l}) - Y_{\theta_n}^{n,\boldsymbol{\theta},\boldsymbol{l}})^2 + \xi_{\theta_n}^{n,\boldsymbol{\theta},\boldsymbol{l}} \end{array}$$

and

$$\begin{array}{rcl} V_{\theta_k}^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k) & = & \mathbb{E}[(H_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - X_T^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2\alpha_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) + \\ \displaystyle\int_{\theta_k}^T\int_E V_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) & + & \pi_{\theta_{k+1}}^{k,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k).\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}),\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})d\theta_{k+1}|\mathcal{F}_{\theta_k}] \\ & = & v_{\theta_k}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_{\theta_k}^{k,x,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - Y_{\theta_k}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_{\theta_k}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}. \end{array}$$

These relations prove that $\pi^* = (\pi^{k,*})_{k=0,\ldots,n}$ is an optimal trading strategy.

**Step 5:** For the verification Theorem 2.2, we have for $k = \{0,1,\ldots,n\}$, $H \equiv 0$ and $t \in [\theta_k, T]$ that

$$V_t^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k) = v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}X_t^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)^2$$

since the value process $V^k$ is unique, we get that the process $v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ is unique too.

$Y^{n,\boldsymbol{\theta},\boldsymbol{l}}$ is unique since we have the formula (2.29). Assume that $Y^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}$ is unique, from (2.33) and since $v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ and $v^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}$ are unique we obtain that $Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ is also unique.

By (2.34), since $V_t^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k)$, $v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ and $Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ are unique, we obtain that $\xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ is then unique.

□

# 3 Numerical Applications

We consider a special case where there is only one default event and such that $\mu^0$, $\sigma^0$ and $\gamma^0$ are constants; $\mu^1(\theta,l)$ and $\sigma^1(\theta,l)$ are only deterministic functions of $\theta$, and the default time $\tau$ is independent of $\mathbb{F}$, so that $\alpha_t(\theta,l)$ is simply a known deterministic function $\alpha(\theta)$ of $\theta\in\mathbb{R}^+$, and the survival probability $G(t)=\mathbb{P}\left[\tau>t|\mathcal{F}_t\right]=\mathbb{P}\left[\tau>t\right]=\int_t^\infty\alpha(\theta)d\theta$ is a deterministic function. We assume that the survival probability follows an exponential distribution with constant default intensity $\lambda$. So there is a constant $\lambda>0$ such that $G(t)=e^{-\lambda t}$ and thus the density function is $\alpha(\theta)=\lambda e^{-\lambda\theta}$. Moreover we will take $\gamma^0>0$ (loss at default) and we consider functions $\mu^1(\theta,l)$ and $\sigma^1(\theta,l)$ which for all $\theta\in[0,T]$ have the form $\mu^1(\theta,l)=\mu^0\left(\frac{\theta}{T}\right)$ and $\sigma^1(\theta,l)=\sigma^0\left(2-\frac{\theta}{T}\right)$. See [6] for the economic interpretation. Here there is no mark, so we will not denote the dependence in $l$. In this case, we have

$$\mathbb{E}\left[(Z_T^Q)^2\right]=\exp\left(-(T-\theta)\left(\frac{\mu^0}{\sigma^0(2\frac{T}{\theta}-1)}\right)^2\right)$$

which gives $v_t^{1,\theta}=\lambda\exp\left(-\lambda t+(T-\theta)\left(\frac{\mu^0}{\sigma^0(2\frac{T}{\theta}-1)}\right)^2\right)$. We take two constant payoffs $H^0$ and $H^1$ such that $H^0>H^1$. This corresponds to the case of a zero coupon with a risk of default. Then our system of BSDEs becomes a system of ordinary differentials equations (ODEs) in this model and has explicit solutions. We adopt for this example another quadratic form which is given by $V_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(x)=v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(2)}x^2-2v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}x+v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(0)}$ with $k=\{0,1\}$ (i.e. 0 for the before default and 1 for the after). We can obtain the terms $v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(2)}$, $v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}$ and $v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(0)}$ using our classical quadratic decomposition form since we have $v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(2)}=v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$, $Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}=\frac{v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}}{v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(2)}}$ and $\xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}=v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(0)}-(Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(2)}$.

We will take here the particular time $t=\theta$. By dynamic programming on the corresponding value function $V_t^0$ given in (1.15), we obtain in our Markovian framework that $V^0$ satisfies the Hamilton-Jacobi-Bellman equation given by

$$\frac{\partial V_t^0(x)}{\partial t}+\inf_{\pi\in\mathbb{R}^{[0,T]}}\left\{\mu^0\pi_t\frac{\partial V_t^0(x)}{\partial x}+\frac{1}{2}\frac{\partial^2V_t^0(x)}{\partial x^2}(\sigma^0)^2\pi_t^2+V_t^1(x+\gamma^0\pi_t)\right\}=0 \qquad (3.51)$$

As, for $i\in\{0,1\}$, we have a quadratic decomposition form of $V_t^i(x)$ given by $V_t^i(x)=v_t^{i,t,(2)}x^2-2v_t^{i,t,(1)}x+v_t^{i,t,(0)}$, we then obtain that the optimal strategy is

$$\pi_t^{0,*}=2\frac{-\gamma^0\left(v_t^{1,t,(2)}x-v_t^{1,t,(1)}\right)+\mu^0\left(-v_t^{0,t,(2)}x+v_t^{0,t,(1)}\right)}{(\sigma^0)^2v_t^{0,t,(2)}+(\gamma^0)^2v_t^{1,t,(2)}}$$

Injecting this in 3.51, we get

$$\left(\frac{\partial v_t^{0,t,(2)}}{\partial t} + v_t^{1,t,(2)}\right) x^2 - 2\left(\frac{\partial v_t^{0,t,(1)}}{\partial t} + v_t^{1,t,(1)}\right) x + \left(\frac{\partial v_t^{0,t,(0)}}{\partial t} + v_t^{1,t,(0)}\right) = \frac{\left[\left(\mu^0 v_t^{0,t,(2)} + \gamma^0 v_t^{1,t,(2)}\right) x - \gamma^0 v_t^{1,t,(1)} - \mu^0 v_t^{1,t,(2)}\right]^2}{(\sigma^0)^2 v_t^{0,t,(2)} + (\gamma^0)^2 v_t^{1,t,(2)}}$$

Then, identifying the different coefficients in $x$, we obtain the following ODEs

$$\begin{aligned}
\frac{\partial v_t^{0,t,(2)}}{\partial t} &= -v_t^{1,t,(2)} + \frac{\left(\mu^0 v_t^{0,t,(2)} + \gamma^0 v_t^{1,t,(2)}\right)^2}{(\sigma^0)^2 v_t^{0,t,(2)} + (\gamma^0)^2 v_t^{1,t,(2)}}, & v_T^{0,T,(2)} &= G(T) = e^{-\lambda T} \\
\frac{\partial v_t^{0,t,(1)}}{\partial t} &= -v_t^{1,t,(1)} + \frac{\left(\mu^0 v_t^{0,t,(2)} + \gamma^0 v_t^{1,t,(2)}\right)\left(\mu^0 v_t^{0,t,(1)} + \gamma^0 v_t^{1,t,(1)}\right)}{(\sigma^0)^2 v_t^{0,t,(2)} + (\gamma^0)^2 v_t^{1,t,(2)}}, & v_T^{0,T,(1)} &= H_0 v_T^{0,T,(2)} \\
\frac{\partial v_t^{0,t,(0)}}{\partial t} &= -v_t^{1,t,(0)} + \frac{\left(\mu^0 v_t^{0,t,(1)} + \gamma^0 v_t^{1,t,(1)}\right)^2}{(\sigma^0)^2 v_t^{0,t,(2)} + (\gamma^0)^2 v_t^{1,t,(2)}}, & v_T^{0,T,(0)} &= H_0^2 v_T^{0,T,(2)}
\end{aligned}$$

The first ODE corresponds to the first BSDE in this Markovian framework. In fact, in this particular case where all coefficients and terminal conditions are deterministic, the predictable component $\beta^0$ of the pair $(v^{0,(2)}, \beta^0)$ solution of the first BSDE equals to zero. Equivalently, the two last ODEs are related to the two last BSDEs in this particular setting. Therefore we can verify numerically the characteristics of the triple $(v^{0,(2)}, Y^0, \xi^0)$ appearing in 2.20 and plotting the solutions of the ODEs.

For the simulations, we take $\mu^0 = 0.2$, $\sigma^0 = 0.05$, $H^0 = 1.2$, $H^1 = 0.9$ and maturity $T = 1$. From Figure 1, we first find that there exists $\delta, \bar{\delta} > 0$ such that $\delta \le v_t^0 \le \bar{\delta} \le 1$. This inequality verifies what we proved in Theorem 2.1 point 1. Furthermore, from the quadratic decomposition form of $V^0$, we have:

$$v_0^0 = V_0^0(1) = \min_{\pi \in \mathcal{A}} \mathbb{E}\left[X_T^{1,\pi}\right]^2.$$

Therefore $v^0$ is related to the minimal variance of a portfolio investment on the asset $S$ with initial wealth $x = 1$. Consequently, to understand the impact of asset parameters on the minimal variance, we shall plot the coefficient $v^0$ with respect to time $t$. Firstly, let study the minimal variance with respect to the jump due to default. We recall that the variance of the portfolio is divided in two parts, the continuous part driven by a Brownian motion and the jump part driven by the default indicator process. In Figure 1, we clearly find that the minimal variance with no jump part ($\gamma = 0$) is least than the minimal variance part with jump part. In others words, the jump part, due to default, increases the minimal variance. We are interested too in understanding the variation of the minimal variance with respect to the intensity parameter. Hence in Figure 2, we find that the minimal variance increases with the intensity parameter. This is an expected result since when the intensity increases, the corresponding probability of default increases too. And so, the occurrence of jump increases and implies an increasing of the variance.

In Figure (4), we observe that the values of the process $Y_0^0$ is quite stable with respect to $\lambda$ for each value of $\gamma$. We recall now that the process $\xi^0$ represents the incompleteness of the market. Hence, in Figure (5), we observe first, since the payoff have a jump between values $H^0$ and $H^1$, that if we take a non vanishing jump in the asset dynamic $S$ (i.e. taking $\gamma \neq 0$) the values of

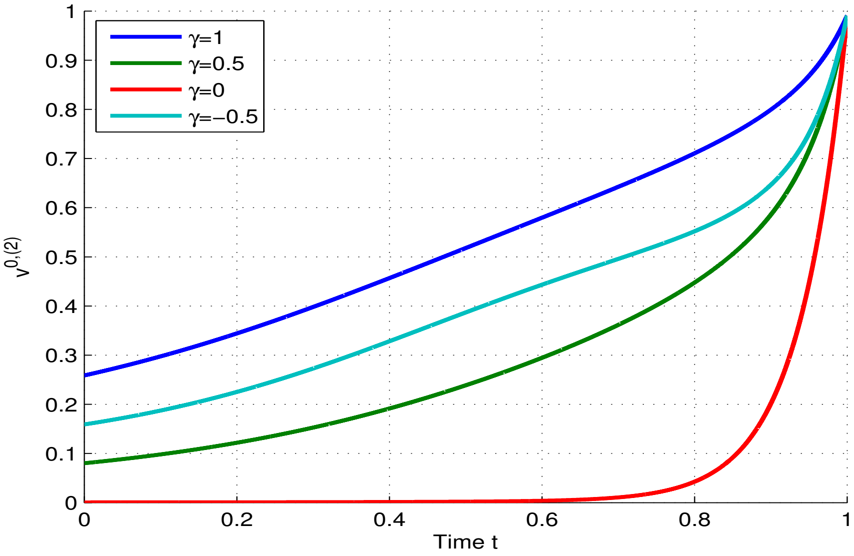


Figure 1: $v_t^0$ in function of time $t \in [0,T]$ with $T = 1$ and $\lambda = 0.01$ for different values of $\gamma$.

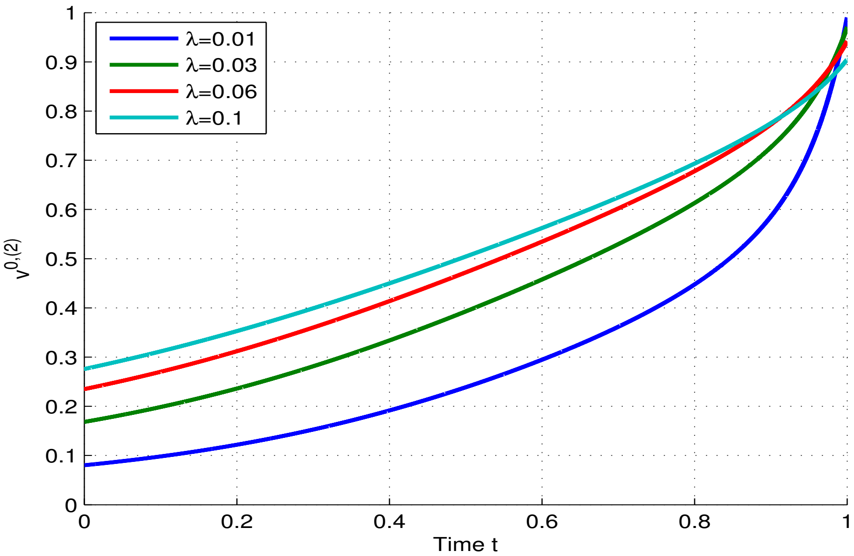


Figure 2: $v_t^0$ in function of time $t \in [0,T]$ with $T = 1$ and $\gamma = 0.5$ for different values of $\lambda$.

$\xi_0^0$ is quite close to zero. This shows that our hedging strategy covers well the model. Whereas, if we take a $\gamma = 0$ then the dynamic of the asset price $S$ doesn't jump when the default occurs although the payoff still jump, we observe that the value of the process $\xi$ increases with respect to the probability of jump. Since we are in a default risk model with jump in the payoff, taking $\gamma = 0$ means to use a continuous asset dynamics $S$ and so to use a Black and Scholes hedging strategy. Hence it is natural to obtain values of $\xi^0$ bigger than in the cases with $\gamma \neq 0$. In a financial example, assuming that the payoff $H$ is a CDO with multiple defaults, then assuming

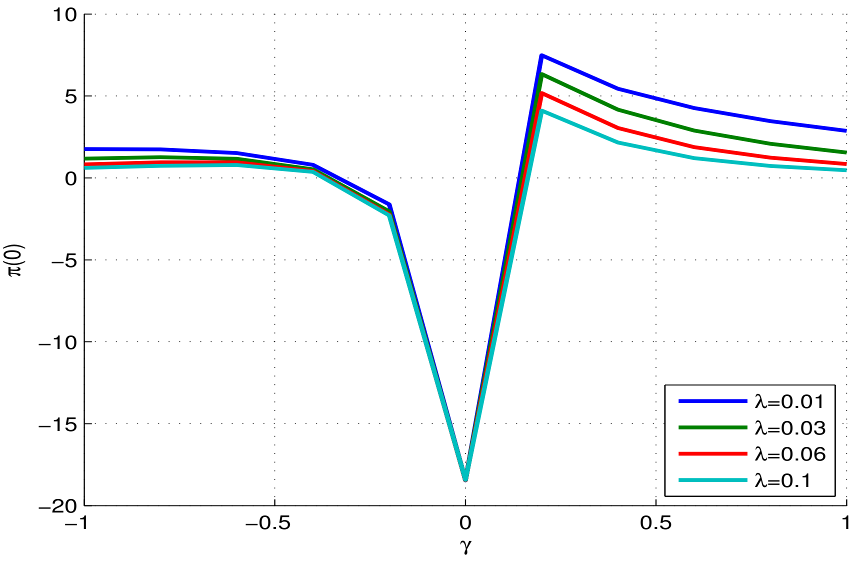


Figure 3: $\pi(0)$ in function of $\gamma$ for different values of $\lambda$.

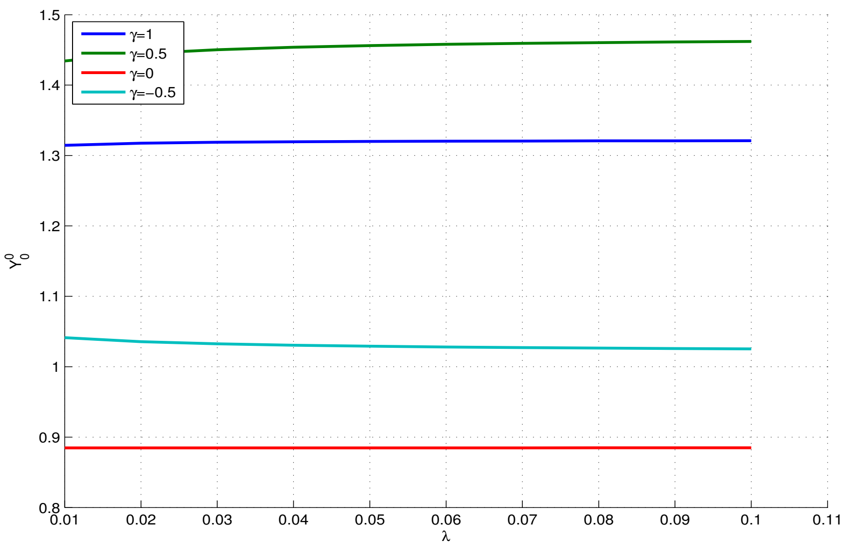


Figure 4: Graphe of $Y_0^0$ in function of $\gamma$ for different values of $\lambda$.

that $S$ is a Black and Scholes model gives less good result in term of hedging than assuming that $S$ is a CDS.

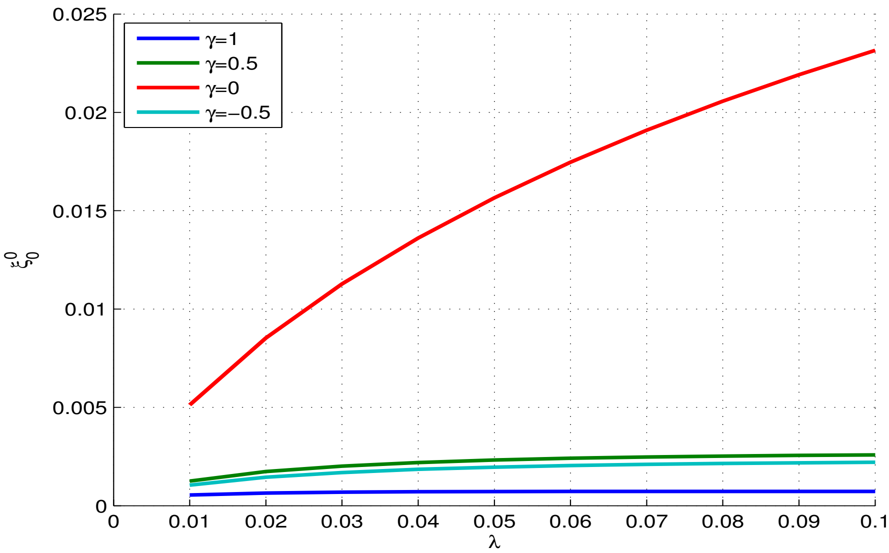


Figure 5: Graphe of $\xi_0^0$ in function of $\gamma$ for different values of $\lambda$.

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# Appendix A

**Proposition 3.1.** *Let $A$ be an adapted increasing continuous process such that there exists a constant $C > 0$ satisfying for any $0 \leq s \leq t$*

$$\mathbb{E}\big[(A_t - A_s)|\mathcal{F}_s\big] \leq C\ ,$$

*then this process $A$ also satisfies*

$$\mathbb{E}\big[\exp\left(\delta(A_t - A_s)\right)|\mathcal{F}_s\big] \leq \frac{1}{1-\delta C}\ , \qquad \forall\, 0 < \delta < \frac{1}{C}\ .$$

*Proof.* Let $A$ an increasing continuous adapted process satisfying $\mathbb{E}[(A_t - A_s)|\mathcal{F}_s] \leq C$. We first prove by iteration that $\mathbb{E}[(A_t - A_s)^p|\mathcal{F}_s] \leq p!C^p$ for any $p \in \mathbb{N}$. For that we assume that for $p \geq 2$, $\mathbb{E}[(A_t - A_s)^{p-1}|\mathcal{F}_s] \leq (p-1)!K^{p-1}$. Let recall that for any increasing continuous adapted process $A$ we have $(A_t - A_s)^p = p\int_s^t (A_t - A_u)^{p-1}dA_u$ for $s \leq t$, consequently we get

$$\begin{aligned}
\mathbb{E}\big[(A_t - A_s)^p|\mathcal{F}_t\big] &= p\mathbb{E}\Big[\int_s^t (A_t - A_u)^{p-1}dA_u|\mathcal{F}_s\Big] = p\mathbb{E}\Big[\int_s^t \mathbb{E}[(A_t - A_u)^{p-1}|\,\mathcal{F}_u]dA_u|\mathcal{F}_s\Big] \\
&\leq (p-1)!C^{p-1}\mathbb{E}[A_t - A_s||\mathcal{F}_s] \leq p!C^p.
\end{aligned}$$

Therefore, we get for any $0 < \delta < \frac{1}{C}$, $\mathbb{E}\Big[\sum_{p\geq 0}\frac{1}{p!}\delta^p(A_t - A_s)^p\Big|\mathcal{F}_s\Big] \leq \sum_{p\geq 0}\delta^p C^p$. Then we conclude the expected result. $\square$

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