Soft ELECTRE TRI outranking method based on belief functions

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Abstract—**Main decisions problems can be described into choice, ranking or sorting of a set of alternatives. The classical ELECTRE TRI (ET) method is a multicriteria-based outranking sorting method which allows to assign alternatives into a set of predetermined categories. ET method deals with situations where indifference is not transitive and solutions can sometimes appear uncomparable. ET suffers from two main drawbacks: 1) it requires an arbitrary choice of λ-cut step to perform the outranking of alternatives versus profiles of categories, and 2) an arbitrary choice of attitude for final assignment of alternatives into the categories. ET finally gives a final binary (hard) assignment of alternatives into categories. In this paper we develop a soft version of ET method based on belief functions which circumvents the aforementioned drawbacks of ET and allows to obtain both a soft (probabilistic) assignment of alternatives into categories and an indicator of the consistency of the soft solution. This Soft-ET approach is applied on a concrete example to show how it works and to compare it with the classical ET method.**
Keywords: ELECTRE TRI, information fusion, belief functions, outranking methods, multicriteria analysis.

I. INTRODUCTION

Multi-criteria decision analysis aims to choose, sort or rank alternatives or solutions according to criteria involved in the decision-making process. The main steps of a multi-criteria analysis consist in identifying decision purposes, defining criteria, eliciting preferences between criteria, evaluating alternatives or solutions and analyzing sensitivity with regard to weights, thresholds, etc. A difference has to be done between total and partial aggregation methods:

- Total aggregation methods such as the Multi-Attribute Utility Theory (M.A.U.T.) [1], [2] synthesizes in a unique value the partial utility related to each criterion and chosen by the decision-maker (DM). Each partial utility function transforms any quantitative evaluation of criterion into an utility value. The additive method is the simplest method to aggregate those utilities.
- Partial aggregation methods which are not based on the principle of preference transitivity. The ELECTRE TRI (ET) outranking method inspired by Roy [3] and finalized by Yu in [4] belongs to this family and it is the support of the research work presented in this paper.

ELECTRE TRI (electre tree) is an evolution of the ELECTRE methods introduced in 1960's by Roy [5] which remain widespread methods used in operational research. The acronym ELECTRE stands for "ELimination Et Choix Traduisant la REalité (Elimination and Choice Expressing the Reality). ET is simpler and more general than the previous ELECTRE methods which have specificities given their context of applications. A good introduction to ET methods with substantial references and detailed historical survey can be found in [6] and additional references in [7]. This paper proposes a methodology inspired by the ET method able to help decision based on imperfect information for soft assignment of alternatives into a given set of categories defined by predeterminate profiles. Our method, called "Soft ELECTRE TRI" (or just SET for short), is based on belief functions. It allows to circumvent the problem of arbitrary choice of λ-cut of the outranking step of ET, and the ad-choice of attitude in the final assignment step of ET as well. Contrariwise to ET which solves the hard assignment problem, SET proposes a new solution for solving the assignment problem in a soft manner. This paper is organized as follows. In Section II, we recall the principles of ET method with its main steps. In Section III, we present in details our new SET method with emphasize on its differences with classical ET. In Section IV, we apply ET and SET on a concrete example proposed by Maystre [8] to show how they work and to make a comparison between the two approaches. Section V concludes this paper and proposes some perspectives of this work.

II. THE ELECTRE TRI (ET) METHOD

Outranking methods like the ET method presented in this section are relevant for Multi-Criteria Decision Analysis (MCDA) [6] when:

- alternatives are evaluated on an ordinal scale;
- criteria are strongly heterogeneous by nature (e.g. comfort, price, pollution);
- compensation of the loss on one criterion by a gain on an another is unacceptable;
- small differences of evaluations are not individually significant while the accumulation of several of these differences may become significant.

We are concerned with an assignment problem in complex situations where several given alternatives have to be assigned to known categories based on multiple criteria. The categories

are defined by profiles values (bounds) for each criteria involved in the problem under consideration as depicted in Fig. 1 below.
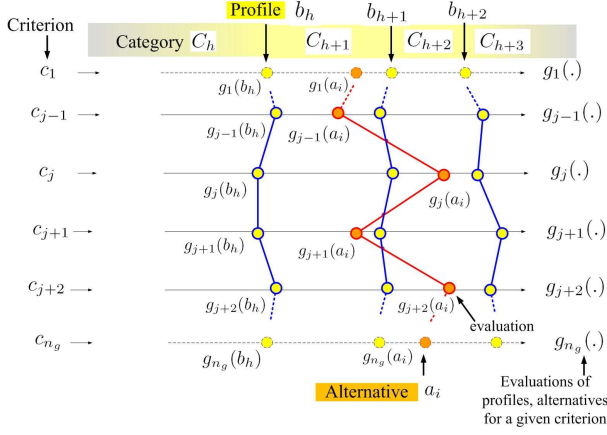

Figure 1: ET aims to assign a category to alternatives.

The ET method is a multicriteria-based outranking sorting method proposing a hard assignment of alternatives a_i in categories C_h. More precisely, the alternatives $a_i \in \mathbf{A}$, $i = 1, \ldots, n_a$ are committed to ordered categories $C_h \in \mathbf{C}$, $h = 1, \ldots, n_h$ according to criteria c_j, $j \in \mathbf{J} = \{1, \ldots, n_g\}$. Each category C_h is delimited by the set of its lower and upper limits b_{h-1} and b_h with respect to their evaluations $g_j(b_{h-1})$ and $g_j(b_h)$ for each criterion c_j ($g_j(.)$ represents the evaluations of alternatives, profiles for a given criterion c_j). By convention, $b_0 \leq b_1 \ldots \leq b_{n_h}$. b_0 is the lower (minimal) profile bound and b_{n_h} is the upper (maximal) profile bound. The overall profile b_h is defined through the set of values $\{g_1(b_h), g_2(b_h), \ldots, g_{n_g}(b_h)\}$ represented by the vertical lines joining the yellow dots in Fig. 1. The outranking relations are based on the calculation of partial concordance and discordance indices from which global concordance and credibility indices [4], [9] are derived based on an arbitrary λ-cut strategy. The final assignment (sorting procedure) of alternatives to categories operated by ET is a hard (binary) assignment based on an arbitrary selected attitude choice (optimistic or pessimistic). ET method can be summarized by the following steps:

- ET-Step 1: Computation of partial concordance indices $c_j(a_i, b_h)$ and $c_j(b_h, a_i)$), and partial discordances indices $d_j(a_i, b_h)$ and $d_j(b_h, a_i)$);
- ET-Step 2: Computation of the global (overall) concordance indices $c(a_i, b_h)$ and $c(b_h, a_i)$ to obtain credibility indices $\rho(a_i, b_h)$ and $\rho(b_h, a_i)$;
- ET-Step 3: Computation of the fuzzy outranking relation grounded on the credibility indices $\rho(a_i, b_h)$ and $\rho(b_h, a_i)$; and apply a λ-cut to get the crisp outranking relation;
- ET-Step 4: Final hard (binary) assignment of a_i into C_h is based on the crisp outranking relation and in adopting either a pessimistic (conjunctive), or an optimistic (dis-

junctive) attitude.

Let's explain a bit more in details the steps of ET and the computation of the indices necessary for the implementation of the ET method.

A. ET-Step 1: Partial indices

In ET method, the partial concordance index $c_j(a_i, b_h)$ (resp. $c_j(b_h, a_i)$) expresses to which extent the evaluations of a_i and b_h (respectively b_h and a_i)) are concordant with the assertion "a_i is at least as good as b_h" (respectively "b_h is at least as good as a_i"). $c_j(a_i, b_h) \in [0, 1]$, based on a given criterion $g_j(.)$, is computed from the difference of the criterion evaluated for the profile b_h, and the same criterion evaluated for the alternative a_i. If the difference[1] $g_j(b_h) - g_j(a_i)$ is less (or equal) to a given indifference threshold $q_j(b_h)$ then a_i and b_h are considered indifferent based on the criterion $g_j(.)$. If the difference $g_j(b_h) - g_j(a_i)$ is strictly greater to given preference threshold $p_j(b_h)$ then a_i and b_h are considered different based on $g_j(.)$. When $g_j(b_h) - g_j(a_i) \in [q_j(b_h), p_j(b_h)]$, the partial concordance index $c_j(a_i, b_h)$ is computed from a linear interpolation corresponding to a weak difference. Mathematically, the partial concordance indices $c_j(a_i, b_h)$ and $c_j(b_h, a_i)$ are obtained by:

$$c_j(a_i, b_h) \triangleq \begin{cases} 0 & \text{if} \quad g_j(b_h) - g_j(a_i) \geq p_j(b_h) \\ 1 & \text{if} \quad g_j(b_h) - g_j(a_i) < q_j(b_h) \\ \frac{g_j(a_i) + p_j(b_h) - g_j(b_h)}{p_j(b_h) - q_j(b_h)} & \text{otherwise} \end{cases} \quad (1)$$

and

$$c_j(b_h, a_i) \triangleq \begin{cases} 0 & \text{if} \quad g_j(a_i) - g_j(b_h) \geq p_j(b_h) \\ 1 & \text{if} \quad g_j(a_i) - g_j(b_h) < q_j(b_h) \\ \frac{g_j(b_h) + p_j(b_h) - g_j(a_i)}{p_j(b_h) - q_j(b_h)} & \text{otherwise.} \end{cases} \quad (2)$$

The partial discordance index $d_j(a_i, b_h)$ (resp. $d_j(b_h, a_i)$) expresses to which extent the evaluations of a_i and b_h (resp. b_h and a_i) is opposed to the assertion "a_i is at least as good as b_h" (resp. "b_h is at least as good as a_i"). These indices depend on a possible veto condition expressed by the choice of a veto threshold $v_j(b_h)$ (such as $v_j(b_h) \geq p_j(b_h) \geq q_j(b_h) \geq 0$) imposed on some criterion $g_j(.)$. They are defined by [4], [9]:

$$d_j(a_i, b_h) \triangleq \begin{cases} 0 & \text{if} \quad g_j(b_h) - g_j(a_i) < p_j(b_h) \\ 1 & \text{if} \quad g_j(b_h) - g_j(a_i) \geq v_j(b_h) \\ \frac{g_j(b_h) - g_j(a_i) - p_j(b_h)}{v_j(b_h) - p_j(b_h)} & \text{otherwise} \end{cases} \quad (3)$$

and

$$d_j(b_h, a_i) \triangleq \begin{cases} 0 & \text{if} \quad g_j(a_i) - g_j(b_h) \leq p_j(b_h) \\ 1 & \text{if} \quad g_j(a_i) - g_j(b_h) > v_j(b_h) \\ \frac{g_j(a_i) - g_j(b_h) - p_j(b_h)}{v_j(b_h) - p_j(b_h)} & \text{otherwise.} \end{cases} \quad (4)$$

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[1]For convenience, we assume here an increasing preference order. A decreasing preference order [9] can be managed similarly by multiplying criterion values by -1.

B. ET-Step 2: Global concordance and credibility indices

- **The global concordance indices**: The global concordance index $c(a_i, b_h)$ (respectively $c(b_h, a_i)$) expresses to which extent the evaluations of a_i and b_h on all criteria (respectively b_h and a_i) are concordant with the assertions "a_i outranks b_h" (respectively "b_h outranks a_i"). In ET method, $c(a_i, b_h)$ (resp. $c(b_h, a_i)$) is computed by the weighted average of partial concordance indices $c_j(a_i, b_h)$ (resp. $c_j(b_h, a_i)$). That is

$$c(a_i, b_h) = \sum_{j=1}^{n_g} w_j c_j(a_i, b_h) \tag{5}$$

and

$$c(b_h, a_i) = \sum_{j=1}^{n_g} w_j c_j(b_h, a_i) \tag{6}$$

where the weights $w_j \in [0, 1]$ represent the relative importance of each criterion $g_j(.)$ in the evaluation of the global concordance indices. The weights add to one. Since all $c_j(a_i, b_h)$ and $c_j(a_i, b_h)$ belong to $[0; 1]$, $c(a_i, b_h)$ and $c(b_h, a_i)$ given by (5) and (6) also belong to $[0; 1]$.

- **The global credibility indices**: The degree of credibility of the outranking relation denoted as $\rho(a_i, b_h)$ (respectively $\rho(b_h, a_i)$) expresses to which extent "a_i outranks b_h" (respectively "b_h outranks a_i") according to the global concordance index $c(a_i, b_h)$ and the discordance indices $d_j(a_i, b_h)$ for all criteria (respectively $c(b_h, a_i)$ and $d_j(b_h, a_i)$). In ET method, these credibility indices $\rho(a_i, b_h)$ (resp. $\rho(b_h, a_i)$) are computed by discounting (weakening) the global concordance indices $c(a_i, b_h)$ given by (5) (resp. $c(b_h, a_i)$ given by (6)) by a discounting factor $\alpha(a_i, b_h)$ in $[0; 1]$ (resp. $\alpha(b_h, a_i)$) as follows:

$$\begin{cases} \rho(a_i, b_h) = c(a_i, b_h)\alpha(a_i, b_h) \\ \rho(b_h, a_i) = c(b_h, a_i)\alpha(b_h, a_i) \end{cases} \tag{7}$$

The discounting factors $\alpha(a_i, b_h)$ and $\alpha(b_h, a_i)$ are defined by [9], [10]:

$$\alpha(a_i, b_h) \triangleq \begin{cases} 1 & \text{if } \mathbf{V}_1 = \emptyset \\ \prod_{j \in \mathbf{V}_1} \frac{1 - d_j(a_i, b_h)}{1 - c(a_i, b_h)} & \text{if } \mathbf{V}_1 \neq \emptyset \end{cases} \tag{8}$$

$$\alpha(b_h, a_i) \triangleq \begin{cases} 1 & \text{if } \mathbf{V}_2 = \emptyset \\ \prod_{j \in \mathbf{V}_2} \frac{1 - d_j(b_h, a_i)}{1 - c(b_h, a_i)} & \text{if } \mathbf{V}_2 \neq \emptyset \end{cases} \tag{9}$$

where $\mathbf{V}_1$ (resp. $\mathbf{V}_2$) is the set of indexes j where the partial discordance indices $d_j(a_i, b_h)$ (reps. $d_j(b_h, a_i)$) is greater than the global concordance index $c(a_i, b_h)$ (resp. $c(b_h, a_i)$), that is:

$$\mathbf{V}_1 \triangleq \{j \in \mathbf{J} | d_j(a_i, b_h) > c(a_i, b_h)\} \tag{10}$$

$$\mathbf{V}_2 \triangleq \{j \in \mathbf{J} | d_j(b_h, a_i) > c(b_h, a_i)\} \tag{11}$$

C. ET-Step 3: Fuzzy and crisp outranking process

Outranking relations result from the transformation of fuzzy outranking relation (corresponding to credibility indices) into a crisp outranking relation[2] S done by means of a λ-cut [9]. λ is called *cutting level*. λ is the smallest value of the credibility index $\rho(a_i, b_h)$ compatible with the assertion "a_i outranks b_h". Similarly λ is the smallest value of the credibility index $\rho(b_h, a_i)$ compatible with the assertion "b_h outranks a_i". In practice the choice of λ value is not easy and is done arbitrary or based on a sensitivity analysis. More precisely, the crisp outranking relation S is defined by

$$\begin{cases} \rho(a_i, b_h) \geq \lambda \Longrightarrow a_i \ S \ b_h \\ \rho(b_h, a_i) \geq \lambda \Longrightarrow b_h \ S \ a_i \end{cases} \tag{12}$$

Binary relations of preference ($>$), indifference (I), incomparability (R) are defined according to (13):

$$\begin{cases} a_i I b_h \Longleftrightarrow a_i \ S \ b_h \text{ and } b_h \ S \ a_i \\ a_i > b_h \Longleftrightarrow a_i \ S \ b_h \text{ and not } b_h \ S \ a_i \\ a_i < b_h \Longleftrightarrow \text{not } a_i \ S \ b_h \text{ and } b_h \ S \ a_i \\ a_i R b_h \Longleftrightarrow \text{not } a_i \ S \ b_h \text{ and not } b_h \ S \ a_i \end{cases} \tag{13}$$

D. ET-Step 4: Hard assignment procedure

Based on outranking relations between all pairs of alternatives and profiles of categories, two attitudes can be used in ET to assign each alternative a_i into a category C_h [6]. These attitudes yields to a hard assignment solution where each alternative belongs or doesn't belong to a category (binary assignment) and there is no measure of the confidence of the assignment in this last step of ET method. The pessimistic and optimistic hard assignments are realized as follows:

- Pessimistic hard assignment: a_i is compared with b_k, b_{k-1}, b_{k-2}, ..., until a_i outranks b_h where $h \leq k$. The alternative a_i is then assigned to the highest category C_h, that is $a_i \rightarrow C_h$, if $\rho(a_i, b_h) \geq \lambda$.
- Optimistic hard assignment: a_i is compared successively to b_1, b_2, ... b_h, ... until b_h outranks a_i. The alternative a_i is assigned to the lowest category C_h, $a_i \rightarrow C_h$, for which the upper profile b_h is preferred to a_i.

III. THE NEW SOFT ELECTRE TRI (SET) METHOD

The objective and motivation of this paper are to improve the appealing ET method in order to provide a soft assignment procedure of alternatives into categories, and to eliminate the drawback concerning both the choice of λ-cut level in ET-Step 3 and the choice of attitude in ET-Step 4. Soft assignment reflects the confidence one has in the assignment which can be a very useful property in applications requiring multi criteria decision analysis. To achieve such purpose and due to long experience in working with belief functions (BF), it has appeared clearly that BF can be very useful for developing a "soft-assignment" version of the classical ET presented in the previous section. We call this new method the "Soft ELECTRE

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[2]It is denoted S because Outranking translates to "Surclassement" in French.

TRI" method (SET for short) and we present it in details in this section.

Before going further, it is necessary to recall briefly the definition of a mass of belief $m(.)$ (also called basic belief assignment, or bba), a credibility function $\mathrm{Bel}(.)$ and the plausibility function $\mathrm{Pl}(.)$ defined over a finite set $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ of mutually exhaustive and exclusive hypotheses. Belief functions have been introduced by Shafer in his development of Dempster-Shafer Theory (DST), see [11] for details. In DST, Θ is called the frame of discernment of the problem under consideration. By convention the power-set (i.e. the set of all subsets of Θ) is denoted 2^Θ since its cardinality is $2^{|\Theta|}$. A basic belief assignment provided by a source of evidence is a mapping $m(.): 2^\Theta \to [0,1]$ satisfying

$$m(\emptyset) = 0 \qquad \text{and} \qquad \sum_{X \in 2^\Theta} m(X) = 1 \qquad (14)$$

The measures of credibility and plausibility of any proposition $X \in 2^\Theta$ are defined from $m(.)$ by

$$\mathrm{Bel}(X) \triangleq \sum_{\substack{Y \subseteq X \\ Y \in 2^\Theta}} m(Y) \qquad (15)$$

$$\mathrm{Pl}(X) \triangleq \sum_{\substack{Y \cap X \neq \emptyset \\ Y \in 2^\Theta}} m(Y) \qquad (16)$$

$\mathrm{Bel}(X)$ and $\mathrm{Pl}(X)$ are usually interpreted as lower and upper bounds of the unknown probability of X. $U(X) = \mathrm{Pl}(X) - \mathrm{Bel}(X)$ reflects the uncertainty on X. The belief functions are well adapted to model uncertainty expressed by a given source of evidence. For information fusion purposes, many solutions have been proposed in the literature [12] to combine bba's efficiently for pooling evidences arising from several sources.

As for the classical ET method, there are four main steps in our new SET method. However, the SET steps are different from the ET steps. The four steps of SET, that are actually very specific and improves the ET steps, are:

- SET-Step 1: Computation of partial concordance indices $c_j(a_i, b_h)$ and $c_j(b_h, a_i)$, partial discordances indices $d_j(a_i, b_h)$ and $d_j(b_h, a_i)$, and also partial uncertainty indices $u_j(a_i, b_h)$ and $u_j(b_h, a_i)$ thanks to a smooth sigmoidal model for generating bba's [13].
- SET-Step 2: Computation of the global (overall) concordance indices $c(a_i, b_h)$, $c(b_h, a_i)$, discordance indices $d(a_i, b_h)$, $d(b_h, a_i)$, and uncertainty indices $u(a_i, b_h)$, $u(b_h, a_i)$;
- SET-Step 3: Computation of the probabilized outranking relations grounded on the global indices of SET-Step 2. The probabilization is directly obtained and thus eliminates the arbitrary λ-cut strategy necessary in ET.
- SET-Step 4: Final soft assignment of a_i into C_h based on combinatorics of probabilized outranking relations.

Let's explain in details the four steps of SET and the computation of the indices necessary for the implementation of the SET method.

A. SET-Step 1: Partial indices

In SET, a sigmoid model is proposed to replace the original truncated trapezoidal model for computing concordance and discordance indices of the ET method. The sigmoidal model has been presented in details in [13] and is only briefly recalled here. We consider a binary frame of discernment[3] $\Theta \triangleq \{c, \bar{c}\}$ where c means that the alternative a_i is concordant with the assertion "a_i is at least as good as profile b_h", and $\bar{c}$ means that the alternative a_i is opposed (discordant) to this assertion. We can compute a basic belief assignment (bba) $m_{ih}(.)$ defined on 2^Θ for each pair (a_i, b_h). $m_{ih}(.)$ is defined from the combination (fusion) of the local bba's $m_{ih}^j(.)$ evaluated from each possible criteria $g_j(.)$ as follows: $m_{ih}^j(.) = [m_1 \oplus m_2](.)$ is obtained by the fusion[4] (denoted symbolically by $\oplus$) of the two following simple bba's defined by:

focal element	$m_1(.)$	$m_2(.)$
c	$f_{s_c, t_c}(g)$	0
$\bar{c}$	0	$f_{-s_{\bar{c}}, t_{\bar{c}}}(g)$
$c \cup \bar{c}$	$1 - f_{s_c, t_c}(g)$	$1 - f_{-s_{\bar{c}}, t_{\bar{c}}}(g)$

Table I: Construction of $m_1(.)$ and $m_2(.)$.

where $f_{s,t}(g) \triangleq 1/(1 + e^{-s(g-t)})$ is the sigmoid function; g is the criterion magnitude of the alternative under consideration; t is the abscissa of the inflection point of the sigmoid. The abscisses of inflection points are given by $t_c = g_j(b_h) - \frac{1}{2}(p_j(b_h) + q_j(b_h))$ and $t_{\bar{c}} = g_j(b_h) - \frac{1}{2}(p_j(b_h) + v_j(b_h))$ and the parameters s_c and $s_{\bar{c}}$ are given by[5] $s_c = 4/(p_j(b_h) - q_j(b_h))$ and $s_{\bar{c}} = 4/(v_j(b_h) - p_j(b_h))$.

From the setting of threshold parameters $p_j(b_h)$, $q_j(b_h)$ and $v_j(b_h)$ (the same as for ET method), it is easy to compute the parameters of the sigmoids (t_c, s_c) and $(t_{\bar{c}}, s_{\bar{c}})$, and thus to get the values of bba's $m_1(.)$ and $m_2(.)$ to compute $m_{ih}^j(.)$. We recommend to use the PCR5 fusion rule[6] since it offers a better management of conflicting bba's yielding to more specific results than with other rules. Based on this sigmoidal modeling, we get now from $m_{ih}^j(.)$ a fully consistent and efficient representation of local concordance $c_j(a_i, b_h)$, local discordance $d_j(a_i, b_h)$ and the local uncertainty $u_j(a_i, b_h)$ by considering:

$$\begin{cases} c_j(a_i, b_h) \triangleq m_{ih}^j(c) \in [0,1] \\ d_j(a_i, b_h) \triangleq m_{ih}^j(\bar{c}) \in [0,1] \\ u_j(a_i, b_h) \triangleq m_{ih}^j(c \cup \bar{c}) \in [0,1]. \end{cases} \qquad (17)$$

Of course, a similar approach must be adapted (not reported here due to space limitation restraint) to

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[3]Here we assume that Shafer's model holds, that is $c \cap \bar{c} = \emptyset$.

[4]with averaging rule, PCR5 rule, or Dempster-Shafer rule [14].

[5]The coefficient 4 appearing in s_c and $s_{\bar{c}}$ expressions comes from the fact that for a sigmoid of parameter s, the tangent at its inflection point is $s/4$.

[6]see [15] for details on PCR5 with many examples.

compute $c_j(b_h, a_i) = m_{hi}^j(c)$, $d_j(b_h, a_i) = m_{hi}^j(\bar{c})$ and $u_j(b_h, a_i) = m_{hi}^j(c \cup \bar{c})$.

Example 1: Let's consider only one alternative a_i and $g_j(.)$ in range $[0, 100]$, and let's take $g_j(b_h) = 50$ and the following thresholds: $q_j(b_h) = 20$ (indifference threshold), $p_j(b_h) = 25$ (preference threshold) and $v_j(b_h) = 40$ (veto threshold) for the profile bound b_h. Then, the inflection points of the sigmoids $f_1(g) \triangleq f_{s_c, t_c}(g)$ and $f_2(g) \triangleq f_{-s_{\bar{c}}, t_{\bar{c}}}(g)$ have the following abscisses: $t_c = 50 - (25 + 20)/2 = 27.5$ and $t_{\bar{c}} = 50 - (25 + 40)/2 = 17.5$ and parameters: $s_c = 4/(25 - 20) = 4/5 = 0.8$ and $s_{\bar{c}} = 4/(40 - 25) = 4/15 \approx 0.2666$. The construction of the consistent bba $m_{ih}^j(.)$ is obtained by the PCR5 fusion of the bba's $m_1(.)$ and $m_2(.)$ given in Table I. The result is shown in Fig. 2.
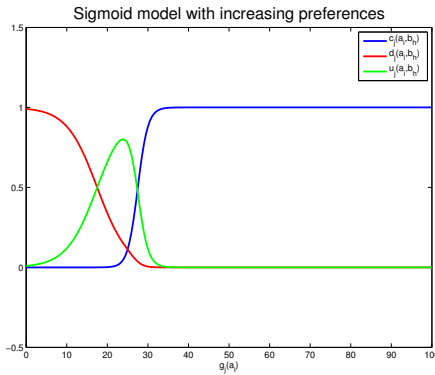

Figure 2: $m_{ih}^j(.)$ corresponding to partial indices.

The blue curve corresponds to $c_j(a_i, b_h)$, the red plot corresponds to $d_j(a_i, b_h)$ and the green plot to $u_j(a_i, b_h)$ when $g_j(a_i)$ varies in $[0; 100]$. $c_j(b_h, a_i)$, $d_j(b_h, a_i)$ and $u_j(b_h, a_i)$ can easily be obtained by mirroring (horizontal flip) the curves around the vertical axis at the mid-range value $g_j(a_i) = 50$.

B. SET-Step 2: Global indices

As explained in SET-Step 1, the partial indices are encapsulated in bba's $m_{ih}^j(.)$ for alternative a_i versus profile b_h ($a_i$vs.b_h), and encapsulated in bba's $m_{hi}^j(.)$ for profile b_h versus alternative a_i ($b_h$vs.a_i). In SET, the global indices $c(a_i, b_h)$, $d(a_i, b_h)$ and $u(a_i, b_h)$ are obtained by the fusion of the n_g bba's $m_{ih}^j(.)$. Similarly, the global indices $c(b_h, a_i)$, $d(b_h, a_i)$ and $u(b_h, a_i)$ are obtained by the fusion of the n_g bba's $m_{hi}^j(.)$. More precisely, one must compute:

$$\begin{cases} m_{ih}(.) = [m_{ih}^1 \oplus m_{ih}^2 \oplus \ldots \oplus m_{ih}^{n_g}](.) \\ m_{hi}(.) = [m_{hi}^1 \oplus m_{hi}^2 \oplus \ldots \oplus m_{hi}^{n_g}](.) \end{cases} \quad (18)$$

To take into account the weighting factor w_j of the criterion valued by $g_j(.)$, we suggest to use as fusion operator $\oplus$ either:

- the weighting averaging fusion rule (as in ET method) which is simple and compatible with probability calculus and Bayesian reasoning,
- or the more sophisticated operator defined by the PCR5 fusion rule adapted for importance discounting presented

in details in [16] which belongs to the family of non-Bayesian fusion operators.

Once the bba's $m_{ih}(.)$ and $m_{hi}(.)$ have been computed, the global indices are defined by:

$$\begin{cases} c(a_i, b_h) \triangleq m_{ih}(c)\alpha(a_i, b_h) \\ d(a_i, b_h) \triangleq m_{ih}(\bar{c})\beta(a_i, b_h) \\ u(a_i, b_h) \triangleq 1 - c(a_i, b_h) - d(a_i, b_h). \end{cases} \quad (19)$$

The discounting factors $\alpha(a_i, b_h)$ and $\beta(a_i, b_h)$ are defined by

$$\alpha(a_i, b_h) \triangleq \begin{cases} 1 & \text{if } \mathbf{V}_\alpha = \emptyset \\ \prod_{j \in \mathbf{V}_\alpha} \frac{1 - d_j(a_i, b_h)}{1 - m_{ih}(c)} & \text{if } \mathbf{V}_\alpha \neq \emptyset \end{cases} \quad (20)$$

$$\beta(a_i, b_h) \triangleq \begin{cases} 1 & \text{if } \mathbf{V}_\beta = \emptyset \\ \prod_{j \in \mathbf{V}_\beta} \frac{1 - c_j(a_i, b_h)}{1 - m_{ih}(\bar{c})} & \text{if } \mathbf{V}_\beta \neq \emptyset \end{cases} \quad (21)$$

with $\begin{cases} \mathbf{V}_\alpha \triangleq \{j \in \mathbf{J} | d_j(a_i, b_h) > m_{ih}(c)\} \\ \mathbf{V}_\beta \triangleq \{j \in \mathbf{J} | c_j(b_h, a_i) > m_{ih}(\bar{c})\} \end{cases} \quad (22)$

$c(b_h, a_i)$, $d(b_h, a_i)$ and $u(b_h, a_i)$ are similarly computed using dual formulas of (19)–(22).

The belief and plausibility of the outranking propositions $X = "a_i > b_h"$ and $Y = "b_h > a_i"$ are then given by

$$\begin{cases} \text{Bel}(X) = c(a_i, b_h) \\ \text{Bel}(Y) = c(b_h, a_i) \end{cases} \quad (23)$$

and $\begin{cases} \text{Pl}(X) = 1 - d(a_i, b_h) = c(a_i, b_h) + u(a_i, b_h) \\ \text{Pl}(Y) = 1 - d(b_h, a_i) = c(b_h, a_i) + u(b_h, a_i) \end{cases} \quad (24)$

C. SET-Step 3: Probabilized outranking

We have seen in SET-Step 2 that the outrankings $X = "a_i > b_h"$ and $Y = "b_h > a_i"$ can be characterized by their imprecise probabilities $P(X) \in [\text{Bel}(X); \text{Pl}(X)]$ and $P(Y) \in [\text{Bel}(Y); \text{Pl}(Y)]$. Figure 3 shows an example with $P(X) \in [0.2; 0.8]$ and $P(Y) \in [0.1; 0.5]$
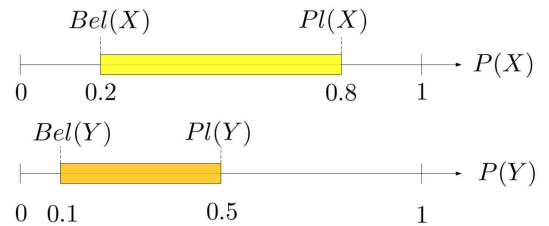
Figure 3: Imprecise probabilities of outrankings.

Solving the outranking problem consists in choosing (deciding) if finally X dominates Y (in such case we must decide X as being the valid outranking), or if Y dominates X (in such case we decide Y as being the valid outranking). Unfortunately, such hard (binary) assignment cannot be done in general[7] because it must be drawn from the unknown probabilities $P(X)$ in $[\text{Bel}(X); \text{Pl}(X)]$ and $P(Y)$

[7]but in cases where the bounds of probabilities $P(X)$ and $P(Y)$ do not overlap.

in $[\text{Bel}(Y); \text{Pl}(Y)]$ where a partial overlapping is possible between intervals $[\text{Bel}(X); \text{Pl}(X)]$ and $[\text{Bel}(Y); \text{Pl}(Y)]$ (see Fig. 3). A soft (probabilized) outranking solution is possible by computing the probability that X dominates Y (or that Y dominates X) by assuming uniform distribution of unknown probabilities between their lower and upper bounds. To get the probabilized outrankings, we just need to compute $P_{X>Y} \triangleq P(P(X) > P(Y))$ and $P_{Y>X} \triangleq P(P(Y) > P(X))$ which are precisely computable by the ratio of two polygonal areas, or can be estimated using sampling techniques.

Figure 4: Probabilization of outranking.

More precisely $\begin{cases} P_{X>Y} = A(X)/(A(X) + A(Y)) \\ P_{Y>X} = A(Y)/(A(X) + A(Y)) \end{cases}$ (25)
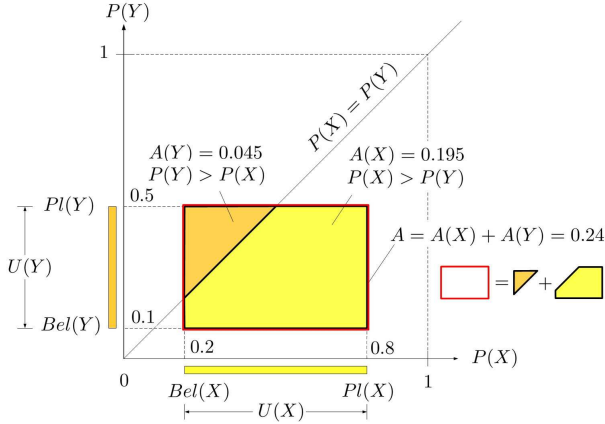
where $A(X)$ is the partial area of the rectangle $A = U(X) \times U(Y)$ under the line $P(X) = P(Y)$ (yellow area in Fig. 4) and $A(Y)$ is the area of the rectangle $A = U(X) \times U(Y)$ above the line $P(X) = P(Y)$ (orange area in Fig. 4). Of course, $A = A(X) + A(Y)$ and $P_{X>Y} = 1 - P_{Y>X}$. As a final result for the example of Fig. 3, and according to (25) and Fig. 4, we finally get the following probabilized outrankings:

$\begin{cases} a_i > b_h \text{ with probability} P_{X>Y} = 0.195/0.24 = 0.8125 \\ b_h > a_i \text{ with probability} P_{Y>X} = 0.045/0.24 = 0.1825 \end{cases}$

For notation convenience, we denote the probabilities of outrankings as $P_{ih} \triangleq P_{X>Y}$ with $X = "a_i > b_h"$ and $Y = "b_h > a_i"$. Reciprocally, we denote $P_{hi} \triangleq P_{Y>X} = 1 - P_{ih}$.

D. SET-Step 4: Soft assignment procedure

From the probabilized outrankings obtained in SET-Step 3, we are now able to make directly the soft assignment of alternatives a_i to categories C_h defined by their profiles b_h. This is easily obtained by the combinatorics of all possible sequences of outrankings taking into account their probabilities. Moreover, this soft assignment mechanism provides also the probability $\delta_i \triangleq P(a_i \to \emptyset)$ reflecting the impossibility to make a coherent outranking. Our soft assignment procedure doesn't require arbitrary choice of

attitude contrariwise to what is proposed in the classical ET method. For simplicity, we present the soft assignment procedure in the example 2 below, which can be adapted to any number $n_h \geq 2$ of categories.

Example 2: Let's consider one alternative a_i to be assigned to categories C_1, C_2 and C_3 based on multiple criteria (taking into account indifference, preference and veto conditions) and intermediate profiles b_1 and b_2. Because b_0 and b_3 are the min and max profiles, one has always $P(X_{i0} = "a_i > b_0") = 1$ and $P(X_{i3} = "a_i > b_3") = 0$. Let's assume that at the SET-Step 3 one gets the following soft outranking probabilities P_{ih} as given in Table II.

Profiles $b_h \to$ Outranking probas $\downarrow$	b_0	b_1	b_2	b_3
P_{ih}	1	0.7	0.2	0

Table II: Soft outranking probabilities.

From combinatorics, only the following outranking sequences $S_k(a_i)$, $k = 1, 2, 3, 4$ can occur with non null probabilities $P(S_k(a_i))$ as listed in Table III, where $P(S_k(a_i))$

Profiles $b_h \to$ Outrank sequences $\downarrow$	b_0	b_1	b_2	b_3	$P(S_k(a_i))$ $\downarrow$
$S_1(a_i)$	>	>	>	<	0.14
$S_2(a_i)$	>	>	<	<	0.56
$S_3(a_i)$	>	<	<	<	0.24
$S_4(a_i)$	>	<	>	<	0.06

Table III: Probabilities of outranking sequences.

have been computed by the product of the probability of each outranking involved in the sequence, that is:

$P(S_1(a_i)) = 1 \times 0.7 \times 0.2 \times 1 = 0.14$
$P(S_2(a_i)) = 1 \times 0.7 \times (1 - 0.2) \times 1 = 0.56$
$P(S_3(a_i)) = 1 \times (1 - 0.7) \times (1 - 0.2) \times 1 = 0.24$
$P(S_4(a_i)) = 1 \times (1 - 0.7) \times 0.2 \times 1 = 0.06$

The assignment of a_i into a category C_h delimited by bounds b_{h-1} and b_h depends on the occurrence of the outranking sequences. Given $S_1(a_i)$ with probability $P(S_1(a_i)) = 0.14$, a_i must be assigned to C_3 because a_i outranks b_0, b_1 and b_2; Given $S_2(a_i)$ with probability 0.56, a_i must be assigned to C_2 because a_i outranks only b_0 and b_1; Given $S_3(a_i)$ with probability 0.24, a_i must be assigned to C_1 because a_i outranks only b_0. Given $S_4(a_i)$ with probability 0.06, a_i cannot be reasonably assigned to categories because of inherent inconsistency of the outranking sequence $S_4(a_i)$ since a_i cannot outperform b_2 and simultaneously underperform b_1 because by profile ordering one has $b_2 > b_1$. Therefore the inconsistency indicator is given by $\delta_i = P(a_i \to \emptyset) = P(S_4(a_i)) = 0.06$. Finally, the soft assignment probabilities $P(a_i \to C_h)$ and the inconsistency indicator obtained by SET-Step 4 are given in Table IV.

Categories $C_h \rightarrow$ Assignment probas $a_i \downarrow$	C_1	C_2	C_3	$\emptyset$
$P(a_i \rightarrow C_h)$	0.24	0.56	0.14	$\delta_i = 0.06$

Table IV: SET Soft Assignment result.

IV. APPLICATION EXAMPLE : ENVIRONMENTAL CONTEXT

In this section, we compare ET and SET methods applied to an assignment problem related to an environmental context proposed originally in [8]. It corresponds to the choice of the location of an urban waste resource recovery disposal which aims to re-use the recyclable part of urban waste produced by several communities. Indeed, this disposal must collect at least $20000 m^3$ of urban waste per year to be economically viable. It must be a collective unit and the best possible location has to be identified. Each community will have to bring its urban waste production to the disposal: the transport costs are valuated in tons by kilometer per year ($t.km/year$). Building such a disposal is generally not easily accepted by population, particularly when the environmental inconveniences are already high. This initial environmental status is measured by a specific criterion. Building an urban waste disposal implies to use a wide area that could be used for other activities such as a sport terrain, touristic equipments, a natural zone, etc. This competition with other activities is measured by a specific criterion.

A. Alternatives, criteria and profiles definition

In our example, 7 possible locations (alternatives/choices) a_i, $i = 1, 2, \ldots, 7$, for urban waste resource recovery disposal are compared according to the following 5 criteria g_j, $j = 1, 2, \ldots, 5$:

g_1 = Terrain price (decreasing preference);

g_2 = Transport costs (decreasing preference);

g_3 = Environment status (increasing preference);

g_4 = Impacted population (increasing preference);

g_5 = Competition activities (increasing preference).

- Price of terrain (g_1) is expressed in $\text{\euro}/m^2$ with decreasing preferences (the lower is the price, the higher is the preference);
- Transport costs (g_2) are expressed in $t.km/year$ with decreasing preferences (the lower is the cost, the higher is the preference);
- The environment status (g_3) corresponds to the initial environmental inconvenience level expressed by population with an increasing direction of preferences. The higher is the environment status, the lower are the initial environmental inconveniences. It is rated with an integer between 0 and 10 (highest environment status corresponding to the lowest initial environmental inconveniences);
- Impacted population (g_4) is an integrated criterion to measure negative effects based on subjective and qualitative criteria. It corresponds to the status of the environment with an increasing direction of preferences. The

higher is the evaluation, the lower are the negative effects. It is rated with an real number between 0 (great number of impacted people) and 10 (very few people impacted);

- Activities competition (g_5) is an integrated criterion, evaluated by a real number, that measures the competition level between activities with an increasing direction of preferences. The higher is the evaluation, the lower is the competition with other activities on the planned location (tourism, sport, natural environment ...).

The evaluations of the 7 alternatives are summarized in Table V, and he alternatives (possible locations) are compared to the 2 decision profiles b_1 and b_2 described in Table VI. The weights, indifference, preference and veto thresholds for criteria g_j are described in Table VII.

Criteria $g_j \rightarrow$ Choices $a_i \downarrow$	g_1 ($\text{\euro}/m^2$)	g_2 ($t \cdot km/\text{year}$)
a_1	-120	-284
a_2	-150	-269
a_3	-100	-413
a_4	-60	-596
a_5	-30	-1321
a_6	-80	-734
a_7	-45	-982

(a) Choices a_i and criteria g_1 and g_2.

Criteria $g_j \rightarrow$ Choices $a_i \downarrow$	g_3 $\{0, 1, \ldots, 10\}$	g_4 $[0, 10]$	g_5 $\{0, 1, \ldots, 100\}$
a_1	5	3.5	18
a_2	2	4.5	24
a_3	4	5.5	17
a_4	6	8.0	20
a_5	8	7.5	16
a_6	5	4.0	21
a_7	7	8.5	13

(b) Choices a_i and criteria g_3, g_4 and g_5.

Table V: Inputs of ET (7 alternatives according to 5 criteria).

Profiles $b_h \rightarrow$ Criteria $g_j \downarrow$	b_1	b_2
g_1: $\text{\euro}/m^2$	-100	-50
g_2: $t \cdot km/\text{year}$	-1000	-500
g_3: $\{0, 1, \ldots, 10\}$	4	7
g_4: $[0, 10]$	4	7
g_5: $\{0, 1, \ldots, 100\}$	15	20

Table VI: Evaluation profiles.

Thresholds $\rightarrow$ Criteria $g_j \downarrow$	w_j (weight)	q_j (indifference)	p_j (preference)	v_j (veto)
g_1: $\text{\euro}/m^2$	0.25	15	40	100
g_2: $t \cdot km/\text{year}$	0.45	80	350	850
g_3: $\{0, 1, \ldots, 10\}$	0.10	1	3	5
g_4: $[0, 10]$	0.12	0.5	3.5	4.5
g_5: $\{0, 1, \ldots, 100\}$	0.08	1	5	8

Table VII: Thresholds.

B. Results of classical ELECTRE TRI

After applying ET-Steps 1 and 3 of the classical ET method described in Section II with a $\lambda = 0.75$ for the λ-cut strategy, one gets the outranking relations listed in Table VIII.

The final hard assignments obtained by ET method using the pessimistic and optimistic attitudes are listed in Table IX.

C. Results of the new Soft ELECTRE TRI

After applying SET-Steps 1 and 3 of the SET method[8] described in Section III, one gets the probabilities of soft outrankings listed in Table X.

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[8]We have used here the PCR5 fusion rule with importance discounting [16], and a sampling technique to compute the probabilities P_{ih}.

	b_0	b_1	b_2	b_3
a_1	$>$	$>$	$<$	$<$
a_2	$>$	R	R	$<$
a_3	$>$	$>$	$<$	$<$
a_4	$>$	$>$	I	$<$
a_5	$>$	R	$<$	$<$
a_6	$>$	$>$	$<$	$<$
a_7	$>$	$>$	$<$	$<$

Table VIII: Outranking relations obtained with ET ($\lambda = 0.75$).

	C_1	C_2	C_3
a_1	0	1	0
a_2	1	0	0
a_3	0	1	0
a_4	0	1	0
a_5	1	0	0
a_6	0	1	0
a_7	0	1	0

(a) Pessimistic attitude.

	C_1	C_2	C_3
a_1	0	1	0
a_2	0	0	1
a_3	0	1	0
a_4	0	0	1
a_5	0	1	0
a_6	0	1	0
a_7	0	1	0

(b) Optimistic attitude.

Table IX: Hard assignments obtained with ET ($\lambda = 0.75$).

Profiles $b_h \rightarrow$ Outranking probas $\downarrow$	b_0	b_1	b_2	b_3
P_{1h}	1	0.9858	0.6211	0
P_{2h}	1	0.8908	0.1812	0
P_{3h}	1	0.9999	0.0570	0
P_{4h}	1	1.0000	0.0807	0
P_{5h}	1	0.2142	0.0145	0
P_{6h}	1	0.9996	0.0006	0
P_{7h}	1	0.9975	0.0106	0

Table X: Probabilities of soft outranking relations by SET.

The final soft assignments obtained by the SET method are listed in Table XI.

	C_1	C_2	C_3	$\emptyset$
a_1	0.0054	0.3735	0.6123	$\delta_1 = 0.0088$
a_2	0.0894	0.7294	0.1614	$\delta_2 = 0.0198$
a_3	0.0001	0.9429	0.0570	$\delta_3 = 0$
a_4	0	0.9193	0.0807	$\delta_4 = 0$
a_5	0.7744	0.2111	0.0031	$\delta_5 = 0.0114$
a_6	0.0004	0.9990	0.0006	$\delta_6 = 0$
a_7	0.0025	0.9869	0.0106	$\delta_7 = 0$

Table XI: SET Soft Assignment matrix $[P(a_i \rightarrow C_h)]$.

D. Discussion

From Table XI, we can get a hard assignment solution (if needed) by assigning each alternative to the category corresponding to the maximum of $P(a_i \rightarrow C_h)$, $h = 1, 2, \ldots$. With SET, it is also theoretically possible to "assign" a_i to none category if δ_i (inconsistency level) is too high. The soft assignments for a_i, $i = 3, \ldots 7$ (see Tables IX,XI) are compatible with the hard assignments with the pessimistic or the optimistic attitudes. In fact, only the soft assignments for a_1 and a_2 having the highest probabilities $P(a_1 \rightarrow C_3) = 0.6123$ and $P(a_2 \rightarrow C_2) = 0.7294$ appear incompatible with ET hard assignments (pessimistic or optimistic). The discrepancy

between these soft and hard assignments solutions is not due to SET method but comes from the arbitrary choice of the level of the λ-cut strategy used in ET method. Another arbitrary choice of λ-cut will generate different ET hard assignments which can in fact become fully compatible with SET soft assignments. For example, if one takes $\lambda = 0.5$, it can be verified that SET soft assignments are now compatible with ET hard assignments for all alternatives in this example. The soft assignments approach of SET is interesting since it doesn't depend on λ values even if the influence of both sigmoids parameters definition, choice of fusion rule, probabilisation method . . . could be further studied.

V. CONCLUSIONS

A new outranking sorting method, called Soft ELECTRE TRI (SET), inspired from the classical ELECTRE TRI and based on beliefs functions and advanced fusion techniques is proposed. SET method uses the same inputs as ET (same criteria and thresholds definitions) but in a more effective way and provides a soft (probabilized) assignment solution. SET eliminates the inherent problem of classical ET due to the arbitrary choice of a λ-cut strategy which forces to adopt either a pessimistic or optimistic attitude for the final hard assignment of alternatives to categories. The interest of SET over ET method is demonstrated on a preexisting environmental context scenario.

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